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Transient and Asymptotic Behavior of a Random-Utility Based Stochastic Search Process in Continuous Space and Time

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TRANSIENT AND ASYMPTOTIC BEHAVIOR OF A RANDOM-UTILITY BASED STOCHASTIC SEARCH PROCESS IN CONTINUOUS SPACE AND TIME*

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Contributions to the Metropolitan Study:6

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LIST OF CONTRIBUTIONS TO THE METROPOLITAN STUDY

1. Anas, A., and L.S. Duann (1983) Dynamic Forecasting of Travel Demand. Collaborative Paper, CP-83-45, International Institute for Applied Systems Analysis, Laxenburg, Austria.

2. Casti, J. (1983) Emergent Novelty, Dynamical Systems and the Modeling of Spatial Processes. Research Report, (forthcoming), International Institute for Applied Systems Analysis, Laxenburg, Austria.

3. Lesse, P.F. (1983) The Statistical Dynamics of Socio-Economic Systems. Collaborative Paper CP-83-51, International Institute for Applied Systems Analysis, Laxenburg, Austria.

4. Haag, G., and W. Weidlich (1983) An Evaluable Theory for a Class of Migration Problems. Collaborative Paper (forthcoming), International Institute for Applied Systems Analysis, Laxenburg, Austria.

5. Nijkamp, P., and U. Schubert (1983) Structural Change in Urban Systems. Collaborative Paper (forthcoming), International Institute for Applied Systems Analysis, Laxenburg, Austria.

6. Leonardi, G. (1983) Transient and Asymptotic Behavior of a Random-Utility Based Stochastic Search Process in Continuous Space and Time. Working Paper (forthcoming) International Institute for Applied Systems Analysis, Laxenburg, Austria.

FOREWORD

Contributions to the Metropolitan Study:6

The project "Nested Dynamics of Metropolitan Processes and Policies" was initiated by the Regional and Urban Development Group in 1982, and the work on this collaborative study started in 1983. The series of contributions to the study is a means of conveying information between the collaborators in the network of the project.

This paper examines search and choice behavior of individual agents in an environment in which alternatives become available stochastically. Such a process may, for example, relate households searching for dwellings, individuals searching for workplaces, shopping and service centers, etc. The results are based on asymptotic properties of maxima of sequences of random variables, and hold under comparatively weak assumptions. Altogether the paper enlarges the platform which is common to various model specifications that are used by different groups in the Metropolitan Study.

In particular, both preference maximizing and satisfying behavior are shown to be asymptotically equivalent. Moreover, the asymptotic (and average) properties of the search process are described by a logit-type model, which on an aggregate level, is formally equivalent to solutions obtained through entropy-maximizing procedures. Hence, in a certain sense the latter type of solutions may also be conceived as being compatible with the search process described. Finally, the paper illustrates a fruitful approach to the problem of determining consistent rules for aggregating processes defined on the micro-level to aggregate descriptions of such processes.

Börje Johansson
Acting Leader
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November, 1983

ABSTRACT

The paper explores the properties of some simple search and choice behaviors, by exploiting the asymptotic properties of maxima of sequences of random variables. Heterogeneity in the preference is introduced by means of additive random utilities, and the actor is assumed to choose points in a plane region, by sampling them according to a stochastic process. It is shown that asymptotic convergence to a Logit model holds under considerably weaker assumptions than those commonly found in the literature to justify it. This asymptotic property is treated in details for utility-maximizing behavior, and outlined for satisfycing behavior. The asymptotic equivalance of the two behaviors suggests that progress in widening the family of asymptotically Logit-equivalance behavors can be made with further research.

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TRANSIENT AND ASYMPTOTIC BEHAVIOR OF A RANDOM-UTILITY BASED STOCHASTIC SEARCH PROCESS IN CONTINUOUS SPACE AND TIME

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

This paper is a follow-up to a previous one (Leonardi, 1982) addressing the general problem of weakening the disaggregate assumptions giving rise to Logit models. The specific assumption weakened here (as in the previous paper) is the form of the random utility distribution. It is shown how, by replacing the usually assumed Gumbel distribution with the broader family of distributions having an asymptotically constant hazard rate, the Logit model arises quite naturally as an asymptotic approximation to a suitably defined search behavior.

The general method outlined in Leonardi (1982) is as follows:

i) the choice behavior is formulated as a search, with sampling from the set of alternatives, evaluating the sampled alternatives, choosing the best one,

ii) the search process is formulated as a stochastic process,

iii) the limiting behavior of the stochastic search process is analyzed by using the asymptotic theory of extremes (Galambos, 1978).

While in Leonardi (1982) the above method was applied to a discrete choice space-discrete time process, here the theory is extended to cover the case of a continuous choice space-continuous time process. Namely, the *environment* in which the choice is made consists of a plane region (the *choice space*), which may be thought of as a geographic one, a *deterministic utility* evaluation, associating a value to each point in the region, and a density of alternatives, specified by a *probability measure* defined on the plane region. The *choice behavior* for a given actor in the above environment is described by a *probability distribution* for the random part of the utility evaluation and by a *stochastic point process* (a Poisson process is used in the paper), giving the distribution of the sample size as a function of the time spent in the search. The mechanics of the search are simple: at each random point in time the actor draws a point of the region with probability proportional to the density of alternatives at that point, evaluates the point by drawing a random utility term and adding it to the deterministic one, compares this value with those obtained from the previously generated random points, and eventually updates the best alternative found so far.

The above search model is defined in section 2 and 3, and its general transient properties are derived in section 4 and 5. In section 6, the main limiting results are obtained by letting the search time go to infinity. The main result of section 6 is theorem 6.2, stating the asymptotic convergence of the above search process to a Logit model.

In section 7 a further extension is outlined, showing how similar asymptotic results can be obtained by replacing utility-maximizing with satisfycing behavior, and suggesting that a broader family of Logit-convergent micro-behaviors can be found.

2. THE ENVIRONMENT

Let the following objects be given:

$\Gamma \subseteq \mathbb{R}^2$ — a (usually bounded) subset of the plane; it defines the *choice space*, and each $r \in \Gamma$ is a possible choice;

W — a probability measure on Γ; it defines the *density of alternatives*;

$\Lambda = [a,b] \subseteq \mathbb{R}$ — an interval on the real line, $a \geq \infty$, $b < \infty$; this is the *deterministic utility* space;

$v : \Gamma \rightarrow \Lambda$ — a function which maps each $r \in \Gamma$ into a $v(r) \in \Lambda$; $v(r)$ is the *deterministic utility* of $r \in \Gamma$;

$L_2(\Gamma, W) = \{g : \int_\Gamma g^2 dW < \infty\}$ — the Hilbert space of square-integrable functions $g : \Gamma \rightarrow \mathbb{R}$, in the measure W; the norm and scalar product in L_2 are defined as:

$$\| g \| = \left(\int_\Gamma g^2 dW\right)^{1/2} \quad , \quad \forall g \in L_2$$

$$(g,f) = \int_\Gamma g\, f\, dW \quad , \quad \forall g, f \in L_2$$

L_2^* — the conjugate space of L_2, i.e. the set of all continuous linear operators on L_2; if $g^* \in L_2^*$, the value of g^* applied at $g \in L_2$ is denoted by $< g, g^* >$.

The following proposition (stated without proof) and definition will be useful.

<u>Proposition 2.1</u> (Riesz isomorphism theorem for Hilbert spaces): There is one and only one $\bar{g} \in L_2$ such that $< g, g^* > = (g, \bar{g}) = \int_\Gamma g \bar{g} \, dW$, $\forall g^* \in {L_2}^*$.

Proposition 2.1 (a classic result in functional analysis) states an isomorphism between L_2^* and L_2, such that each linear operator $g^* \in L_2^*$ can be represented by a $\bar{g} \in L_2$, and its application to a $g \in L_2$ can be represented by a scalar product $(g, \bar{g})$.

<u>Definition 2.1</u> (Gateaux derivative):
Let $H : L_2 \to \mathbb{R}$ be a functional on L_2 such that, for $g \in L_2$

$$\lim_{\lambda \to 0} \frac{H(g+\lambda f) - H(g)}{\lambda} = < f, u^* >, \quad \forall f \ L_2$$

Let further $H'(g) \in L_2$ be such that

$< f, u^* > = (f, H'(g))$ according to the isomorphism stated in proposition 1,

then $H'(g)$ is called the *Gateaux derivative* of H at g.

<u>Note 2.1</u> Due to the Riesz isomorphism, u^* and $H'(g)$ can be interchangeably called the Gateaux derivative of H. Choosing $H'(g)$ is a matter od convenience, since it makes an explicit representation of u^* available. In the case where $H(g + \lambda f)$ is differentiable with respect to λ, one calls $H'(g)$ the Gateaux derivative of H at g if:

$$\frac{d}{d\lambda} H(g + \lambda f) \Big|_{\lambda = 0} = (f, H'(g)) \quad . \tag{1}$$

<u>Note 2.2</u> It is easily checked that $v \in L_2$. Indeed:

$$\int_\Gamma v^2 dW < b^2 < \infty$$

3. THE SEARCH AND CHOICE BEHAVIOR

Let the search and choice behavior be defined by the following objects and assumptions:

$F(x)$ — a probability distribution on $\mathbb{R}$; this is the *random utility distribution.* The density $F'(x)$ is assumed to exist for all $x \in \mathbb{R}$,

$R_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ — a Poisson process with intensity λ; $R_n(t)$ is the probability that n trials are made in a time interval $[o,t)$,

<u>Assumption 3.1</u> (The sampling process) at each trial in $[o,t)$ an $r \in \Gamma$ is drawn according to the measure W, i.e. $Pr\{r \in A\} = W(A)$, $\forall A \subseteq \Gamma$

<u>Assumption 3.2</u> (The evaluation) any drawn alternative $r \in \Gamma$ is given a utility $\tilde{u} = v(r) + \tilde{\theta}$ where $\tilde{\theta}$ is a random variable with distribution $F(x)$ (the random utilities are thus *independent identically distributed*--i.i.d.--at each trial). The distribution of $\tilde{u}$ is given by:

$$G(x) = P_r\{\tilde{u} < x\} = \int_{r \in \Gamma} F[x - v(r)]\,dW(r)$$

<u>Assumption 3.3</u> (Utility maximizing behavior) the actor is assumed *utility maximizer*, i.e., if after n trials he has drawn alternatives $r_1, \ldots, r_k, \ldots, r_n$, with utilities $\tilde{u}_1, \ldots, \tilde{u}_k, \ldots, \tilde{u}_n$, he chooses an alternative i, $1 \leq i \leq n$, such that

$$\tilde{u}_i = \max_{1 \leq k \leq n} \tilde{u}_k$$

<u>Note 3.1</u> In order to keep the choice unique, it can be assumed that when there are two or more maxima, the one drawn at the earliest trial is kept. This assumption is not

really needed, as long as the probability distribution involved are smooth enough to consider the occurrence of a tie an event of zero measure.

4. THE DISCRETE SEARCH AND CHOICE PROCESS

The search and choice behavior as a function of the number of trials is now analyzed. Define:

$\tilde{x}_n = \max\limits_{1 \leq k \leq n} \tilde{u}_k$	the utility of the best alternative found after n trials
$\tilde{s}_n \in \Gamma$	the best alternative found after n trials

The process is described by the following three objects:

$Q_n(x) = \Pr\{\tilde{x}_n < x\}$	the distribution of the maximum utility found after n trials
$P_n(r)$	the density of choice after n trials; it is defined in such a way that $\Pr\{\tilde{s}_n \in A\} = \int_{r \in A} P_n(r)\,dW(r)$, $\forall A \subseteq \Gamma$ (2)
$V_n(v) = \int_{-\infty}^{\infty} x\,dQ_n(x)$	the expected utility after n trials; $V_n(v)$ will be regarded as a functional mapping v into a real number, i.e. $V_n : L_2 \rightarrow \mathbb{R}$

A closed form for $Q_n(x)$ and $P_n(r)$ is given in the following proposition:

<u>Proposition 4.1</u> for $n < \infty$

$$Q_n(x) = G^n(x) \tag{3}$$

$$P_n(r) = n \int_{-\infty}^{\infty} G^{n-1}(x)\,dF[x-v(r)] \tag{4}$$

Proof. To prove equation (3), note that the occurrence of the event

$$\tilde{x}_n < x$$

is equivalent to the joint occurrence of the events

$$\tilde{u}_1 < x,\ldots,\tilde{u}_n < x$$

and since the $\tilde{u}_k$, k=1,...,n are i.i.d. random variables with distribution G(x), equation (3) follows.

To prove equation (4), the event $\tilde{s}_n \in A$ can occur if, and only if, an $r \in A$ has been drawn at some trial, with a utility greater than the ones found in all other trials. The probability that this occurs for some trial k = 1,...,n is

$$\int_{r\in A} \int_{-\infty}^{\infty} G^{n-1}(x)\ dF[x-v(r)]\ dW(r)$$

and multiplying this by the number of trials n yields:

$$P_r\{\tilde{s}_n \in A\} = \int_{r\in A} \{\, n \int_{-\infty}^{\infty} G^{n-1}(x)\ dF[x-v(r)]\} \ dW(r)$$

and comparison with equation (2) yields equation (4). Q.E.D.

An important property of $V_n(v)$ is stated in the following proposition:

Proposition 4.2 for $n < \infty$

$$P_n(r) \in L_2 \tag{5}$$

$$P_n(r) = V_n'(v) \tag{6}$$

Proof. To prove that $P_n(r)$ is square integrable, one uses the Cauchy inequality in the following form:

$$\{n\int_{-\infty}^{\infty} G^{n-1}(x)\ dF[x-v(r)]\}^2 \leqslant n^2\int_{-\infty}^{\infty} G^{2(n-1)}(x)\ dF[x-v(r)]$$

From this, the definition of $G(x)$ and equation (4) it follows:

$$\int_{r\in\Gamma} P_n^2(r)\ dW(r) \leqslant n^2\int_{-\infty}^{\infty} G^{2(n-1)}(x)\ dG(x) = \frac{n^2}{2n-1} < \infty$$

To prove equation (6), from the definition of $V_n(v)$ and $G(x)$ it follows for all $f\in L_2$:

$$\frac{d}{d\lambda}V_n(v+\lambda f) = -\int_{-\infty}^{\infty} xd\{nG^{n-1}(x)\int_{r\in\Gamma} F'[x-v(r)-\lambda f(r)]f(r)dW(r)\}$$

and by using the rule of integration by parts and substituting from equation (4):

$$\frac{d}{d\lambda}V_n(v+\lambda f)\Big|_{\lambda=0} = \int_{r\in\Gamma}\{n\int_{-\infty}^{\infty} G^{n-1}(x)dF[x-v(r)]\}f(r)dW(r) =$$
$$= (f, P_n)$$

Comparison of this result with equation (1) establishes equation (6). Q.E.D.

Note 4.1 Equation (6) is the continuous-space counterpart of the *integrability conditions* property, already known for random utility models in a discrete choice space, and extensively discussed in Williams (1977), Ben Akiva and Lerman (1979), Daly (1979), Leonardi (1981, 1982).

In the general economic theory of demand, these are known as the *Hotelling necessary conditions* for the existence of a consumer surplus function (Hotelling, 1938). Equation (6) can thus be restated by saying that $V_n(v)$ is the *consumer surplus* associated with the *demand function* $P_n(r)$.

5. THE CONTINUOUS TIME PROCESS

In analogy with the discrete search process, the behavior of the continuous time process is analyzed by means of the following three objects:

$$Q(x,t) = \sum_{n=0}^{\infty} Q_n(x)\, R_n(t)$$ the distribution of the maximum utility found in a time interval $[0,t)$,

$$P(r,t) = \sum_{n=0}^{\infty} P_n(r)\, R_n(t)$$ the density of choice for a time interval $[0,t)$,

$$V(v,t) = \sum_{n=0}^{\infty} V_n(v)\, R_n(t)$$ the expected utility for a time interval $[0,t)$.

Use of equations (3) and (4) and easy calculations yield the following proposition, stated without proof:

<u>Proposition 5.1</u> for $t < \infty$

$$Q(x,t) = e^{-\lambda t[1-G(x)]} \tag{7}$$

$$P(r,t) = \lambda t \int_{-\infty}^{\infty} e^{-\lambda t[1-G(x)]} dF[x-v(r)] = \int_{-\infty}^{\infty} \frac{F'[x-v(r)]}{G'(x)}\, dQ(x,t) \tag{8}$$

<u>Note 5.1</u> For $t < \infty$, $Q(x,v)$ is not a proper distribution. Indeed

$$Q(-\infty, t) = e^{-\lambda t} = R_0(t) > 0 \quad .$$

A similar comment applies to the choice density, since

$$\int_{r\in\Gamma} P(r,t)\ dW(r) = 1 - e^{-\lambda t} < 1 \quad .$$

This is because, in a finite time interval [0,t), there is always a non-zero probability that *no alternative* is drawn, and therefore no choice is made.

From propositions 4.2 and 5.1, the following proposition follows, whose proof is obvious.

Proposition 5.2. For $t < \infty$

$$V(v,t) = \lambda t \int_{-\infty}^{\infty} x e^{-\lambda t[1-G(x)]} dG(x) \tag{9}$$

$$P(r,t) \in L_2 \tag{10}$$

$$P(r,t) = V'(v,t) \quad . \tag{11}$$

6. SOME ASYMPTOTIC RESULTS

The purpose of this section is to explore the behavior of the continuous time process as $t \to \infty$. The following additional definition and assumption will be used.

Definition 6.1

$\rho(x) = \dfrac{F'(x)}{1-F(x)} = \dfrac{-d}{dx} \log\ [1-F(x)]$ is the *hazard rate* of the distribution $F(x)$.

Assumption 6.1

$$\lim_{x\to\infty} \rho(x) = \beta, 0 < \beta < \infty$$

Note 6.1 Assumption 6.1 implies the property:

$$\lim_{x\to\infty} \frac{1-F(x+y)}{1-F(x)} = e^{-\beta y} \quad , \quad -\infty < y < \infty \tag{12}$$

Indeed, it is true (and easily checked from definition 6.1) that

$$1 - F(x+y) = [1-F(x)] \exp [-\int_x^{x+y} \rho(z)dz] =$$

$$= [1-F(x)] \exp [-\int_0^{y} \rho(z+x)dz] \quad . \qquad (13)$$

Applying the mean value theorem for integrals, there is some $\xi \in [0,y)$ for which

$$\int_0^y \rho(z+x)dz = y\,\rho\,(\xi+x) \qquad (14)$$

Replacing the estimate (14) on the right-hand side of (13) yields

$$1 - F(x+y) = [1-F(x)]\, e^{-y\rho\,(\xi+x)}$$

or

$$\frac{1 - F(x+y)}{1 - F(x)} = e^{-y\rho\,(\xi+x)}$$

and, taking the limit as $x \to \infty$,

$$\lim_{x\to\infty} \rho\,(\xi+x) = \beta$$

$$\lim_{x\to\infty} \frac{1 - F(x+y)}{1 - F(x)} = e^{-\beta y} \quad .$$

The asymptotic results which follow make use of the following objects:

$$\phi = \int_{r\in\Gamma} e^{\beta v(r)}\, dW(r) \qquad (15)$$

$$\psi = \frac{1}{\beta} \log \phi \qquad (16)$$

a(t) root of the equation

$$1 - F[a(t)] = 1/\lambda t \tag{17}$$

or

$$a(t) = F^{-1}(1-1/\lambda t) \tag{18}$$

Note 6.2. ϕ and ψ can be regarded as functionals on L_2, i.e.:

$$\phi : L_2 \to \mathbb{R} \quad , \quad \psi : L_2 \to \mathbb{R}$$

In this case, their value for a specific $v \in L_2$ will be denoted by $\phi(v)$ and $\psi(v)$. From equation (18) it follows that:

$$\lim_{t\to\infty} a(t) = F^{-1}(1) = \infty \quad . \tag{19}$$

Theorem 6.1. (Asymptotic form of the maximum utility distribution). Under assumption 6.1,

$$\lim_{t\to\infty} Q[a(t) + \psi + x,t] = \exp(-e^{-\beta x}), \ -\infty < x < \infty$$

Proof. Due to definitions (15) and (16) and properties (12) and (19):

$$\lim_{t\to\infty} \frac{1 - F[a(t) + \psi + x - v(r)]}{1 - F[a(t)]} =$$

$$= e^{-\beta[\psi + x - v(r)]} = e^{-\beta x} \frac{e^{\beta v(r)}}{\phi}$$

From this and equation (17) it follows:

$$\lim_{t\to\infty} \lambda t\{1-F[a(t) + \psi + x - v(r)]\} = e^{-\beta x} \frac{e^{\beta v(r)}}{\phi} \tag{20}$$

This and definition (15) imply:

$$\lim_{t\to\infty} \lambda t\{1-G[a(t) + \psi + x]\} = \frac{e^{-\beta x}}{\phi} \int_{r\in\Gamma} e^{\beta v(r)} dW(r) = e^{-\beta x} \qquad (21)$$

Result (21) and equation (7) finally yield

$$\lim_{t\to\infty} Q[a(t) + \psi + x,t] = \exp(-e^{-\beta x}) \quad \underline{\text{Q.E.D.}}$$

<u>Theorem 6.2</u> (Asymptotic form of the choice density). Under assumption 6.1,

$$\lim_{t\to\infty} P(r,t) = \frac{e^{\beta v(r)}}{\phi}$$

<u>Proof</u>. From definition 6.1:

$$F'(x) = \rho(x)[1-F(x)]$$

On the other hand, it is of course true that

$$\frac{F'[x-v(r)]}{G[(x)} = \frac{\lambda t F'[x-v(r)]}{\lambda t G'(x)}$$

Therefore, the right-hand side of (8) can be written as:

$$\int_{-\infty}^{\infty} \frac{\rho[a(t) + \psi + x - v(r)]\lambda t\{1-F[a(t) + \psi + x-v(r)]\}}{\int_{r\in\Gamma} \rho[a(t) + \psi + x - v(r)]\lambda t\{1-F[a(t) + \psi + x-v(r)]\}dW(r)} dQ[a(t)+\psi+x,t]$$

Now taking the limit for $t \to \infty$ and using assumption 6.1, definition (15), theorem 6.1 and equation (20):

$$\lim_{t\to\infty} P(r,t) = \int_{-\infty}^{\infty} \frac{\beta e^{-\beta x} e^{\beta v(r)}}{\beta e^{-\beta x} \int_{r\in\Gamma} e^{\beta v(r)} dW(r)} d[\exp(-e^{-\beta x})] =$$

$$= \frac{e^{\beta v(r)}}{\phi} \int_{-\infty}^{\infty} d[\exp(-e^{-\beta x})] = \frac{e^{\beta v(r)}}{\phi} \quad \underline{\text{Q.E.D.}}$$

Corollary 6.1 (Asymptotic form of the expected utility).

Theorem 6.1 implies:

$$\lim_{t\to\infty} [V(v,t) - a(t)] = \psi(v) + \gamma/\beta$$

where γ is Euler's constant.

Proof. By definition

$$V(v,t) = \sum_{n=0}^{\infty} V_n(v) R_n(t) = \int_{-\infty}^{\infty} x d[\sum_{n=0}^{\infty} Q_n(x,t) R_n(t)] = \int_{-\infty}^{\infty} x dQ(x,t)$$

Now changing variables of integration

$$V(v,t) = \int_{-\infty}^{\infty} [a(t) + \psi + x]\, dQ\, [a(t) + \psi + x,t]$$

and using theorem 6.1.

$$\lim_{t\to\infty} [V(v,t) - a(t)] = \psi + \int_{-\infty}^{\infty} x d\, [\exp(-e^{-\beta x})] + \lim_{t\to\infty} [a(t) - a(t)] = \psi + \gamma/\beta$$

Q.E.D.

Corollary 6.2 (Asymptotic validity of the relationship between expected utility and choice density).

Theorems 6.1 and 6.2 imply:

$$\lim_{t\to\infty} P(r,t) \in L_2$$

$$\lim_{t\to\infty} P(r,t) = \psi'(v)$$

Proof. From theorem 6.2:

$$\int_{r\in\Gamma} \left[\frac{e^{\beta v(r)}}{\phi}\right]^2 dW(r) \leq \frac{e^{2\beta b}}{\phi^2} < \infty$$

hence the asymptotic choice density is square integrable. From definitions (15) and (16).

$$\psi(v+\lambda f) = \frac{1}{\beta} \log \int_{r\in\Gamma} e^{\beta v(r)} e^{\lambda\beta f(r)} dW(r)$$

therefore:

$$\frac{d}{d\lambda} \psi(v+\lambda f) = \frac{\int_{r\in\Gamma} e^{\beta v(r)} e^{\lambda\beta f(r)} f(r)\, dW(r)}{\int_{r\in\Gamma} e^{\beta v(r)} e^{\lambda\beta f(r)} dW(r)}$$

and

$$\frac{d}{d\lambda} \psi(v+\lambda f)\Big|_{\lambda=0} = \int_{r\in\Gamma} \frac{e^{\beta v(r)}}{\phi} f(r)\, dW(r) = (f, \frac{e^{\beta v}}{\phi})$$ Q.E.D.

Discussion of theorems 6.1 and 6.2 and corollaries 6.1 and 6.2.

The above results extend to continuous space and continuous time the results proved for discrete space and discrete time in Leonardi (1982). The main result is theorem 6.2, stating that the choice density is asymtotically approximated by a Logit form, even with no specific assumption on the form of F(x). The crucial assumption used here is assumption 6.1, which is much weaker than the one commonly used to derive a Logit model, namely, $F(x) = \exp(-e^{-\beta x})$, a Gumbel extreme value distribution (see Domencich and McFadden, 1975, for instance). Actually, the Gumbel distribution appears in theorem 6.1, but as an *asymptotic result*, not as an assumption. Since the family of distributions satisfying assumption 6.1 is very broad, a Logit-type choice behavior can be expected to be produced by a wide variety of random-utility evaluation processes. It might also be observed that there is a formal equivalence between the results obtained here and Boltzman Statistical Mechanics. Indeed, theorem 6.2 defines a *Boltzman Distribution*, ϕ can be identified with the *Partition Function* of statistical mechanics, while ψ can be identified with the *Thermodynamic Potential* (up to a multiplicative constant). This statistical mechanics analogy is developed in Leonardi (1977), although it must be stressed that it is based on totally different assumptions than the random-utility ones.

7. A NOTE ON THE ASYMPTOTIC EQUIVALENCE BETWEEN UTILITY MAXIMIZING AND SATISFYCING BEHAVIOR

It has been shown in the previous section how the Logit choice model is obtained as an asymptotic approximation to a utility maximizing choice over a plane region, when the random utility distribution satisfies assumption 6.1. It is interesting to explore to what extent the utility maximizing assumption is crucial to this result, by comparing it with other popular behavioral assumptions. Here the comparison with the so-called "satisfycing behavior" assumption will be outlined, and it will be shown that, under suitable conditions, the two behaviors are asymptotically indistinguishable.

Let assumptions 3.1 and 3.2 be kept, but assumption 3.3 be replaced by:

Assumption 7.1 (satisfycing behavior) the actor is assumed *satisfycer*, i.e., a real number y (threshold utility or aspiration level) exists such that, if after n trials he has drawn alternatives $r_1,\ldots,r_k,\ldots,r_n$, with utilities $\tilde{u}_1,\ldots,\tilde{u}_k,\ldots,\tilde{u}_n$, the search stops if and only if

$$\tilde{u}_n \geq y$$

$$\tilde{u}_k < y \quad , \qquad k = 1,\ldots,n-1 \qquad .$$

In other words, a choice is made as soon as an alternative whose utility exceeds the threshold y is found.

The search behavior considered in assumption 7.1 is a somewhat simplified version of more general satisfycing models, where the threshold y might itself be changing during the search. However, the results to be derived do not depend that much on the detailed mechanism for updating y, provided it can be assumed that it is, or becomes in the long run, large.

On a purely intuitive ground, it is clear that, whenever the search stops, the alternative which is chosen (the last one) has the highest utility among all these of the sample generated so far. Therefore, if the threshold y becomes large enough, it becomes unlikely that the search stops soon, and the sample size to be generated is likely to become large. One is thus led again to make probability statements on the maximum over a large sequence of random variables, and an asymptotic equivalence between utility maximizing for n (or t) $\rightarrow \infty$ and satisfycing for $y \rightarrow \infty$ can be expected. This conjecture is proved rigorously in the next theorems.

Define:

P(r,y)	The density of choice when a threshold utility y is used.

A closed form for P(r,y) is given in the following theorem:

Theorem 7.1 for $y < \infty$

$$P(r,y) = \frac{1-F[y-v(r)]}{1-G(y)} \tag{22}$$

Proof. Let $\tilde{s}\in\Gamma$ be the chosen alternative. The event $\tilde{s}\in A\subseteq\Gamma$ can occur after n trials if and only if an $r\in A$ has been drawn at the nth trial, with a utility greater than or equal to y, while alternatives have been drawn in the previous n-1 trials, with utilities less than y. Using the distribution G(r) defined in assumption 3.2, the probability of this event is:

$$\int_{r\in A} \{1-F[y-v(r)]\}\, G^{n-1}(y)\, dW(r) \quad . \tag{23}$$

Summation of (23) over $n = 1,\ldots,\infty$ yields:

$$\Pr\{\tilde{S}\in A\} = \int_{r\in A} \{1-F[y-v(r)]\}\left[\sum_{n=0}^{\infty} G^{n}(y)\right] dW(r) =$$

$$= \int_{r\in A} \frac{1-F[y-v(r)]}{1-G(y)}\, dW(r)$$

and this establishes equation (22). Q.E.D.

The asymptotic result is straightforward:

Theorem 7.2. (Asymptotic form of the choice density).
Under assumption 6.1.

$$\lim_{y\to\infty} P(r,y) = \frac{e^{\beta v(r)}}{\phi}$$

Proof. Using the definition of G(x), equation (22) can be rewritten as

$$P(r,y) = \frac{\dfrac{1-F[y-v(r)]}{1-F(y)}}{\displaystyle\int_{r\in\Gamma} \frac{1-F[y-v(r)]}{1-F(y)}\, dW(r)}$$

and assumption 6.1, used in the form specified by equation (12) of note 6.1, yields immediately:

$$\lim_{y\to\infty} P(r,y) = \frac{e^{\beta v(r)}}{\displaystyle\int_{r\in\Gamma} e^{\beta v(r)} dW(r)}$$

which, due to definition (15), establishes the theorem. Q.E.D.

The result in theorem 7.2 is identical to that in theorem 6.2, although the proof is different and, in a sense, simpler and more elegant. The issue raised by the results in this section, that is the asymptotic equivalence of different behavioral assumptions, is worth further research developments, and of course, is not exausted by the relatively simplified examples given here. What is pointed out is the possibility of obtaining stable asymptotic results not only by changing or generalizing specific assumptions within the same behavioral structure (the issue explored in section 6) but even by changing or generalizing the behavioral structure itself.

8. CONCLUDING REMARKS

A considerable difference of opinions exists on the theoretical underpinnings of logit-type models. A broad two-fold classification can be made dividing them in two: the disaggregate school (typically represented by Domencich and McFadden, 1975, or Ben-Akiva and Lerman, 1979, although rooted in the work of Luce, 1959, and Manski, 1973), which would insist on justifying such models on very specific micro-level behavioral assumptions, as well as very detailed functional and parametric specifications of the underlying probabilistic structure; and the aggregate approaches, the statistical mechanics analogue (entropy maximizing) being the most popular one (Wilson, 1970, and more recently, Lesse, 1983, for some very interesting theoretical developments), and the "cost-efficiency" principle of Smith (1978) being its macro-economic counterpart.

Loosely speaking, while the first school stresses the dependency of choice patterns on the specific behavioral assumptions, the second one stresses what seems to be the opposite, that is the relative insensitivity of choice patterns observed at the aggregate level from specific behaviors at the disaggregate level.

This paper is a contribution towards reconciliation of the two extremes, although the point of view adopted here is quite different from both. Rather than overspecifying micro-behavioral assumptions, or neglecting them at all, it has been shown how a wide, but still micro-economically sensible, family of behaviors can be mapped into a single asymptotic model.

The concept of "asymptotic" is proposed here as a replacement for the more restrictive "aggregate". A key argument to the derivation of the results, both in sections 6 and 7, has been identifying some quantity in the system which becomes large (the sample size, the time spent in search, or the threshold utility level).

The results of section 7 are particularly important for future developments, since they suggest that the family of micro-level behaviors generating the same asymptotic model can be considerably broader than what one obtains by just generalizing some functional forms. In other words, different *decision criteria*, and not only different probability distributions, may lead to the same asymptotic model.

Two final notes concern space and dynamics. The role of space, in the geographic sense, does not appear as crucial in the results of this paper. However, it should be stressed that the continuous nature of the choice set *is* crucial to the derivations (and indeed it corrects some otherwise artificial results in Leonardi, 1982), and geographic space is perhaps the only known phenomenon in nature whose description in continuous terms is not a mathematical artifact.

What is really missing in the geography implied in the paper is the effect of space on the state of knowledge of the actor in the choice process. This knowledge is summarized by a measure W, which does not change over time. It is plausible to think of a learning mechanism which updates W as alternatives over the region are explored; it is also plausible to think of the metric of space (distance) playing a role in determining or constraining the learning mechanism. One future development should therefore be to produce asymptotic results, similar to those given here, for choice processes including a suitably general family of geography-dependent learning mechanisms on the choice set.

As for dynamics, a comparison between utility maximizing and satisfycing behavior is interesting. Although both behaviors lead to the same asymptotic form, they imply a different economy of description when used in a dynamic framework. The utility maximizing behaviors needs to keep track of the previously found maximum utility, in order to make the next move. The satisfycing behavior does not need to keep track of any information on previously tested alternatives, since, if the search is still going on, by definition they where below the threshold level, and any newly drawn alternative need not be compared with them, but only with the threshold.

The satisfycing assumption seems therefore superior, in terms of economy of description, to the utility maximizing one, since it leads to a simple Markovian structure in the choice dynamics, without the need to expand the description of the state of the system by including utility distributions in it. The use of this property as a simplifying device is under study for applications to housing and labor mobility, and the models proposed in Leonardi (1983a,b,c) are under revision from this point of view.

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