International Institute for
Applied Systems Analysis
IIASA www.iiasa.ac.at

Chances of Survival in a Chaotic Environment

Yashin, A.I.

IIASA Working Paper

WP-83-100

October 1983

CHANCES OF SURVIVAL IN A CHAOTIC ENVIRONMENT

A.I. Yashin

October 1983
WP-83-100

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
2361 Laxenburg, Austria

FOREWORD

The Core Concepts group of the System and Decision Sciences Area is concerned with the study of fundamental systems concepts, one of which is *heterogeneity*. Much systems work mistakenly treats populations as homogeneous, disregarding the fact that different elements of the population often react in different ways to the same set of conditions.

In this paper, Anatoli Yashin examines mathematically how individual differences in frailty ("susceptibility" to death), defined as a quadratic function of the environmental factors, affect the mortality rates in a population. He goes on to show how our chances of survival depend on the extent of our knowledge about the processes affecting death.

Andrzej Wierzbicki
Chairman
Systems and Decision Sciences

CHANCES OF SURVIVAL IN A CHAOTIC ENVIRONMENT

A.I. Yashin

1. Introduction

When there are unexpected changes in crucial social, economic, or physical variables, the natural human response is to look for and try to analyze the factors responsible for the change. For example, sociologists and psychologists studying human behavior try to understand the motivation mechanisms that cause people to change their place of work, their place of residence, or their life-style. Health-care managers try to find environmental, social and economic factors influencing the incidence of particular diseases or causes of death, and use them to explain changes in the disease spectrum. Ecologists attempt to link industrial development with ecological changes. Geologists try to identify environmental factors which could warn them of earthquakes or volcanic eruptions. Engineers attempt to find the particular conditions responsible for the failure of components. Economists may study the social and political environment hoping to find an explanation of structural changes in the economy.

The resulting investigations involve the collection of a wide variety of data associated not only with the primary event itself (the unexpected change) but also with possible related processes. This additional information provides a more detailed description of hazard rates, thus increasing our knowledge of the chances of a certain event occurring under various circumstances.

The number and variety of factors and processes influencing a given phenomenon are often such that it is impossible to take all of them into account or to exercise any control over their individual behavior. Thus, in these multicausal cases a description of the combined effects of all of the related processes is sometimes more helpful in understanding the mechanisms generating unexpected change than descriptions of the individual effects.

The probabilistic laws of large numbers and limit theorems provide a formal basis for these "macrodescriptions"; different formulations may be used under different conditions. Some forms of the limit theorems produce random variables or stochastic processes with Gaussian distributions; experience has shown that in many situations this Gaussian approximation of the uncertainties is justified.

In this paper we will concentrate on the concept of frailty introduced and studied in [1,2]. We will assume that the influence of many external factors on the changes (transitions from one state to another) experienced by individuals may be represented by a Gaussian random variable or a Gaussian stochastic process. Frailty will be defined as a quadratic function of the environmental factors. We will consider here one particular change: the transition from life to death. As before, differences in frailty will imply individual differences in "susceptibility" to death under specified circumstances. We will also assume that the process is monitored by an observer whose aim is to evaluate the age-specific mortality rates for the observable cohort.

Making some additional assumptions, it can be shown that the conditional distribution of the unobservable environmental parameters or processes is also Gaussian. This situation recalls the well-known generalization of the Kalman filter scheme [3,4,5,6]. A similar problem was studied in [7]; in this case the mortality rate was assumed to be influenced by the values of some randomly evolving physiological factors. The purpose of this paper is to show how our chances of survival depend on the level of our knowledge about the processes affecting death.

2. Evaluation of Mortality Rates

Assume that the frailty of an individual can be described in terms of a random variable $\mathbf{Z} = Y^2$ where Y is a Gaussian random variable with mean m_0 and variance γ_0. Let $\sigma(\mathbf{Z})$ be a σ-algebra in Ω generated by the random variable $\mathbf{Z}$. Denote by $F(t,\mathbf{Z}) = \mathbf{P}(\mathbf{T} \le t \mid \sigma(\mathbf{Z}))$ the $\sigma(\mathbf{Z})$-conditional distribution function of termination times $\mathbf{T}$. Assume that $F(t,\mathbf{Z})$ has the form

$$F(t,\mathbf{Z}) = 1 - e^{-\mathbf{Z}\int_0^t \lambda(u)du}$$

where $\lambda(t)$, $t \ge 0$, may be interpreted in some applications as the age-specific mortality rate for an average individual [2]. Using $\bar{\lambda}(t)$ to denote the observed age-specific mortality rate determining the nonconditional distribution $\bar{F}(t)$ of death times $\mathbf{T}$, we have [2]

$$\bar{\lambda}(t) = \bar{\mathbf{Z}}(t)\,\lambda(t)$$

where

$$\bar{\mathbf{Z}}(t) = \mathbf{E}(\mathbf{Z} \mid \mathbf{T} > t)$$

is the conditional mathematical expectation of $\mathbf{Z}$ given the event $\{\mathbf{T} > t\}$.

The form of the $\bar{\lambda}(t)$ depends on the conditional distribution of frailty $\mathbf{Z}$. It turns out that if the frailty $\mathbf{Z}$ is generated by Gaussian random variable Y, it is possible to calculate the conditional distribution of Y and find an analytical form for $\bar{\mathbf{Z}}(t)$. Moreover, this conditional distribution is Gaussian, as shown by the following theorem.

Theorem 1. *Let* $\mathbf{Z} = Y^2$, *where* Y *is a Gaussian random variable with mean* a *and variance* σ^2. *Then the conditional distribution of* Y *given the event* $\{\mathbf{T} > t\}$ *is also Gaussian, with a mean* m_t *and variance* γ_t *that satisfy the equations*

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} = -2\,\lambda(t)\, m_t\, \gamma_t, \quad m_0 = a \tag{1}$$

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = -2\,\lambda(t)\, \gamma_t^2, \quad \gamma_0 = \sigma^2 \quad . \tag{2}$$

Proof. From Bayes' rule the conditional density of random variable Y may be represented in the form:

$$\mathbf{P}(x \mid \mathbf{T} > t) = \frac{h(x)\, \mathbf{P}(\mathbf{T} > t \mid Y = x)}{\mathbf{P}(\mathbf{T} > t)}$$

where (from the definitions of $\mathbf{Z}$ and $\mathbf{T}$)

$$h(x) = \frac{1}{(2\pi\gamma_0)^{1/2}} e^{-\frac{(x-m_0)^2}{2\gamma_0}}$$

$$\mathbf{P}(\mathbf{T} > t \mid x) = e^{-x^2 \int_0^t \lambda(u)\mathrm{d}u}$$

and

$$\mathbf{P}(x \mid \mathrm{T} > t) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{P}(Y \leq x \mid \mathrm{T} > t) \quad .$$

Substituting the formulas for $h(x)$ and $\mathbf{P}(\mathrm{T} > t)$ into the equation for $\mathbf{P}(x \mid \mathrm{T} > t)$ leads to

$$\mathbf{P}(x \mid \mathrm{T} > t) = f(t)\, \mathrm{e}^{-\frac{[x(2\sigma^2 K(t)+1) - a]^2}{2\sigma^2(2\sigma^2 K(t)+1)}}$$

where

$$K(t) = \int_0^t \lambda(u)\mathrm{d}u$$

and $f(t)$ is some function which does not depend on x and which acts as a normalizing factor. It is evident that this formulation of the conditional density $\mathbf{P}(x \mid \mathrm{T} > t)$ corresponds to a Gaussian distribution with $a/[2\sigma^2 K(t)+1]$ and $\sigma^2/[2\sigma^2 K(t)+1]$ as mean and variance, respectively. Substituting these values for m_t and γ_t, it is not difficult to check that they satisfy the equations given in the theorem.

Assume now that the environment evolves over time. Denote by $Y(t)$, $t \geq 0$, the continuous time process describing the evolution of the random environmental factors.

Let process $Y(t)$, $t \geq 0$, satisfy the linear stochastic differential equation

$$\mathrm{d}Y(t) = a_0(t) + a_1(t)Y(t)\,\mathrm{d}t + b(t)\,\mathrm{d}w(t)\,,\ Y(0) = Y_0 \tag{3}$$

where Y_0 is a Gaussian random variable with mean m_0 and variance γ_0, $w(t)$ is an H-adapted Wiener process, $\mathrm{H} = (H_t)_{t\geq 0}$ is some nondecreasing right continuous family of σ-algebras, and H_0 is completed by sets of $\mathbf{P}$-zero measure from $H = H_\infty$. Denote by H^y the family of σ-algebras in Ω generated by the values of the random process $Y(u)$:

$$\mathbf{H}^y = (H_t^y)_{t \geq 0}, \quad H_t^y = \bigcap_{u>t} \sigma \{ Y(v), \quad v \leq u \} \quad .$$

Assume that process $Y(t)$ determines the rate of occurrence of some unexpected event, characterized by the random time of occurrence $\mathbf{T}$:

$$\mathbf{P} (\mathbf{T} > t \mid H_t^y) = e^{-\int_0^t Y^2(u) \lambda (u) \, du} \quad . \tag{4}$$

Notice that process $\mathbf{Z}(u) = Y^2(u)$, $u \geq 0$, may be interpreted as frailty changing stochastically over time. Using the terminology of the martingale theory one could say that the process

$$A(t) = \int_0^{t \Delta \mathbf{T}} \lambda(u) \, Y^2(u) \, du$$

is an $\mathbf{H}^y$-predictable compensator of the life-cycle process

$$X_t = \mathbf{I} (\mathbf{T} < t), \quad t \geq 0 \quad .$$

This means that the process

$$M_t = \mathbf{I} (\mathbf{T} < t) - A(t), \quad t \geq 0$$

is an $\mathbf{H}^y$-adapted martingale. If the termination time $\mathbf{T}$ is viewed as the time of death the process $Y^2(t)$, $t \geq 0$, may be regarded as the age-specific mortality rate for an individual with history $Y_0^t = \{ Y(u), \quad 0 \leq u \leq t \}$.

Letting $\bar{\lambda}(t)$, $t \geq 0$, denote the observed age-specific mortality rate we have [2]:

$$\bar{\lambda}(t) = \lambda(t) \, \bar{\mathbf{Z}}(t), \quad t \geq 0 \quad ,$$

where

$$\bar{\mathbf{Z}}(t) = \mathbf{E} \{ Y^2(t) \mid \mathbf{T} > t \} \quad .$$

In order to calculate the observed mortality rate $\bar{\lambda}(t)$, $t \geq 0$, it is necessary to compute the second moment of the conditional distribution of the $Y(u)$ given $\{ \mathbf{T} > t \}$. It turns out that this moment may be calculated easily using the result postulated in the next theorem.

Theorem 2. *Assume that process $Y(t)$ and termination time $\mathbf{T}$ are related by formulas (3) and (4). Then the conditional distribution of $Y(t)$ given $\{\mathbf{T} > t\}$ is Gaussian. The mean m_t and variance γ_t of this distribution are given by the following equations:*

$$\frac{dm_t}{dt} = a_0(t) + a_1(t)\, m_t - 2m_t \gamma_t \lambda(t) , \quad m_0 \quad , \tag{5}$$

$$\frac{d\gamma_t}{dt} = 2a_1(t)\, \gamma_t + b^2(t) - 2\lambda(t)\, \gamma_t^2 , \quad \gamma_0 \quad . \tag{6}$$

The formula for $\bar{\lambda}(t)$ is then

$$\bar{\lambda}(t) = \lambda(t)\, (\, m_t^2 + \gamma_t \,) \quad .$$

The proof of this theorem is given in the Appendix.

3. Population Structure in a Random Environment

Assume that a population may be represented as a collection of several groups of individuals (defined on the basis of sex, ethnic group, etc.). Introduce a random variable U taking a finite number of possible values $(1,2,\ldots,K)$ with a priori probabilities $p_1,p_2,\ldots,p_K$. Let the age-specific mortality rate of the average individual depend on the value of the random variable U; each value of U is associated with a particular population group. In this case the survival probability of a person from group U with a history H_t^Y of environmental or physiological characteristics up to time t may be written as follows:

$$\mathbf{P}(\,\mathbf{T} > t \mid H_t^Y \vee \sigma(U)) = e^{-\int_0^t Y^2(u)\lambda(U,u)\, du} \quad .$$

If the observer takes into account the differences between people belonging to different population groups he should produce K different patterns of age-specific mortality rates $\bar{\lambda}(i,t)$, $i = \overline{1,K}$. These mortality rates correspond to the conditional survival probabilities

$$\mathbf{P}(\mathbf{T} > t \mid U = i) = e^{-\int_0^t \bar{\lambda}(i,u)\,du}, \quad i = \overline{1,K} .$$

In order to calculate $\bar{\lambda}(i,t)$, $i = \overline{1,K}$ it is necessary to have K different estimations $m_t(i)$, $\gamma_t(i)$ that are solutions of the following equations:

$$\frac{dm_t(i)}{dt} = a_0(t) + a_1(t)\, m_t(i) - 2m_t(i)\, \gamma_t(i)\, \lambda(i,t), \quad m_0(i), \quad i = \overline{1,K}$$

$$\frac{d\gamma_t(i)}{dt} = 2a_1(t)\, \gamma_t(i) + b^2(t) - 2\lambda(i,t)\, \gamma_t^2(i), \quad \gamma_0(i), \quad i = \overline{1,K} .$$

The formula for $\bar{\lambda}(i,t)$ is

$$\bar{\lambda}(i,t) = \lambda(i,t)\,(\, m_t^2(i) + \gamma_t(i)\,), \quad i = \overline{1,K} .$$

Note that the evolution of the environmental or physiological factors may also be dependent on the population group. In this situation we have K different processes influencing the mortality rates in K population groups:

$$dY_i(t) = a_0(i,t) + a_1(i,t)\, Y_i(t)\, dt + b\,(i,t)\, dw_i(t), \quad Y_i(0) = Y_{i,0}$$

where the $Y_{i,0}$ are Gaussian random variables with means $m_0(i)$ and variances $\gamma_0(i)$, and the $w_i(t)$ are H-adapted Wiener processes. The formula for $\bar{\lambda}(i,t)$ will be the same as before, but the equations for $m_t(i)$ and $\gamma_t(i)$ will contain different parameters $a_0(i,t)$, $a_1(i,t)$, $b(i,t)$:

$$\frac{dm_t(i)}{dt} = a_0(i,t) + a_1(i,t)\, m_t(i) - 2m_t(i)\, \gamma_t(i)\, \lambda(i,t), \quad m_0(i), \quad i = \overline{1,K}$$

$$\frac{d\gamma_t(i)}{dt} = 2a_1(i,t)\, \gamma_t(i) + b^2(i,t) - 2\lambda(i,t)\, \gamma_t^2(i), \quad \gamma_0(i), \quad i = \overline{1,K} .$$

If the observer does not differentiate between people from different groups, the observed age-specific mortality rate $\bar{\lambda}(t)$ will depend on the proportion $\pi_i(t)$, $i = \overline{1,K}$, of individuals in the different groups. These proportions coincide with the conditional probabilities of the events $\{ U = i \}$, $i = \overline{1,K}$, given $\{ \mathbf{T} > t \}$, and can be shown to satisfy the following equations:

$$\pi_j(t) = \pi_j(0) + \int_0^t \pi_j(u) \left(\bar{\lambda}(j,u) - \sum_{i=1}^{K} \bar{\lambda}(i,u)\, \pi_i(u) \right) du \quad .$$

In this case $\bar{\lambda}(t)$ may be represented as follows:

$$\bar{\lambda}(t) = \sum_{i=1}^{K} \bar{\lambda}(i,t)\, \pi_i(t) \quad .$$

Appendix: Proof of Theorem 2

Introduce the conditional characteristic function $f_t(\alpha)$ defined as follows:

$$f_t(\alpha) = \mathbf{E}\,(e^{i\alpha Y(t)} \mid \mathbf{T} > t), \quad t \ge 0 \quad .$$

According to Bayes' rule, this can be approximated by

$$f_t(\alpha) = \mathbf{E}'\,(e^{i\alpha Y(t)}\, \varphi(t))$$

where

$$\varphi(t) = e^{-\int_0^t \lambda(u)\, (Y^2(u) - \overline{Y^2(u)})\, du} \quad ,$$

$$\overline{Y^2(u)} = \mathbf{E}\,(Y^2(u) \mid \mathbf{T} > u) \quad ,$$

and $\mathbf{E}'$ denotes the mathematical expectation with respect to the marginal probability measure corresponding to the Wiener process $W(t)$.

Before proceeding further we must recall Ito's differential rule [3], which is summarized in the following lemma.

Lemma. *Let* $w(t)$ *be an* H-*adapted Wiener process and* $A(t)$ *and* $B(t)$ *be* H-*adapted random functions such that*

$$\int_0^t |A(u)| \, du < \infty \qquad \text{P-a.s.}$$

$$\int_0^t B^2(u) \, du < \infty \qquad \text{P-a.s.}$$

Define the process X_t *by the equality*

$$X_t = X_0 + \int_0^t A(u) \, du + \int_0^t B(u) \, dw(u)$$

where X_0 *is some integrable and* H_0*-measurable random variable. Let function* $F(x,t)$ *be twice (continuously) differentiable in the first variable* x *and once (continuously) differentiable in the second variable* t. *Then* $F(X_t,t)$ *may be represented as follows:*

$$F(X_t,t) = F(X_0,0) + \int_0^t F_t'(X_u,u) \, du + \int_0^t F_x'(X_u,u) \, A(u) \, du$$
$$+ \int_0^t F_x'(X_u,u) \, B(u) \, dw(u) + \frac{1}{2}\int_0^t F_{x,x}''(X_u,u) \, B^2(u) \, du \quad .$$

Using this result we have for the product $e^{i\alpha Y(t)}\varphi(t)$:

$$e^{i\alpha Y(t)}\varphi(t) = e^{i\alpha Y(0)} + \int_0^t (i\alpha) \, e^{i\alpha Y(u)} \, \varphi(u) \, (\alpha_0(u) + \alpha_1(u) \, Y(u))du$$
$$+ \int_0^t (i\alpha) \, e^{i\alpha Y(u)} \, \varphi(u) \, b(u)dw_u - \frac{\alpha^2}{2}\int_0^t e^{i\alpha Y(u)} \, \varphi(u) \, b^2(u)du$$
$$+ \int_0^t e^{i\alpha Y(u)} \, \varphi(u) \, \lambda(u)(Y^2(u) - \overline{Y^2(u)})du \quad .$$

Taking the mathematical expectation of both sides of this equality leads to

$$f_t(\alpha) = f_0(\alpha) + i\alpha\int_0^t a_0(u)\, f_u(\alpha)\,\mathrm{d}u + i\alpha\int_0^t a_1(u)\,\mathbf{E}[\, e^{i\alpha Y(u)}\varphi(u)\, Y(u)\,]\,\mathrm{d}u$$
$$- \frac{\alpha^2}{2}\int_0^t b^2(u)\, f_u(\alpha)\,\mathrm{d}u - \int_0^t \lambda(u)\,\mathbf{E}[\, e^{i\alpha Y(u)}\,\varphi(u)\, Y^2(u)\,]\,\mathrm{d}u$$
$$+ \int_0^t f_u(\alpha)\,\overline{Y^2(u)}\,\lambda(u)\,\mathrm{d}u \quad .$$

Notice that $f_0(\alpha)$ has the form:

$$f_0(\alpha) = e^{i\alpha m_0 - \frac{1}{2}\alpha^2\gamma_0} \quad .$$

This particular form and the equation for $f_t(\alpha)$ suggest that we should search for an $f_t(\alpha)$ in the same form:

$$f_t(\alpha) = e^{i\alpha m_t - \frac{1}{2}\alpha^2\gamma_t} \tag{A.1}$$

where m_t and γ_t satisfy ordinary differential equations

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} = g(t), \quad m_0 \tag{A.2}$$

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = G(t), \quad \gamma_0 \quad . \tag{A.3}$$

(We assume that the equations for m_t and γ_t have unique solutions.) The functions $g(t)$ and $G(t)$ can be found from the equation for $f_t(\alpha)$ as follows. First, note that the following equalities hold:

$$f_t' = \mathbf{E}\,(\, i\, e^{i\alpha Y(t)}\,\varphi(t)\; Y(t))$$

$$f_t'' = -\mathbf{E}\,(\, e^{i\alpha Y(t)}\,\varphi(t)\; Y^2(t))$$

where f_t' and f_t'' denote the first and second derivatives, respectively, of the function $f_t(\alpha)$ with respect to α .

Applying these formulas to the equation for $f_t(\alpha)$ we obtain (omitting the dependence of f_t on α for simplicity):

$$f_t = f_0 + i\alpha\int_0^t a_0(u)\, f_u \mathrm{d}u + \alpha\int_0^t f_u' a_1(u)\mathrm{d}u$$
$$-\frac{\alpha^2}{2}\int_0^t f_u\, b^2(u)\mathrm{d}u + \int_0^t \lambda(u)\, f_u''\, \mathrm{d}u + \int_0^t \lambda(u)\, \overline{Y^2(u)}\, f_u \mathrm{d}u \quad .$$

Derivatives f_t' and f_t'' may be calculated from equation (A.1):

$$f_t' = f_t(im_t - \alpha\gamma_t)$$

$$f_t'' = f_t(im_t - \alpha\gamma_t)^2 - f_t\gamma_t \quad .$$

Substituting these derivatives into the equation for $f_t(\alpha)$, differentiating with respect to t and using equations (A.2) and (A.3) for m_t and γ_t we obtain:

$$f_t\ [\, i\alpha\, g(t) - \frac{1}{2}\alpha_2\, G(t)\,] = i\alpha a_0(t)\, f_t + \alpha f_t(\, im_t - \alpha\gamma_t\,) a_1(t)$$
$$-\frac{\alpha^2}{2} f_t b^2(t) + \lambda(t)\, f_t[(im_t - \alpha\gamma_t)^2 - \gamma_t] + \lambda(t)\, f_t(\gamma_t + m_t^2) \quad .$$

Taking the real and imaginary parts of this equation yields

$$g(t) = a_0(t) + a_1(t)m_t - 2\lambda(t)m_t\gamma_t \tag{A.4}$$

$$G(t) = 2a_1(t)\gamma_t + b^2(t) - 2\lambda(t)\,\gamma_t^2 \tag{A.5}$$

which lead to the equations for m_t and γ_t described in the theorem.

Notice that the above form of the $f_t(\alpha)$ is equivalent to the Gaussian law for conditional distribution of the $Y(t)$ given the event $\{\mathbf{T} > t\}$.

We now have to show that equation (A.3) with $G(t)$ given by (A.5) has a unique solution. To do this we first assume that $\gamma_{1,t}$ and $\gamma_{2,t}$ are two different solutions of equation (A.3). We then have

$$|\gamma_{1,t} - \gamma_{2,t}| \le 2\int_0^t a_1(u)\,|\gamma_{1,u} - \gamma_{2,u}|\,\mathrm{d}u \tag{A.6}$$

$$+ 2\int_0^t \lambda(u)(\gamma_{1,u} - \gamma_{2,u})|\gamma_{1,u} - \gamma_{2,u}|\,du \quad .$$

Let $r(u)$ denote the function

$$r(u) = 2[a_1(u) + \lambda(u)(\gamma_{1,u} + \gamma_{2,u})] \quad .$$

Then inequality (A.6) may be rewritten in the form:

$$|\gamma_{1,t} - \gamma_{2,t}| \leq \int_0^t r(u)|\gamma_{1,u} - \gamma_{2,u}|\,du \quad .$$

The Grenuolli-Bellman lemma shows that $\gamma_{1,t}$ and $\gamma_{2,t}$ must coincide, and this completes the proof.

REFERENCES

1. J.W. Vaupel, K. Manton, and E. Stallard, "The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality," *Demography* 16, pp. 439-454 (1979).

2. J.W. Vaupel and A.I. Yashin, *The Deviant Dynamics of Death in Heterogeneous Populations*, RR-83-1, International Institute for Applied Systems Analysis, Laxenburg, Austria (1982).

3. R.S. Liptzer and A.N. Shiryaev, *Statistics of Random Processes*, Springer-Verlag, Berlin and New York (1978).

4. R.S. Liptzer, "Gaussian Martingales and Generalization of the Kalman-Bucy Filter", *Theory of Probability and Applications* (in Russian), 20(2), pp.34-45 (1975).

5. A.I. Yashin, "Conditional Gaussian Estimation of Characteristics of the Dynamic Stochastic Systems," *Automatic and Remote Control*, (5), pp. 57-67 (1980).

6. A.I. Yashin, "A New Proof and New Results in Conditional Gaussian Estimation Procedures", *Proceedings of the 6th European Meeting on Cybernetics and Systems Research*, April 1982, North-Holland Publishing Company (1982).

7. M.A. Woodbury and K.G. Manton, "A Random Walk Model of Human Mortality and Aging", *Theoretical Population Biology*, 11(1), pp.37-48 (1977).