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PREFACE

The main purpose of this paper is to discuss the numerical procedures of optimization with respect to unknown distribution functions which are based on ideas of duality. Dual problem is formulated as minimax type problem without concavity of "inner" problem of maximization. Numerical procedure allowed to avoid difficlties concerning "inner" problem solution is proposed.

STOCHASTIC OPTIMIZATION PROBLEMS WITH PARTIALLY KNOWN DISTRIBUTIONS FUNCTIONS

Y. Ermoliev and C. Nedeva

1. Introduction

The conventional stochastic programming problem may be formulated with some generality as minimization of the function

$$F^0(x) = E_y f^0(x,y) = \int f^0(x,y)dH(y) \tag{1}$$

subject to

$$F^i(x) = E_y f^i(x,y) = \int f^i(x,y)dH(y) \leq 0,\ i = \overline{1,m} \tag{2}$$

$$x \in X, \int_Y dH(y) = 1, \tag{3}$$

where $y = (y_1, y_2, \ldots, y_r) \in Y, Y \supset R^n$ is a vector of random parameters; $H(y)$ is a given distribution function; and $f^\nu(x,\cdot), \nu = \overline{0,m}$ are random functions possessing all the properties necessary for expressions (1), and (2) to be meaningful.

In practice, we often do not have exact information about $H(y)$, except some of its characteristics, for instance, bounds for the mean value or other moments. Such information can usually be written in terms of constraints of the type

$$Q^k(H) = E_y q^k(y) = \int q^k(y) dH(y) \le 0,\ k = \overline{1,l} \tag{4}$$

$$\int_Y dH(y) = 1 \ . \tag{5}$$

In particular, we could have the following constraints on joint moments:

$$c_{\tau_1,\tau_2,\ldots,\tau_r} \le E y_1^{\tau_1} \cdots y_r^{\tau_r} \le C_{\tau_1,\tau_2,\ldots,\tau_r} \tag{6}$$

where $C_{\tau_1,\tau_2,\ldots,\tau_r}$, $c_{\tau_1,\tau_2,\ldots,\tau_r}$ are given constants.

Constraints (4)-(6) are a special case of constraints (2) and (3). It seems therefore reasonable to consider the following problem: find a vector x which minimizes

$$\max_{H \in K(x)} \int f^0(x,y) dH(y), \tag{7}$$

where $K(x)$ is the class of functions H satisfying constraints (2), (3) for fixed x.

The main purpose of this paper is to find a method of solving such problems using procedures based on duality ideas. Other approaches to optimization problems with randomized strategies have been examined by Fromovitz [1], Ermoliev [2, 3], Kaplinski and Propoi [4], and Golodnikov [5].

2. Optimization with Respect to Distribution Functions

Let us begin with solution procedures for the "inner" problem of maximization. Suppose we have to solve a much simpler problem: find the distribution function H maximizing

$$Q^0(H) = Eq^0(y) = \int q^0(y)dH(y) \tag{8}$$

subject to

$$Q^i(H) = Eq^k(y) = \int q^k(y)dH(y) \le 0,\ k = \overline{1,l} \tag{9}$$

$$\int_Y dH(y) = 1. \tag{10}$$

This is a generalization of the known moments problem. If (9) were replaced by special constraints (6), it could also be regarded as the problem of evaluating system reliability subject to given upper and lower bounds for central joint moments. In particular, the case in which

$$q^0(y) = \begin{cases} 1, & \text{if } y \in A \\ 0, & \text{if } y \notin A \end{cases}$$

where A is a subset of Y, leads to

$$Q^0(H) = P\{y \in A\} \tag{11}$$

and problem (8)-(10) becomes the problem of evaluating (11) with respect to a distribution from a given family of distributions. By solving this problem we could obtain generalizations of the well-known Chebyshev inequality.

Problem (8)-(10) appear to be solvable by means of a modification of the revised simplex method (see [3] and [6]). The following fact makes this possible. Consider the set

$$Z = \{z \mid z = (q^0(y), q^1(y), \ldots, q^l(y)), y \in Y\} \ .$$

Suppose this set is compact. This will be true, for instance, if Y is compact and functions $q^\nu(y), \nu = \overline{0,l}$ are continuous. Consider also the set

$$\text{co}Z = \{z \mid z = \sum_{t=1}^{N} p_t Z^t, Z^t \in Z, \sum_{t=1}^{N} p_t = 1, p_t \ge 0\} \ ,$$

where N is an arbitrary finite number. The following statement is proved in [3].

Lemma 1

$$\mathrm{co}Z = \{Q = (Q^0(H), Q^1(H), \dots, Q^l(H)) \mid H \geq 0, \int_Y dH = 1\} .$$

Therefore problem (8)-(10) is equivalent to maximizing z subject to

$$z \in (z_0, z_1, \dots, z_l) \in \mathrm{co}Z,\ z_k \leq 0,\ k = \overline{1,l}.$$

According to the Caratheodory theorem each point of co$Z$ is a convex combination of at most $l + 2$ points $z \in Z$:

$$\mathrm{co}Z = \{z = \sum_{j=1}^{l+2} q^\nu(y^j)p_j,\ \nu = \overline{0,l},\ p_j \geq 0,\ \sum_{j=1}^{l+2} p_j = 1,\ y^j \in Y\} .$$

Each point on the boundary of co$Z$ can be represented as a convex combination of at most $l + 1$ points of Z. Therefore, it is easy to see that problem (8)-(10) is equivalent to the problem of finding $l + 1$ points $y^j \in Y$, $j = \overline{1,l+1}$ and $l + 1$ real numbers $p_1, p_2, \dots, p_{l+1}$ such that

$$\sum_{j=1}^{l+1} q^0(y^j)p_j = \max \tag{12}$$

subject to

$$\sum_{j=1}^{l+1} q^k(y^j)p_j \leq 0,\ k = \overline{1,l}, \tag{13}$$

$$\sum_{j=1}^{l+1} p_j = 1,\ \ p_j \geq 0,\ \ j = \overline{1,l+1} . \tag{14}$$

Consider arbitrary $l + 1$ points $\bar{y}^j$, $j = \overline{1,l+1}$ and for the fixed set $\{\bar{y}^1, \bar{y}^2, \dots, \bar{y}^{l+1}\}$ find a solution $\bar{p} = (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_{l+1})$ of problem (12)-(14) with respect to p. Assume that $\bar{p}$ exists and $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{l+1})$ is a solution associated with it. The dual problem is then to minimize

$$u_{l+1} \tag{15}$$

subject to

$$q^0(\bar{y}^j) - \sum_{k=1}^{l} q^k(\bar{y}^j)u_k - u_{l+1} \le 0,\ j = \overline{1,l+1} \tag{16}$$

$$u_k \ge 0,\ k = \overline{1,l} \tag{17}$$

Now let y be an arbitrary point of Y. Consider the following augmented problem of maximization with respect to $(p_1, p_2, \ldots, p_{l+1}, p)$: maximize

$$\sum_{j=1}^{l+1} q^0(\bar{y}^j)p_j + q^0(y)p \tag{18}$$

subject to

$$\sum_{j=1}^{l+1} q^k(\bar{y}^j)p_j + q^k(y)p \le 0,\ k = \overline{1,l}, \tag{19}$$

$$\sum_{j=1}^{l+1} p_j + p = 1. \tag{20}$$

It is clear that if there exists a point $y = y^*$ such that

$$q^0(y^*) - \sum_{k=1}^{l} q^k(y^*)\bar{u}_k - \bar{u}_{l+1} > 0$$

then the solution $\bar{p}$ could be improved by dropping one of the columns $(q^0(\bar{y}^j), q^1(\bar{y}^j), \ldots, q^l(\bar{y}^j), 1)$ $j = \overline{1,l+1}$ from the basis and entering instead the column $(q^0(y^*), q^1(y^*), \ldots, q^l(y^*), 1)$ $j = \overline{1,l+1}$ following the revised simplex method. Point y^* could be defined as

$$y^* = \arg\max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} \bar{u}_k q^k(y)] \tag{21}$$

Theorem 1. (Optimality condition) Let $\bar{p}$ be a solution of problem (12)-(14) for fixed $\{\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^{l+1}\}$ and $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{l+1})$ be the associated dual variables. The pair $\{\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^{l+1}\}, \bar{p}$ is an optimal solution of problem (12)-(14) if and only if

$$q^0(y) - \sum_{k=1}^{l} \bar{u}_k q^k(y) - \bar{u}_{l+1} \le 0, \; y \in Y. \tag{22}$$

The proof of this theorem follows immediately from the facts mentioned above and the following inequality.

Let $y^{1,s}, y^{2,s}, \ldots, y^{l+1,s}$ be a set of points $y^{j,s} \in Y$. Suppose that $p^s = (p_1^s, p_2^s, \ldots, p_{l+1}^s)$ is a solution of problem (12)-(14) with respect to p for $y^j = y^{j,s}$ and that $u^s = (u_1^s, u_2^s, \ldots, u_{l+1}^s)$ is the corresponding solution of the dual problem (15)-(17). For an arbitrary pair of solutions $(\bar{p}, \bar{u})$ the following lemma holds:

Lemma 2

$$\delta_s + \max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} u_k^s q^k(y) - u_{l+1}^s] \ge \bar{\delta} \ge \delta_s, \tag{23}$$

where

$$\bar{\delta} = \sum_{j=1}^{l+1} q^0(\bar{y}^j) \bar{p}_j, \; \delta_s = \sum_{j=1}^{l+1} q^0(y^{j,s}) p_j^s \; .$$

In fact,

$$\bar{\delta} = \max \{z_0 \,|\, (z_0, z_1, \ldots, z_l) \in \mathrm{co} Z, \; z_k \le 0, \; k = \overline{1,l}\}$$

Since $u_1^s \ge 0, 1, \ldots, u_l^s \ge 0$, $u_{l+1}^s = \delta_s$, then

$$\left[z_0 - \sum_{k=1}^{l} u_k^s z_k - u_{l+1}^s \right] + \delta_s \ge z_0.$$

and we have

$$\max \{z_0 - \sum_{k=1}^{l} u_k^s z_k - u_{l+1}^s \,|\, (z_0, z_1, \ldots, z_l) \in \mathrm{co} Z, \; z_k \le 0, \; k = \overline{1,l}\} + \delta_s \ge$$

$$\max\{z_0 \mid (z_0, z_1, \ldots, z_l) \in \mathrm{co}Z,\ z_k \le 0,\ k = \overline{1,l}\} = \bar{\delta} \ .$$

But on the other hand

$$\max\{z_0 - \sum_{k=1}^{l} u_k^s z_k - u_{l+1}^s \mid (z_0, z_1, \ldots, z_l) \in \mathrm{co}Z,\ z_k \le 0,\ k = \overline{1,l}\} + \delta_s \le$$

$$\max\{z_0 - \sum_{k=1}^{l} u_k z_k - u_{l+1}^s \mid (z_0, z_1, \ldots, z_l) \in \mathrm{co}Z\} + \delta_s =$$

$$\max_{y \in Y} [q^0(y) + \sum_{k=1}^{l} u_k^s q^k(y) + u_{l+1}^s] + \delta_s$$

This proves the desired inequality. Consider now the following minimax-type problem:

$$\min_{u \ge 0} \max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)] \tag{24}$$

This problem is dual to (8)-(10) or (12)-(14).

Theorem 2. Let a solution of either problem (8)-(10) or problem (24) exist. Then a solution of the other problem exists and the optimal values of the objective functions of both problems are equal.

Proof. Fix $\bar{y}$, and consider problems (12)-(14), (15)-(17).

1. Let pair $\{\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^{l+1}\}$, $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{l+1})$ be a solution of the primal problem (12)-(14) equivalent to (8)-(10). For fixed $y^i = \bar{y}^j$, $j = \overline{1,l+1}$ consider the pair of dual problems (12)-(14) and (15)-(17). Let $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{l+1})$ be a solution of problem (15)-(17). According to condition (22) point $u = \bar{u}$ satisfies the inequalities

$$q^0(y) - \sum_{k=1}^{l} u_k q^k(y) - u_{l+1} \le 0, \quad y \in Y, \quad u_k \ge 0, \quad k = \overline{1,l} \tag{25}$$

and we also have

$$\sum_{j=1}^{l+1} q^0(\bar{y}^j)\bar{p}_j = \bar{u}_{l+1}. \tag{26}$$

Since an arbitrary feasible solution $u = (u_1, u_2, \ldots, u_{l+1})$ of dual problem (15)-(17) satisfies the inequality

$$\sum_{j=1}^{l+1} q^0(\bar{y}^j)\bar{p}_j \le \bar{u}_{l+1}. \tag{27}$$

then a feasible solution $u = (u_1, u_2, \ldots, u_{l+1})$ of equation (25) will certainly satisfy the same inequality. From this and (26), (27) we can conclude that $\bar{u}$ is an optimal solution of the problem of minimizing u_{l+1} subject to (25). This problem is equivalent to problem (24) and from (26) the optimal values of the objective functions of the primal problem (12)-(14) and the dual problem (24) are equal.

2. Suppose that $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{l+1})$ minimizes u_{l+1} subject to (25), where $\tilde{u}_k > 0,\ k = \overline{1,\tau}$; $\tilde{u}_k = 0,\ k = \overline{\tau+1,l}$. Then there exist points $\tilde{y}^j, j = \overline{1,\tau+1}$ such that

$$q^0(\tilde{y}^j) - \sum_{k=1}^{\tau+1} \tilde{u}_k q^k(\tilde{y}^j) - \tilde{u}_{l+1} = 0,\ j = \overline{1,\tau+1},$$

where vectors $(q^1(\tilde{y}^j), q^2(\tilde{y}^j), \ldots, q^\tau(\tilde{y}^j), 1),\ j = \overline{1,\tau}$ are linearly independent. Therefore the point $\tilde{u}$ also minimizes u_{l+1} subject to

$$q^0(\tilde{y}^j) - \sum_{k=1}^{\tau+1} u_k q^k(\tilde{y}^j) - u_{l+1} => 0,\quad j = \overline{1,\tau+1},\quad u_k \ge 0,\quad k = \overline{1,l}$$

According to duality theory there exists a solution $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\tau+1}, 0 \ldots, 0)$ to the following dual problem: maximize (with respect to p)

$$\sum_{j=1}^{l+1} q^0(\tilde{y}^j) p_j$$

subject to

$$\sum_{j=1}^{l+1} q^k(\tilde{y}^j)p_j \le 0, \quad k = \overline{1,l}$$

$$\sum_{y=1}^{l+1} p_j = 1, \quad p_j \ge 0, \quad j = \overline{1,l+1}.$$

Since from (25)

$$q^0(y) - \sum_{k=1}^{l} \tilde{u}_k q^k(y) - \tilde{u}_{l+1} \le 0, \quad y \in Y,$$

then, from Theorem 1, the pair $\{\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^{l+1}\}$, $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{l+1})$ is a solution of the original problem (12)-(14) with the optimal value of the objective function (12) equal to $\tilde{u}_{l+1}$. Therefore Theorem 2 is true.

It is important to note that if it is assured that the functions $q^0(y), q^k(y)$ are concave and the set Y is convex, problem (24) is dual to the nonlinear programming problem

$$\max \; \{q^0(y) \,|\, q^k(y) \le 0, \quad y \in Y, \quad k = \overline{1,l}\}. \tag{28}$$

Thus (24) and (28) would remain dual in the general case if the concept of a solution in a mixed stratgy of problem (24) is accepted.

3. Existence of Solutions. Connections With Gaming Problems

The existence of solutions to the dual problem follows from nondegeneracy assumptions. For instance, assume that there exists a distribution $\hat{H}(y)$ which satisfies the following generalized Slater condition (see Ermoliev 1970):

$$\int_Y q^k(y) d\hat{H}(y) < 0, \, k = \overline{1,l}, \; \int_Y d\hat{H}(y) = 1. \tag{29a}$$

In accordance with Lemma 1, the left-hand side of the inequality in (29a) belongs to co$Z$. Thus, (29a) is equivalent to the existence of a set

$\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^{l+1}$ of points in Y and a vector $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{l+1})$ with nonnegative components such that

$$\sum_{j=1}^{l+1} q^k(\tilde{y}^j)\tilde{p}_j < 0, \quad k = \overline{1,l}, \quad \sum_{j=1}^{l+1} \tilde{p}_j = 1. \tag{29b}$$

Now consider the Lagrange function

$$\Phi(H,u) = \int q^0(y)dH(y) - \sum_{k=1}^{l} u_k \int q^k(y)dH(y)$$

and let a pair (H^*,u^*) be a saddle point of $\Phi(H,u)$:

$$\Phi(H^*,u) \geq \Phi(H^*,u^*) \geq \Phi(H,u^*) \tag{30}$$

$$u \geq 0, \quad H(y) \geq 0, \quad \int_Y dH(y) = 1.$$

The problem of finding a saddle point (H^*,u^*) could be viewed as a gaming problem with the following payoff function:

$$\varphi(y,u) = q^0(y) - \sum_{k=1}^{l} u_k q^k(y)$$

and mixed strategy for the first player. It follows from (29a) that for an arbitrary saddle point $(H^*,u^*), u^* = (u_1^*, u_2^*, \ldots, u_l^*)$ we have

$$u_k^* \leq \left[\int q^0(y)dH^*(y) - \int q^0(y)d\tilde{H}(y)\right] / \int q^k(y)d\tilde{H}(y) \ .$$

Thus the second component of point (H^*,u^*) is bounded. If we now assume that Y is a compact set and $q^0(y), q^k(y), k = \overline{1,l}$ are continuous functions, then general results from two-person game theory suggest the existence of an optimal strategy (H^*,u^*) satisfying (30) with distribution H^* concentrated over at most $l+1$ points of Y. Then the following three quantities (31-33) are all equal to each other:

$$\max_H \min_{u \geq 0} \int_Y [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)] dH(y), \tag{31}$$

$$\max \{ \min_{u \geq 0} \sum_{j=1}^{l+1} [q^0(y^j) - \sum_{k=1}^{l} u_k q^k(y^j)] p_j \mid y^j \in Y, \sum_{j=1}^{l+1} p_j = 1, j = \overline{1,l+1} \} \tag{32}$$

$$\max_H \min_{u \geq 0} \int_Y [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)] dH(y). \tag{33}$$

Since

$$\max_H \min_{u \geq 0} \int_Y [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)] dH(y) =$$

$$\max_H \begin{cases} \int_Y q^0(y) dH(y), & \text{if } \int_Y q^k(y) dH(y) \leq 0, \quad k = \overline{1,l} \\ -\infty, & \text{otherwise} \end{cases}$$

then problem (31) is equivalent to the original problem (8)-(10). In the same way, problem (32) is equivalent to problem (12)-(14). It is easy to see that

$$\min_{u \geq 0} \max_H \int_Y [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)] dH(y) = \min_{u \geq 0} \max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)]$$

and, therefore problem (33) is equivalent to the dual problem (24).

4. Computational Methods

Algorithm 1

Theorem 1 makes it possible to create a numerical "procedure" based on the general idea of the revised simplex method (see Ermoliev 1970).

Fix $(l+1)$ points $y^{0,1}, y^{0,2}, \ldots, y^{0,l+1}$ in Y and solve problem (12)-(14) with respect to p for $y^j = y^{0,j}$, $j = \overline{1,l+1}$. Suppose that a solution $p^0 = (p_1^0, p_2^0, \ldots, p_{l+1}^0)$ to this problem exists. Let $u^0 = (u_1^0, u_2^0, \ldots, u_l^0)$ be a solution of dual problem (15)-(17) with respect to u. The vector u^0 satisfies the constraints (25) for $y \in \{y^{0,1}, y^{0,2}, \ldots, y^{0,l+1}\}$. If u^0 satisfies the constraints (25) for all $y \in Y$, then the pair $\{y^{0,1}, y^{0,2}, \ldots, y^{0,l+1}\}$ and $p^0 = (p_1^0, p_2^0, \ldots, p_{l+1}^0)$ is a solution of the original problem (12)-(14). If this is not the case, we can improve the solution $\{y^{0,1}, y^{0,2}, \ldots, y^{0,l+1}\}, p^0$ by introducing a new point y^0 such that

$$y^0 = \arg \max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} u_i^0 q^i(y)] .$$

Denote by $p^1 = (p_1^1, p_2^1, \ldots, p_{l+1}^1)$ a solution of the augmented problem (18)-(20) with respect to p for fixed $\bar{y}^j = y^{0,j}$, $y = y^0$. We shall us $y^{1,1}, y^{1,2}, \ldots, y^{1,l+1}$ to denote those points $y^{0,1}, \ldots, y^{0,l+1}, y^0$ that correspond to the basic variables of solution p^1.

Thus, the first step of the algorithm is terminated and we can then pass to the next step: determination of u^1, y^1, etc. In general, after the s-th iteration we have points $y^{s,1}, y^{s,2}, \ldots, y^{s,l+1}$, a solution $p^s = (p_1^s, p_2^1, \ldots, p_{l+1}^s)$ of problem (12)-(14) for $y^j = y^{s,j}$, and a corresponding solution $u^s = (u_1^s, u_2^s, \ldots, u_{l+1}^s)$ of the dual problem (15)-(17). Find

$$y^s = \arg \max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} u_k^s q^k(y)].$$

If

$$\Delta(y^s, u^s) = q^0(y^s) - \sum_{k=1}^{l} u_k^s q^k(y^s) - u_{l+1}^s < 0,$$

then the solution $\{y^{s,1}, y^{s,2}, \ldots, y^{s,l+1}\}$, $p^s = (p_1^s, p_2^s, \ldots, p_{l+1}^s)$ can be improved by solving the augmented problem (18)-(20) for $\bar{y}^j = y^{s,j}$, $y = y^s$. Denote by $y^{s+1,1}, y^{s+1,2}, \ldots, y^{s+1,l+1}$ those points from $\{y^{s,1}, y^{s,2}, \ldots, y^{s,l+1}\} \cup y^0$ that correspond to the basic variables of the obtained solution p^{s+1}. The pair $\{y^{s+1,1},y^{s+1,2},\ldots,y^{s+1,l+1}\}$, p^{s+1} is the new approximate solution to the original problem, and soon if $\Delta(y^s,u^s) \geq 0$, then according to (22) the pair $\{y^{s,1},y^{s,2},\ldots,y^{s,l+1}\},p^s$ is the desired solution.

Algorithm 2

Using Theorem 2 we can create a method which gives a solution to the dual problem of minimizing the function

$$\gamma(u) = \max_{y \in Y} [q^0(y) - \sum_{k=1}^{l} u_k q^k(y)]$$

with respect to $u \geq 0$.

Although the function $\gamma(u)$ is convex, we need a solution (at least an ε-solution) of the "inner" problem in order to minimize $\gamma(u)$. If the functions $q^0(y),q^k(y)$ are nonconcave it becomes difficult to use well-known methods, and therefore we adopt the following approach, which is based on stochastic optimization techniques. Suppose we have to solve the more general problem of minimizing

$$\gamma(u) = \max_{y \in Y} \psi(u,y)$$

subject to $u \in U$, where $\psi(u,y)$ is a convex function with respect to u and U is a convex compact set. The algorithm may then be constructed as follows. Let $P(\cdot)$ be a probabilistic measure on Y, and fix arbitrary $u^0 \in U$, $y^0 \in Y$. Suppose that the s-th iteration we have arrived at some points u^s,y^s. Then the next approximation u^{s+1},y^{s+1} is derived as follows:

(i) based on the probabilistic measure P choose $N \geq 1$ points:

$$y^{1,s},y^{2,s},\ldots,y^{N,s};$$

(ii) Take

$$y^{s+1} = \begin{cases} y^s, \text{ if } \gamma(u^s, y^{\tau,s}) = \max_i \gamma(u^s, y^{i,s}) \le \gamma(u^s, y^s), \\ y^{\tau,s}, \text{ if } \gamma(u^s, y^{\tau,s}) > \gamma(u^s, y^s); \end{cases}$$

(iii) Compute

$$u^{s+1} = \pi[u^s - \rho_s \psi_u(u^s, y^{s+1})], \; s = 0, 1, \ldots,$$

$$\psi_u(u^s, y) \in \{g \mid \psi(u, y) - \psi(u^s, y) \ge \langle g, u - u^s \rangle, \; u \in U\}$$

where ρ_s is the step size; π is the result of the projection operation on U.

As will be shown in a forthcoming article, this procedure converges with probability 1 under rather weak assumptions which include (in addition to those mentioned above):

$$\rho_s \ge 0, \; \rho_s \to 0, \; \sum_{s=0}^{\infty} \rho_s = \infty;$$

$$|\gamma(u, y) - \gamma(u, z)| \le L \| y - z \|, \quad by, z \in Y, \; u \in U;$$

and the assumption that measure P is in some sense nondegenerate.

5. Stochastic Programming Problem

The stochastic programming problem with unknown distribution function introduced earlier is the minimization of

$$\max_{H \in K(x)} \int f^0(x, y) dH(y) \tag{34}$$

with respect to x, where set $K(x)$ is defined by the relations

$$\int f^i(x, y) dH(y) \le 0, \; i = \overline{1, m}, \; x \in X. \tag{35}$$

Suppose that for each $x \in X$ there exists an optimal solution of the maximization problem (34). In practice, this assumption is not as restrictive as it seems. We can always change problem (34)-(35) slightly so that a feasible solution H satisfying (35) for fixed $x \in X$ will exist. To do this it is necessary only to increase the dimensionality of y by introducing new variables $y_1^{\pm}, y_2^{\pm}, \ldots, y_m^{\pm}$ and considering the minimization of

$$\max_{H \in \kappa(x)} \int \left[f^0(x,y) - M \sum_{i=1}^{m} (y_i^+ + y_i^-) \right] dH(y, y^{\pm}) \tag{34'}$$

where $\kappa(x)$ is described by the constraints

$$\int [f^i(x,y) + y_i^+ - y_i^-] dH(y, y^{\pm}) \le 0, \quad i = \overline{1,m}, \quad x \in X, \quad y_i^+ \ge 0, \quad y_i^- \ge 0 \tag{35'}$$

with M sufficiently large. For fixed $x \in X$ there always is a degenerate distribution $H(y, y^{\pm})$ satisfying (35').

According to the duality theorem above, for each fixed $x \in X$

$$\max_{H \in K(x)} \int f^0(x,y) dH(y) = \min_{u \ge 0} \max_{y \in Y} [f^0(x,y) - \sum_{i=1}^{m} u_i f^i(x,y)].$$

The whole problem (34), (35) can then be reduced to a minimax-type problem as follows: minimize the function

$$\max_{y \in Y} [f^0(x,y) - \sum_{i=1}^{m} u_i f^i(x,y)]$$

with respect to $x \in X$, $u \ge 0$. This type of problem can be solved using Algorithm 2.

6. Conclusions

The purpose of this paper is to consider methods for solving optimization problems with unknown distribution functions. The algorithms discussed here have been successfully applied to real problems, dealing, for instance, with superconducting power cable lines, where the class of unknown distributions consisted of distributions with given bounds for the mathematical expectations and the variance of the random parameters. The main advantage of Algorithm 2 is undoubtedly its computational simplicity. In addiion, this method does not require concavity of $f^0(x,y)$, $f^i(x,y)$ as functions of y.

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