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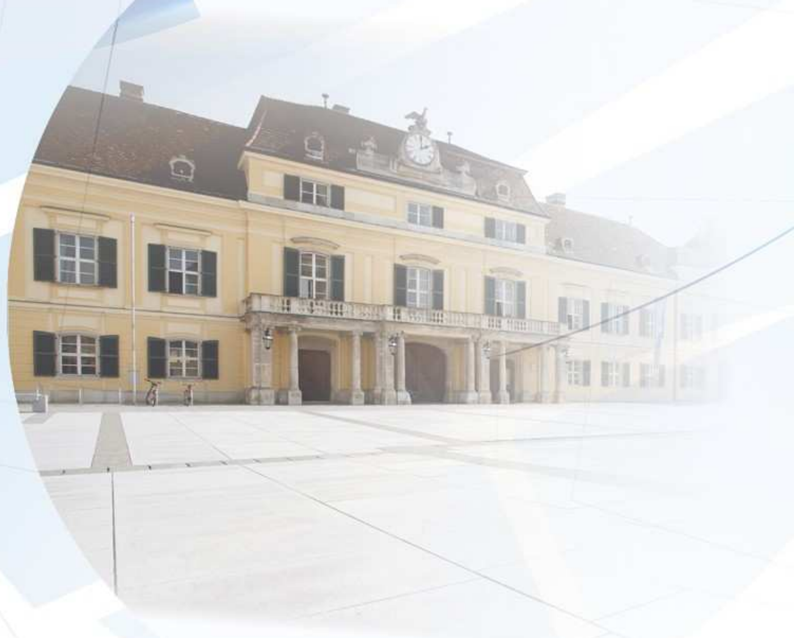

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UNCERTAINTY AND MULTIPLE OBJECTIVES IN STORAGE CONTROL PROBLEMS (A MIN-MAX APPROACH)

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PREFACE

Analysis concerned with problems of the rational use of natural resources almost invariably deals with uncertainties with regard to the future behavoiur of the system in question and with multiple objectives reflecting conflicting goals of the users of the resources. Uncertainty means that the information available is not sufficient to unambiguously predict the future of the system, and the multiplicity of the objectives, on the other hand, calls for establishing rational trade-offs among them. The rationality of the trade-offs is quite often of subjective nature and can not be formally incorporated into mathematical models supporting the analysis, and the information with regard to the future may vary with time. Then the challenge to the analyst is to elaborate a mathematical and computer implemented system that can be used to perform the analysis recognizing both the above aspects of real world problems.

These were the issues addressed during the summer study "Real-Time Forecast versus Real-Time Management of Hydrosystems", organized by the Resources and Environment Area of IIASA in 1981. The general line of research was the elaboration of new approaches to analyzing reservoir regulation problems and to estimating the value of the information reducing the uncertainties. Computationally, the research was based on the hydrosystem of Lake Como, Northern Italy. This paper starts a short series of IIASA publications based upon the results obtained during the study. It describes the theoretical background for the new max-min approach to analysing multiobjective problems with uncertainties and outlines briefly some applications of the approach to water reservoir regulation problems.

Janusz Kindler, Area Chairman
Resources and Environment Area

CONTENTS

1. INTRODUCTION

A storage system may be considered a buffer between an uncontrolled supply of a commodity and a demand for this commodity expressing the needs of various users. The main goal of such systems is the *satisfaction of the demand* performed by releasing suitable amounts of the commodity stored in the system when the supply was abundant. Obviously, the demand can be better satisfied if the storage capacity is unlimited, or, equivalently, if there are no constraints on the storage peaks. However, large capacities are usually quite costly so that a second characteristic goal of such control systems is often the *attenuation of storage peaks.*

A typical example of a storage control system is a regulated lake in which the runoff of the upstream catchment is stored and used later to satisfy agricultural, municipal, and industrial water demands of the downstream users while avoiding too large floods at the lake site. Problems of this type have long since been analyzed (see, for instance, [1]) and many applications can be found in the fields of economics, natural resources management, and industrial engineering. In the majority of the cases uncertain supplies and/or demands are modeled (with more or less adequate justification) by stochastic processes and the objectives are expressed in terms of expected values of suitable indicators. On the contrary, in this paper we suggest a max-min approach aimed at determining control laws that guarantee a given degree of satisfaction of the goals in the case of the worst possible pattern of supplies. The essential feature of the method is that the information available to the decision-maker (DM) to model the future behaviour of the system is given in the form of explicitly described sets of sequences of supplies.

The basic idea of the approach suggested in this paper has been developed at the Computing Center of the USSR Academy of Sciences and implemented there for flood protection problems in the multireservoir Volga River System. More recently, the method has been successfully applied to the multiobjective control problem of Lake Como in Northern Italy (see [4]).

The paper is organized in sections as follows. In the next section the basic characteristics of the problem are described with emphasis on the information structure. Section 3 deals with the problem of demand satisfaction, and Section 4 - with a similar problem of attenuation of storage peaks. Then the results obtained in Section. 3, 4 are used in Section 5 for analyzing the corresponding two-objective problem. The last section contains some comments on the application to Como lake and a few remarks on the possible extensions of the approach.

2. DESCRIPTION OF THE STORAGE CONTROL PROBLEM

In this section we describe the characteristics of the storage control problem that is dealt with in the paper. In doing this we shall sometimes be using the water management terminology although the approach suggested herein can also be applied to other types of systems.

2.1. State Equation and Control Constraints

Let us consider a storage system (water reservoir) and denote by x_t (state variable) the storage (amount of water in the reservoir) at the beginning of t-th time-interval ($t=0,1,\ldots$). We also denote by a_t the supply (the inflow of water) into the system during t-th time-interval, and by r_t (control variable) the release (of water) from the reservoir during the same interval.

Using this notation *the state equation* (mass-balance equation) of the system can be written in the form:

$$x_{t+1} = x_t + a_t - r_t, \quad t \geq 0, \tag{1}$$

where the release r_t is subject to the *the control constraint:*

$$0 \le r_t \le \mathbf{N}(x_t), \quad t \ge 0 \tag{2}$$

with $\mathbf{N}(\cdot)$ being a given function of the state variable (in the case of water reservoirs $\mathbf{N}(\cdot)$ is the so called stage-discharge function). In the following, we assume that this function is differentiable and such that

$$\mathbf{N}(0) = 0, \quad 0 \le \frac{\mathbf{dN}(x)}{\mathbf{d}t} < 1, \quad x \ge 0.$$

Note that these properties imply $\mathbf{N}(x) \le x$ for all $x \ge 0$ and this in turn, together with the nonnegativeness of the supplies and the initial storage, guarantees that $x_t \ge 0$, for any $t > 0$.

2.2. Information Structure and Control Laws

We consider a multistage decision-making process during which the decision-maker at any current instant of time t chooses a value of the corresponding control variable r_t based on the information available to him at that instant. We assume that in the course of the control process at any current instant of time t (beginning of t-th time interval) the DM knows the value x_t of the state variable and also the value a_t of the supply during t-th time interval.

We also assume that at any current instant t the DM models his knowledge with regard to possible future values of the supplies from instant $t+1$ up to infinity in the form of a set of infinite sequences of supplies. We shall denote this set as $\mathbf{A}_{t+1}^{\infty}$ and its elements as $a_{t+1}^{\infty}=(a_{t+1},a_{t+2},\ldots)$. In the sequel we shall use similar notation $\mathbf{A}_{\tau}^{t}$ $(t \ge \tau)$ for a set of finite sequences $a_{\tau}^{t}=(a_{\tau},\ldots,a_t)$.

In order to describe sets $\mathbf{A}_t^{\infty}$ we introduce a "reference set" $\mathbf{A}*_0^{T-1}$ used for "generating" the elements of sets $\mathbf{A}_t^{\infty}$. This set consists of sequences of supplies, each of length T, where T is a time period corresponding to "seasonal" fluctuations of the supplies. In water management problems, for example, the time period T is usually one year and set $\mathbf{A}*_0^{T-1}$ may contain past observations of the inflows for a number of years together with sequences generated using statistical techniques, and possibly also hypothetic sequences added into this set by the DM to make it more representative from his viewpoint. For convenience the periods (years) will be numbered with integer variable k =1,2,3,.....

Using this notation the structure of sets A_t^∞ can be described as follows. For any instant t of the form $t=(k-1)T$, (k =1,2,...) (beginning of a year) corresponding set $A_{(k-1)T}^\infty$ consists of all infinite concatenations of the sequences from "reference" set $A*_0^{T-1}$. This means essentially that at the end of any year the DM does not use any information to model the supplies of the next year except the a priori information that is contained in the "reference" set. This implies

$$A_{(k-1)T}^{kT-1} = A*_0^{T-1}, \quad A_{(k-1)T}^\infty = A_0^\infty, \quad k \geq 1 \tag{3}$$

On the other hand, we assume that the observations made by DM from the beginning of a year up to a current instant t can reduce ambiguity with regard to the future supplies *for the remaining part of the same year.* For example, in the case of lakes where the snowmelt occurs between May and July, if at time t = May 15 of year k snowmelt has already occurred, then all sequences characterized by too high values of inflows after May 15 can be discarded from reference set $A*_0^{T-1}$ before determining set A_{t+1}^{kT-1}. Therefore, we assume that for any current $k \geq 1$ and $t \in [(k-1)T, kT-1)$ set A_{t+1}^{kT-1} may contain only some of the subsequences belonging to set $A*_0^{T-1}$. In conclusion, for any $k \geq 1$ and $t \in [(k-1)T, kT-1)$ set A_{t+1}^∞ consists of all sequences obtained by concatenating a sequence of set A_{t+1}^{kT-1} with a sequence of set A_0^∞.

Assuming that set $A*_0^{T-1}$ is known to the DM prior to starting the control process, we denote by tuple

$$(t, x_t, a_t, A_{t+1}^{kT-1})$$

the information available to the DM at current instant t.

A control law is a rule for calculating a value of the control variable for any instant of time $t \in [(k-1)T, kT-1]$, $k \geq 1$ and for any tuple of information. We shall describe this rule by a function

$$r(t, x_t, a_t, A_{t+1}^{kT-1})$$

and use r as the short notation for this function and r_t as the short notation for the value of this function corresponding to a fixed value of t and the other arguments.

In the sequel we shall also denote by $\varphi(\tau, t, x, r, a_\tau^{t-1})$ the state of the system at instant t with x being the initial state at instant τ, r being the control law and a_τ^{t-1} the sequence of supplies from τ up to $t-1$. Of course, $\varphi(\tau, t, x, r, a_\tau^{t-1}) = x$ for $t = \tau$.

2.3. Goal Constraints

As already pointed out, *the goals* in controlling a storage system consist in satisfying demands for the releases (of water from the reservoir) and in attenuating storage peaks. Formally these goals can be expressed in terms of inequalities which are to be observed when choosing an appropriate control of the system. In our case these inequalities have the following form *(goal constraints):*

$$r_t \geq \alpha r^*_t, \quad t \geq 0 \qquad (4a)$$

$$x_t \leq \beta x^*_t, \quad t \geq 0 \qquad (4b)$$

with r^*_t, x^*_t being prespecified reference values for the minimal release and the maximal storage at which there are no losses or damages at time t. The coefficients α ($\alpha \leq 1$) and β ($\beta \geq 1$) are introduced to make these constraints more flexible. Clearly, the ideal case would be to find a control law which guarantees the satisfaction of the goal constraints with $\alpha=1$ and $\beta=1$; but if such a control law does not exist then constraints (4) can be relaxed by introducing some values $\alpha<1$ and/or $\beta>1$. In these cases the problem of controlling the system is of the multiobjective nature and consists in providing the greatest possible values of α (the lowest possible water deficits) and the lowest possible values of β

(the lowest possible flood damages). We assume that both r^*_t and x^*_t are periodic in the sense that

$$r^*_t = r^*_{t+T}, \quad x^*_t = x^*_{t+T} \tag{5}$$

for any $t \geq 0$, with T being the time period introduced above.

The problem considered in this paper consists in determining couples (x_0, r) of initial states and control laws which guarantee the satisfaction of constraints (2) and (4) for given values of α and β. To be more precise we shall introduce the following

Definition 1 ((α,β)-feasibility of (x_τ, r)). *Given an instant of time τ and set $\mathbf{A}_\tau^\infty$ of possible sequences of supplies a couple (x_τ, r) is called (α,β)-feasible iff storage $\varphi(\tau, t, x_\tau, r, a_\tau^{t-1})$ and the corresponding values r_τ of the control variable satisfy constraints (2) and (4) for all $t \geq \tau$ and all $a_\tau^{t-1} \in \mathbf{A}_\tau^{t-1}$.*

Similarly a couple (x_τ, r) will be called α-feasible [β-feasible] if control constraints (2) and goal constraints (4a) [(4b)] are satisfied for all $t \geq \tau$ and all possible supply sequences. Thus, a couple (x_τ, r) is (α,β)-feasible iff it is both α- and β-feasible.

In the following we determine sets of α-, β-, and (α,β)-feasible couples which will be denoted by $\mathbf{F}_\tau^\alpha$, $\mathbf{F}_\tau^\beta$ and $\mathbf{F}_\tau^{\alpha,\beta}$ respectively.

3. SATISFACTION OF DEMANDS

In this section we suggest a way for determining a set $\mathbf{F}_0^\alpha$ of α-feasible couples (x_0, r) which has the form

$$\mathbf{F}_0^\alpha = \mathbf{X}_0^\alpha \times \mathbf{R}^\alpha,$$

where

$$\mathbf{X}_\tau^\alpha = \{x_\tau \mid \textit{there exists } r \textit{ such that } (x_\tau, r) \textit{ is } \alpha\textit{-feasible}\}, \quad \tau \geq 0$$

and set $\mathbf{R}^\alpha$, which is described later, is such that if $r \in \mathbf{R}^\alpha$ then for any $x_\tau \in \mathbf{X}_\tau^\alpha$ couple (x_τ, r) is α-feasible. Therefore, set $\mathbf{F}_0^\alpha$ can be obtained by separately determining sets $\mathbf{X}_0^\alpha$ and $\mathbf{R}^\alpha$.

In what follows we shall denote by r^{min} the control law which specifies the value αr^*_t for the control variable at any current instant of time t. This control law, that we shall refer to as *minimal release policy*, will be used to determine sets $\mathbf{X}_\tau^\alpha$.

Lemma 1. *For any $\tau \geq 0$ if $x_\tau \in \mathbf{X}_\tau^\alpha$ then couple (x_τ, r^{min}) is α-feasible.*

Proof. If $x_\tau \in \mathbf{X}_\tau^\alpha$ then, by definition, there exists a control law r such that (x_τ, r) is α-feasible. Denote as r_t and x_t the corresponding values of the respective variables for some fixed sequence $a_\tau^\infty \in \mathbf{A}_\tau^\infty$. Let us also denote by r_t^{min} $(= \alpha r^*_t)$ and x_t releases and storages obtained for the same sequence of supplies by applying the minimal release policy. Then, from state equation (1) we have

$$(x_{t+1} - x_{t+1}) = (x_t - x_t) + (r_t - r_t^{min}).$$

But $x_\tau - x_\tau = 0$ and $r_t - r_t^{min} \geq 0$ since r is α-feasible, so that

$$x_t \geq x_t$$

for all $t \geq \tau$. Therefore, using the α-feasibility property of (x_τ, r) and the assumption that $\mathbf{N}(\cdot)$ is nondecreasing we have

$$r_t^{min} = \alpha r^*_t \leq r_t \leq \mathbf{N}(x_t) \leq \mathbf{N}(x_t)$$

for all $t \geq \tau$ and all sequences $a_\tau^\infty \in \mathbf{A}_\tau^\infty$, which shows that r^{min} satisfies control constraints (2) and therefore couple (x_τ, r^{min}) is α-feasible. Q.E.D.

Lemma 1 implies that set $\mathbf{X}_\tau^\alpha$ contains those and only those storages x_τ that satisfy the following conditions:

$$\alpha r^*_t \leq \mathbf{N}(x_t), \quad t \geq \tau, \tag{6a}$$

$$x_{t+1} = x_t + a_t - \alpha r^*_t, \quad t \geq \tau \tag{6b}$$

for all supply sequences $a_\tau^\infty \in \mathbf{A}_\tau^\infty$.

Another property of set $\mathbf{X}_\tau^\alpha$ implied by Lemma 1 is the following:

$$x_\tau \in \mathbf{X}_\tau^\alpha \Rightarrow x \in \mathbf{X}_\tau^\alpha, \quad x \geq x_\tau.$$

Therefore, if we introduce the value

$$x_\tau^\alpha = \min_{x \in \mathbf{X}_\tau^\alpha} x, \tag{7}$$

then set $\mathbf{X}_\tau^\alpha$ can explicitly be described as

$$\mathbf{X}_\tau^\alpha = \{x \mid x \geq x_\tau^\alpha\}. \tag{8}$$

Problems of type (7) are not practically solvable since they involve an infinite number of constraints (see (6)). However using periodicity properties (3), (5) we can conclude that

$$\mathbf{X}_{kT}^\alpha = \mathbf{X}_0^\alpha, \quad k \geq 1, \tag{9}$$

and therefore $x_{kT}^\alpha = x_0^\alpha$ for all $k \geq 1$. This fact allows the following to be proved:

Lemma 2. *For any $k \geq 1$ and any $\tau \in [(k-1)T, kT-1]$ if $x_\tau \in \mathbf{X}_\tau^\alpha$ then*

$$x_{kT} = \varphi(\tau, kT, x_\tau, r^{min}, a_\tau^{kT-1}) \geq x_0^\alpha. \tag{10}$$

for any $a_\tau^{kT-1} \in \mathbf{A}_\tau^{kT-1}$.

Proof. If we assume the contrary, then using (9) we obtain that for some $k \geq 1$, $\tau \in [(k-1)T, kT-1]$, $x_\tau \in \mathbf{X}_\tau^\alpha$ and $a_\tau^{kT-1} \in \mathbf{A}_\tau^{kT-1}$ couple $(\varphi(\tau, kT, x_\tau, r^{min}, a_\tau^{kT-1}), r^{min})$ is not α-feasible and therefore using (3) we conclude that $x_\tau \notin \mathbf{X}_\tau^\alpha$. This contradiction completes the proof.

This result shows that by adding constraint (10) to equations of type (6) we can describe set $\mathbf{X}_\tau^\alpha$ by means of a finite number of constraints. More specifically, we can formulate the following sequence of mathematical programming problems for calculating values x_τ^α, $\tau \geq 0$:

Problem-0.

$$\begin{gathered} x_0^\alpha = \min x_0 \\ \alpha r^*_t \leq \mathbf{N}(x_t), \quad t = 0, \ldots, T-1, \\ x_{t+1} = x_t + a_t - \alpha r^*_t, \quad t = 0, \ldots, T-1, \\ x_T \geq x_0, \quad a_0^{T-1} \in \mathbf{A}^{*T-1}_0. \end{gathered}$$

Problem-τ. $(\tau \in ((k-1)T, kT-1], k \geq 1)$.

$$\begin{gathered} x_\tau^\alpha = \min x_\tau \\ \alpha r^*_t \leq \mathbf{N}(x_t), \quad t = \tau, \ldots, kT-1, \end{gathered}$$

$$x_{t+1} = x_t + a_t - \alpha r^*_t, \quad t=\tau,\ldots,kT-1,$$
$$x_{kT} \geq x_0^\alpha, \quad a_\tau^{kT-1} \in \mathbf{A}_\tau^{kT-1}.$$

Notice that Problem-0 should be solved first since its solution x_0^α appears in the formulation of Problem-τ. Moreover, it is important to point out that Problem-τ (τ>0) must be solved in real-time since set $\mathbf{A}_\tau^{kT-1}$ is known only at time ($\tau-1$). If the real-time computations are not feasible (due to costs involved or some other reasons), then the reference set can be used instead of set $\mathbf{A}_\tau^{kT-1}$ and the corresponding problem may be solved prior to starting the control process. In such cases no advantage is taken of the observations obtained during the course of the process and values x_τ^α computed in this way are generally greater than those computed using the real-time information in the form of sets $\mathbf{A}_\tau^{kT-1}$.

Another property of x_τ^α (needed in Section 5) is that x_τ^α is nondecreasing with α. In fact, as is easily verified, storages x_t obtained by applying to system (1) with $x_\tau = x_\tau^\alpha$ minimal release policy $r_t^{min} = \alpha r^*_t$ with $\alpha<\alpha$, certainly satisfy all the constraints in Problem-τ with α substituted by α, and this implies $x_\tau^\alpha \leq x_\tau^\alpha$.

In practice, the determination of values x_τ^α is very simple. If the number of sequences of set $\mathbf{A}_\tau^{kT-1}$ is finite then it suffices to solve the corresponding Problem-τ for each supply sequence a_τ^{kT-1} and select the maximal of those values (the determination of each of these storages may be done by simulating the behavior of the system for different initial storages x_τ from time τ to time kT for a given supply sequence and the minimal release policy until the lowest initial storage satisfying conditions (6a) and (10) is obtained).

We can now define set $\mathbf{R}^\alpha$ of control laws which was mentioned at the beginning of this section.

Definition 2 (set $\mathbf{R}^\alpha$). *Set $\mathbf{R}^\alpha$ consists of all control laws*

$$r_t = r(t, x_t, a_t, \mathbf{A}_t^{kT-1})$$

satisfying the inequalities

$$\min\{N(x_t), \alpha r^*_t\} \le r_t \le \min\{N(x_t), \max\{x_t + a_t - x^\alpha_{t+1}, \alpha r^*_t\}\} \tag{11}$$

for all $k \ge 1$, $t \in [(k-1)T, kT-1]$, *and all* $x_o \in X^\alpha_o$.

Now we can prove the following

Theorem 1. *If a control law* $r \in R^\alpha$ *then the couple* (x_o, r) *is* α*-feasible for any* $x_o \in X^\alpha_o$.

Proof. Notice first that the right-hand side inequality in (11) implies the satisfaction of control constraint (2) while the left-hand side inequality is equivalent to goal constraint (4a) provided

$$\alpha r^*_t \le N(x_t). \tag{12}$$

Therefore, we should prove that any control law r satisfying (11) gives rise to releases r_t and storages x_t that satisfy (12) for any t, provided $x_o \in X^\alpha_o$.

At time $t=0$ the satisfaction of (12) is guaranteed by the definition of X^α_o (see Problem-0). As for the future, there are two cases: either $x_o + a_o - x^\alpha_1 \ge \alpha r^*_o$, or, on the contrary, $x_o + a_o - x^\alpha_1 < \alpha r^*_o$. In the first case, (11) implies

$$x_1 = x_o + a_o - r_o \ge x^\alpha_1,$$

i.e. $x_1 \in X^\alpha_1$, and therefore the definition of X^α_1 (see Problem-τ for τ=1) guarantees the satisfaction of (12) for t=1. In the second case, (11) implies $r_o = \alpha r^*_o$ and thus (12) for t=1 is implied by the definition of X^α_o (see Problem-0).

If at time t=1 $x_1 \in X^\alpha_1$ we can repeat the above argument and prove that (12) holds for t=2. If, on the contrary, $x_1 \notin X^\alpha_1$ we can consider the following two cases: either $x_1+a_1-x^\alpha_2 \ge \alpha r^*_1$ or $x_1+a_1-x^\alpha_2 < \alpha r^*_1$. In the first case, (11) implies $x_2=x_1+a_1-r_1 \ge x^\alpha_2$ so that the definition of X^α_1 guarantees the satisfaction of (12) at time t=2. In the latter case, (11) implies $r_1 = \alpha r^*_1$ and therefore, recalling that $r_o = \alpha r^*_o$, the satisfaction of (12) for t=2 is guaranteed by the definition of X^α_o.

By recursively using the same arguments the theorem is proved.

From Theorem 1 it follows that the multistage decision-making control process providing for the satisfaction of goal constraints (4a) consists of the following. At any instant of time the DM uses set A_{t+1}^{kT-1} to calculate value x_{t+1}^{a} by solving Problem-τ for $\tau = t+1$ and then chooses any value of r_t (release during time-interval t) satisfying inequalities (11). Of course, the initial state of the system at time $t=0$ must belong to set X_0^a obtained by solving Problem-0.

We can give a useful interpretation of this process. Let us introduce for any state x and for any nonempty set $\mathbf{X}$ distance $\mathbf{d}$ from x to set $\mathbf{X}$ as follows:

$$\mathbf{d}(x,\mathbf{X}) = \min_{y \in \mathbf{X}} |x-y|.$$

Now for any fixed $t \geq 0$ we formulate the problem:

$$\begin{aligned} &\mathbf{d}(\varphi(t,t+1,x_t,r_t,a_t),\mathbf{X}_{t+1}^{a}) \to \min_{r_t} \\ &s.t.\ \ \min\{\mathbf{N}(x_t),\alpha r^*_t\} \leq r_t \leq \mathbf{N}(x_t) \end{aligned} \qquad (13)$$

with x_t, a_t, r^*_t, α being fixed values of the respective variables.

As can be seen the set of all solutions r_t to problem (13) is described by inequalities (11). From here it follows that at any current instant t the behavior of DM is equivalent to the tendency of bringing state x_{t+1} of the system into "target" set $\mathbf{X}_{t+1}^{a}$ determined on the basis of the information currently available.

4. ATTENUATION OF STORAGE PEAKS

The problem of attenuation of storage peaks is similar to the problem of demand satisfaction described in the preceding section. Indeed, while the satisfaction of demand can be facilitated by higher initial storages ($\mathbf{X}_0^a$ is bounded from below) and realized by applying the minimal release policy, it is intuitive that the storage peaks decrease with the initial storage and are attenuated by releasing the maximal amount of the commodity $\mathbf{N}(x_t)$ at any time t. (For example, in a regulated lake the lowest possible flood is obtained by keeping the gates of the regulation dam permanently open).

In this section we are determinig a set $\mathbf{F}_0^\beta$ of β-feasible couples (x_0, r) which has the form similar to that of $\mathbf{F}_0^\alpha$ (Section 3):

$$\mathbf{F}_0^\beta = \mathbf{X}_0^\beta \times \mathbf{R}_\beta$$

and all the results obtained in Section 3 have their dual which can formally be obtained by replacing the set $\mathbf{X}_\tau^\alpha$ with the set

$$\mathbf{X}_\tau^\beta = \{x_\tau \mid \textit{there exists } r \textit{ such that } (x_\tau, r) \textit{ is } \beta\textit{-feasible} \}$$

and r^{min} with the *maximal release policy* r^{max} given by

$$r_t^{max} = r(t, x_t, a_t, \mathbf{A}_{t+1}^{kT-1}) = \mathbf{N}(x_t).$$

The dual of Lemma 1 is the following

Lemma 3. *For any $\tau \geq 0$ if $x_\tau \in \mathbf{X}_\tau^\beta$ then the couple (x_τ, r^{max}) is β-feasible.*

Proof. The proof is a direct consequence of the property $\mathbf{dN}(x)/\mathbf{d}x < 1$ which implies

$$\mathbf{N}(x_t) - \mathbf{N}(x_t) \leq x_t - x_t$$

for all $x_t \geq x_t \geq 0$. In fact, if $x_\tau \in \mathbf{X}_\tau^\beta$ then, by definition, there exists a control law r that gives rise to releases r_t and storages x_t which satisfy constraints (2) and (4b) for $t \geq \tau$. On the other hand, storages x_t and releases $r_t^{max} = \mathbf{N}(x_t)$ obtained at time t by applying the maximal release policy are such that (see eq. (1))

$$x_{t+1} - x_{t+1} = x_t - x_t + \mathbf{N}(x_t) - r_t.$$

But $r_t \leq \mathbf{N}(x_t)$ since (x_τ, r) is β-feasible; hence

$$x_{t+1} - x_{t+1} \geq x_t - x_t + \mathbf{N}(x_t) - \mathbf{N}(x_t).$$

Therefore $x_t \geq x_t$ implies $x_{t+1} \geq x_{t+1}$. Since $x_\tau = x_\tau$ we have $x_t \leq x_t \leq \beta x^*_t$ for all $t \geq \tau$, and this implies the Lemma.

Moreover, it can be shown that to obtain an explicit description of set $\mathbf{X}_\tau^\beta$ it suffices to find the value

$$x_\tau^\beta = \max_{x \in \mathbf{X}_\tau^\beta} x$$

since

$$X_\tau^\beta = \{x \mid 0 \le x \le x_\tau^\beta\}.$$

The following property (dual of Lemma 2) also holds:

Lemma 4. *For any $k \ge 1$ and any $\tau \in [(k-1)T, kT-1]$ if $x_\tau \in X_\tau^\beta$ then*

$$x_{kT} = \varphi(\tau, kT, x_\tau, r^{max}, a_\tau^{kT-1}) \le x_0^\beta$$

for any $a_\tau^{kT-1} \in A_\tau^{kT-1}$.

This property implies that values x_τ^β can be obtained by solving the following mathematical programming problems

Problem-0.

$$x_0^\beta = \max x_0$$
$$x_t \le \beta x^*_t, \quad t=0,\dots,T-1,$$
$$x_{t+1} = x_t + a_t - N(x_t), \quad t=0,\dots,T-1,$$
$$x_T \le x_0, \quad a_0^{T-1} \in A_0^{T-1}.$$

Problem-τ ($\tau \in ((k-1)T, kT-1]$, $k \ge 1$).

$$x_\tau^\beta = \max x_\tau$$
$$x_t \le \beta x^*_t, \quad t=\tau,\dots,kT-1,$$
$$x_{t+1} = x_t + a_t - N(x_t), \quad t=\tau,\dots,kT-1,$$
$$x_{kT} \le x_0^\beta, \quad a_\tau^{kT-1} \in A_\tau^{kT-1}.$$

Now we can define set R^β similar to set R^α.

Definition 3 (set R^β). *Set R^β consists of all control laws*

$$r_t = r(t, x_t, a_t, A_{t+1}^{kT-1})$$

satisfying inequalities

$$\min\{N(x_t), \max\{x_t + a_t - x_{t+1}^\beta, 0\}\} \le r_t \le N(x_t) \tag{14}$$

for all $k \ge 1$, $t \in [(k-1)T, kT-1]$.

The properties of control laws from this set are given by the following

Theorem 2. *If a control law $r \in R^\beta$ then the couple (x_0, r) is β-feasible for any $x_0 \in X_0^\beta$.*

The proof of this Theorem is similar to that of Theorem 1 and is therefore omitted here.

All the comments made in Section 3 and related to Theorem 1 can now be rephrased in terms of Theorem 2. In particular, the decision-making process can be interpreted as the minimization at any time t of the distance (see Section 3) between state x_{t+1} and set X_{t+1}^{β}.

5. TWO-OBJECTIVE PROBLEM AND SEMI-EFFICIENT CONTROL LAWS

In this section we consider a problem of satisfaction of both goal constraints (4) for some fixed values of α and β, and find a set $\mathbf{F}_o^{\alpha,\beta}$ of (α,β)-feasible couples that has the form

$$\mathbf{F}_o^{\alpha,\beta} = \mathbf{X}_o^{\alpha,\beta} \times \mathbf{R}^{\alpha,\beta} \tag{15}$$

where

$$\mathbf{X}_o^{\alpha,\beta} = \mathbf{X}_o^{\alpha} \cap \mathbf{X}_o^{\beta}, \tag{16}$$

$$\mathbf{R}^{\alpha,\beta} = \mathbf{R}^{\alpha} \cap \mathbf{R}^{\beta}. \tag{17}$$

with sets $\mathbf{X}_o^{\alpha}$, $\mathbf{X}_o^{\beta}$, $\mathbf{R}^{\alpha}$, $\mathbf{R}^{\beta}$ defined in the preceding sections.

Obviously, from the previous discussions and from (16) it follows that

$$\mathbf{X}_o^{\alpha,\beta} = \{x \mid x_o^{\alpha} \le x \le x_o^{\beta}\},$$

where x_o^{α} and x_o^{β} are obtained as in Section. 3, 4, while set $\mathbf{R}^{\alpha,\beta}$ is described by the following

Theorem 3. *A control law*

$$r_t = r(t, x_t, a_t, \mathbf{A}_{t+1}^{kT-1})$$

belongs to set $\mathbf{R}^{\alpha,\beta}$ iff it satisfies inequalities

$$\min\{\mathrm{N}(x_t), \max\{x_t + a_t - x_{t+1}^{\beta}, \alpha r^*_t\}\} \le r_t \le \min\{\mathrm{N}(x_t), \max\{x_t + a_t - x_{t+1}^{\alpha}, \alpha r^*_t\}\}$$

for all $k \ge 1$, $t \in [(k-1)T, kT-1]$.

The proof of this theorem is straightforward and consists in combining inequalities (11) and (14) using (17).

Another problem of interest in this context consists in obtaining in some sense "the best" values of α and β which can be guaranteed using control laws from sets of the type $\mathbf{R}^{\alpha,\beta}$. To discuss this problem we introduce the following

Definition 4. *A couple (α^0,β^0) is called semi-efficient iff set $\mathbf{F}_0^{\alpha^0,\beta^0}$ is not empty and for any other couple (α,β) such that $\alpha>\alpha^0$ and $\beta<\beta^0$ corresponding set $\mathbf{F}_0^{\alpha,\beta}$ is empty. For any semi-efficient couple (α^0,β^0) we refer to control laws from set $\mathbf{R}^{\alpha^0,\beta^0}$ as semi-efficient control laws.*

Then the problem we now consider consists in determining a set of semi-efficient control laws and a corresponding set of initial states for the system in question. The procedure for solving this problem suggested here includes the following stages:

a) obtaining a semi-efficient couple (α^0,β^0) of the values of the indicators considered,

b) obtaining set $\mathbf{X}_0^{\alpha^0,\beta^0}$ and set $\mathbf{R}^{\alpha^0,\beta^0}$ of (α^0,β^0)- feasible control laws.

To obtain a semi-efficient couple (α^0,β^0) we use the standard procedure: first we fix some value α^0 of α and then obtain the minimal value β^0 of β such that the inequalities

$$r_t \geq \alpha^0 r^*_t \tag{18a}$$

$$x_t \leq \beta^0 x^*_t \tag{18b}$$

can be satisfied for all $t\geq 0$ and all sequences $a_0^\infty \in \mathbf{A}_0^\infty$ using α_0-feasible control laws. As this analysis should be performed prior to starting the control process the only information with regard to the possible future supplies that is available at this stage is set $\mathbf{A}_0^\infty$ that is defined using the reference set as discussed in Section 2.

Let us fix some value α^0. Then by solving the problem of demand satisfaction as in Section 3 we can obtain the set of control laws of the type $\mathbf{R}^{\alpha^0}$ providing for the satisfaction of inequalities (18a) for all $t\geq 0$ and all sequences $a_0^\infty \in \mathbf{A}_0^\infty$.

It is important to note here that values $x_t^{\alpha^0}$ $t \geq 0$ present in the description of this set as in (11) are obtained at this stage using only the a priori information in the form of set $\mathbf{A}*_0^{T-1}$. To indicate that we denote by $x_t^{\alpha^0}$ the values obtained at this stage of the analysis. Let us also introduce the notation

$$r_t^{max}(\alpha^0) = \mathbf{min}\{\mathbf{N}(x_t), \mathbf{max}\{x_t + a_t - x_{t+1}^{\alpha^0}, \alpha^0 r*_t\}\} \tag{19}$$

for the control law corresponding to the right-hand side of inequalities (11) describing set $\mathbf{R}^{\alpha^0}$ of control laws obtained at this stage. Clearly, as this follows from Definition 4 and from (15)-(17), the value β^0 such that (α^0, β^0) is a semi-efficient couple can be obtained as the minimal β for which the problem of peak attenuation has a solution *in set* $\mathbf{R}^{\alpha^0}$ *of control laws*. Formally this implies that when solving the problem of demand satisfaction at this stage we should use control law (19) as the maximal release policy (but not the stage-discharge policy $\mathbf{N}(\cdot)$). Using the results obtained in Section 4 we can formulate the following problem for determining β^0:

$$\beta^0 = \min_{x_0} \beta$$
$$x_t \leq \beta x*_t, \quad t=0,\ldots,T-1,$$
$$x_{t+1} = x_t + a_t - r_t^0(\alpha^0), \quad t=0,\ldots,T-1,$$
$$x_0^{\alpha^0} \leq x_T \leq x_0, \quad a_0^{T-1} \in \mathbf{A}*_0^{T-1}$$

Having thus obtained β^0, we come to the second stage to find set $\mathbf{R}^{\alpha^0,\beta^0}$ of semi-efficient control laws together with set of initial states $\mathbf{X}_0^{\alpha^0,\beta^0}$. The procedure for that consists in solving the problem of demand satisfaction for $\alpha = \alpha^0$ as in Section 3 and the problem of peak attenuation for $\beta = \beta^0$ as in Section 4. Then using (16), (17) and Theorem 3 we can obtain sets $\mathbf{X}_0^{\alpha^0,\beta^0}$ and $\mathbf{R}^{\alpha^0,\beta^0}$.

6. CONCLUDING REMARKS

A two-objective storage control problem has been considered in this paper and solved using a min-max approach. This approach allows a set of control laws to be determined (see Theorem 3) which guarantees given degrees of satisfaction of the goals in the case of the worst sequence of supplies out of a prespecified set of possible sequences. Using this approach the DM at any time can determine a range of feasible releases.

Operationally the approach is characterized by two optimization stages. The first stage can be performed prior to starting the control process and consists in determining the "highest" degrees of satisfaction of the goals by solving some mathematical programming problem over the whole characteristic time-interval (a year). This is done by using only the a priori information with regard to future supplies. The second stage involves solving a sequence of mathematical programming problems which are defined in real-time since they depend upon the information obtained in the course of the control process. This stage may be relatively costly as its implementation requires a computer at the disposal of the DM. However the use of an on-line computer can be avoided if the DM decides not to use all the current information.

The approach suggested in this paper has been applied to solving problems related to Lake Como (Northern Italy) which has been regulated since 1946. The reference releases r^*_t are in that case the yearly periodic water demands of seven hydroelectric power plants and six agricultural districts covering 120,000 hectares of land, while the reference storage x^*_t is constant and corresponds to the level of the main square of the town of Como which was very frequently flooded during the last decades. The historical values of α and β are respectively 0.30 and 2.10 and correspond to 70 percent shortage of the demand which occurred during the most severe drought of the last years (July 1976) and to the peak of 1.36 meters above the main square in Como (flood of October 1979). The application of the approach has demonstrated that the above values of the goals are not efficient in the sense specified in Section 5 of this paper. In fact, for the historical value α=0.3 the efficient value of β is 1.86 (the peak of the worst flood is lowered by 30 cm), while the efficient value of α corresponding to the historical value β=2.10 is 0.82 (the shortage of demand during summer 1976 period was reduced to 18 percent).

The approach presented in this paper must be slightly modified to make it applicable to some situations differing from those discussed in Section 2. For example, in some cases the storage is constrained from below ($x_t \geq x$), while in other cases the goals are not expressed in pointwise form as in (4) but in an integral form. The first case can be dealt with by suitably redefining the control constraint (2), while the second case can be solved assuming that the control laws are functions also of the current values of the goals. Both cases have been considered in a lake management study [4], in which the constraint $x_t \geq x$ has been introduced to take into account navigational requirements, and the two goals were the minimization of the yearly water deficit and of the number of days of flood per year.

Of course, not all the extensions are as simple as the two mentioned above. For example, the cases of multistorage systems and of uncertain demands, as well as some others, probably require substantial and conceptual modifications of the material presented in this paper.

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