Imperfect Information, Simplistic Modeling and the Robustness of Policy Rules

Snower, D. and Wierzbicki, A.P.

IIASA Working Paper

WP-82-024

March 1982

Working Paper

IMPERFECT INFORMATION, SIMPLISTIC MODELING AND THE ROBUSTNESS OF POLICY RULES

D. Snower and A. Wierzbicki

March 1982
WP-82-24

International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

NOT FOR QUOTATION
WITHOUT PERMISSION
OF THE AUTHOR

IMPERFECT INFORMATION, SIMPLISTIC MODELING AND THE ROBUSTNESS OF POLICY RULES

D. Snower and A. Wierzbicki

March 1982
WP-82-24

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

SUMMARY

The paper presents a methodology for dealing with the problems of imperfect information or simplistic modeling in macroeconomic policy problems. The methodology permits to choose a robust policy from a given set of candidate policies--that is, a policy that makes the social welfare least sensitive to various potential modeling errors. This can be achieved even if the potential modeling errors are related to model structure or delays in model equations--without requiring that the models with more complicated structure or delays are fully solved and optimized. The particular example chosen to illustrate the methodology is a macroeconomic model of intertemporal optimization of monetary control of inflation and unemployment. The conclusions for this particular model are two-fold. Firstly, neglected delays or other modeling errors cannot, in general, substantiate rigorously the constant monetary growth rule that is usually advanced because of such modeling inaccuracies. In fact, by choosing an appropriate feedback policy formulation it is possible to obtain reasonable results of an active policy even if the underlying model used for policy derivation is very simple and the economic reality to which the policy is applied is much more complicated. Secondly, rigorous case can be made against 'impetuous' policy making with regard to inflation and unemployment, that is, against policies that by attaching a small weight to unemployment attempt to approach rapidly long-run targets for inflation. Such a policy strategy may induce instability, either through delay effects, or by making the macroeconomic system very sensitive to other modeling errors.

IMPERFECT INFORMATION, SIMPLISTIC MODELING, AND THE ROBUSTNESS OF POLICY RULES

D. Snower and A. Wierzbicki

1. INTRODUCTION

This paper is concerned with the formulation of macroeconomic policy rules from macro-models which are inaccurate representations of economic reality. The models are inaccurate due to imperfect information or because they are rough approximations of known economic mechanisms. Rough approximations, viz., "simplistic models", may be used in order to keep the analytical or computational derivations of policy rules manageable. The policy rules are meant to optimize the policy maker's objective function. Under conditions of imperfect information or simplistic modeling, the policy maker is aware that policy rules which are optimal with regard to his model are not necessarily optimal with regard to the actual economic system he aims to control. How should the policy rules be devised in the light of the inaccuracies of the underlying model?

If the policy maker faces probabilistic risk rather than uncertainty (i.e., his imperfect knowledge is representable by a model whose true parameter distributions are known), then the policy rules may be derived from a stochastic optimization problem. Yet such problems are often notoriously difficult to solve. Besides, macroeconomic policy makers seldom, if ever, have perfect information on the parameter distributions of the models they use and these models are generally simplistic. Uncertainty and simplistic modeling call for a different approach to the formation of policy rules.

This paper presents such an approach. Given uncertainty or simplistic modeling, policy makers are commonly interested in devising policy rules which are not only optimal with regard to their model, but also insensitive to particular errors in model specification. In this context, imperfect information and simplistic modeling pose analogous difficulties for the formulation of optimal policy rules. Regardless of whether errors in model specification are attributable to imperfect information or deliberate simplification of economic relations, our aim is to find policy rules which are not sensitive to these errors. To do so, we formulate a number of different policy rules, all of which are optimal with regard to the policy maker's objective function and model, but which are not all equally sensitive to changes in model specification.

Naturally, *all* macro-economic models are simplified representations of actual economic activities. The rationale for such simplifications is that amendments to the models which introduce greater realism at the expense of greater complexity do not affect qualitatively the conclusions of the analysis at hand. In the formulation of optimal macro-economic policy rules, modeling simplifications are commonly regarded as acceptable if they have a negligible impact on the properties of the policy rules. On Occam's Rasor grounds, in fact, such simplifications are desirable.

It would appear, at first sight, that the application of Occam's Rasor requires that policy rules be derived first from a complex model which is the closest representation of economic reality which the model-builder is capable of creating and then successively from simpler models. Simplifications which lead to close approximations of the former policy rules are accepted; the rest are rejected. Of course, in practice macro-economic models are not constructed in this manner; but to the degree to which they are not, their structure cannot be rationalized on Occam's Rasor grounds.

The basic insight of this paper is twofold. First, Occam's Rasor may be used not only as a criterion for the construction of simplistic models, but also as a criterion for the formulation of optimal policy rules. The aim of our analysis is to find a number of policy rules which are optimal for a given model and to choose the policy rule which provides the strongest Occam's Rasor rationale for that model. Second, Occam Rasor can be applied without explicitly deriving the policy rules from a more complex and realistic counterpart of the model.

In other words, (i) the policy rules themselves, if appropriately chosen, can make modeling simplifications harmless for the formulation of these policy rules, and (ii) it is possible to establish whether a simplification is harmless without explicitly comparing the policy implications of a "realistic" model with its simplistic counterparts.

The economic literature contains numerous attempts to derive optimal policy rules in the context of models which are incorrectly specified. Perhaps the most prominent attempt is the monetarist argument that the money supply should grow at a constant percentage rate per annum, because the magnitude and timing of the effect of a money supply change on aggregate demand are difficult to predict.[1] However, the monetarists have not explained precisely how the constant monetary growth rule may be deduced from the assumption of modeling inaccuracy. Presumably, they do not intend to suggest that such a rule invariably emerges as the optimal solution of a stochastic optimization problem in which the relation between the money supply and aggregate demand is described by parameters with known distributions.

Nor does the constant monetary growth rule necessarily emerge from our methodology for the choice of optimal policy rules, as we will show. To illustrate our methodology we will derive monetary policy rules from a model containing an expectations-augmented Phillips curve. In particular, we assume that the policy maker's objective function depends on unemployment and expected inflation. According to the expectations-augmented Phillips curve, actual inflation depends inversely on the unemployment rate and positively on the expected inflation rate. Inflationary expectations are generated by an adaptive mechanism. The policy maker can influence the rate of unemployment by changing the growth rate of the money supply.

A rise in this growth rate decreases the unemployment rate in the short run (and thereby raises the value of the policy objective function) and increases the expected inflation rate in the medium run (and thereby lowering the value of the policy objective function). The policy maker presumes that the Phillips curve or the adaptive expectations mechanism are incorrectly specified. How should the optimal monetary rule be formulated?

This policy problem merely serves an illustrative purpose in our analysis of the formulation of policy rules. In general, our analysis pertains to any dynamic model in which (a) a present policy impulse affects the value of the policy objective function (henceforth called, euphemistically, the "social welfare function") at present and in the future, (b) there is an intertemporal tradeoff between these effects (such that a present social welfare gain is associated with a future welfare loss, and vice versa), and (c) the model is an inaccurate representation of actual economic processes.

A diverse assortment of important macroeconomic policy problems share these properties. In the standard theory of optimal economic growth, there is a tradeoff between the production of nondurable consumption and investment goods, and social welfare depends on the flow of consumption through time. If the policy maker stimulates durable consumption, social welfare rises in the short run, but falls in the longer run (since the capital stock whereby future consumption goods can be produced, grows more slowly than it would have done in the absence of the consumption stimulus). The policy maker may be aware that his depiction of the production possibility frontier is an inaccurate representation of the actual tradeoff between consumption and investment good production. In the basic theory of optimal resource depletion control, a policy stimulus to the production of nondurable consumption goods implies an increase in the rate of resource depletion. Thus, social welfare rises in the short run, but falls in the longer run (since the resources, necessary for the production of future consumption goods, are depleted at a more rapid rate). Similarly, in the theory of optimal pollution control, a consumption stimulus gives rise to a larger flow of durable pollutants. Social welfare rises in the short run on account of the consumption stimulus, and falls in the longer run on account of the argumented pollutant stock. The policy maker may seek the optimal consumption trajectory in the context of an inaccurate model of the relation between consumption and resource depletion or between consumption and pollution. This list of examples could be extended considerably. Our analysis of policy rules applies equally well to all of them. Our choice of monetary policy rules to control inflation and unemployment is to be understood as a concrete illustration of a methodology with rather wide application.

The expectations-augmented Phillips curve in our model embodies the natural rate hypothesis. In other words, the unemployment rate is solely related to errors in inflationary expectations. Correct inflationary expectations are associated with a unique rate of unemployment, the "natural" rate. The lower the rate of unemployment, the greater the actual rate of inflation relative to the expected rate of inflation. The natural rate of hypothesis has received considerable empirical support (e.g., Gordon 1972, Turnowsky 1972, Vanderkamp 1972, Parkin 1973, Mackay & Hart 1974, Parkin, Summer & Ward 1976, Lucas & Rapping 1969, Darby 1976) and has been given various logically distinct, but not mutually exclusive microfoundations (e.g., the "misperceived real wage" paradigm of Friedman 1968, Lucas 1972, 1973, and Sargent 1973; the "job search" paradigm of Alchian 1970, McCall 1970, Mortensen 1970, Gronau 1971, Parsons 1973, Salop 1973, Lucas & Prescott 1974, and Siven 1974; and the "price setting" paradigm of Phelps 1970).

On the other hand, the adaptive expectations mechanism, which we use to generate inflationary expectations, has not been given much attention in the macroeconomic literature since the theory of rational expectations came into widespread use. However, several reasons may be given for our use of adaptive expectations. Adaptive expectations might be considered as a realistic approximation of the ideal process of rational expectations. It is generally recognized that if the structure of the macro-economic model changes and economic agents gain information on this change through a costly process of learning, then their expectations cannot be expected to be rational during this process. For such circumstances, adaptive expectations mechanisms (possibly with flexible adjustment coefficients) can be motivated on theoretical (e.g., Friedman 1979, Shiller 1978, Taylor 1975, and De Canio 1976) and empirical grounds (e.g., Lawson 1980, Lahiri 1976, and Turnowsky 1970).

In our analysis we consider monetary policy which, inter alia, takes the form of closed-loop control of unemployment. Here monetary impulses cannot be specified at present for all future points in time. Instead, the growth rate of the money supply will depend on how the state variable of our model (the expected rate of inflation) evolves through time. Yet since our model (by assumption) is an inaccurate representation of actual economic activities, the evolution of the state variable cannot be precisely foreseen. Consequently, the growth rate of the money supply is not perfectly predictable either. Provided that the monetary authority is able to change the money growth rate faster than the public is able to learn of this change, the public cannot be expected to have rational expectations contingent on the monetary authority's information set.

We assume that the public forms its expectations adaptively instead. For simplicity, the adjustment coefficient of the adaptive expectations mechanism is held constant through time.[2] Admittedly, we also use this mechanism when monetary policy takes the form of open-loop control. Yet, if the public knows the functional form of the expectations-augmented Phillips curve and of the monetary authority's objective function, then it can perfectly predict open-loop control policies and thus can be expected to have rational expectations contingent on the monetary authority's information set. If rational expectations are assumed, however, our policy exercise becomes rather uninteresting, for then monetary policy rules are no longer able to affect the unemployment rate. The natural rate hypothesis makes the unemployment rate depend on errors in inflationary expectations, while the rational expectations hypothesis ensures (in the context of our analysis) that such errors do not occur. Thus, systematic monetary policy is impotent. (See, for example, Sargent and Wallace 1975, 1976, Sargent 1973, 1976, and Barro 1976).[3)]

In our analysis, serving as it does primarily illustrative purposes, the assumption of adaptive expectations is retained even under open-loop policies. As noted, the analysis also applies to the choice of policy rules in macroeconomic models centering around the tradeoffs between consumption and pollution, consumption and resource depletion, and consumption and capital accumulation. In these latter models, government policies are commonly assumed to affect consumption either directly (via government consumption expenditures) or indirectly (via taxes or environmental controls). Here the assumption of rational expectations does not necessarily make policy rules ineffective with regard to real economic variables. (See, for example, McCallum and Whitaker 1979, Buiter 1977, and Tobin and Buiter 1980). To maintain the applicability of our analysis to these policy problems and to keep the structure of our model monolithic, the assumption of adaptive expectations is made for all our policy exercises.

The problem of choosing policy rules under imperfect information or simplistic modeling is approached in accordance with the methodology suggested by Wierzbicki (1977). Two macroeconomic models are considered:

(1) The *basic model* is used by the policy maker to devise optimal policy rules. It may be a simplified version of a more realistic model or it may be inaccurate because of the policy maker's imperfect information about macroeconomic activity.

(2) The *extended model* is our proxy for "actual" macroeconomic activity. This model is unknown to the policy maker or intractable for the derivation of policy rules. It serves to indicate various ways in which economic reality may differ from the basic model. Policy rules are derived with regard to the basic, not the extended, model. Thus, the extended model simply provides a concrete illustration of the uncertainty the policy maker faces or of the need for constructing simplistic models.

A policy maker who is uncertain as to the accuracy of his model may wish to test his policies on a proxy of economic reality before implementing them. The extended model is such a proxy.[4)]

We will consider three ways in which the basic model may be an inaccurate representation of the extended model:

(a) Mistaken parameter estimates: The (nonzero) coefficients of the expectations-augmented Phillips curve in the basic model differ from those in the extended model. This inaccuracy is attributable to imperfect information rather than to simplistic modeling.

(b) Mistaken functional specification: The functional form of the expectations-augmented Phillips curve in the basic model differs from that in the extended model.[5)] Both imperfect information and simplistic modeling may be the source of this inaccuracy.

(c) Mistaken delay estimates[6)]: The basic model's Phillips curve has a different delay structure than that of the extended model. As a simple example, the basic model may ignore delays in the relation between inflation and unemployment, while the extended model takes them into account. This, too, can be attributed to both imperfect information and simplistic modeling.

As we shall see, it is quite simple to extend out analysis to include these modeling inaccuracies not only with regard to the expectations-augmented Phillips curve, but also with regard to the relation between the rate of growth of the money supply and the unemployment rate. Modeling inaccuracies of this sort are, as noted, a major reason why monetarists advocate constant monetary growth rules. Since the relation between the money supply on the one hand and inflation and unemployment on the other hand is difficult to predict both in magnitude and time structure--so the argument runs--the money supply should be expanded at a constant percentage rate per annum. We will examine this argument in the light of our methodology for the choice of policy rules. In other words, given that one or more of the modeling mistakes above is made, we will specify a number of policy rules which are optimal with regard to the basic model and then choose the rule which makes social welfare least sensitive to the postulated mistakes. We call this rule "robust" with regard to the modeling errors. It will be shown that constant monetary growth rules do not necessarily emerge from this exercise. If there is a case to be made for such rules, then this depends very much on some crucial parameters of the basic model.

As the analysis below indicates, the trajectories of inflation and unemployment which are induced by a given policy rule may be described in terms of two components: (a) the long-run optimal stationary levels of inflation and unemployment, and (b) the rate at which inflation and unemployment approach their respective long-run levels through time. Some values of crucial parameters of the basic model (mostly, a larger weight given to inflation versus unemployment in the social welfare functional) induce faster rates of approach than others. It will be shown that those parameter values which cause the rates of approach exceed certain threshold levels run the danger of making the macroeconomic system unstable. In this case, the system's dynamic behaviour becomes very sensitive to modeling errors and thus the policy maker has little chance to maintain both inflation and unemployment near their target paths. This might be considered as an argument for a constant monetary growth rule because of the basic difficulty of achieving anything better by a more active policy. On the other hand, social preferences with parameters which induce slower rates of approach result in policy rules which do not have this undesirable property, and a constant monetary growth rule cannot be substantiated in such a case. This is, however, not an argument for or against a constant monetary growth rule, but much rather an argument against trying to reach long-run targets for inflation and unemployment in a short period of time. It appears that this argument is not a theoretical curio, but a case of immediate and far-reaching policy implications. Over the past years there has been a heated controversy in a number of mature market economies--the United States, Great Britain, Germany and others--about how fast a government should attempt to reduce the rate of inflation to its long-run target level. Thus far, a government's degree of "impatience" with inflation has been viewed largely as a question of taste. The more weight a government attaches to inflation relative to unemployment in its policy objective function, the faster it should drive the rate of inflation towards the long-run inflation target. Our analysis suggests that "taste" is not the end of the matter. We indicate that "impatient" governments run the risk of macroeconomic instability. In other words, a case--unrelated to the policy maker's preferences--is to be made for the less impetuous policy directives.

The paper is organized as follows. Section 2 presents the underlying macroeconomic model and describes various monetary policy rules. Section 3 provides analytical solutions to the policy problems. Section 4 describes the methodology of our robustness analysis, i.e., provides the criteria for the choice of policy rules. Section 5 evaluates the various policy rules by means of these criteria. Finally, Section 6 contains a brief overview.

2. STATEMENT OF THE POLICY PROBLEM

The basic model consists of three analytical building blocks: (i) a relation between the expected rate of inflation and the rate of unemployment, (ii) a relation between the rate of growth of the money supply and the unemployment rate, and (iii) a social welfare functional which depends on the expected rate of inflation and the rate of unemployment.

The first building block is composed of an expectations-augmented Phillips curve and an adaptive expectations mechanism.

Let x be the expected rate of inflation, x_a the actual rate of inflation, u the actual rate of unemployment, and u_n the natural rate of unemployment. Then the Phillips curve is

$$x_a = x - A \cdot (u - u_n) \quad , \tag{1}$$

where A is a positive constant. Inflationary expectations are generated by

$$\dot{x} = B \cdot (x_a - x) \quad , \tag{2}$$

where B is also a positive constant and $\dot{x}$ is the rate of change of x through time. Substituting (1) into (2),

$$\dot{x} = -C \cdot (u - u_n) \quad , \tag{3}$$

where $C = A \cdot B > 0$.

The second building block is composed of a quantity theory of money and a variant of Okun's Law.[7)] Let M be the stock of money, V the income velocity of circulation, P the price level, and Q the production of goods and services. Then

$$M \cdot V \equiv P \cdot Q \tag{4}$$

Suppose that V is constant. Let g be the growth in production and m the growth in the money supply, which is continually equal to the growth in money demand. Then

$$m = x_a + g \tag{5}$$

Let g_n be the trend rate of production growth. Then our variant of Okun's Law can be expressed as

$$\dot{u} = \delta \cdot (g_n - g) \tag{6}$$

where δ is a positive constant. Substituting (6) into (5),

$$m = x_a + g_n - \frac{1}{\delta} \cdot \dot{u} \qquad (7)$$

Substituting (1) into (7),

$$m = x - A \cdot (u-u_n) + g_n - \frac{1}{\delta} \cdot \dot{u} \qquad (8)$$

The objectives of the policy maker are represented by a "social welfare" functional. At every instant of time social welfare depends on the expected inflation rate and the unemployment rate as follows:

$$1 - \frac{1}{2} \cdot x^2 - \frac{q}{2} \cdot u^2 \qquad (9)$$

where q is a positive constant. A rise in the expected inflation rate affects social welfare adversely because it induces economic agents to economize on money balances and therefore to bear the higher transactions costs of exchanging money for interest-bearing assets. A rise in the unemployment rate also lowers social welfare since the marginal utility of consumption is always assumed to be larger than the marginal disutility of the labour required to produce one unit of the consumption good. The marginal utilities of both expected inflation and unemployment are assumed to be negative and declining.

From (3) it is apparent that a fall in the current rate of unemployment raises the future rate of change of expected inflation. Thus a current welfare gain is associated with a future welfare loss (and vice versa). The policy maker faces an intertemporal optimization problem. We assume that he maximizes social welfare from the present time (time t=0) to the infinite future. His rate of time discount, r, is constant and social welfare occuring at different points in time enters his objective function additively.[8)] Hence, the social welfare functional may be written as:

$$W = \int_0^\infty e^{-rt} \cdot (1-\frac{1}{2} \cdot x^2-\frac{q}{2} \cdot u^2)dt \qquad (10)$$

In sum, the policy problem is to maximize (10) subject to (3) and (8), where m is the control variable and x and u are the state variables. Note the m is not an argument of the social welfare functional; there are no policy instrument adjustment costs. Assume for the moment, that m can be changed instantaneously by infinitely large amounts; thereby giving rise to instantaneous, finite changes of u, as determined via Equation (8). Then, the policy problem may be restated in the following, simpler form:

$$W = \int_0^\infty e^{-rt} \cdot (1 - \tfrac{1}{2} \cdot x^2 - \tfrac{q}{2} \cdot u^2)\,dt. \tag{11}$$

$$\text{subject to } \dot{x} = -C \cdot (u - u_n) \quad ,$$

where x is the only state variable and u may be termed a "surrogate control variable". However, it is well known (see, for example, Markus and Lee 1967) that the optimal control u for the problem (11) is a differentiable function of time. Thus, the optimal $\dot{u}$ is well defined and given this $\dot{u}$ along with the optimal trajectories for u and x, we can compute the optimal m from (8). Consequently, m is continuous with respect to time and so the discontinuous changes in m, assumed above, are not really needed.[9)] Hence, the trajectory of m which keeps u on the path prescribed by optimization problem (11) is optimal with regard to the maximization of (10) subject to (3) and (8). Thus, u may be used as a control variable in problem (11) even though it enters as a state variable in the other problem.

Problem (11) serves as our basic model. The extended model may differ from the basic one in various ways. In the case of mistaken parameter estimates, the differential equation of the extended model may be written as

$$\dot{x} = -\tilde{C} \cdot (u - \tilde{u}_n) \quad , \tag{12}$$

where $(\tilde{C}-C)$ or $(\tilde{u}_n - u_n)$ is a measure of the mistake in parameter estimation. For the sake of brevity, we will consider only the case of a mistake estimate of u_n.

In the case of mistaken functional specification, we choose the following expectations-augmented Phillips curve for our extended model:

$$x_a = x - A[u - u_n - \frac{\gamma}{2}(u-u_n)^2] \tag{13}$$

where γ is a parameter (not necessarily positive). (Note that $x=x_a$ wherever $u=u_n$). Substituting (2) into (13), we obtain

$$\dot{x} = -C \cdot (u-u_n-\frac{\gamma}{2}(u-u_n)^2) \tag{14}$$

In the case of mistaken delay estimates, the following differential equation is used for the extended model:

$$\dot{x} = -C \cdot (u(t-\tau) - u_n) \quad , \tag{15}$$

where τ denotes the length of time whereby $\dot{x}$ is delayed behind u.[10)]

We now come to the crux of our policy exercise. How should the policy rule for u (and hence for m) be devised, given that the policy maker knows that his basic model contains one or more of the modeling errors above? We assume that, on account of imperfect information or simplistic modeling, the policy maker is unable to correct these errors, and therefore, unable to derive his policy rule by optimizing over the extended model.

As noted, our methodology for the choice of policy rule may be summarized in two steps. First, derive a number of policy rules, each of which are optimal with respect to the basic model. Second, find the rule that when applied to the extended model, makes social welfare minimally sensitive to the modeling errors above. This is the rule to be adopted in the light of imperfect information or simplistic modeling. Needless to say, the sensitivity analysis must be undertaken without solving the control problem in terms of the extended model.

A "policy" is a mapping from a set of variables in the extended model (or their counterparts in economic reality) into the set of control variables. Every policy which maximizes the social welfare functional with respect to the basic model we shall call a "basic-optimal" policy. A number of different policies, as we shall see, may all be basic-optimal; yet, they may not have the same welfare implications when applied to the extended model.

The following are a sample of basic optimal policies.

(1) *Open-loop control:* Find the basic-optimal trajectory of the control variable. Manipulate the actual level of u (which is generated by the extended model, not the basic model) so that u remains on this trajectory.

Here, the time path of the unemployment rate is predetermined from the present till the infinite future; it does not depend on discrepancies between the behaviour of the basic and extended model. In other words, the money supply must be manipulated to keep unemployment on the predetermined, basic-optimal path.

(2) *Classical closed-loop control:* Find the basic-optimal relation between u and x through time. For the x generated by the extended model at every point in time, set the actual level of u in accordance with the above relation.

Hence, the time path of the unemployment rate depends on the observed value of x (rather than on the value of x generated by the basic model). In other words, the money supply must be manipulated to maintain the predetermined dynamic relation between the expected inflation rate and the unemployment rate.

(3) *State trajectory tracking control:* Find the basic-optimal trajectory of the state variable, x. Set u such that x remains on this trajectory. Here, the money supply is adjusted to keep the expected inflation rate on a predetermined path.

In Section 3, it is shown that policies (1)-(3) are special cases drawn from a family of basic-optimal policies with infinitely many members.

(4) *Open-loop, Hamiltonian-optimizing control:* Find the basic-optimal trajectory of the costate variable. Set u such that the current

value Hamiltonian, defined in terms of the above costate variable and the Phillips curve of the extended model, is maximized.

Here, the policy is to maximize, at each point of time, the observed difference between the current welfare and the social cost of accumulation of inflationary expectations (i.e., the costate variable).

(5) *Closed-loop, Hamiltonian-optimizing control:* Find the basic-optimal relation between the costate variable and the expected inflation x through time. For the x generated by the extended model at every point in time, set the costate variable in accordance with the above relation. Set u such that the current value Hamiltonian, defined in terms of the above costate variable and the Phillips curve of the extended model, is maximized.

(6) *Open-loop, benefit-cost control:* Find the basic-optimal trajectory of the costate variable. Define the benefit-cost ratio at a particular point in time to be social welfare (at that point in time) divided by the social cost of accumulation of inflationary expectations (at that point in time), as given by the above costate variable and the Phillips curve of the extended model. Set u such that this benefit-cost ratio is maximized at every point in time.

(7) *Closed-loop, benefit-cost control:* Find the basic-optimal relation between the costate variable and x through time. For the x generated by the extended model at every point in time, set the costate variable in accordance with the above relation. Set u such that the benefit-cost ratio, defined in terms of the above costate variable and the Phillips curve of the extended model, is maximized at every point in time.

As for policies (1)-(3), policies (4)-(7) can be combined to generate yet further policy options. All above policies are equivalent and optimal if the extended model is identical to the basic one.

What now remains to be done is to specify these policies rigorously and to find those policies which are least sensitive to the various modeling errors considered above. This is the subject of the following three sections.

3. DERIVATION OF THE OPTIMAL SOLUTION AND ALTERNATIVE POLICIES

The current-value Hamiltonian function for the problem (11) has the form

$$H = 1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t) + \zeta(t) \cdot C(u(t) - u_n) \qquad (16)$$

where $\zeta(t)$ is the costate variable, i.e., the social cost of accumulating inflationary expectations. The first-order necessary condition is

$$H_u = 0 \Leftrightarrow u(t) = \frac{C}{q}\zeta(t) \quad , \qquad (17)$$

which is also a sufficient condition since H is strongly concave. The costate equation has the form

$$\dot{\zeta}(t) - r\zeta(t) = H_x \Leftrightarrow \dot{\zeta}(t) = r\zeta(t) - x(t) \qquad (18)$$

and the state equation (3), after substituting (17), takes the form

$$\dot{x}(t) = -H_\zeta \Rightarrow \dot{x}(t) = -C \cdot (\frac{C}{q}\zeta(t) - u_n) \ ; \ x(0) = x_0 \qquad (19)$$

Various methods can be used for solving the system of canonical equations (18), (19) and for establishing the existence of optimal solutions to (11). To derive alternative policy rules, however, it is convenient to use the Riccati substitution (see, for example, Althans and Falb 1966):

$$\zeta(t) = K(t)x(t) + M(t) \qquad (20)$$

in which (as shown in the appendix) K(t) is defined by the Riccati equation

$$\dot{K}(t) = \frac{C^2}{q} K^2(t) + rK(t) - 1 \qquad (21)$$

and M(t) by the auxiliary equation

$$\dot{M}(t) = (\frac{C^2}{q} K(t) + r)M(t) - Cu_n K(t) \qquad (22)$$

One of the sufficient conditions for the existence of optimal solutions, if the Hamiltonian function is strongly concave, is that the Riccati equation (21), when solved backward in time, has a bounded solution. For the equation (21), such a solution does exist (see the appendix) and has the following form:

$$K(t) = K = ((1+4\frac{C^2}{r^2 q})^{\frac{1}{2}} - 1)\frac{rq}{2C^2} > 0 \qquad (23)$$

The corresponding solution of (22) is

$$M(t) = M = Cu_n K^2 \qquad (24)$$

Now, substituting (20), (23), (24) into (9), (10) and solving the differential equations we obtain the optimal solutions for the problem (11)

$$\hat{x}(t) = (x_0 - \hat{x}_\infty)e^{-\upsilon t} + \frac{rqu_n}{C} \quad ; \quad \lim_{t\to\infty} \hat{x}(t) = \hat{x}_\infty = \frac{rqu_n}{C} > 0 \qquad (25)$$

$$\hat{\zeta}(t) = K(x_0 - \hat{x}_\infty)e^{-\upsilon t} + \frac{qu_n}{C} \quad ; \quad \lim_{t\to\infty} \hat{\zeta}(t) = \hat{\zeta}_\infty = \frac{qu_n}{C} > 0 \qquad (26)$$

$$\hat{u}(t) = \frac{CK}{q}(x_0 - \hat{x}_\infty)e^{-\upsilon t} + u_n \quad ; \quad \lim_{t\to\infty} \hat{u}(t) = \hat{u}_\infty = u_n \qquad (27)$$

where $\hat{x}_\infty$, $\hat{\zeta}_\infty$, $\hat{u}_\infty$ denote optimal long-run solutions and

$$\upsilon = \frac{C^2 K}{q} = \frac{r}{q}((1+4\frac{C^2}{r^2 q})^{\frac{1}{2}} - 1) > 0 \qquad (28)$$

is a coefficient measuring the speed with which inflation and unemployment approach their respective long-run optimal values.

This speed attains its maximal value when $r \to 0$:

$$\upsilon_{max} = \lim_{r \to 0} \upsilon = \frac{C}{q^{\frac{1}{2}}} \tag{29}$$

The ratio of this maximal speed to the depreciation rate r

$$\nu = \frac{\upsilon_{max}}{r} = \frac{C}{rq^{\frac{1}{2}}} \tag{30}$$

or, more precisely, its squared value ν^2, plays an important role as a basic aggregated parameter in the analysis that follows. Observe, for example, that the ratio υ/r, as specified by (28), is a function of ν^2 alone, $\upsilon/r = \frac{1}{2}((1+4\nu^2)^{\frac{1}{2}} - 1)$. Suppose the depreciation rate r is given and fixed, and the parameter ν^2 is changed by choosing an appropriate weighting coefficient q; if $q \to \infty$, then $\nu^2 \to 0$, if $q \to 0$, then $\nu^2 \to \infty$. The graph of υ/r as a function of ν^2 is given in Figure 1.

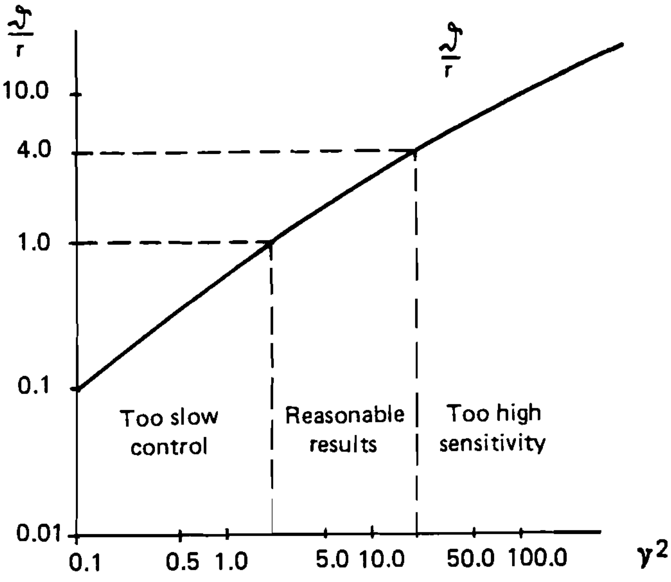

Figure 1. Dependence of the relative speed of control υ/r, on the aggregated parameter $\nu^2 = \frac{C}{r^2 q}$.

Since a reasonable value of the depreciation rate is r=0.1/year, parameter values that result in a ratio $\upsilon/r < 1$ might be considered as not acceptable socially: the speed of approaching long-term solutions is too slow in such a case. The ratio $\upsilon/r=1$ is obtained by $\nu^2=2$, $\upsilon/r=4$ by $\nu^2=20$. We shall show that all policies become very sensitive to modeling errors (i.e., become rather impracticable) for ν^2 much larger than 20, while they remain robust for ν^2 between 2 and 20. For this range of parameters ($\nu^2 \geq 2$) we can also reasonably approximate υ by $\upsilon_{max} = C/q^{\frac{1}{2}}$ and υ/r by ν (see the appendix).

The optimal value of the welfare functional (10) can be determined as a quadratic function of $x_0-\hat{x}_\infty$

$$\hat{W} = \hat{W}_0 - \Delta\hat{W} - \frac{1}{2}\Delta^2\hat{W} \tag{31}$$

where

$$\hat{W}_0 = \frac{1}{r}(1-\frac{1}{2}\ qu_n^2(1+\frac{qr^2}{C^2})) \tag{32}$$

$$\Delta\hat{W} = \frac{qu_n}{C}(x_0-\hat{x}_\infty) \tag{33}$$

$$\Delta^2\hat{W} = K(x_0-\hat{x}_\infty)^2 \tag{34}$$

The term $\hat{W}_0$ stands for social welfare in equilibrium (characterized, as noted above, by the absence of expectational errors) $\Delta\hat{W}$ is the first-order approximation of welfare losses due to an initial disequilibrium (characterized by the difference between the initial and long-run optimal rate of expected inflation), and $\Delta^2 W$ is the second-order approximation of such losses. Simplistic modeling or imperfect information increase such losses further; however, the additional losses are always of second-order form and shall be thus compared with the term $\Delta^2\hat{W}$.

Moreover, the maximal value of the Hamiltonian function, interpreted as a shadow price for passing time is

$$\hat{\zeta}_t(t) = 1 - \frac{1}{2}\hat{x}^2(t) - \frac{q}{2}\hat{u}(t) + \hat{\zeta}(t) \cdot C \cdot (\hat{u}(t) - u_n) \qquad (35)$$

Before proceeding to the definition of alternative policy rules, consider the following short-hand description of our basic and extended models. Define the vectors of parameters which distinguish the extended from the basic model as

$$\underline{\alpha} = (\beta, \gamma, \tau) \quad ; \qquad \underline{a} = (0,0,0) \qquad (36)$$

and let $\tilde{u}_n = u_n + \beta$. The extended model is

$$W = \int_0^\infty e^{-rt}(1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t))dt \qquad (37)$$

subject to $\dot{x}(t) = -C(u(t-\tau) - u_n - \beta - \frac{\gamma}{2}(u(t-\tau) - u_n - \beta)^2)$.

We denote this model by $M(\underline{\alpha})$. Clearly, the extended model is identical to the basic model (11) if $\underline{\alpha} = \underline{a}$. We denote the basic model by $M(\underline{a})$. The variables of the basic model depend on the parameter $\underline{a}$ and will be denoted $\hat{x}(t,\underline{a}), \hat{u}(t,\underline{a})$, etc. We approach the solution to the extended model as that of the basic model through techniques related to the implicit function theorem.

Now consider alternative policies, all of which are optimal when applied to the basic model, but which might yield different solutions when applied to the extended one. The simplest policy is the open-loop optimal control:

$$u^0(t,\underline{a}) = \hat{u}(t,\underline{a}) \qquad (38)$$

obtained from the basic model and applied to the extended one.

Another policy is the classical closed-loop optimal control defined by a function $\hat{u}(t,\underline{a})$ which depends solely on the current $\hat{x}(t,\underline{a})$ and not on the initial value x_0. When comparing (25), (27) it is easy to see that $\hat{u}(t,\underline{a}) = \frac{CK}{q}\hat{x}(t,\underline{a}) + (1-rK)u_n$; when implementing this policy rule in the extended model, however, $\hat{x}(t,\underline{a})$ is substituted by expected inflation $x(t)$ taken from the extended model. Thus, the classical closed-loop optimal control is

$$u^c(x(t),\underline{a}) = \frac{CK}{q} x(t) + (1-rK)u_n \tag{39}$$

Another alternative policy is the optimal trajectory tracking control:

$$u^t(x(t),\underline{a}) = \{u(t) \text{ such that } x(t) = \hat{x}(t,\underline{a})\} \tag{40}$$

which means that the rate of expected inflation $x(t)$ which emerges from the extended model is maintained at the predetermined path $\hat{x}(t,\underline{a})$.

All these alternative policy rules (38), (39), (40) are members of an infinite family of closed-loop controls parameterized by a coefficient λ:

$$u^\lambda(x(t),\underline{a}) = \hat{u}(t,\underline{a}) + \lambda \cdot \frac{CK}{q} \cdot (x(t) - \hat{x}(t,\underline{a})) \tag{41}$$

If $\lambda=0$, then $u^\lambda(x(t),\underline{a}) = u^0(t,\underline{a})$. If $\lambda=1$, then $u^\lambda(x(t),\underline{a}) = u^c(x(t),\underline{a})$. If the extended model (37) taken together with the policy rule (41) remain stable as $\lambda\to\infty$ which can be shown to be the case if $\tau=0$, then it is easy to check that $u^\lambda(x(t),\underline{a}) \to u^t(x(t),\underline{a})$.

For intermediate values of λ, however, we have a parametric policy rule which dictates that a linear combination of unemployment and expected inflation, observed from the extended model, $u(t) - \lambda \cdot \frac{C \cdot K}{q} x(t)$ follow the predetermined path $\hat{u}(t,\underline{a}) - \lambda \cdot \frac{CK}{q} \hat{x}(t,\underline{a})$.

Aside from this infinite number of policy rules, these are yet further possibilities. Since the optimal control maximizes the Hamiltonian function (16), the policy maker may employ in the open-loop Hamiltonian maximizing feedback (see Wierzbicki 1977):

$$u^{ho}(\hat{\zeta}(t,\underline{a}),\underline{\alpha}) = \operatorname{argmax}\ (1-\tfrac{1}{2}x^2(t)-\tfrac{q}{2}u^2(t) - \hat{\zeta}(t,\underline{a})\cdot f(u(t),\underline{\alpha})) \tag{42}$$

with u(t), x(t) are taken from the extended model, and $f(u(t),\underline{\alpha}) = \dot{x}(t)$ is the derivative of state variable as measured in the extended model. In other words, we compute the value of the current welfare function and, using the Phillips curve of the extended model as well as the shadow price for the accumulation of inflationary expectations $\hat{\zeta}(t,\underline{a})$, we maximize the difference between the current welfare and the cost of future inflation. However, we need not use a predetermined shadow price $\hat{\zeta}(t,\underline{a})$; since we also know its close-loop form $\hat{\zeta}(t,\underline{a}) = K\hat{x}(t,\underline{a}) + M$, we could also use the measured x(t) for corrections of this shadow price. This results in the closed-loop Hamiltonian maximizing feedback:

$$u^{hl}(\hat{\zeta}(x(t),\underline{a}),\underline{\alpha}) = \underset{u(t)}{\operatorname{argmax}}\ (1-\tfrac{1}{2}x^2(t) - \tfrac{q}{2}u^2(t) - (Kx(t) + M)f(u(t),\underline{\alpha})) \tag{43}$$

The Hamiltonian function (16) can also be rewritten in a different form. For example, the relation

$$u^{fo}(\hat{\zeta}(t,\underline{a}),\hat{\zeta}_t(t,\underline{a}),\underline{\alpha}) = \underset{u(t)}{\operatorname{argmax}}\ \frac{1-\frac{1}{2}x^2(t) - \frac{q}{2}u^2(t)}{\hat{\zeta}(t,\underline{a})\cdot f(u(t),\underline{\alpha})+\hat{\zeta}_t(t,\underline{a})} \tag{44}$$

yields also $u^{fo}(\hat{\zeta}(t,\underline{a}),\hat{\zeta}_\tau(t,\underline{a}),\underline{a}) = \hat{u}(t,\underline{a})$ at $\underline{\alpha}=\underline{a}$; while it might yield different solution when $\underline{\alpha} \neq \underline{a}$. This is the open-loop, benefit-cost control. Observe that if $\hat{\zeta}_t(t,\underline{a}) \equiv 0$ would hold, then $\hat{\zeta}(t,\underline{a})$ would not affect the maximization in (44) and the resulting $u^{fo}(\hat{\zeta}(t,\underline{a}),0,\underline{\alpha}) = \hat{u}(t,\underline{\alpha})$ would be optimal for the extended model no matter what errors $\underline{a} \neq \underline{\alpha}$ were made in the basic model. Thus, the benefit-cost control is perfectly robust if the cost of passing time is negligible. However, in the example considered here, $\hat{\zeta}_t(t,\underline{a}) \neq 0$ and it will be shown that this control policy has some undesirable properties.

If $\hat{\zeta}_t(t,\underline{a}) \neq 0$, then the errors in determining the shadow prices can be corrected in a close-loop structure

$$u^{fc}(\hat{\zeta}(x(t),\underline{a}),\hat{\zeta}_t(x(t),\underline{a}),\underline{\alpha}) = \underset{u(t)}{\operatorname{argmax}} \frac{1-\frac{1}{2}x^2(t)-\frac{q}{2}u^2(t)}{(Kx(t)+M)f(u(t),\underline{\alpha})+\hat{\zeta}_t(x(t),\underline{a})} \tag{45}$$

where $\hat{\zeta}_t(x(t),\underline{a})$ is determined as in (35) but with $u(t)$ and $\zeta(t)$ substituted by (17), (20); $x(t)$ is taken from the extended model.

Clearly, it would be possible to generate yet other policy rules, each involving observations from variables from the extended model and a scheme for influencing these variables which is basic-optimal (i.e., yield optimal solutions whenever $\underline{a}=\underline{\alpha}$). However, we restrict our attention to the policy rules listed above. If the probability distributions of the model parameters were known, then a dual stochastic optimal control and estimation problem[11)] could be formulated and possibly solved. Yet, such problems are notoriously difficult. In this paper we concentrate on the derivation of policy rules when the parameter distributions are not known. Such conditions call for a different methodology, to which the following section is devoted.

4. METHODOLOGY OF ROBUSTNESS ANALYSIS

Consider a given policy rule u^i, mapping the variables measured in the extended model into the control actions for this model. This policy is defined with the help of the basic model and thus depends on parameters $\underline{a}$ (see Figure 2).

Suppose it is at least conceptually possible to solve the extended model under this policy rule, thus obtaining $x^i(t,\underline{\alpha},\underline{a})$ and $u^i(t,\underline{\alpha},\underline{a})$; these results depend on the policy rule i as well as on the parameters $\underline{\alpha},\underline{a}$. Similarly, suppose we could compute the social welfare functional of the extended model under this policy rule:

$$W^i(\underline{\alpha},\underline{a}) = \int_0^\infty e^{-rt}\left(1-\frac{1}{2}(x^i(t,\underline{\alpha},\underline{a}))^2 - \frac{q}{2}(u^i(t,\underline{\alpha},\underline{a}))^2\right)dt \qquad (46)$$

If we were able to optimize the extended model and compute the corresponding social welfare functional $\hat{W}(\underline{\alpha})$, we would find that $W^i(\underline{\alpha},\underline{a}) \leq \hat{W}(\underline{\alpha})$, since the differences between the basic model and the extended one ($\underline{a} \neq \underline{\alpha}$) imply that the policy rule may not be optimal for the extended model. Thus, as a measure of the robustness of a policy rule, we use the welfare loss of applying this rule to the extended model:

$$S^i(\underline{\alpha},\underline{a}) = \hat{W}(\underline{\alpha}) - W^i(\underline{\alpha},\underline{a}) \qquad (47)$$

However, since the extended model is more complicated than the basic one, a direct computation of $\hat{W}(\underline{\alpha})$ and $S^i(\underline{\alpha},\underline{a})$ may be impossible. On the other hand, the function $S^i(\underline{\alpha},\underline{a})$ has several useful properties that facilitate its approximation *even if only the solutions of the basic model, but not those of the extended model are known.* First, $S^i(\underline{\alpha},\underline{a})$ is non-negative:

$$S^i(\underline{\alpha},\underline{a}) \geq 0 \quad ; \quad S^i(\underline{\alpha},\underline{a}) = 0 \text{ for all } \underline{\alpha} = \underline{a} \qquad (48)$$

Therefore, if S^i is differentiable, its first-order derivatives are zero for all $\underline{\alpha}=\underline{a}$:

$$S^i_{\underline{\alpha}}(\underline{a},\underline{a}) = S^i_{\underline{a}}(\underline{a},\underline{a}) = 0 \qquad (49)$$

It follows further (see Wierzbicki 1977) that if S^i is twice differentiable, its second-order derivatives have a specific symmetry property:

$$S^i_{\underline{a}\,\underline{a}}(\underline{a},\underline{a}) = -S^i_{\underline{\alpha}\,\underline{a}}(\underline{a},\underline{a}) = -S^i_{\underline{a}\,\underline{\alpha}}(\underline{a},\underline{a}) = S^i_{\underline{\alpha}\,\underline{\alpha}}(\underline{a},\underline{a}) \;:=\; \hat{S}^i_{\underline{a}\,\underline{a}} \qquad (50)$$

Therefore, $S^i(\underline{\alpha},\underline{a})$ can be approximated by

$$S^i(\underline{\alpha},\underline{a}) = \frac{1}{2}(\underline{\alpha}-\underline{a})^T \hat{S}^i_{\underline{a}\,\underline{a}}(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||^2) \tag{51}$$

where $o(\cdot)$ is a function converging to zero faster than its argument. As a next step, we need a method for computing $\hat{S}^i_{\underline{a}\,\underline{a}}$. If we could approximate the differences between $x^i(t,\underline{\alpha},\underline{a})$, $u^i(t,\underline{\alpha},\underline{a})$ and $\hat{x}(t,\underline{\alpha}),\hat{u}(t,\underline{\alpha})$ which would be optimal for the extended model

$$\begin{aligned} x^i(t,\underline{\alpha},\underline{a}) - \hat{x}(t,\underline{\alpha}) &= \tilde{x}^i(t)\cdot(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||) \\ u^i(t,\underline{\alpha},\underline{a}) - \hat{u}(t,\underline{\alpha}) &= \tilde{u}^i(t)\cdot(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||) \quad , \end{aligned} \tag{52}$$

then we could easily [12] determine the quadratic form of the approximation (51):

$$\begin{aligned} &\frac{1}{2}(\underline{\alpha}-\underline{a})^T \hat{S}^i_{\underline{a}\,\underline{a}}(\underline{\alpha}-\underline{a}) \\ &= \frac{1}{2}\int_0^\infty e^{-rt}((\tilde{x}^i(t)\cdot(\underline{\alpha}-\underline{a})^2 + q(\tilde{u}^i(t)\cdot(\underline{\alpha}-\underline{a}))^2)dt \end{aligned} \tag{53}$$

However, $\tilde{x}^i(t)$ and $\tilde{u}^i(t)$, called *extended structural variations* [13], are usually not directly computable. It is usually simpler to compute the *basic structural variations*

$$\begin{aligned} x^i(t,\underline{\alpha},\underline{a}) - \hat{x}(t,\underline{a}) &= \bar{x}^i(t)\cdot(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||) \\ u^i(t,\underline{\alpha},\underline{a}) - \hat{u}(t,\underline{a}) &= \bar{u}^i(t)\cdot(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||) \quad . \end{aligned} \tag{54}$$

These approximate the difference between the extended model (undergiven policy rule, or in given *control structure*) and the optimal solutions in the basic mode. We wish to express the extended structural variations, in (53), via the basic structural variations minus the basic optimal variations:

$$\begin{aligned} \tilde{x}^i(t) &= \bar{x}^i(t) - \hat{\bar{x}}(t) \\ \tilde{u}^i(t) &= \bar{u}^i(t) - \hat{\bar{u}}(t) \quad . \end{aligned} \tag{55}$$

For this purpose, we must compute the basic optimal variations:

$$
\begin{aligned}
\hat{x}(t,\underline{\alpha}) - \hat{x}(t,\underline{a}) &= \hat{\tilde{x}}(t)\cdot(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||) \\
\hat{u}(t,\underline{\alpha}) - \hat{u}(t,\underline{a}) &= \hat{\tilde{u}}(t)\cdot(\underline{\alpha}-\underline{a}) + o(||\underline{\alpha}-\underline{a}||)
\end{aligned}
\qquad (56)
$$

The interrelations among the optimizations of the basic model, the extended model, and the solution of the basic model under given policy are illustrated in Figure 2. The usefulness of the method above lies in the fact that *although the extended model might be difficult to optimize or to solve explicitly under given policy rule, the variational equations that determine the basic optimal variations and basic structural variations are usually much simpler than the extended model itself.* The reason is that these variational equations are solved along the solutions of the basic model, at the parameter value $\underline{a}$. For example, if the extended model contains delayed variables, the variational equations related to the influence of the delay are not delayed themselves.

Observe now that if $\underline{\alpha}$ is a column vector of p parameters, $\tilde{x}^i(t)$ and $\tilde{u}^i(t)$ are, in fact, row vectors of p variations corresponding to these parameter changes; thus $x^{iT}(t)x^i(t)$ and $u^{iT}(t)u^i(t)$ are pxp matrices. The matrix $\hat{S}^i_{\underline{a}\,\underline{a}}$ thus has the form[14)]

$$\hat{S}^i_{\underline{a}\,\underline{a}} = \int_0^\infty e^{-rt}(x^{iT}(t)x^i(t) + qu^{iT}(t)u^i(t))dt \qquad (57)$$

In order to analyse fully the approximation (51), we should actually compute the matrix $\hat{S}^i_{\underline{a}\,\underline{a}}$, estimate independently some bounds on the changes $\underline{\alpha}-\underline{a}$, and approximate upper bounds for (51) by eigenvalue analysis. Though such an analysis is straightforward, yet, for the sake of simplicity, we shall only compute the diagonal elements of $\hat{S}^i_{\underline{a}\,\underline{a}}$, use some upper bounds on the elements of $\underline{\alpha}-\underline{a}$, and compare the results for each element of this vector independently. The reason for this simplification is that we do not wish to concentrate on joint effects of the changes of various components of $\underline{\alpha}$. Rather, we would like to investigate the separate effects of each policy rule on the welfare loss approximation. It will be shown that these effects differ widely from one policy rule to the other.[15)]

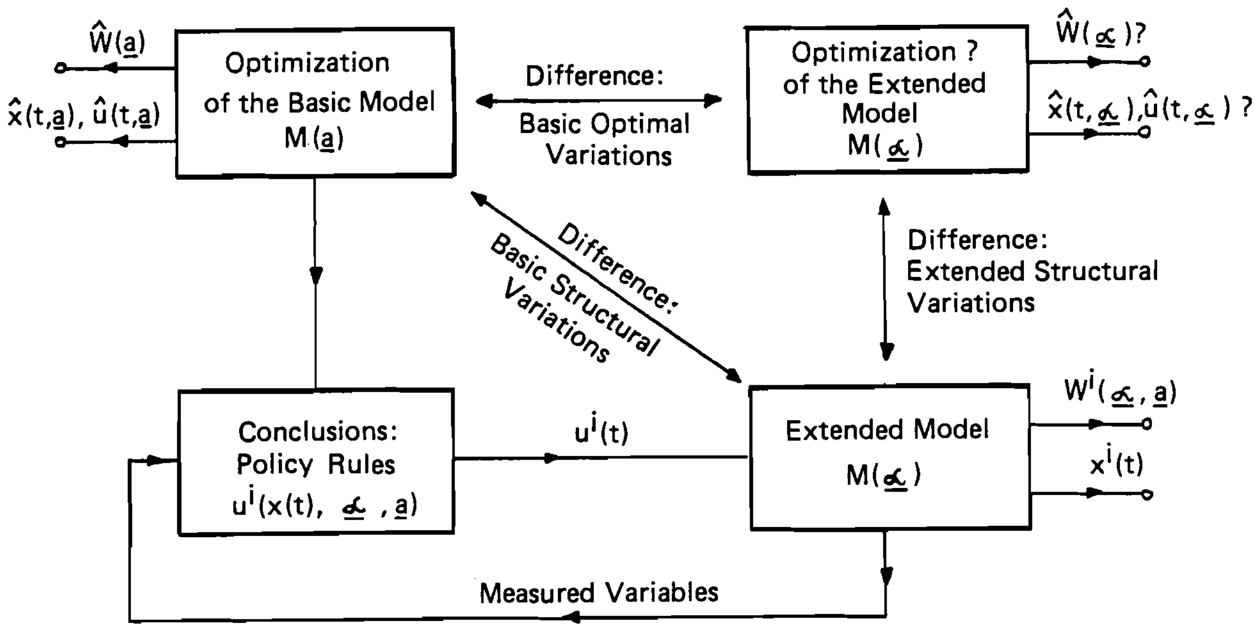

Figure 2. Block-diagram representation of the relations between the basic model, an extended model together with a policy rule, and an optimized extended model.

5. COMPARISON OF POLICY RULES ON EXTENDED MODELS

5a. Bounds of Parameter Changes

If the methods discussed here were to be applied empirically, bounds on parameter changes β, γ, τ would have to be estimated by econometric methods. In this theoretical paper, however, we simply assume relative bounds on these parameters. The assumed bound on the changes of β is

$$|\beta|_{max} = \frac{1}{2}\frac{C}{rq}|x_0 - \hat{x}_\infty| \qquad (58)$$

i.e., the error in evaluating u_n, is related to the divergence between the actual and long-term expected rates of inflation, and thus also to the divergence between the actual and natural rate of unemployment (since $\beta=\tilde{u}_n-u_n$ and $u_n=\hat{u}_\infty=\frac{C}{rq}\hat{x}_\infty$ see the appendix). We assume also that the error in γu_n^2 is of similar nature

$$|\gamma|_{max} = \frac{1}{4}\frac{C}{rqu_n^2}|x_0 - \hat{x}_\infty| \tag{59}$$

Finally, for the delay τ we assume a relative bound

$$\tau_{max} = \frac{1}{4\upsilon_{max}} = \frac{q^{\frac{1}{2}}}{4C} \tag{60}$$

since the relative effects of an overlooked delay τ are characterized by the number $\upsilon\tau$ (see the appendix).

5b. Sensitivity to Errors in Estimating Natural Unemployment Rate

In this simple case we know the optimal solutions for the extended model, just substituting u_n by $\tilde{u}_n$ in (25), (26), (27). Thus, the basic optimal variations are obtained immediately

$$\hat{x}(t,\beta) - \hat{x}(t,\underline{a}) = \beta(1-e^{-\upsilon t})\cdot\frac{rq}{C} \quad ; \quad \hat{\hat{x}}(t) = \frac{rq}{C}(1-e^{-\upsilon t})$$
$$\hat{u}(t,\beta) - \hat{u}(t,\underline{a}) = \beta(1-rKe^{-\upsilon t}) \quad ; \quad \hat{\hat{u}}(t) = 1 - rKe^{-\upsilon t} \tag{61}$$

with $\beta=\tilde{u}_n-u_n$; we assume $\beta\neq 0$, $\gamma=0$, $\tau=0$ in this subsection and denote $\hat{x}(t,\underline{\alpha})=\hat{x}(t,\beta)$, $\hat{u}(t,\underline{\alpha})=\hat{u}(t,\beta)$ in this case. The basic structural variations depend on an assumed policy rule. Consider first the family of closed-loop controls (41) and substitute it into the extended state equation (12) to obtain (assuming $\tilde{C}=C$):

$$\dot{x}^\lambda(t) = -C(u^\lambda(t) - \tilde{u}_n) = -C(\hat{u}(t,\underline{a}) - \tilde{u}_n + \lambda\frac{CK}{q}(x^\lambda(t,\underline{a}))) \quad ;$$
$$x^\lambda(0) = x_0 \tag{62}$$

Again, in this relatively simple case we can solve the extended model analytically:

$$x^{\lambda}(t) = \beta \frac{q}{\lambda CK}(1 - e^{-\lambda \upsilon t}) + \hat{x}(t,\underline{a}) \tag{63}$$

and determine the basic structural variations

$$\begin{aligned} \frac{1}{\beta}(x^{\lambda}(t) - \hat{x}(t,\underline{a})) &= \bar{x}^{\lambda}(t) = \frac{q}{\lambda CK}(1 - e^{-\lambda \upsilon t}) \\ \frac{1}{\beta}(u^{\lambda}(t) - \hat{u}(t,\underline{a})) &= \bar{u}^{\lambda}(t) = 1 - e^{-\lambda \upsilon t} \end{aligned} \tag{64}$$

which, in turn, imply together with (61), (56) the extended structural variations

$$\begin{aligned} \tilde{x}^{\lambda}(t) &= \frac{rq}{C}\left(\frac{1}{\lambda rK}(1 - e^{-\lambda \upsilon t}) - (1-e^{-\upsilon t})\right) \\ \tilde{u}^{\lambda}(t) &= rKe^{-\upsilon t} - e^{-\lambda \upsilon t} \end{aligned} \tag{65}$$

Now we can compute the second-order derivative of the welfare losses $S^{\lambda}(\underline{\alpha},\underline{a})$ with respect to β:

$$\begin{aligned} \hat{S}^{\lambda}_{\beta\beta} &= \int_0^{\infty} e^{-rt}((\tilde{x}^{\lambda}(t))^2 + q(\tilde{u}^{\lambda}(t))^2)dt \\ &= \frac{q}{r}(1-rK)^2\left(2\,\frac{(1-\lambda)(1+rK+\lambda(1-2rK))}{(rK+\lambda(1-rK))(rK+2\lambda(1-rK))} + 1\right) \end{aligned} \tag{66}$$

In this simple case the function $S^{\lambda}(\underline{\alpha},\underline{a})$ is a quadratic function of β; hence the welfare losses associated with β are $\frac{1}{2}\beta^2\hat{S}^{\lambda}_{\beta\beta}$. In order to obtain a coefficient of robustness which does not depend on units of measurement, we use the ratio of $\beta^2_{max}\,\hat{S}^{\lambda}_{\beta\beta}$ and $\Delta^2\hat{W}$ (the ratio of losses due to inexact parameter estimation to natural losses due to an initial disequilibrium

$$\hat{R}^{\lambda}_{\beta} = \frac{\beta^2_{max}\ \hat{S}^{\lambda}_{\beta\beta}}{\Delta^2 \hat{W}} = \frac{(1-rK)^3}{4(rK)^3}\left(1+2\ \frac{(1-\lambda)(1+rK+\lambda(1-2rK))}{(rK+\lambda(1-rK))(rK+2\lambda(1-rK))}\right) \tag{67}$$

If $\lambda=0$, for the open-loop policy, this robustness coefficient takes the form

$$\hat{R}^{0}_{\beta} = \frac{(1-rK)^3(2+2rK+(rK)^2)}{4(rK)^5} \tag{68}$$

For $\lambda=1$, the classical closed-loop policy, we obtain

$$\hat{R}^{c}_{\beta} = \frac{(1-rK)^3}{4(rK)^3} \tag{69}$$

and for $\lambda\to\infty$, the optimal trajectory tracking policy:

$$\hat{R}^{t}_{\beta} = \frac{1-rK}{4rK} \tag{70}$$

We can also determine the feedback coefficient $\hat{\lambda}_{\beta}$ that minimizes (67) and thus provides for the best robustness in this policy family:

$$\hat{\lambda}_{\beta} = \frac{(((rK)^2+2rK-2)+9(rK)^2(1-rK))^{\frac{1}{2}}-((rK)^2+2rK-2}{3rK(1-rK)} \tag{71}$$

All these results depend on the parameters rK, which are determined, via (23), by the parameters $\frac{c^2}{rq} = \nu^2$, the squared ratio of the maximal speed of controlling inflation and unemployment, $\upsilon_{max} = \frac{c}{q^{\frac{1}{2}}}$, to the time discount rate r. Thus, when analysing robustness coefficient graphically, we shall employ the parameter ν^2 instead of $rK=((1+4\nu^2)^{\frac{1}{2}}-1)/2\nu^2$.

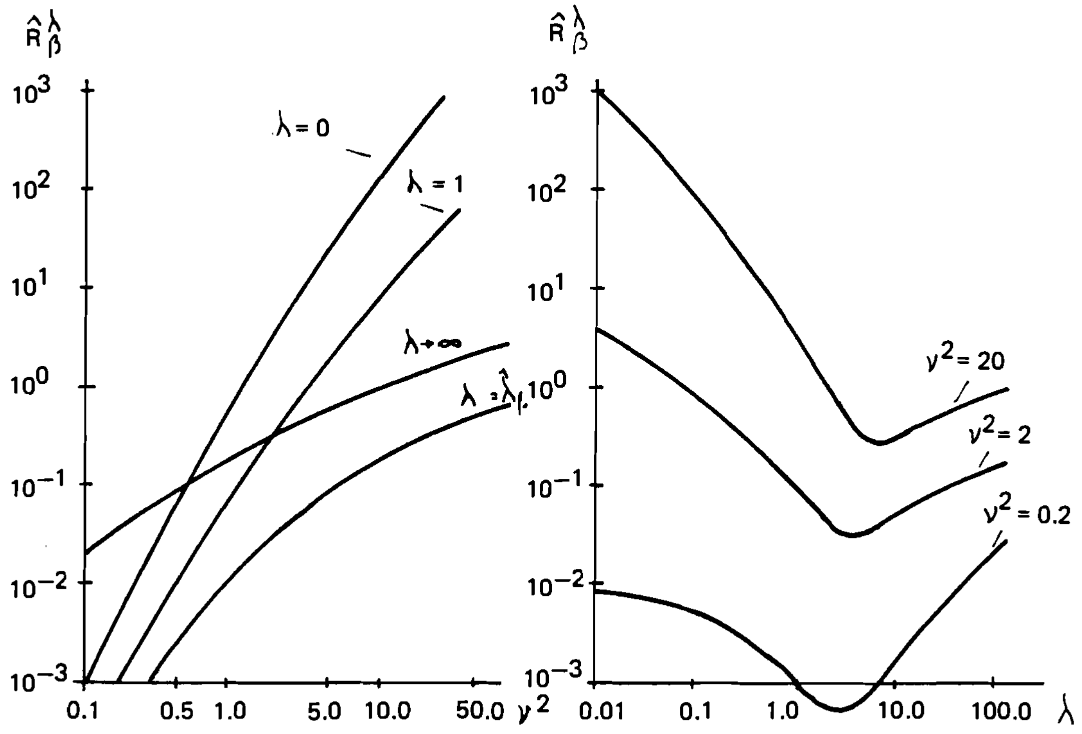

Figure 3. Dependence of the robustness coefficient $\hat{R}^{\lambda}_{\beta}$ on ν^2 and λ.

The graphs of $\hat{R}^{\lambda}_{\beta}$ as a function of ν^2 and λ, presented in Figure 3 indicate dramatic changes of robustness--upto 10^3 times and over--depending on the choice of λ. This can be interpreted in the following way. When the attempt to maintain precisely a predetermined path of unemployment (at $\lambda=0$) or a predetermined path of expected inflation (at $\lambda\to\infty$) is made, even slight errors in the evaluation of the natural unemployment rate u_n can result in large welfare losses. However, if an appropriate combination of unemployment and expected inflation, $\hat{u}(t,\underline{a})-\lambda \frac{CK}{q} \hat{x}(t,\underline{a})$ with λ close to $\hat{\lambda}_{\beta}$, is chosen as a policy target, small errors in the evaluation of u_n do not cause significant welfare losses. We, however, see that even the choice of λ cannot reduce welfare losses sufficiently, if the parameter ν^2 becomes much larger than 20--which would correspond to the desire of obtaining fast results in controlling inflation (high ν/r, see Figure 1) by attaching a small weight q to unemployment.

Consider now the open-loop Hamiltonian maximizing policy:

$$u^{ho}(\hat{\zeta}(t,\underline{a}),\underline{\alpha})$$
$$= \underset{u(t)}{\operatorname{argmax}} \; (1-\tfrac{1}{2}x^2(t) - \tfrac{q}{2}u^2(t) + \hat{\zeta}(t,\underline{a}) \cdot C \cdot (u(t) - \tilde{u}_n)) \tag{72}$$

The fact that we measure the accurate current speed of change of inflationary expectations, $\dot{x}(t) = -C(u(t) - \tilde{u}_n)$, does not influence the maximization in (72); no matter what $\tilde{u}_n$ is taken, we obtain

$$u^{ho}(\hat{\zeta}(t,\underline{a}),\underline{\alpha}) = \frac{C}{q}\hat{\zeta}(t,\underline{a}) = \hat{u}(t,\underline{a}) = u^o(t,\underline{a}) \tag{73}$$

Thus, the open-loop Hamiltonian maximizing policy is equivalent to the simple open-loop policy, if no changes of the functional form of the Phillips curve are considered. Similarly, it can be shown for this case that

$$u^{hc}(\hat{\zeta}(x(t),\underline{a}),\underline{\alpha}) = u^c(t,\underline{a}) \tag{74}$$

the closed-loop Hamiltonian maximizing policy is equivalent to the classical closed-loop policy. Since the simple open-loop and the classical closed-loop are not the best choices from our given set of policy options, it is not desirable to pursue Hamiltonian maximizing policies.

The benefit-to-cost optimizing policies perform even more poorly. By setting $f(u(t),\underline{\alpha}) = -C(u(t)-u_n-\beta)$ and computing the maximum of (43) we obtain[16)]

$$u^{fo}(t) = \beta + \hat{u}(t,\underline{a}) + \frac{U(\hat{x},\hat{u},\underline{a})}{q\hat{u}(t,\underline{a})} - \left(\left(\beta + \hat{u}(t,\underline{a}) + \frac{U(\hat{x},\hat{u},\underline{a})}{q\hat{u}(t,\underline{a})}\right)^2 - \frac{2}{q}\left(1 - \frac{1}{2}(x^{fo}(t))^2\right)\right)^{\frac{1}{2}} \tag{75}$$

where $u^{fo}(t)=u^{fo}(\hat{\zeta}(t,\underline{a}),\ \hat{\zeta}_t(t,\underline{a}),\underline{\alpha})$ and $U(\hat{x},\hat{u},\underline{a})=1-\frac{1}{2}\hat{x}^2(t,\underline{a}) - \frac{q}{2}\hat{u}^2(t,\underline{a})$ for the sake of notational economy; $x^{fo}(t)$ denotes here the state $x(t)$ measured in the extended model under the policy rule (75). Now, if we substitute (75) into the extended model $\dot{x}^{fo}(t)=-C(u^{fo}(t)-u_n-\beta)$, we obtain a nonlinear differential equation

$$\dot{x}^{fo}(t) = -C(\hat{u}(t,\underline{a}) - u_n + \frac{U(\hat{x},\hat{u},\underline{a})}{q\hat{u}(t,\underline{a})} - ((\beta + \hat{u}(t,\underline{a}) + \frac{U(\hat{x},\hat{u},\underline{a})}{q\hat{u}(t,\underline{a})})^2 - \frac{2}{q}(1 - \frac{1}{2}(x^{fo}(t))^2))^{\frac{1}{2}}) \quad (76)$$

As noted in the preceding section, we do not have to solve this equation. It is sufficient to linearize it in β at $\beta=0$, $x^{fo}(t)=\hat{x}(t,\underline{a})$ to obtain the equations for basic structural variations[17].

$$\dot{\bar{x}}^{fo}(t) = C\frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})}\bar{x}^{fo}(t) + C(\frac{qu^2(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})} + 1 \quad ;$$

$$\bar{x}^{fo}(0) = 0 \quad (77)$$

$$\bar{u}^{fo}(t) = -\frac{q\hat{u}^2(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})} - \frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})}\bar{x}^{fo}(t)$$

When taking into account (56), (61), we derive also the equations for extended structural variations

$$\dot{\tilde{x}}^{fo}(t) = C\frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})}\tilde{x}^{fo}(t) + C\eta^{fo}(t,\underline{a}) \quad ; \quad \tilde{x}^{fo}(0) = 0$$

$$\tilde{u}^{fo}(t) = -\frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})}\tilde{x}^{fo}(t) - \eta^{fo}(t,\underline{a}) \quad (78)$$

$$\eta^{fo}(t,\underline{a}) = 1 + \frac{q\hat{u}^2(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})} - rKe^{-\upsilon t} + \frac{rq}{C}\frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})}(1-e^{-\upsilon t})$$

The differential equations in both (77) and (78) are unstable if $U(\hat{x},\hat{u},\underline{a})>0$ which occurs when $\hat{x}(t,\underline{a}),\hat{u}(t,\underline{a})$ are sufficiently small. Thus, also the equation (76) is unstable[18], if

$U(\hat{x},\hat{u},\underline{a})>0$. In other words, the benefit-to-cost maximizing policy invariably leads to unstable results wherever there are errors in the evaluation of the natural unemployment rate. On the other hand, this does not imply that the welfare loss under this policy is necessarily infinite; it might be finite if $r \gg C\,\frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})}$ for all t. However, even in such a case, when computing $\tilde{x}^{fo}(t)$, $\tilde{u}^{fo}(t)$ under somewhat simplifying assumption that $x_0=\hat{x}_\infty$ (see the appendix) it can be shown that

$$(\tilde{x}^{fo}(t))^2 > (\tilde{x}^{o}(t))^2 \quad ; \quad (\tilde{u}^{fo}(t)^2 > (\tilde{u}^{o}(t))^2 \qquad (79)$$

which implies that the welfare loss under the benefit-to-cost maximizing policy is larger than under the simple open-loop policy. Since a similar result can be derived for the closed-loop benefit-to-cost maximizing policy, we conclude that these policies are not desirable ways of dealing with the problems of inflation and unemployment within our analytical context.[19)]

5c. Sensitivity to Mistaken Functional Specification

We assume here that $\underline{\alpha}=(0,\gamma,0)$, i.e., the extended model takes the form:

$$\dot{x}(t) = -C(u(t) - u_n - \frac{\gamma}{2}(u(t) - u_n)^2) \quad ; \quad x(0) = x_0 \qquad (80)$$

The problem of maximizing (10) subject to (80) does not admit an analytic solution; however, this problem has solutions for sufficiently small γ and these solutions are differentiable in γ. This can be seen by writing the Hamiltonian function for this problem

$$H(\zeta(t),x(t),u(t)) = 1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t) + \zeta(t)\cdot C$$
$$\cdot (u(t) - u_n - \frac{\gamma}{2}(u(t) - u_n)^2) \qquad (81)$$

and the necessary conditions of optimality

$$H_u = 0 \Leftrightarrow qu(t) + \gamma C\zeta(t)(u(t) - u_n) - C\zeta(t) = 0 \tag{82}$$

$$\dot{\zeta}(t) - r\zeta(t) = H_x \Leftrightarrow \dot{\zeta}(t) = r\zeta(t) - x(t) \tag{83}$$

$$\dot{x}(t) = -H_\zeta \Leftrightarrow \dot{x}(t) = -C(u(t) - u_n - \frac{\gamma}{2}(u(t) - u_n)^2) \quad ;$$

$$x(0) = x_0 \tag{84}$$

and observing that the Hamiltonian function remains concave for sufficiently small γ and a corresponding Riccati equation has a backward stable solution which depends differentiably on the parameter γ; similarly, the solutions of (84), (82) depend then differentiably on γ. However, we omit these details here, and show only how to derive basic variational equations.

Denote the solutions of this problem by $\hat{u}(t,\gamma)=\hat{u}(t,\underline{a})+\gamma\hat{\bar{u}}(t)+o(\gamma)$, $\hat{x}(t,\gamma)=\hat{x}(t,\underline{a})+\gamma\hat{\bar{x}}(t)+o(\gamma)$, $\hat{\zeta}(t,\gamma)=\hat{\zeta}(t,\underline{a})+\gamma\hat{\bar{\zeta}}(t)+o(\gamma)$ and rewrite the equations (82), (83), (84) as

$$[q\hat{u}(t,\underline{a}) - C\hat{\zeta}(t,\underline{a})] + \gamma(q\hat{\bar{u}}(t) - C\hat{\bar{\zeta}}(t) + C\hat{\zeta}(t,\underline{a})(\hat{u}(t,\underline{a}) - u_n)) + o(\gamma) = 0 \tag{85}$$

$$[\dot{\hat{\zeta}}(t,\underline{a}) - r\hat{\zeta}(t,\underline{a}) + \hat{x}(t,\underline{a})] + \gamma\dot{\hat{\bar{\zeta}}}(t) = \gamma(r\hat{\bar{\zeta}}(t) - \hat{\bar{x}}(t)) + o(\gamma) \tag{86}$$

$$[\dot{\hat{x}}(t,\underline{a}) + C(\hat{u}(t,\underline{a}) - u_n)] + \gamma\dot{\hat{\bar{x}}}(t) = -\gamma C(\hat{\bar{u}}(t) - \frac{1}{2}(\hat{u}(t,\underline{a}) - u_n)^2) + o(\gamma) \quad ; \quad \hat{\bar{x}}(0) = 0 \tag{87}$$

Since the expressions in square brackets are zeros--cf. (17), (18), (19)--we subdivide the remainders by γ and let $\gamma \to 0$ to obtain

$$\hat{\bar{u}}(t) = \frac{C}{q}\hat{\bar{\zeta}}(t) - \hat{u}(t,\underline{a}) \cdot (\hat{u}(t,\underline{a}) - u_n) \tag{88}$$

$$\dot{\hat{\hat{\zeta}}}(t) = r\hat{\hat{\zeta}}(t) - \hat{\hat{x}}(t) \tag{89}$$

$$\dot{\hat{\hat{x}}}(t) = -C(\frac{C}{q}\hat{\hat{\zeta}}(t) - \frac{1}{2}(\hat{u}(t,\underline{a}) - u_n)^2 - \hat{u}(t,\underline{a})\cdot(\hat{u}(t,\underline{a}) - u_n)) \quad ;$$

$$\hat{\hat{x}}(0) = 0 \tag{90}$$

since $\frac{C}{q}\hat{\zeta}(t,\underline{a})=\hat{u}(t,\underline{a})$; the last equation is obtained by taking into account (88). Observe that, if $\hat{u}(t,\underline{a})=u_n$, under initial equilibrium conditions, then $\hat{\hat{\zeta}}(t)\equiv 0$, $\hat{\hat{x}}(t)\equiv 0$. If $\hat{u}(t,\underline{a})-u_n$ increases, which might be caused by an increase of $x_0-\hat{x}_\infty$, then also $\hat{\hat{\zeta}}(t)$ and $\hat{\hat{x}}(t)$ will, in general, increase. Thus, the sensitivity of optimal solutions to the parameter γ increases with the distance from equilibrium. While it is possible to solve the equations (89), (90) in their general form, we can significantly simplify computations by assuming approximately that $\hat{u}(t,\underline{a})-u_n \approx u_n$, that is, by solving these equations at a standard distance from equilibrium. In this case, we obtain

$$\hat{\hat{x}}(t) \approx \frac{5}{2}u_n^2\frac{rq}{C}(1-e^{-\upsilon t}) \quad ; \quad \hat{\hat{\zeta}}(t) \approx \frac{5}{2}u_n^2\frac{q}{C}(1-rKe^{-\upsilon t}) \quad ;$$

$$\hat{\hat{u}}(t) \approx \frac{1}{2}u_n^2(1-5rKe^{-\upsilon t}) \tag{91}$$

Consider now the family of closed-loop policies (41). The extended model equation under these policies takes the form

$$\dot{x}^\lambda(t) = -C(\hat{u}(t,\underline{a}) - u_n + \lambda\frac{KC}{q}(x^\lambda(t) - \hat{x}(t,\underline{a}))$$
$$-\frac{1}{2}\gamma(\hat{u},(t,\underline{a}) - u_n + \lambda\frac{KC}{q}(x^\lambda(t) - \hat{x}(t,\underline{a}))^2) \tag{92}$$

Again, the solutions of this equation are differentiable functions of γ, $x^\lambda(t)=\hat{x}(t,\underline{a})+\gamma\bar{x}^\lambda(t)+o(\gamma)$, $u^\lambda(t)=\hat{u}(t,\underline{a})+\gamma\bar{u}^\lambda(t)+o(\gamma)$. When linearizing (92) by the same technique as applied for (82), (83), (84), we obtain

$$\dot{\bar{x}}^{\lambda}(t) = -\lambda \frac{KC^2}{q}\bar{x}^{\lambda}(t) + \frac{1}{2}C(\hat{u}(t,\underline{a}) - u_n)^2 \quad ; \quad \bar{x}^{\lambda}(0) = 0 \tag{93}$$

Under the approximative assumption $\hat{u}(t,\underline{a})-u_n \approx u_n$, (93) yields

$$\bar{x}^{\lambda}(t) \approx \frac{1}{2}\frac{u_n^2 q}{\lambda KC}(1-e^{-\lambda\upsilon t}) \quad ; \quad \bar{u}^{\lambda}(t) \approx \frac{1}{2}u_n^2(1-e^{-\lambda\upsilon t}) \tag{94}$$

The basic structural variations $\bar{x}^{\lambda}(t),\bar{u}^{\lambda}(t)$ and the basic optimal variations $\hat{\hat{x}}(t),\hat{\hat{u}}(t)$ determine extended structural variations

$$\tilde{x}^{\lambda}(t) \approx \frac{1}{2}u_n^2\frac{rq}{C}\left(\frac{1}{\lambda rK}(1-e^{-\lambda\upsilon t}) - 5(1-e^{-\upsilon t})\right)$$

$$\tilde{u}^{\lambda}(t) \approx \frac{1}{2}u_n^2(5rKe^{-\upsilon t} - e^{-\lambda\upsilon t}) \tag{95}$$

and, in turn, the second-order derivative of the welfare loss

$$\hat{S}^{\lambda}_{\gamma\gamma} = \int_0^{\infty} e^{-rt}((\tilde{x}^{\lambda}(t))^2 + q(\tilde{u}^{\lambda}(t))^2)dt$$

$$\approx \frac{25u_n^4\cdot rK^2q}{2}\left(1 - \frac{1}{50r^2K^2}\cdot\frac{9r^2K^2+2rK-2+19\lambda(1-rK)rK}{(rK+\lambda(1-rK))(rK+2\lambda(1-rK))}\right) \tag{96}$$

A dimensionless robustness coefficient is

$$\hat{R}^{\lambda}_{\gamma} = \frac{|\gamma|^2_{max}\ \hat{S}^{\lambda}_{\gamma\gamma}}{\Delta^2\hat{W}}$$

$$\approx \frac{25}{32}\ \frac{1-rK}{rK}\left(1 - \frac{1}{50r^2K^2}\cdot\frac{9r^2K^2+2rK-2+19\lambda(1-rK)rK}{(rK+\lambda(1-rK))(rK+2\lambda(1-rK))}\right) \tag{97}$$

and it takes the following forms for $\lambda=0$ (open-loop policy), $\lambda=1$ (classical closed-loop policy) and $\lambda\to\infty$ (optimal trajectory tracking):

$$\hat{R}^{0}_{\gamma} \approx \frac{1-rK}{64r^5K^5}(50r^4K^4 - 9r^2K^2 - 2rK + 2) \tag{98}$$

$$\hat{R}^c_\gamma \approx \frac{1 - rK}{64r^3K^3(2-rK)} (-50r^3K^3+91r^2K^2-2rK+2) \tag{99}$$

$$\hat{R}^t_\gamma \approx \frac{25}{32} \frac{1-rK}{rK} \tag{100}$$

The feedback coefficient $\hat{\lambda}_\gamma$ that minimizes (97), thus providing for the most robust policy from this family, can be determined as

$$\hat{\lambda}_\gamma \approx \begin{cases} 0, \text{ if } 4r^2K^2+3rK-3>0 \Leftarrow rK>0.758 \Leftrightarrow \nu^2<0.42 \\ \dfrac{((9r^2K^2+2rK-2)^2 - 19r^2K^2(4r^2K^2+3rK-3))^{\frac{1}{2}} - (9r^2K^2+2rK-2)}{19rK(1-rK)}, \\ \text{if } \nu^2>0.42. \end{cases} \tag{101}$$

Similarly as in the previous subsection, we represent these results as a function of the parameter $\nu^2=\frac{1-rK}{r^2K^2}$ rather than rK (see Figure 4). The dependence of the robustness coefficient $\hat{R}^\lambda_\gamma$ on λ is rather weak--except for larger ν^2, when the application

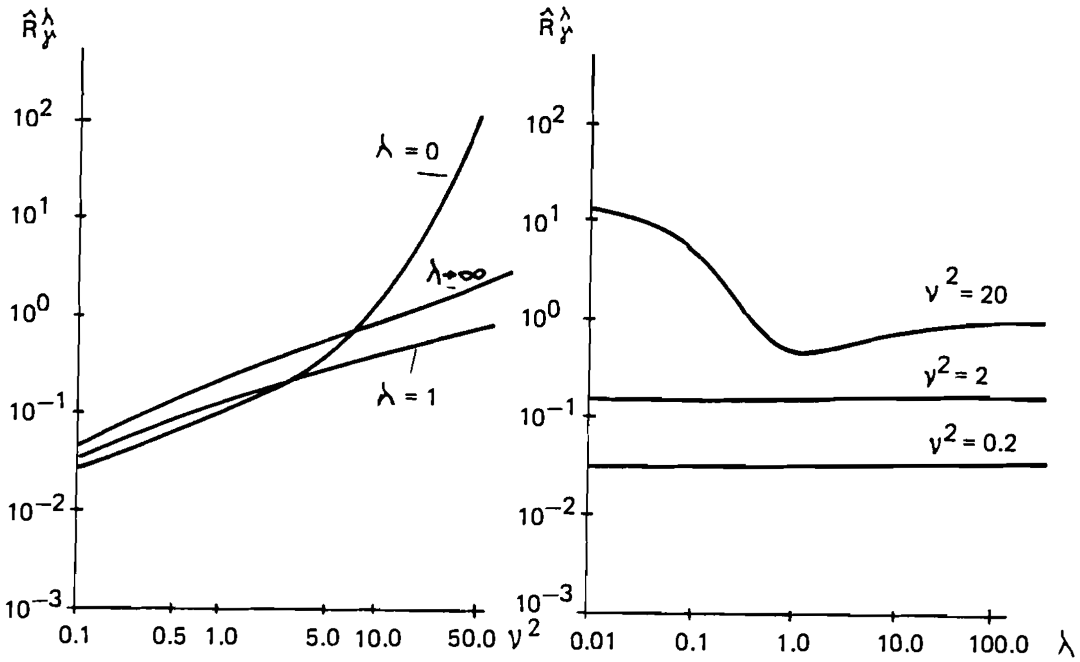

Figure 4. Dependence of the Robustness Coefficient $\hat{R}^\lambda_\gamma$ on ν^2 and λ.

of $\lambda \ll \hat{\lambda}_\gamma$ is not advisable.

However, since $\hat{\lambda}_\beta > \hat{\lambda}_\gamma$ (see Figure 4a) and $\hat{R}_\gamma^\lambda$ does not rise steeply for $\lambda > \hat{\lambda}_\gamma$, we may presume that the feedback coefficient λ should be chosen to $\hat{\lambda}_\beta$, to provide for greater robustness with respect to the uncertainty about natural unemployment rate than with respect to the uncertainty about the functional form of the Phillips curve.

A simplified form for a compromise $\hat{\lambda}$ (that is close to $\hat{\lambda}_\beta$ but satisfies $\hat{\lambda}_\gamma < \hat{\lambda} < \hat{\lambda}_\beta$) can be obtained by assuming

$$\hat{\lambda} = \frac{1}{rK} \tag{102}$$

(see Figure 5a). If this particular $\hat{\lambda}$ is chosen, then the policy target $\hat{u}(t,\underline{a}) - \hat{\lambda}\,\frac{CK}{q}\,\hat{x}(t,\underline{a}) = u(t) - \hat{\lambda}\,\frac{CK}{q}\,x(t)$ can be

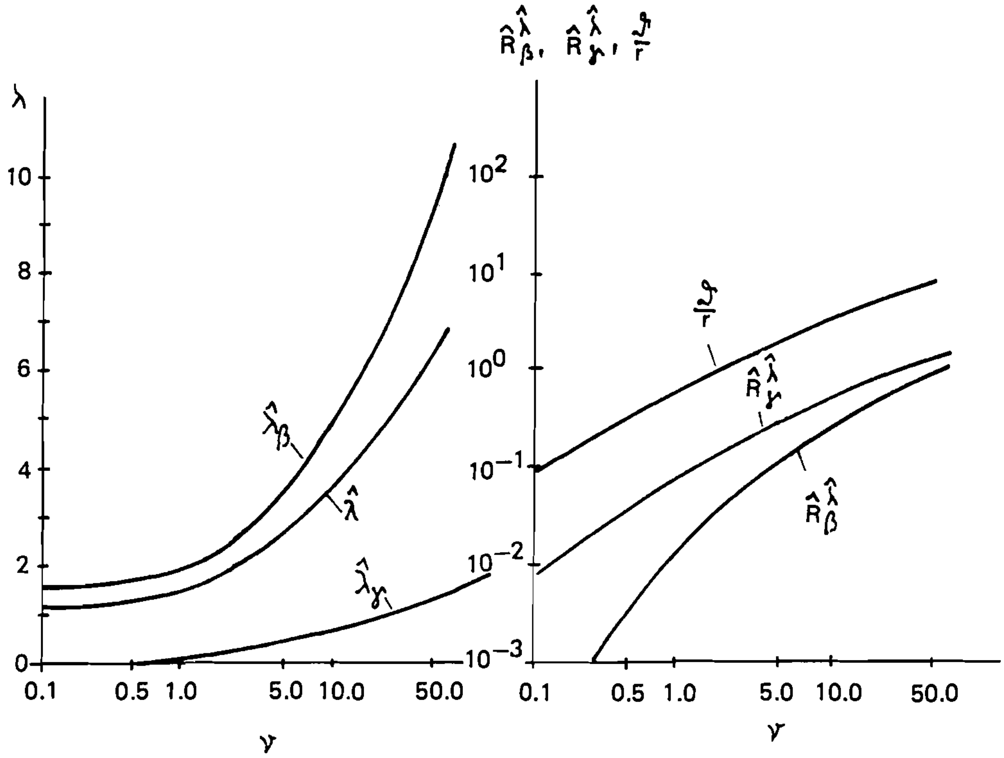

Figure 5. a) Comparison of Optimal Feedback Coefficients $\hat{\lambda}_\beta$, $\hat{\lambda}$, and $\hat{\lambda}_\gamma$;

b) Comparison of Robustness Coefficients $\hat{R}_\beta^{\hat{\lambda}}$ and $\hat{R}_\gamma^{\hat{\lambda}}$ at $\lambda = \hat{\lambda}$ and of the Relative Speed υ/r of Controlling Inflation Towards Its Long-term Value.

transformed to the form

$$u(t) - \frac{C}{rq}x(t) = \varepsilon \cdot (u_n - \frac{C}{rq}x_0)e^{-\upsilon t} \tag{103}$$

where

$$\varepsilon = 1 - rK = \frac{2\nu^2-1+(4\nu^2+1)^{\frac{1}{2}}}{2\nu^2} \quad ; \quad \nu^2 = \frac{C^2}{r^2 q} \quad ; \quad \upsilon = r\cdot\frac{(4\nu^2+1)^{\frac{1}{2}}-1}{2} \tag{104}$$

This means that a robust policy is to choose money supply rate m as to keep the combination of unemployment and the expected inflation, $u(t)-\frac{C}{rq}x(t)$, at the predetermined trajectory $\varepsilon(u_n-\frac{C}{rq}x_0)e^{-\upsilon t}$. The robustness coefficients $R_\beta^{\hat{\lambda}}$ and $R_\gamma^{\hat{\lambda}}$ corresponding to this policy are shown as functions of ν^2 in Figure 5b; parallel to this, Figure 5b also displays the coefficient $\frac{\upsilon}{r}$ -- the relative speed of controlling inflation towards its long-term value--from Figure 1.

The graphs in Figure 4b should be interpreted as follows. Once a coefficient $\hat{\lambda}$ and thus a robust policy form has been chosen, the robustness coefficients depend on basic parameters of the model. The aggregate parameter ν^2 can be inversely influenced by the choice of weighting coefficient q at the unemployment rate u(t) in the social welfare functional (10). If this coefficient is large enough, such that the resulting ν^2 is small, say, $\nu^2 \leq 2$, then the attainable robustness coefficients are quite good. However, for $\nu^2 \leq 2$ we have the relative speed $\upsilon/r \leq 1$, which means that we control inflation slower than the time discount rate--a solution that is socially not acceptable. To increase the relative speed, we must go to $\nu^2 > 2$; at $\nu^2 = 20$ we obtain $\upsilon/r = 4$, which with r=0.1/year would give $\upsilon = 0.4$/year, still a moderate speed of controlling inflation (a reduction of circa 40% of the original distance $x_0 - \hat{x}_\infty$ in a year). However, the robustness of the solution suffers from the choice of the higher relative speed. If some impetous decision-maker would promise his electorate to bring the inflation down in a year (say to achieve $\upsilon/r\approx 10$) he would have to go as far as to $\nu^2\approx 100$, where the robustness coefficients are rather bad; errors in model specification might then easily result in social welfare losses several times higher than the social welfare losses due to the initial difference $x_0-\hat{x}_\infty$.

Admittedly, the results shown in Figure 5b are based on the assumed bounds $|\beta|_{max}$ and $|\gamma|_{max}$. While these bounds do not influence the conclusions on the choice of the feedback parameter λ, they do influence the actual values of robustness coefficients. Thus, an impetous decision-maker could require more precise econometric estimations of the bounds on parameter uncertainty. Our theoretical exercise does not permit us to draw conclusions about the desirable degree of accuracy for parameter evaluations; it provides only possible guidelines for the econometric pursuit of such conclusions. However, given an attainable degree of accuracy, the results will have the same qualitative character: the faster we try to achieve the long-term goals, the more sensitive to model errors are the results of our policy.

For the sake of brevity, we omit here the discussion of Hamiltonian-maximizing and benefit-to-cost maximizing policies, although the former policies might make sense in the case of a mistaken functional speicification.

5d. Sensitivity to Delays

We assume here $\underline{\alpha}=(0,0,\tau)$, that is, the extended model takes the form

$$\dot{x}(t) = -C(u(t-\tau) - u_n) \quad ;$$

$$x(0) = x_0 \quad , \quad u(t) \text{ - given for } t\in[-\tau;0) \qquad (105)$$

The problem of maximizing (10) subject to (105) is, in fact, not very difficult to solve (see the appendix). However, this is due to the special form of the problem and we shall proceed as if we did not know its explicit solutions. We can do so, because we know its solutions if $\tau=0$ (the basic model solutions) and we need only linear approximations of its solutions for small τ. We denote the 'unknown' solutions by $\hat{x}(t,\tau)$, $\hat{u}(t,\tau)$, $\hat{\zeta}(t,\tau)$; assuming their differentiability[20] in τ at $\tau=0$, we postulate $\hat{x}(t,\tau)=\hat{x}(t,\underline{a})+\tau\hat{\bar{x}}(t)+o(\tau)$, $\hat{u}(t,\tau)=\hat{u}(t,\underline{a})+\tau\hat{\bar{u}}(t)+o(\tau)$, $\hat{\zeta}(t,\tau)=\hat{\zeta}(t,\underline{a})+\tau\hat{\bar{\zeta}}(t)+o(\tau)$.

The necessary conditions of optimality for problems with delays in control (see, for example, Wierzbicki 1970) require that the following shifted Hamiltonian function is maximized

$$\tilde{H}(\zeta(t+\tau),\ x(t)\ ,u(t)) = 1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t) + \zeta(t+\tau)\cdot C(u(t) - u_n) \tag{106}$$

where the delay in control is compensated by a forward time shift (of both $\zeta(t)$ and $u(t-\tau)$); the adjoint equation for the shadow price $\zeta(t)$, however, remains in its classical form if there is no delay in state variables. Thus, the necessary conditions of optimality are:

$$\tilde{H}_u = 0 \Leftrightarrow \hat{u}(t,\tau) = \frac{C}{q}\hat{\zeta}(t+\tau,\tau) \tag{107}$$

$$\dot{\hat{\zeta}}(t,\tau) - r\hat{\zeta}(t,\tau) = H_x \Leftrightarrow \dot{\hat{\zeta}}(t,\tau) = r\hat{\zeta}(t,\tau) - \hat{x}(t,\tau) \tag{108}$$

$$\begin{gathered}\dot{\hat{x}}(t,\tau) = -H_\zeta \Leftrightarrow \dot{\hat{x}}(t,\tau) = -C(\hat{u}(t-\tau,\tau) - u_n)\ ; \\ \hat{x}(0,\tau) = x_0\ ,\ \hat{u}(t,\tau) \text{ - given for } t\in[-\tau;0)\end{gathered} \tag{109}$$

Now subtract the terms $\hat{u}(t,\underline{a}) = \frac{C}{q}\hat{\zeta}(t,\underline{a})$ from both sides of (107) and use the postulated forms of $\hat{u}(t,\tau)$, $\hat{\zeta}(t,\tau)$ to obtain:

$$\begin{aligned}\tau\hat{\bar{u}}(t) + o(\tau) &= \frac{C}{q}(\hat{\zeta}(t+\tau,\tau) - \hat{\zeta}(t,\tau) + \hat{\zeta}(t,\tau) - \hat{\zeta}(t,\underline{a})) \\ &= \tau\frac{C}{q}\left(\frac{\hat{\zeta}(t+\tau,\tau)-\hat{\zeta}(t,\tau)}{\tau} + \hat{\bar{\zeta}}(t) + \frac{o(\tau)}{\tau}\right)\end{aligned} \tag{110}$$

When subdividing by τ and letting $\tau\to 0$ (observe that $\dot{\hat{\zeta}}(t,0)=\dot{\hat{\zeta}}(t,\underline{a})$) the equation for the basic optimal control variation is obtained:

$$\hat{\hat{u}}(t) = \frac{C}{q}(\dot{\hat{\zeta}}(t,\underline{a}) + \hat{\hat{\zeta}}(t)) \tag{111}$$

The same technique can be used for deriving the equation for the basic optimal state variation, which takes the form

$$\dot{\hat{\hat{x}}}(t) = -C(\hat{\hat{u}}(t) - \dot{\hat{u}}(t,\underline{a})) \quad ; \quad \hat{\hat{x}}(0) = 0 \ , \tag{112}$$

while the equation for the basic optimal costate variation is easily derived as

$$\dot{\hat{\hat{\zeta}}}(t) = r\hat{\hat{\zeta}}(t) - \hat{\hat{x}}(t) \tag{113}$$

All these variational equations *do not contain delayed terms*, although they approximate a problem with delays; they are therefore easy to solve.

Observe now that $\dot{\hat{u}}(t,\underline{a})=\frac{C}{q}\dot{\hat{\zeta}}(t,\underline{a})$; substituting this equation into (111), (112), we obtain:

$$\dot{\hat{\hat{x}}}(t) = -\frac{C^2}{q}\hat{\hat{\zeta}}(t) \tag{114}$$

Since the system of equations (113), (114) is homogeneous, its solutions are of the form $\hat{\hat{\zeta}}(t)=K\hat{\hat{x}}(t)$ where K is defined as in (21), (23); but if $\dot{\hat{\hat{x}}}(t)=\frac{C^2}{q}K\hat{\hat{x}}(t)$ and $\hat{\hat{x}}(0)=0$, then $\hat{\hat{x}}(t)\equiv 0$ and $\hat{\hat{\zeta}}(t)\equiv 0$. Thus, we have the basic optimal variations

$$\hat{\hat{u}}(t) = -\frac{C}{q}\dot{\hat{\zeta}}(t,\underline{a}) = -\upsilon\frac{CK}{q}(x_0 - \hat{x}_\infty)e^{-\upsilon t} \quad ; \quad \hat{\hat{x}}(t) \equiv 0 \ ,$$

$$\hat{\hat{\zeta}}(t) \equiv 0 \tag{115}$$

Consider now the family (41) of closed-loop policies. The extended model equation under these policies takes the form:

$$\dot{x}^{\lambda}(t) = -C(\hat{u}(t-\tau,\underline{a}) - u_n + \lambda \frac{CK}{q}(x^{\lambda}(t-\tau) - \hat{x}(t-\tau,\underline{a}))) \quad ;$$

$$x^{\lambda}(t) - \text{given by (105) for } t\in[0;\tau] \tag{116}$$

Provided that the equation (116) *is stable,* we can approximate its solution as a differentiable function of time. By subtracting the basic model equation from both sides of (116), subdividing by τ and letting $\tau\to 0$, we obtain the equation for basic structural variations

$$\dot{\bar{x}}^{\lambda}(t) = -\lambda\upsilon\bar{x}^{\lambda}(t) + C\dot{\hat{u}}(t,\underline{a}) \quad ; \quad \bar{x}^{\lambda}(0) = 0 \tag{117}$$

which yields the solution

$$\bar{x}^{\lambda}(t) = \frac{\upsilon}{\lambda-1}(x_0 - \hat{x}_{\infty})(e^{-\lambda\upsilon t} - e^{-\upsilon t}) \tag{118}$$

and while taking into account (41)

$$\bar{u}^{\lambda}(t) = \lambda\frac{CK}{q}\bar{x}^{\lambda}(t) = \frac{\lambda}{\lambda-1}\upsilon\frac{CK}{q}(x_0 - \hat{x}_{\infty})(e^{-\lambda\upsilon t} - e^{-\upsilon t}) \tag{119}$$

Thus, the extended structural variations are

$$\tilde{x}^{\lambda}(t) = \bar{x}^{\lambda}(t) - \hat{\bar{x}}(t) = \frac{\upsilon}{\lambda-1}(x_0 - \hat{x}_{\infty})(e^{-\lambda\upsilon t} - e^{-\upsilon t})$$

$$\tag{120}$$

$$\tilde{u}^{\lambda}(t) = \bar{u}^{\lambda}(t) - \hat{\bar{u}}(t) = \frac{CK}{q}\upsilon(x_0 - \hat{x}_{\infty})(\frac{\lambda}{1-\lambda}e^{-\lambda\upsilon t} - \frac{1}{1-\lambda}e^{-\upsilon t})$$

and the second-order derivative of the welfare loss is determined as

$$\hat{S}^{\lambda}_{\tau\tau} = \int_0^{\infty} e^{-rt}((\tilde{x}^{\lambda}(t))^2 + q(\tilde{u}^{\lambda}(t))^2)dt$$

$$= r(x_0 - \hat{x}_{\infty})^2 \frac{(1-rK)^3}{rK(rK+2\lambda(1-rK))} \tag{121}$$

If we assume $\tau_{max} = \frac{1}{4\upsilon_{max}} = \frac{q^{\frac{1}{2}}}{4C}$, then the robustness coefficient takes the form

$$\hat{R}^{\lambda}_{\tau} = \frac{\tau^2_{max}\ \hat{S}^{\lambda}_{\tau\tau}}{\Delta^2 \hat{W}} = \frac{(1-rK)^2}{4(rK+2\lambda(1-rK))} \tag{122}$$

Observe that this robustness coefficient decreases monotonically with λ. However, if we assume $\lambda=\hat{\lambda}$, we obtain relatively small values of $\hat{R}^{\hat{\lambda}}_{\tau}$ as shown in Figure 6.

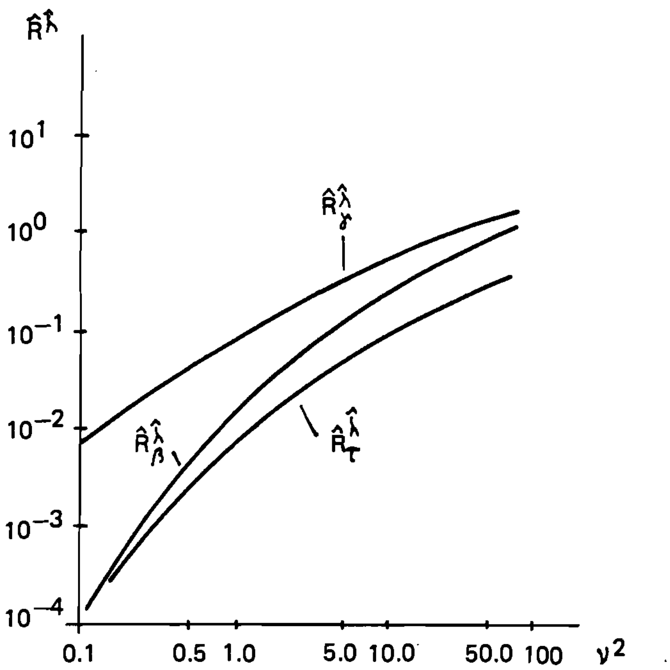

Figure 6. Comparison of Robustness Coefficients $\hat{R}^{\lambda}_{\beta}, \hat{R}^{\lambda}_{\tau}$ and $\hat{R}^{\lambda}_{\tau}$, with $\lambda=\hat{\lambda}$.

Thus, we can conclude that *the omission of delays in the basic model does not jeopardise the successful application of closed-loop policies to the extended model with delays*--provided that the extended model remains stable and the approximation of the welfare loss remains valid (observe that a correctly computed $\hat{S}^{\lambda}_{\tau\tau}$ would increase sharply, even to infinity, if the extended model becomes unstable).

Therefore, we must investigate stability conditions of the equation (116). Since this equation is linear, it is sufficient to analyse the stability of its homogeneous part:

$$\dot{x}(t) = -\lambda\upsilon x(t-\tau) \quad ; \quad x(t) = 0 \text{ for } t < 0 \ , \ x(0) = x_0 \qquad (123)$$

When applying Laplace transformation to both sides of this equation, we obtain

$$s \cdot X(s) = -\lambda\upsilon e^{-\tau s} X(s) \qquad (124)$$

and we can use (see the appendix) the classical Nyquist criterion for the stability of a feedback system. The result is that the equation (116) is stable, if

$$\tau\upsilon < \frac{\pi}{2\lambda} \qquad (125)$$

(where π=3.14 ...). Therefore, the use of a sufficiently large feedback coefficient λ would destroy the stability of the system. Moreover, it is known from the classical theory of feedback systems, that the condition (125) should be satisfied with 50% margin

$$\tau\upsilon < \frac{\pi}{4\lambda} \qquad (126)$$

in order to obtain approximately aperiodic solutions of (116), which is necessary for a satisfactory approximation of $x^{\lambda}(t)$ by $\hat{x}(t,\underline{a})+\tau\bar{x}^{\lambda}(t)$, since both $\hat{x}(t,\underline{a})$ and $\bar{x}^{\lambda}(t)$ are aperiodic.

We can investigate further the condition (125) if we assume $\lambda=\hat{\lambda}$, consider $\hat{\lambda}$ as a function (102) of ν, $\hat{\lambda}(\nu) = \dfrac{2\nu^2}{(1+4\nu^2)^{\frac{1}{2}} - 1}$, consider $\frac{\upsilon}{r}$ as a function of ν, $\frac{\upsilon}{r} = \dfrac{(1+4\nu^2)^{\frac{1}{2}}}{2}$, and transform the condition (125) to

$$\tau \cdot r < \frac{\pi}{2\hat{\lambda}(\nu) \cdot \frac{\upsilon}{r}} = \frac{\pi}{2\nu^2} \qquad (127)$$

or, if taken with 50% margin

$$\tau \cdot r < \frac{\pi}{4\nu^2} \tag{128}$$

Since τ and r might be taken as given parameters--say, $r=0.1$/year and $\tau=0.25$ years as an estimate of delay[21]--hence the inequality (128) specifies really the highest value of the parameter ν^2 that is admissible for a stable implementation of the closed-loop policy (41) with $\lambda=\hat{\lambda}$. For example, if $r=0.1$/year and $\tau=0.25$ years, we obtain $\nu^2<31.4$, $\upsilon_{max}=r\cdot\nu<0.560$/year. If $\upsilon_{max}=C/q^{\frac{1}{2}}$ is higher than this value--say, because of a very small weight q related to the unemployment in the social welfare functional--then the closed-loop policy with $\lambda=\hat{\lambda}$ is not desirable. However, for these high values of ν^2 all other policies are also very sensitive to all types of errors (see Figures 3, 4, 5). Thus, if the weight q is small, resulting in relative high speed υ of controlling inflation or, equivalently, in a hight value of the parameter ν^2, a monetary control of inflation and unemployment cannot really be effective--either because of delayed effects endangering stability or because of incomplete information and various modeling errors.

On the other hand, υ, υ_{max} and ν^2 can always be diminished by increasing the weight q related to the unemployment. After all, a relative speed of $\upsilon_{max}=0.56$/year, $\upsilon=0.51$/year is not that small: $e^{-0.51}\approx 0.60$, which means that the long-term solution is being approached at a rate of circa 40% per year. In short, these results suggest that policy directions aimed at achieving the long-run target rates of inflation and unemployment rapidly run the risk of either inducing instability or becoming particularly sensitive to unpredictable modeling errors. Policy directions which aim for a slow, gradual achievement of the long-run targets have a greater chance of success.

6. OVERVIEW

In the analysis above we have presented a methodology for the formulation and assessment of macroeconomic policy rules under conditions of uncertainty or simplistic modeling. We believe that these conditions are prevalent in macroeconomic policy making: rarely do policy makers have accurate knowledge of the parameter distributions in their models and usually these models are recognized as rough approximations of the macroeconomic phenomena which the policy makers face. The policy maker is assumed to know which parts of his macro-economic model may be subject to error--which parameter estimates and delay estimates may be mistaken and which functions may be misspecified--but he does not know how these errors are distributed.

In this setting, the policy maker formulates a number of different policies each of which optimize his objective function (viz., the "social welfare function") subject to the constraints represented in his model. The policy which is actually chosen is the one which makes social welfare minimally sensitive to the given modeling errors. This is not a narrowly prescribed, watertight exercise, because the number of policies to choose from is limited only by the policy maker's imagination. Our methodology does not reduce the policy maker to a spiritless automaton which solves a given optimization program over and over again. The policy maker always faces the possibility of discovering new policy options which may make social welfare even less sensitive to his modeling errors than the policies he currently employs. It does not appear possible to find a universally "best" policy with regard to our methodology. All that our methodology does is provide criteria for the choice of a policy from a given set of candidate policies.

The traditional way of deriving how sensitive social welfare is to policies derived from modeling errors is (a) to maximize social welfare with respect to the "wrong" model and thereby derive the "wrong" policy, (b) to maximize social welfare with respect to a hypothetically "right" model and thereby derive the "right" policy, (c) to find the level of social welfare associated with the "right" model and the "wrong" policy, and

(d) to take the difference between social welfare under (b) and (c). In our methodology, we essentially follow this way. However, we change the interpretation of the 'right' and 'wrong' models by replacing them with 'extended' and 'basic' models, and we simplify the steps (b, c, d) by model linearization. The extended model is not 'right', the basic model is not 'wrong'--for if the policy maker were able to find the socially optimal policy with regard to the 'right' model, there would be no reason for him to use the 'wrong' model in the first place. Thus, the basic (not 'wrong') model is really, for a given purpose, the best representation of the problem at hand, and the extended (not 'right') model simulates possible, not actual errors in the basic model. Moreover, when applying our methodology, we do not have to solve the more complicated extended model--neither to optimize it exactly, nor to determine exactly the impact of the basic-optimal (not 'wrong') policy in this model. Instead of this, we apply linear approximations to the extended model solutions in both cases, and approximate thus the social welfare loss as required in the step (d). Our methodology permits thus the policy maker to choose a robust policy (viz., from a given set of candidate policies, the one that makes the social welfare least sensitive to modeling errors) without requiring that the social welfare optimization problem be solved with reference to the more complicated, extended model.

Our methodology applies to all macroeconomic problems in which policies affect social welfare at present and in the future and there is a tradeoff between these welfare effects. As noted in Section 1, a large number of macro-economic problems share these properties. The problem of inflation and unemployment which we chose to analyse performs simply an illustrative purpose. In fact, there is no reason why our methodology should be applied solely to macro problems. Microeconomic problems with the properties above--say, a firm maximizing the discounted stream of its present and future profits subject to uncertain technological conditions--are usually amenable.

Yet, within the confines of the particular problem of inflation and unemployment treated in this paper, we examined the monetarist case for constant money growth rules and developed a rather sceptical stance in this regard. Furthermore, we were able to formulate a rigorous argument against "impetuous" policy making (i.e., policies which cause target variables to approach their long-run optimal levels rapidly). Such a policy strategy may induce instability, either through delay effects, or by making the macroeconomic system very sensitive to modeling errors. Although one could suspect that such a conclusion is valid--by comparing, say, the effects of inflation and unemployment policies in countries such as Austria and the USA or Great Britain--a rigorous argument using model uncertainty and policy robustness considerations adds to a further understanding of this problem.

APPENDIX: MATHEMATICAL DERIVATIONS

Solution of the Basic Model

For the problem:

$$\underset{x,u}{\text{maximize}}\ W(x,u) = \int_0^\infty e^{-rt}\left(1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t)\right)dt \qquad (A1)$$

subject to differential constraint--the state equation:

$$\dot{x}(t) = -C(u(t) - u_n) \quad ; \quad x(0) = x_0 \qquad (A2)$$

we have the (current value) Hamiltonian function:

$$H = 1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t) + \zeta(t)\cdot C\cdot(u(t) - u_n) \qquad (A3)$$

and first-order optimality conditions

$$H_u = 0 \Leftrightarrow u(t) = \frac{C}{q}\zeta(t) \qquad (A4)$$

$$\dot{\zeta}(t) - r\zeta(t) = H_x \Leftrightarrow \dot{\zeta}(t) = r\zeta(t) - x(t) \qquad (A5)$$

together with the state equation (A2). We do not know yet whether a solution of the problem (A1), (A2) exists. However, if we substitute (A4) into (A2) and combine with (A5), we obtain the system of canonical equations:

$$\begin{cases} \dot{x}(t) = -C(\frac{C}{q}\zeta(t) - u_n) \quad ; \quad x(0) = x_0 \\ \dot{\zeta}(t) = r\zeta(t) - x(t) \end{cases} \tag{A6}$$

which can be solved by using the Riccati substitution:

$$\zeta(t) = K(t)x(t) + M(t) \quad ; \quad \dot{\zeta}(t) = \dot{K}(t)x(t) + K(t)\dot{x} + \dot{M}(t) \tag{A7}$$

where $K(t)$ and $M(t)$ are supposed to be such that (A7) holds independently of x_0. By substituting (A7) into the second equation in (A6) while taking into account the first one, we obtain

$$\dot{K}(t)x(t) + K(t)(-C(\frac{C}{q}(K(t)x(t) + M(t)) - u_n)) + \dot{M}(t) = r(K(t)x(t) + M(t)) - x(t) \tag{A8}$$

or, equivalently

$$(\dot{K}(t) - \frac{C^2}{q}K^2(t) - rK(t) + 1)x(t) = (r + \frac{C^2}{q}K(t))M(t) - Cu_nK(t) - \dot{M}(t) \tag{A9}$$

Since this equation should hold independently of x_0, thus, also of $x(t)$, both of its sides must be zero. In this way, we derived the Riccati equation:

$$\dot{K}(t) = \frac{C^2}{q}K^2(t) + rK(t) - 1 \tag{A10}$$

and the auxiliary equation

$$\dot{M}(t) = (\frac{C^2}{q}K(t) + r)M(t) - Cu_nK(t) \tag{A11}$$

These equations are useful for many aspects of analysis of optimal solutions of the problem (A1), (A2): not only they define closed-loop control laws and the second-order derivative K(t) of the (current value) Bellman optimal value function, but also they can be used to check sufficient conditions for optimality. One of the various forms of sufficient conditions of the existence and optimality of solutions to (A1), (A2) is as follows (see, for example, Wierzbicki 1977):

1) The solutions should satisfy the necessary conditions (A4), (A5), (A2).
2) The second-order derivative H_{uu} of the Hamiltonian function (A3) should be strictly negative for all t (which is, indeed, the case).
3) The Riccati equations (A10), (A11) should possess bounded solutions when solved backward in time--which, in the case of constant coefficients and an infinite time interval, means that they should have a stationary solution that is stable backwards in time.

The last two conditions are sufficient for strict negative definiteness of the Hessian operator of the welfare functional reduced to the space of control trajectories. Now, the stationary solutions of the Riccati equation (A10) result from the following equation

$$0 = \frac{c^2}{q} K^2 + rK - 1 \quad ; \quad K(t) = K = \text{const} \qquad \text{(A12)}$$

and have the form

$$K = \frac{rq}{2c^2} \left(-1 \pm \left(1 + 4 \frac{c^2}{r^2 q}\right)^{\frac{1}{2}}\right) \qquad \text{(A13)}$$

Of these two solutions, only the positive one is backward-stable. We can check this by applying the first Lapunov theorem: a solution of a nonlinear differential equation is asymptotically stable, if a linearized equation at this solution is asymptotically stable, and unstable, if the linearized equation is unstable. To check backward-stability, we change the sign of time, and linearize (A10) at K(t)=K to obtain

$$\frac{d\bar{K}(t)}{d(-t)} = -\left(\frac{2C^2}{q}K+r\right)\bar{K}(t) + 1 \tag{A14}$$

where $\bar{K}(t)$ is a variation of K(t). This equation is asymptotically stable if $\frac{2C^2}{q}K + r > 0$; but $\frac{2C^2}{q}K + r = \frac{1}{K} + \frac{C^2K}{q}$ from (A12), hence we must have K>0. Hence, the backward stable stationary solution of (A10) is

$$K = \frac{rq}{2C^2}\left(-1 + \left(1 + 4\frac{C^2}{r^2q}\right)^{\frac{1}{2}}\right) = \frac{1}{r}\cdot\frac{(1+4\nu^2)^{\frac{1}{2}} - 1}{2\nu^2} > 0 \tag{A15}$$

(where we introduced a composite parameter $\nu^2 = \frac{C^2}{r^2q}$, for reasons explained in the main text in the paper). The equation (A11) has a stationary solution

$$M = \frac{Cu_nK}{\frac{C^2}{q}K+r} = Cu_nK^2 \tag{A16}$$

which is backward-stable if $\frac{C^2}{q}K + r = \frac{1}{K} > 0$, hence if K>0.

We can now substitute the values of K and M into (A7), (A6) to obtain

$$\dot{x}(t) = -\frac{C^2}{q}Kx(t) - \frac{C^2}{q}M + Cu_n \quad ; \quad x(0) = x_0 \tag{A17}$$

Since $Cu_n - \frac{C^2}{q}M = Cu_n\left(1 - \frac{C^2}{q}K^2\right) = rKCu_n$, the integration of (A17) yields

$$\hat{x}(t) = x_0e^{-\upsilon t} + (1-e^{-\upsilon t})\frac{rKCu_n}{\upsilon} \tag{A18}$$

where $\upsilon = \frac{C^2}{q}K$; if we denote $\frac{rKCu_n}{\upsilon} = \frac{rqu_n}{C}$ by $\hat{x}_\infty$, then

$$\hat{x}(t) = (x_0 - \hat{x}_\infty)e^{-\upsilon t} + \hat{x}_\infty \tag{A19}$$

By substituting this into (A7) we obtain

$$\hat{\zeta}(t) = K(x_0 - \hat{x}_\infty)e^{-\upsilon t} + \hat{\zeta}_\infty \tag{A20}$$

where $\hat{\zeta}_\infty = K\hat{x}_\infty + M = \frac{rKqu_n}{C} + Cu_n K^2 = \frac{qu_n}{C}(rK + \frac{C^2K^2}{q}) = \frac{qu_n}{C}$.
By substituting this into (A4), we obtain

$$\hat{u}(t) = \frac{CK}{q}(x_0 - \hat{x}_\infty)e^{-\upsilon t} + u_n \tag{A21}$$

Now, the solutions (A19), (A20), (A21) are indeed the solutions of the problem (A1), (A2), since they satisfy the first-order necessary conditions and they correspond to a backward stable solution of the Riccati equation, while the second derivative of the Hamiltonian function there is strictly negative (thus, the welfare functional W, treated as a functional of control trajectory u alone, while the dependence on state trajectory x is reduced by solving the state equation (A2) and substituting into (A1), has its second-order derivative--the Hessian operator--strictly negative definite; this is sufficient for the existence of a unique maximum of the welfare functional. The control trajectory u is here considered to be an element of a Hilbert space with the norm $||u||=(\int_0^\infty e^{-rt}u^2(t)dt)^{\frac{1}{2}})$.

When analysing the obtained optimal solutions, we observe that they depend actually on three parameters. These parameters are: the discount rate r; the natural unemployment rate u_n; and the composite parameter $\nu^2 = \frac{C^2}{r^2 q}$ that, at given r, could be taken as representing the influence of the weighting coefficient q and the squared coefficient C^2.

Thus, the Riccati coefficient K, defined by (A14), depends on r and ν^2, while rK depends on ν^2 alone. At a given value of r, K decreases monotonically as a function of ν^2 from the value $\frac{1}{r}$ at $\nu \to 0$ (say, $q \to \infty$) to 0 as $\nu \to \infty$ (say, $q \to 0$). On the other hand, as a function of r, taking into account the dependence of ν^2 on r (at given C,q) K also decreases monotonically from the value $\frac{q^{\frac{1}{2}}}{C}$ at $r \to 0$ to 0 as $r \to \infty$ (we could set $K \approx 1/r$ for large enough r such that $r^2 >> 4C^2/q$). The relative speed coefficient υ has the form:

$$\upsilon = r \cdot \frac{(1+4\nu^2)^{\frac{1}{2}} - 1}{2} \tag{A22}$$

Thus, if r is given, υ increases monotonically as a function of ν^2 from the value 0 at $\nu=0$ (say, $q \to \infty$) to ∞ as $\nu \to \infty$ ($q \to 0$; we could set $\upsilon \approx r\nu$ for $\nu >> 1/2$). This means that increasing the weight q to unemployment necessarily decreases the relative speed of control. On the other hand, as a function of r at given C,q, the speed coefficient υ decreases monotonically, from the value

$$\upsilon_{max} = \frac{C}{q^{\frac{1}{2}}} \tag{A23}$$

at r=0, to the value 0 as $r \to \infty$ (we could set $\upsilon \approx \frac{C^2}{qr}$ for $r >> 2C/q^{\frac{1}{2}}$). Thus, a larger discount rate also decreases the relative speed of control.

However, if we assume that it is socially desirable to have the relative speed coefficient υ at least as large as the discount rate r, $\upsilon/r \geq 1$, then we easily obtain from (A22) that this could happen only if $\nu^2 \geq 2$ --and, from (A14) that this corresponds to $K \leq \frac{1}{2r}$. For this range of $\nu^2 \geq 2$ ($\nu >> 1/2$) we can set $\upsilon \approx r\nu = \frac{C}{q^{\frac{1}{2}}} = \upsilon_{max}$, thus υ practically does not depend on r and changes only with C and q, decreasing with an increased q.

The optimal value of the social welfare function, after appropriate integrations (omitted here) has the form

$$W = W(x,u) = \frac{1}{r}\left(1 - \frac{1}{2} q u_n^2 \left(1 + \frac{1}{\nu^2}\right)\right) - \hat{\zeta}_\infty (x_0 - \hat{x}_\infty) - \frac{1}{2} K (x_0 - \hat{x}_\infty)^2 \tag{A24}$$

Observe that this is a quadratic function of x_0, with

$$\frac{\partial \hat{W}}{\partial x_0} = -\hat{\zeta}_\infty - K(x_0-\hat{x}_\infty) = -\hat{\zeta}(0) \quad ; \quad \Delta\hat{W} = -\hat{\zeta}_\infty(x_0-\hat{x}_\infty) \tag{A25}$$

and

$$\frac{\partial^2 \hat{W}}{\partial x_0^2} = -K \quad ; \quad \Delta^2\hat{W} = -K(x_0-x_\infty)^2 \tag{A26}$$

Thus, the costate variable determines the first derivative of the optimal value function, while the Riccati coefficient--its second-order derivative. These are known properties of those variables, and are presented here only in order to check the correctness of a rather long integration in (A25). The forms ΔW and $\Delta^2\hat{W}$ represent the first and second-order parts of welfare losses due to the initial disequilibrium $x_0-\hat{x}_\infty$.

Finally, observe that we can represent the optimal control $\hat{u}(t)$ as a function of $\hat{x}(t)$ and not as a function of x_0:

$$\hat{u}(t) = \frac{CK}{q}\hat{x}(t) - \frac{CK}{q}\hat{x}_\infty + u_n = \frac{CK}{q}\hat{x}(t) + (1-rK)u_n \tag{A27}$$

which gives the classical version of the closed-loop optimal control. The feedback coefficient $\frac{CK}{q} = \frac{\upsilon}{C}$ has the same properties as the relative speed coefficient υ, scaled down by C. Thus, for $\upsilon/r \geq 1$, $\nu^2 \geq 2$, we have $\frac{CK}{q} \approx \frac{\upsilon_{max}}{C} = \frac{1}{q^{\frac{1}{2}}}$. This has an important interpretation: since $\hat{u}(t)-u_n = \frac{\upsilon}{0}(\hat{x}(t)-\hat{x}_\infty)$, hence at optimal solutions, the ratio of the (out-of-equilibrium) unemployment $\hat{u}(t)-u_n$ to the (out-of-equilibrium) expected inflation $\hat{x}(t)-\hat{x}_\infty$ approaches, but never exceeds, the value $1/q^{\frac{1}{2}}$.

Bounds on Parameter Changes

In this theoretical exercise, bounds on parameter changes must be assumed heuristically, however, with some rational arguments. We observe that first $\hat{x}_\infty = \frac{rq}{C} u_n$. If we are uncertain about the actual natural rate of unemployment, $\tilde{u}_n$, the largest mistake we can reasonably postulate is related to the assumption that the current expected inflation x_0 is equal to its long-term optimal value, $x_0 = \frac{rq}{C}\tilde{u}_n$, $\tilde{u}_n = \frac{C}{rq} x_0$, $\beta = \tilde{u}_n - u_n = \frac{C}{rq} x_0 - u_n = \frac{C}{rq}(x_0 - \hat{x}_\infty)$. Such a mistake is not very probable: observing a real economic process, we can usually tell whether we are far or close to the long-term equilibrium. Thus, we assume that we can estimate at least half of the distance from the long-term equilibrium, which gives half as large bound on the uncertainty parameter β:

$$|\beta|_{max} = |\hat{u}_n - u_n|_{max} = \frac{1}{2}\frac{C}{rq}|x_0 - \hat{x}_\infty| \tag{A28}$$

To obtain a comparable bound on the parameter γ, we assume that we observe correctly the variable

$$\frac{x - xa}{A} = \Delta = u - \tilde{u}_n - \frac{\gamma}{2}(u - \tilde{u}_n)^2 \tag{A29}$$

which would give $\gamma = 2\,\dfrac{\Delta - u + \tilde{u}_n}{(u-\tilde{u}_n)^2}$. We are uncertain about both $\Delta - u + \tilde{u}_n$ and $(u-\tilde{u}_n)^2$. We can assume $|\Delta - u + \tilde{u}_n| < |\beta|_{max}$, since this uncertainty is related to the uncertainty in $\tilde{u}_n$. However, the larger is $(u-\tilde{u}_n)^2$, the better conclusions we can obtain about γ; in a probabilistic estimation, this fact would correspond to the impact of the variance of the explaining variable. Thus, we must assume some minimal 'variance' $(u-\tilde{u}_n)^2$; if we take, arbitrarily, a rather small 'variance' $(u-\tilde{u}_n)^2 \approx 4u_n^2$, then we obtain the bound

$$|\gamma|_{max} = \frac{1}{2}\,\frac{|\beta|_{max}}{u_n^2} = \frac{1}{4}\,\frac{C}{rq}\,\frac{|x - x_\infty|}{u_n^2} \tag{A30}$$

To obtain a bound on the maximal neglected delay τ_{max} we note that if the postulated optimal change of the expected inflation has the form $\hat{x}(t)=(x_0-\hat{x}_\infty)e^{-\upsilon t}+\hat{x}_\infty$, and a real economic process would exhibit a delay τ in the impact of unemployment on inflation, we would observe actual $\tilde{x}(t)=x_0$ for $t\leq\tau$. Thus, the observed difference has the form:

$$\tilde{x}(t) - \hat{x}(t) = (x_0-\hat{x}_\infty)(1-e^{-\upsilon t}) \approx (x_0-\hat{x}_\infty)\upsilon t \quad , \qquad t\leq\tau \tag{A31}$$

Now we assume arbitrarily that we might overlook this difference until its relative value becomes 25%, that is, until $\frac{\tilde{x}(t)-\hat{x}(t)}{x_0-\hat{x}_\infty} = \frac{1}{4}$. Thus, the maximal delay that we can overlook is bounded by

$$\tau_{max} = \frac{1}{4\upsilon} \approx \frac{1}{4\upsilon_{max}} = \frac{q^{\frac{1}{2}}}{4C} \tag{A32}$$

(here we assume that we are interested in the range of solutions where $\upsilon/r\geq 1$, $\nu^2 \geq 2$, and $\upsilon \approx \upsilon_{max}$).

Sensitivity to Errors in Estimating Natural Unemployment Rate

In this case, we assume that policies derived from the basic model with parameter u_n are applied to an extended model, which differs from the basic one only by a changed parameter $\tilde{u}_n$, where $\tilde{u}_n-u_n=\beta$; the parameter $\underline{\alpha}=(\beta,0,0)$, the parameter $\underline{a}=(0,0,0)$. Hence, in this simple case we know the optimal solutions for the extended model which will be denoted here by $\hat{x}(t,\underline{\alpha})=x(t,\beta)$, $\hat{u}(t,\underline{\alpha})=\hat{u}(t,\beta)$; in particular

$$\begin{aligned} \hat{x}(t,\beta) &= x_0e^{-\upsilon t} + (1-e^{-\upsilon t}) \cdot \frac{rq}{C} \cdot \tilde{u}_n \\ \hat{u}(t,\beta) &= \frac{CK}{q} x_0e^{-\upsilon t} + \tilde{u}_n - rK \cdot e^{-\upsilon t} \cdot \tilde{u}_n \end{aligned} \tag{A33}$$

If we subtract from these solutions the solutions (A18), (A21) for the basic model--denoted here by $\hat{x}(t,\underline{a})$, $\hat{u}(t,\underline{a})$--we obtain

$$\hat{x}(t,\beta) - \hat{x}(t,\underline{a}) = (\tilde{u}_n - u_n) \cdot \frac{rq}{C}(1-e^{-\upsilon t}) \qquad \text{(A34)}$$

$$\hat{u}(t,\beta) - \hat{u}(t,\underline{a}) = (\tilde{u}_n - u_n) \cdot (1-rKe^{-\upsilon t}) \qquad \text{(A35)}$$

By the definition of basic optimal variations $\hat{\hat{x}}(t)$, $\hat{\hat{u}}(t)$--see Equation (56)--we obtain in this case

$$\hat{\hat{x}}(t) = \frac{rq}{C}(1-e^{-\upsilon t}) \quad ; \quad \hat{\hat{u}}(t) = 1-rKe^{-\upsilon t} \qquad \text{(A36)}$$

Now we consider the family of closed-loop control policies:

$$u^{\lambda}(x(t),\underline{a}) = \hat{u}(t,\underline{a}) + \lambda\frac{CK}{q}(x(t) - \hat{x}(t,\underline{a})) \qquad \text{(A37)}$$

and suppose such a policy is implemented to the extended model with $\tilde{u}_n \neq u_n$; we denote the state and control under such implementation by $x^{\lambda}(t)$, $u^{\lambda}(t)$:

$$\dot{x}^{\lambda}(t) = -C(\hat{u}(t,\underline{a}) - \tilde{u}_n + \lambda\frac{CK}{q}(x^{\lambda}(t) - (t,\underline{a}))) \quad ;$$

$$x^{\lambda}(0) = x_0 \qquad \text{(A38)}$$

Now, $\hat{x}(t,\underline{a})$ satisfies $\dot{\hat{x}}(t,a) = -C(\hat{u}(t,\underline{a}) - u_n)$; by subtracting this from both sides of Equation (A38), we obtain

$$\dot{x}^{\lambda}(t) - \dot{\hat{x}}(t,\underline{a}) = -\lambda\upsilon(x^{\lambda}(t) - \hat{x}(t,\underline{a})) + C\beta \quad ;$$

$$x^{\lambda}(0) - \hat{x}(0,\underline{a}) = 0 \qquad \text{(A39)}$$

where $\upsilon = \frac{C^2K}{q}$ and $\beta = \tilde{u}_n - u_n$. The solution of this equation is

$$x^{\lambda}(t) - \hat{x}(t,\underline{a}) = \beta C \cdot \frac{1}{\lambda\upsilon}(1-e^{-\lambda\upsilon t}) \qquad \text{(A40)}$$

which, when set into (A37), yields

$$u^{\lambda}(t) - \hat{u}(t,\underline{a}) = \beta(1-e^{-\lambda\upsilon t}) \tag{A41}$$

Thus, the basic structural variations--see Equation (54)--we have the form

$$\bar{x}^{\lambda}(t) = \frac{q}{\lambda CK}(1-e^{-\lambda\upsilon t}) \quad ; \quad \bar{u}^{\lambda}(t) = 1-e^{-\lambda\upsilon t} \tag{A42}$$

and the extended structural variation--see Equation (55)--are

$$\tilde{x}^{\lambda}(t) = \frac{rq}{C}\left(\frac{1}{\lambda rK}(1-e^{-\lambda\upsilon t}) - (1-e^{-\upsilon t})\right) \quad ;$$

$$\tilde{u}^{\lambda}(t) = rKe^{-\upsilon t}-e^{-\lambda\upsilon t} \tag{A43}$$

To compute the second-order derivative $\hat{S}^{\lambda}_{\beta\beta}$ of the welfare losses--see Equation (53)--we have to integrate the expression

$$\hat{S}^{\lambda}_{\beta\beta} = \int_0^{\infty} e^{-rt}((\tilde{x}^{\lambda}(t))^2 + q(\tilde{u}^{\lambda}(t))^2)dt \tag{A44}$$

which after rather long integrations and transformations (omitted here for brevity's sake; however, these transformations have been checked by assuming $\lambda=0,1$, or ∞ and performing independent, simpler integrations and transformations) results in

$$\hat{S}^{\lambda}_{\beta\beta} = \frac{q}{r}(1-rK)^2 \left(2\,\frac{(1-\lambda)(1+rK+\lambda(1-2rK))}{(rK+\lambda(1-rK))(rK+2\lambda(1-rK))} + 1\right) \tag{A45}$$

Now, the robustness coefficient $\hat{R}^{\lambda}_{\beta} = \beta^2_{max}\,\hat{S}^{\lambda}_{\beta\beta}/\Delta^2\hat{W}$ is easily determined

$$\hat{R}^{\lambda}_{\beta} = \frac{(1-rK)^3}{4(rK)^3}\left(1 + 2\,\frac{(1-\lambda)(1+rK+\lambda(1-2rK)}{(rK+\lambda(1-rK))(rK+2\lambda(1-rK))}\right) \tag{A46}$$

The equation $\frac{\partial \hat{R}_\beta^\lambda}{\partial \lambda} = 0$ is a quadratic equation in λ (with the full form omitted for the sake of brevity); we choose by checking the sign of $\frac{\partial^2 \hat{R}_\beta^\lambda}{\partial \lambda^2}$ the root that indeed minimizes $\hat{R}_\beta^\lambda$:

$$\hat{\lambda}_\beta = \frac{(((rK)^2+2rK-2) + 9(rK)^2(1-rK))^{\frac{1}{2}} - ((rK)^2 + 2rK-2)}{3rK(1-rK)} \tag{A47}$$

While taking into account that $rK = \frac{(1+4\nu^2)^{\frac{1}{2}} - 1}{2\nu^2}$ is a monotonically decreasing function of ν^2, with $rK \to 1$ as $\nu^2 \to 0$, we can show that $\hat{\lambda}_\beta$ is a monotonically decreasing function of rK and thus increasing function of ν^2, with $\hat{\lambda}_\beta \to 1.5$ as $\nu^2 \to 0$. The graphs of $\hat{\lambda}_\beta$ and $\hat{R}_\beta^\lambda$ are given in Figures 3, 5 in the main text.

The open and the closed-loop Hamiltonian maximizing policies as well as the open and closed-loop benefit-to-cost maximizing policies are considered in the main text.

Here we show only that the welfare losses under the open-loop benefit-to-cost maximizing policy are larger than the corresponding welfare losses under the open-loop policy. With this aim, we recall Equation (78) for the extended structural variations:

$$\dot{\tilde{x}}^{fo}(t) = C \cdot \kappa^{fo}(t,\underline{a}) \cdot \tilde{x}^{fo}(t) + C\eta^{fo}(t,\underline{a}) \quad ; \quad \tilde{x}^{fo}(0) = 0 \quad ;$$

$$\tilde{u}^{fo}(t) = \kappa^{fo}(t,\underline{a}) \cdot \tilde{x}^{fo}(t,\underline{a}) - \eta^{fo}(t,\underline{a}) \tag{A48}$$

where

$$\kappa^{fo}(t,\underline{a}) = \frac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})} \quad ;$$

$$\eta^{fo}(t,\underline{a}) = 1 + \frac{q\hat{u}^2(t,\underline{a})}{U(\hat{x},\hat{u},\underline{a})} - rKe^{-\upsilon t} + rqC \cdot \ {}^{fo}(t,\underline{a}) \cdot (1-e^{-\upsilon t}) \quad ;$$

$$U(\hat{x},\hat{u},\underline{a}) = 1 - \frac{1}{2}\hat{x}^2(t,\underline{a}) - \frac{q}{2}\hat{u}^2(t,\underline{a}) \tag{A49}$$

Since the structural variations under the open-loop policy have the form (see Equation (A43), where we should set $\lambda=0$):

$$\tilde{x}^{o}(t) = Ct - \frac{rq}{C}(1-e^{-\upsilon t}) > 0 \quad ; \quad \tilde{u}^{o}(t) = rKe^{-\upsilon t} - 1 < 0 \tag{A50}$$

and do not depend on x_0, when computing $\tilde{x}^{fo}(t)$, $\tilde{u}^{fo}(t)$ we can assume any reasonable x_0 that does not increase these variations unnecessarily. The most reasonable assumption is $x_0=\hat{x}_\infty$, which means that we will compute $\tilde{x}^{fo}(t)$, $\tilde{u}^{fo}(t)$ along the long-term equilibrium for the basic model. In such a case, we obtain

$$\kappa^{fo} = \frac{rqu_n^2}{C(1 - \frac{1}{2}qu_n^2 - \frac{1}{2}\frac{r^2q^2u_n^2}{C^2}}) \quad ; \tag{A51}$$

$$\eta^{fo} = 1 - rKe^{-\upsilon t} + \kappa^{fo}(\frac{C}{r} + rqC(1-e^{-\upsilon t}))$$

$$U(\hat{x},\hat{u},\underline{a}) = 1 - \frac{1}{2}\frac{r^2q^2u_n^2}{C^2} - \frac{1}{2}qu_n^2 = 1 - \frac{1}{2}qu_n^2(1 + \frac{1}{\nu^2})$$

Observe that for reasonable values of q, u_n, ν^2 we have $U(\hat{x},\hat{u},\underline{a}) \approx 1$ and $\kappa^{fo} << r/C$, $\eta_{fo}=-\tilde{u}^{o}(t)+\varepsilon_1(t)$, where $\varepsilon_1(t)$ is a small positive function. Thus, also $\tilde{u}^{fo}(t)=\tilde{u}^{o}(t)-\varepsilon_2(t)$, where $\varepsilon_2(t)$ is a small positive function; but $\tilde{u}^{o}(t)$ is negative, hence $(\tilde{u}^{fo}(t))^2 > (\tilde{u}^{o}(t))^2$. The Equation (A47) can be rewritten as

$$\dot{\tilde{x}}^{fo}(t) = -C\tilde{u}^{fo}(t) \quad ; \quad \tilde{x}^{fo}(0) = 0 \tag{A52}$$

while $\tilde{x}^{o}(t) = -C\tilde{u}^{o}(t)$, $\tilde{x}^{o}(0) = 0$. Thus, the solution of (A52) has also the form $\tilde{x}^{fo}(t)=\tilde{x}^{o}(t)+\varepsilon_3(t)$, where $\varepsilon_3(t)$ is a small positive function, and $(\tilde{x}^{fo}(t))^2 > (\tilde{x}^{o}(t))^2$. Therefore, we obtain the inequality

$$\int_0^{\infty} e^{-rt}((\tilde{x}^{fo}(t))^2 + q(\tilde{u}^{fo}(t))^2)dt > \int_0^{\infty} e^{-rt}((\tilde{x}^{o}(t))^2 + q(\tilde{u}^{o}(t))^2)\,dt \qquad \text{(A53)}$$

which means that the welfare loss with the open-loop benefit-to-cost maximizing policy is larger than the loss with the open-loop policy. In an analogous way, the same result can be established for the closed-loop benefit-to-cost maximizing policy versus the closed-loop policy.

Sensitivity to Mistaken Functional Specification

We assume here $\beta=0$, $\gamma\neq 0$, $\tau=0$. Thus, the extended model has the form

$$\dot{x}(t) = -C(u(t) - u_n - \frac{\gamma}{2}(u(t) - u_n)^2) \quad ; \quad x(0) = x_0 \qquad \text{(A54)}$$

First we must derive the basic optimal variations. The Hamiltonian function for the optimization problem related to the extended model has the form

$$H = 1 - \frac{1}{2}x^2(t) - \frac{q}{2}u^2(t) + \zeta(t)\cdot C\cdot(u(t) - u_n - \frac{\gamma}{2}(u(t) - u_n)^2) \qquad \text{(A55)}$$

and yields the necessary conditions of optimality

$$H_u = 0 \Leftrightarrow qu(t) + \gamma C\zeta(t)(u(t) - u_n) - C\zeta(t) = 0 \qquad \text{(A56)}$$

$$\dot{\zeta}(t) - r\zeta(t) = H_x \Leftrightarrow \dot{\zeta}(t) = r\zeta(t) - x(t) \qquad \text{(A57)}$$

$$\dot{x}(t) = -H_{\zeta} \Leftrightarrow \dot{x}(t) = -C(u(t) - u_n - \frac{\gamma}{2}(u(t) - u_n)^2) \qquad \text{(A58)}$$

We denote solutions to the extended optimization problem by $\hat{u}(t,\gamma) = \hat{u}(t,\underline{a}) + \gamma\hat{\hat{u}}(t) + o(\gamma)$, $\hat{x}(t,\gamma) = \hat{x}(t,\underline{a}) + \gamma\hat{\hat{x}}(t) + o(\gamma)$, $\hat{\zeta}(t,\gamma) = \hat{\zeta}(t,\underline{a}) + \gamma\hat{\hat{\zeta}}(t) + o(\gamma)$. We substitute the solutions into (A56÷58), subtract the relations for $\hat{u}(t,\underline{a})$, $\hat{x}(t,\underline{a})$, $\hat{\zeta}(t,\underline{a})$ as defined by the basic model (A4), (A5), (A2) and obtain as explained in the main text, the set of differential equations for basic optimal variations:

$$\hat{\hat{u}}(t) = \frac{C}{q}\hat{\hat{\zeta}}(t) - \Psi \tag{A59}$$

$$\dot{\hat{\hat{\zeta}}}(t) = r\hat{\hat{\zeta}}(t) - \hat{\hat{x}}(t) \tag{A60}$$

$$\dot{\hat{\hat{x}}}(t) = -\frac{C^2}{q}\hat{\hat{\zeta}}(t) + C(\phi+\Psi) \quad ; \quad \hat{\hat{x}}(0) = 0 \tag{A61}$$

where

$$\Psi = \hat{u}(t,\underline{a})(\hat{u}(t,\underline{a}) - u_n) \quad ; \quad \phi = \frac{1}{2}(\hat{u}(t,\underline{a}) - u_n)^2 \tag{A62}$$

Observe that when $\hat{u}(t,\underline{a})-u_n=0$, then $\Psi=0$, $\phi=0$ and the set of equations (A59÷61) has trivial solutions $\hat{\hat{u}}(t)=0$, $\hat{\hat{\zeta}}(t)=0$, $\hat{\hat{x}}(t)=0$. If $\hat{u}(t,\underline{a})-u_n = \frac{CK}{q}(x_0-\hat{x}_\infty)e^{-\upsilon t}$ is positive and increases--say, if x_0 increases above $\hat{x}_\infty$--then Ψ and ϕ increase with it monotonically. The solutions of the linear set of equations (A60); (A61) will increase then together with $\phi+\Psi$ (more precisely, a norm of the solutions increases with the norm of $\phi+\Psi$, where we could take as a norm $||\phi+\Psi|| = (\int_0^\infty e^{-rt}(\phi(t)+\Psi(t))^2 dt)^{\frac{1}{2}}$. A general way of solving Equations (A59÷A61) consists of a Riccati substitution

$$\hat{\hat{\zeta}}(t) = K(t)\hat{\hat{x}}(t) + \overline{M}(t) \tag{A63}$$

where K(t) satisfies the Riccati equation (A10), since the homogeneous parts of Equations (A60), (A61) and that of Equation (A6) are identical; thus, we take the backward stable stationary solution of (A10), $K(t)=K$ as defined by Equation (15). The auxiliary equation for $\overline{M}(t)$ is different; however, since Equations (A60), (A61) have the same form as Equation (A6) except for the fact that $\phi+\Psi$ substitutes u_n, we obtain the equation similar to (A11):

$$\dot{\overline{M}}(t) = (\frac{C^2}{q} K + r)\overline{M}(t) - CK(\phi+\Psi) \tag{A64}$$

where $\frac{C^2}{q} K + r = 1/K$ from (A12). The general difference between (A11) and (A64) is that $\phi+\Psi$ is a function of time, while u_n is a constant. Thus, to obtain a general backward stable, bounded solution to (A64) we have to consider its general solution:

$$\overline{M}(t) = \overline{M}(o)e^{t/K} + e^{t/K} \cdot CK^2 \int_0^t e^{-\tau/K}(\phi(\tau) + \Psi(\tau))d\tau \tag{A65}$$

and, after the integration, we choose $\overline{M}(0)$ in such a way that the unbounded terms with $e^{t/K}$ vanish from the solution. While all this can be done, the solutions have rather complicated forms and result in very tedious integrations when further computing the second-order derivative of welfare losses $\hat{S}^{\lambda}_{\gamma\gamma}$. Thus, we are going to simplify the computations by accepting approximate expressions for all variations. The simplification is based on the observation that all variations depend monotonically on the norm of $\hat{u}(t,\underline{a})-u_n = \frac{CK}{q}(x_0-x_\infty)e^{-\upsilon t}$; if we substitute this expression by a constant function of time, say, by $\hat{u}(t,\underline{a})-u_n \approx \frac{CK}{q}(x_0-x_\infty)$, we would only estimate all variations and their impact on $\hat{S}^{\lambda}_{\gamma\gamma}$ from above. However, when estimating the bound on γ_{max}, we assumed a 'variance' $(u-u_n)^2$ by $4u_n^2$. Correspondingly, we should assume some value of x_0; if we take x_0 such that $\hat{u}(t,\underline{a})-u_n \approx u_n$ on average, the 'variance' $(u-u_n)^2$ might be considered to be close to $4u_n^2$ because of the actual time-dependence of $\hat{u}(t,\underline{a})$. Clearly, the choice of the 'variance' and the bound γ_{max} or the corresponding 'average' $\hat{u}(t,\underline{a})-u_n \approx u_n$ are arbitrary and affect the absolute value of the finally derived robustness coefficient $\hat{R}^{\lambda}_{\gamma}$. For a more precise estimation of robustness coefficient, we would have to estimate econometrically γ_{max} and

x_0, determine the precise of the functions Ψ and ϕ, and integrate all variational equations for this precise form, using numerical integration if necessary. Here, in order to provide for an example, we perform only approximate calculations under a reasonable set of assumptions.

Under the approximate assumption $\hat{u}(t,\underline{a})-u_n \approx u_n$, the functions Ψ and ϕ take the form:

$$\Psi = 2u_n^2 \quad ; \quad \phi = \frac{1}{2} u_n^2 \tag{A66}$$

and the equation (A64) has a stationary, backward stable solution

$$\overline{M}(t) = \overline{M} = CK^2 \cdot \frac{5}{2} u_n^2 \tag{A67}$$

Now, we substitute (A63) into (A61) to obtain

$$\dot{\hat{x}}(t) = -\frac{C^2K}{q}\hat{x}(t) - \frac{C^2}{q}\overline{M}(t) + C(\phi+\Psi) \quad ; \quad \hat{x}(0) = 0 \tag{A68}$$

Since $1 - \frac{C^2K^2}{q} = rK$ and thus $C(\phi+\Psi) - \frac{C^2}{q}\overline{M} = rKC \cdot \frac{5}{2}u_n^2$, the solution of (A68) is

$$\hat{x}(t) = \frac{rq}{C} \cdot \frac{5}{2} u_n^2 (1-e^{-\upsilon t}) \tag{A69}$$

with $\upsilon = \frac{C^2K}{q}$. Correspondingly, we use (A63) to obtain

$$\hat{\zeta}(t) = \frac{5}{2} u_n^2 \left(\frac{rqK}{C}(1-e^{-\upsilon t}) + CK^2\right) = \frac{5}{2} u_n^2 \frac{q}{c}(1-rKe^{-\upsilon t}) \tag{A70}$$

and (A59) yields

$$\hat{u}(t) = \frac{1}{2} u_n^2 (1-5rKe^{-\upsilon t}) \tag{A71}$$

We now turn to the basic structural variations corresponding to the family of closed-loop policies (A37). By substituting (A37) into (A53), we obtain

$$\dot{x}^{\lambda}(t) = -C(\hat{u}(t,\underline{a}) - u_n + \lambda\frac{KC}{q}(x^{\lambda}(t) - \hat{x}(t,\underline{a}) - \frac{1}{2}\gamma(\hat{u}(t,\underline{a})$$

$$-u_n + \lambda\frac{KC}{q}(x^{\lambda}(t) - \hat{x}(t,\underline{a}))^2) \qquad \text{(A72)}$$

We assume the solutions in the form $x^{\lambda}(t)=\hat{x}(t,\underline{a})+\gamma\bar{x}^{\lambda}(t)+o(\gamma)$, $u^{\lambda}(t)=\hat{u}(t,\underline{a})+\gamma\bar{u}^{\lambda}(t)+o(\gamma)$, substitute them into (A72) and subtract the equation of the basic model, subdivide by γ and let $\gamma\to 0$. All this yields the equation for the basic structural variations:

$$\dot{\bar{x}}^{\lambda}(t) = \lambda\frac{KC^2}{q}\bar{x}^{\lambda}(t) + C\phi \quad ; \quad \bar{x}^{\lambda}(0) = 0 \qquad \text{(A73)}$$

Again, we would have to integrate this equation for a precise form of the function ϕ; however, assuming the approximate form (A66) we obtain

$$\bar{x}^{\lambda}(t) = \frac{1}{2}u_n^2 \cdot \frac{q}{\lambda KC}(1-e^{-\lambda\upsilon t}) \qquad \text{(A74)}$$

From (A37), we have $\bar{u}^{\lambda}(t)=\lambda\frac{CK}{q}\bar{x}^{\lambda}(t)$; therefore

$$\bar{u}^{\lambda}(t) = \frac{1}{2}u_n^2(1-e^{-\lambda\upsilon t}) \qquad \text{(A75)}$$

The extended structural variations are obtained as

$$\tilde{x}^{\lambda}(t) = \bar{x}^{\lambda}(t) - \hat{\bar{x}}(t) = \frac{1}{2}u_n^2\frac{rq}{C}\left(\frac{1}{\lambda rK}(1-e^{-\lambda\upsilon t}) - 5(1-e^{-\upsilon t})\right)$$

$$\text{(A76)}$$

$$\tilde{u}^{\lambda}(t) = \bar{u}^{\lambda}(t) - \hat{\bar{u}}(t) = \frac{1}{2}u_n^2(5rKe^{-\upsilon t} - e^{-\lambda\upsilon t}) \qquad \text{(A77)}$$

Even under the simplifying assumptions of constant ϕ and Ψ, the integration of the second-order derivative $\hat{S}^{\lambda}_{\gamma\gamma}$ of the welfare losses is rather long:

$$\hat{S}^{\lambda}_{\gamma\gamma} = \int_0^{\infty} e^{-rt}((\tilde{x}^{\lambda}(t))^2 + q(\tilde{u}^{\lambda}(t))^2)dt$$

$$= \frac{25u_n^2 rK^2 q}{2}(1 - \frac{1}{50r^2K^2}\,\frac{9r^2K^2+2rK-2+19\lambda(1-rK)rK}{(rK+\lambda(1-rK)(rK+2\lambda(1-2rK))}) \qquad \text{(A78)}$$

(Again, this expression has been checked by independent integrations for separate $\lambda=0$, 1, or ∞). Now, it is easy to determine the robustness coefficient

$$\hat{R}^{\lambda}_{\gamma} = \frac{|\gamma|^2_{max}\hat{S}^{\lambda}_{\gamma\gamma}}{\Delta^2\hat{W}} = \frac{25}{32}\,\frac{1-rK}{rK}(1 - \frac{1}{50r^2K^2}\,\frac{9r^2K^2+2rK-2+19\lambda(1-rK)rK}{(rK+\lambda(1-rK))(rK+2\lambda(1-2rK)})$$

(A79)

The equation $\frac{\partial\hat{R}^{\lambda}_{\gamma}}{\partial\lambda} = 0$ is again a quadratic equation in λ; however, its root that indeed minimizes $\hat{R}^{\lambda}_{\gamma}$ might become negative, and we have the constraint $\lambda\geq 0$ imposed by stability considerations for the closed-loop policy (A37). Thus, we obtain

$$\hat{\lambda}_{\gamma} = \begin{cases} 0, \text{ if } 4r^2K^2 + 3rK - 3 > 0 \Leftarrow rK > 0.758 \Leftrightarrow \nu^2 < 0.42 \\ \dfrac{((9r^2K^2+2rK-2)^2-19r^2K^2(4r^2K^2+3rK-3))^{\frac{1}{2}}-(9r^2K^2+2rK-2)}{19rK(1-rK)} \\ \qquad \text{if } \nu^2>0.42 \end{cases} \qquad \text{(A80)}$$

which yields smaller values of $\hat{\lambda}_{\gamma}$ (see Figure 5a in the main text) than $\hat{\lambda}_{\beta}$ determined by (A47). However, a much more simple expression:

$$\hat{\lambda} = \frac{1}{rK} \tag{A81}$$

yields values such that $\hat{\lambda}_\gamma < \hat{\lambda} < \hat{\lambda}_\beta$ (see Figure 5a; this inequality can be also proven analytically, but we omit these details). The compromise on robustness coefficients $\hat{R}^\lambda_\beta$ and $\hat{R}^\lambda_\gamma$ at $\lambda = \hat{\lambda}$ is quite satisfactory (see Figure 5b).

Sensitivity to Delays

We assume here that $\beta=0$, $\gamma=0$ and $\tau \neq 0$. Thus, the extended model equation is

$$\dot{x}(t) = -C(u(t-\tau) - u_n) \quad ;$$

$$x(0) = x_0 \quad ; \quad u(t) \text{ given for } t \in [-\tau; 0) \tag{A81}$$

When maximizing the welfare functional (A1) under the difference-differential constraint (A81), we use the modified Hamiltonian function

$$H_\Sigma = 1 - \frac{1}{2} x^2(t) - \frac{q}{2} u^2(t) + \zeta(t) \cdot C \cdot (u(t-\tau) - u_n)$$
$$+ 1 - \frac{1}{2} x^2(t+\tau) - \frac{q}{2} u^2(t+\tau) + \zeta(t+\tau) \cdot C \cdot (u(t) - u_n) \tag{A82}$$

However, this function is used for maximization with respect to u(t) alone (*not* to $u(t+\tau)$ *nor* $u(t-\tau)$). Thus, it is sufficient to consider only a part of this function:

$$\tilde{H} = 1 - \frac{1}{2} x^2(t) - \frac{q}{2} u^2(t) + \zeta(t+\tau) \cdot C(u(t) - u_n) \tag{A83}$$

Denote the solutions to the optimization problem of the extended model by $\hat{u}(t,\tau)$, $\hat{x}(t,\tau)$, $\hat{\zeta}(t,\tau)$. The necessary condition of optimality is that the function H_Σ--or equivalently, the function $\tilde{H}$--has a maximum in u(t) at the optimal solution.

Thus:

$$\tilde{H}_u = 0 \Leftrightarrow \hat{u}(t,\tau) = \frac{C}{q}\,\zeta(t+\tau,\tau) \tag{A84}$$

The costate equation for this problem retains its basic form, since there are no delays in the state in this problem:

$$\zeta(t,\tau) - r\hat{\zeta}(t,\tau) = H_x \Leftrightarrow \dot{\hat{\zeta}}(t,\tau) = r\hat{\zeta}(t,\tau) - \hat{x}(t,\tau) \tag{A85}$$

and the state equation, after substituting (A84), takes the form

$$\dot{\hat{x}}(t,\tau) = -H_\zeta \Leftrightarrow \dot{\hat{x}}(t,\tau) = -C\left(\frac{C}{q}\,\hat{\zeta}(t,\tau) - u_n\right) \quad ; \quad \hat{x}(0,\tau) = x_0 \quad ;$$

$$\hat{x}(t,\tau) \text{ given by (A79) for } t \in [0;\tau] \tag{A86}$$

Now, the system of canonical equations (A85), (A86) does not contain delayed terms and is, formally, identical with (A6). Thus, we could try to use the known solutions $\hat{x}(t,\underline{a})$, $\hat{\zeta}(t,\underline{a})$ of (A6) to obtain the solutions of (A85), (A86). This would, however, be incorrect since the state trajectory $\hat{x}(t,\tau)$ for $t \in [0;\tau]$ is predetermined by the given $u(t)$ for $t \in [-\tau;0]$; thus, a correct way is to accept this initial part of the trajectory and start to optimize the state trajectory at $t=\tau$, with a new initial state $x(\tau,\tau)$. Proceeding this way, we can write the optimal solutions for the optimization problem of the extended model:

$$\hat{x}(t,\tau) = (\hat{x}(\tau,\tau) - \hat{x}_\infty)e^{-\upsilon(t-\tau)} + \hat{x}_\infty \quad , \qquad \text{for } t \geq \tau \tag{A87}$$

$$\hat{\zeta}(t,\tau) = \bar{K}(\hat{x}(\tau,\tau) - \hat{x}_\infty)e^{-\upsilon(t-\tau)} + \hat{\zeta}_\infty \quad , \qquad \text{for } t \geq \tau \tag{A88}$$

$$\hat{u}(t,\tau) = \frac{CK}{q}(\hat{x}(\tau,\tau) - \hat{x}_\infty)e^{-\upsilon t} + u_n \quad , \qquad \text{for } t \geq 0 \tag{A89}$$

where $\hat{x}(\tau,\tau)$ must be obtained by solving the equation (A81) for $t \in [0;\tau]$ and $\hat{x}_\infty$, ζ_∞, υ are the same as in the basic optimal solutions.

However, the fact that we can actually optimize explicitly the extended model with delays is due to the particular form of this model (we have only one delayed control in the state equation; we could not get the solutions of this model as easily if, for example, the state equation would contain both delayed and undelayed control). Thus, we shall proceed further as if the explicit optimal solutions (A87÷89) for the extended model were not known.

We postulate $\hat{x}(t,\tau)=\hat{x}(t,\underline{a})+\tau\hat{\bar{x}}(t)+o(\tau)$, $\hat{u}(t,\tau)=\hat{u}(t,\underline{a})+\tau\hat{\bar{u}}(t)+o(\tau)$, $\hat{\zeta}(t,\tau)=\hat{\zeta}(t,\underline{a})+\tau\hat{\bar{\zeta}}(t)+o(\tau)$, and use (A84) to obtain:

$$\hat{u}(t,\tau) - \hat{u}(t,\underline{a}) = \tau\hat{\bar{u}}(t) + o(\tau)$$

$$= \frac{C}{q}(\hat{\zeta}(t+\tau,\tau) - \hat{\zeta}(t,\tau) + \hat{\zeta}(t,\tau) - \hat{\zeta}(t,\underline{a}))$$

$$= \tau\,\frac{C}{q}\left(\frac{\hat{\zeta}(t+\tau,\tau)-\hat{\zeta}(t,\tau)}{\tau} + \hat{\bar{\zeta}}(t) + \frac{o(\tau)}{\tau}\right)$$

(A90)

Observe that $\lim_{\tau\to 0}\frac{1}{\tau}(\hat{\zeta}(t+\tau,\tau)-\hat{\zeta}(t,\tau))=\dot{\hat{\zeta}}(t,0)=\dot{\hat{\zeta}}(t,\underline{a})$. Hence, when subdividing both sides of (A89) by τ and letting $\tau\to 0$, we obtain the equation for the basic optimal control variation:

$$\hat{u}(t) = \frac{C}{q}(\hat{\bar{\zeta}}(t) + \dot{\hat{\zeta}}(t,\underline{a})) \qquad \text{(A91)}$$

Similarly, by subtracting from both sides of (A81) the corresponding sides of (A2), we have

$$\dot{\hat{x}}(t,\tau) - \dot{\hat{x}}(t,\underline{a}) = \tau\dot{\bar{x}}(t) + o(\tau)$$

$$= -C\cdot(\hat{u}(t-\tau,\tau) - \hat{u}(t,\underline{a}))$$

$$= -\tau\cdot C\cdot\left(\frac{\hat{u}(t-\tau,\tau) - \hat{u}(t,\tau)}{\tau} + \hat{\bar{u}}(t) + \frac{o(\tau)}{\tau}\right)$$

(A92)

Since $\lim_{\tau\to 0} \frac{1}{\tau}(\hat{u}(t-\tau,\tau)-\hat{u}(t,\tau))=-\dot{\hat{u}}(t,0)=-\dot{\hat{u}}(t,\underline{a})$ (the control in the basic model is a differentiable function of time), we obtain

$$\dot{\hat{\bar{x}}}(t) = -C(\hat{\bar{u}}(t) - \dot{\hat{u}}(t,\underline{a})) \quad ; \quad \hat{\bar{x}}(0) = 0 \qquad \text{(A93)}$$

The equation for costate variation is easily derived:

$$\dot{\hat{\bar{\zeta}}}(t) = r\hat{\bar{\zeta}}(t) - \hat{\bar{x}}(t) \qquad \text{(A94)}$$

However, since $\frac{C}{q}\dot{\hat{\zeta}}(t,\underline{a})=\dot{\hat{u}}(t,a)$, the substitution of (A91) into (A93) yields

$$\dot{\hat{\bar{x}}}(t) = -\frac{C^2}{q}\hat{\bar{\zeta}}(t) \quad ; \quad \hat{\bar{x}}(0) = 0 \qquad \text{(A95)}$$

Now, the system of canonical equations (A94), (A95) is homogeneous; if we apply the Riccati substitution $\hat{\bar{\zeta}}(t)=K\hat{\bar{x}}(t)+\bar{M}(t)$, we obtain $\bar{M}(t)\equiv 0$. Since, however, $\hat{\bar{x}}(0)=0$, $\hat{\bar{\zeta}}(t)=K\hat{\bar{x}}(t)$ yields necessarily $\hat{\bar{x}}(t)\equiv 0$, $\hat{\bar{\zeta}}(t)\equiv 0$. Thus, we have the basic optimal variations

$$\hat{\bar{u}}(t) = \frac{C}{q}\dot{\hat{\zeta}}(t,\underline{a}) = -\upsilon\frac{CK}{q}(x_0-\hat{x}_\infty)e^{-\upsilon t} \quad ; \quad \hat{\bar{x}}(t) \equiv 0 \quad ,$$

$$\hat{\bar{\zeta}}(t) \equiv 0 \qquad \text{(A96)}$$

If we apply the family (A37) of closed-loop policies to the extended model (A81), we obtain

$$\dot{x}^\lambda(t) = -C(\hat{u}(t-\tau,\underline{a}) - u_n + \lambda\frac{CK}{q}(x^\lambda(t-\tau) - \hat{x}(t-\tau,\underline{a}))) \quad ;$$

$$x^\lambda(t) \text{ given by (A81) for } t\in[0;\tau]$$

We assume that (A96) has stable solutions (in further text, we will analyse the stability conditions for (A96)). Therefore, we can approximate $x^{\lambda}(t)=\hat{x}(t,\underline{a})+\tau\bar{x}^{\lambda}(t)+o(\tau)$. By subtracting (A2) from both sides of (A96), we have

$$\dot{x}^{\lambda}(t)-\dot{\hat{x}}(t,\underline{a})=\tau\dot{\bar{x}}^{\lambda}(t)+o(\tau)$$

$$=-C(\hat{u}(t-\tau,\underline{a})-\hat{u}(t,\underline{a})+\lambda\frac{CK}{q}(\tau\bar{x}^{\lambda}(t-\tau)+o(\tau)))$$

$$=-\tau\cdot C\cdot(\frac{\hat{u}(t-\tau,\underline{a})-\hat{u}(t,\underline{a})}{\tau}+\lambda\frac{CK}{q}(\bar{x}^{\lambda}(t-\tau)+\frac{o(\tau)}{\tau}))$$

(A98)

which yields, if $\tau\to 0$

$$\dot{\bar{x}}^{\lambda}(t)=-\lambda\upsilon\bar{x}^{\lambda}(t)+C\dot{\hat{u}}(t,\underline{a}) \quad ; \quad \bar{x}^{\lambda}(0)=0 \tag{A99}$$

Thus, basic structural variations are determined by integrating (A99)

$$\bar{x}^{\lambda}(t)=\frac{\upsilon}{\lambda-1}(x_0-\hat{x}_{\infty})(e^{-\lambda\upsilon t}-e^{-\upsilon t}) \tag{A100}$$

and by substituting $\bar{u}^{\lambda}(t)=\lambda\frac{CK}{q}\bar{x}^{\lambda}(t)$

$$\bar{u}^{\lambda}(t)=\frac{\lambda}{\lambda-1}\upsilon\frac{CK}{q}(x_0-\hat{x}_{\infty})(e^{-\lambda\upsilon t}-e^{-\upsilon t}) \tag{A101}$$

It now remains to determine the extended structural variations.

$$\tilde{u}^{\lambda}(t)=\bar{u}^{\lambda}(t)-\hat{\bar{u}}(t)=\frac{CK}{q}\upsilon(x_0-\hat{x}_{\infty})(\frac{\lambda}{1-\lambda}e^{-\lambda\upsilon t}-\frac{1}{1-\lambda}e^{-\upsilon t})$$

(A102)

$$\tilde{x}^{\lambda}(t) = \overline{x}^{\lambda}(t) - \hat{\overline{x}}(t) = \frac{\upsilon}{\lambda-1}(x_0-\hat{x}_\infty)(e^{-\lambda\upsilon t} - e^{-\upsilon t}) \qquad \text{(A103)}$$

and to integrate the second-order derivative of the welfare loss:

$$\hat{S}^{\lambda}_{\tau\tau} = \int_0^{\infty} e^{-rt}((\tilde{x}^{\lambda}(t))^2 + q(\tilde{u}^{\lambda}(t))^2)dt$$

$$= r(x_0-\hat{x}_\infty)^2 \frac{(1-rK)^3}{rK(rK+2\lambda(1-rK))} \qquad \text{(A104)}$$

Now, the robustness coefficient $\hat{R}^{\lambda}_{\tau}$ has the form

$$\hat{R}^{\lambda}_{\tau} = \frac{\tau^2_{max}\hat{S}^{\lambda}_{\tau\tau}}{\Delta^2\hat{W}} = \frac{(1-rK)^2}{4(rK+2\lambda(1-rK))} \qquad \text{(A105)}$$

and is a monotonically decreasing function of λ. It would be wrong, however, to assume that we should choose very large λ in order to obtain small values of $\hat{R}^{\lambda}_{\tau}$: very large λ might result in instability of (A96) and, therefore, the aperiodic approximation $\overline{x}^{\lambda}(t)$ defined by (A99) does not hold any longer for large λ.

Since the equation (A96) is linear, its stability is determined by its homogeneous part:

$$\dot{x}(t) = -\lambda\upsilon x(t-\tau) \quad ; \quad x(t) = 0 \text{ for } t < 0,\ x(0) = x_0$$

(A106)

To analyse the stability of this equation, we first apply Laplace transformation to both sides of it:

$$sX(s) = -\lambda\upsilon \cdot e^{-\tau s}X(s) \qquad \text{(A107)}$$

and represent the result as a feedback system, with the feedback error X(s) and the output variable Y(s)=K(s)X(s), determined by the open-loop transfer function $K(s)=\frac{\lambda\upsilon}{s}e^{-\tau s}$:

$$X(\sigma) = 0 - Y(s) \quad ; \quad Y(s) = K(s)X(s) \quad ; \quad K(s) = \frac{\lambda\upsilon}{s}e^{-\tau s}$$

(A108)

Now, we can use the classical Nyquist criterion for the stability of feedback systems. We analyse the argument and the module of the complex variable $K(j\omega)=\frac{\lambda\upsilon}{j\omega}e^{-j\omega\tau}$, where j is the imaginary unit. We have

$$\arg K(j\omega) = -\frac{\pi}{2} - j\omega\tau \quad ; \quad |K(j\omega)| = \frac{\lambda\upsilon}{\omega} \qquad (A109)$$

The Nyquist criterion states (in this simple case) that the feedback system is stable if $|K(j\omega_0)|<1$ for ω_0 such that $\arg K(j\omega_0) = -\pi$. Thus, we have $\omega_0=\frac{\pi}{2\tau}$ and $\frac{\lambda\upsilon}{\omega_0}=\frac{2\lambda\upsilon\tau}{\pi}<1$ guaranteeing stability, which can be rewritten as:

$$\tau\upsilon < \frac{\pi}{2\lambda} \qquad (A110)$$

However, in classical feedback systems theory it is also known that if the Nyquist criterion is satisfied with 50% margin ($|K(j\omega_0)|<0.5$)

$$\tau\upsilon < \frac{\pi}{4\lambda} \qquad (A111)$$

then the dynamic solutions in the feedback system might be reasonably approximated by aperiodic solutions (the system is far enough from its stability boundary, hence, the closed-loop solution might be approximated by aperiodic functions--if the open-loop itself is aperiodic).

Therefore, we conclude that the approximation (A99) of the solutions of the extended model under closed-loop policies might be reasonably applied only if (A111) is satisfied. If we assume that $\lambda=\hat{\lambda} = \frac{1}{rK} = \frac{2\nu^2}{(1+4\nu^2)^{\frac{1}{2}} - 1}$, and observe that $\upsilon/r = \frac{(1+4\nu^2)^{\frac{1}{2}} - 1}{2}$, then we transform condition (A110) to the form

$$\tau \cdot r < \frac{\pi}{2\nu^2} \qquad \text{(A112)}$$

or, with 50% margin

$$\tau \cdot r < \frac{\pi}{4\nu^2} \qquad \text{(A113)}$$

which, given an estimated τ and r, limits the maximal ν^2 and, thus, the maximal relative speed of control, $\frac{\upsilon}{r} = \frac{(1+4\nu^2)^{\frac{1}{2}} - 1}{2}$, that can be assumed (say, by choosing q in $\nu^2 = \frac{c^2}{r^2 q}$) when applying a closed-loop policy with $\lambda=\hat{\lambda}$.

FOOTNOTES

1. The monetarists have also justified their constant monetary growth rule by arguing that policy makers tend to misspecify their monetary rules by attempting to stabilize interest rates rather than growth rates of the money supply. The relevance of this argument has become muted in recent years as a number of governments--including the American and the British--have concentrated increasingly on monetary stocks and less on interest rates in their formulation of monetary policies.

2. It is feasible to relax this assumption in the context of our analysis.

3. However, Snower (1981) has shown that systematic monetary policy may remain effective if the macro-economic model is nonlinear. Yet the subject of optimal policy rules under rational expectations lies beyond the scope of this paper.

4. The extended model serves as a guinea pig, an experimental laboratory field for checking conclusions derived from the basic model. The need for such models in economic analysis is paramount and there are several attempts at using such models in policy-oriented analysis. In this paper, a much more theoretically organised way of using such models is proposed.

5. Inaccuracies (a) and (b) might be formally equivalent. For example, the functional form of the Phillips curve in the extended model may be expanded by Taylor series. The coefficients of the higher-order terms may be set to zero in the basic model. Then the mistaken parameter estimates of the basic model are responsible for its "simplistic" functional specification. Nevertheless, it is heuristically useful to distinguish between (a) and (b). Thus, we denote the discrepancy above as a case of (b) rather than (a).

6. We make a distinction between delays and lags. A dynamic delay is the time after which the first effect of an exogenous impulse are observed. An econometric lag is the time after which the statistically most significant effects are observed. Thus, delays are usually smaller than lags.

7. This combination falls in the tradition of Vanderkamp (1975), Dornbusch and Fischer (1978), and others.

8. For the discussion of other possible formulations of the decision-maker's objective function see Wierzbicki (1980a, b).

9. Except, possibly, at the initial time $t=0$, if the initial values $x(0)=x_0$ and $u(0)=u_0$ do not coincide with an optimal path. In such a case, a sharp change of m might be required to bring them on an optimal path.

10. The case in which m lags τ units of time behind u is analytically uninteresting. The optimal trajectory of $m(t)$ implied by the basic model becomes identical, in this case, with the optimal trajectory of $m(t-\tau)$ of the extended model.

11. Here, by dual control problem we understand (following Feldbaum 1962) a problem of joint estimation of model parameters and optimization of control. Only very simple classes of dual control problems possess known solutions.

12. This is because the state equation, at $\underline{\alpha}=\underline{a}$, is linear in $x(t)$ and $u(t)$. If it were not, the integrand in (53) should be based on the second derivatives of the Hamiltonian function.

13. Strictly speaking, $\tilde{x}^i(t)$ and $\tilde{u}^i(t)$ are extended structural sensitivity *operators*, not *variations*; by variations we should rather understand $\tilde{x}^i(t)\cdot(\underline{\alpha}-\underline{a})$ and $\tilde{u}^i(t)\cdot(\underline{\alpha}-\underline{a})$. However, we shall use in further text the colloquial name variations for $\tilde{x}^i(t)$ and $\tilde{u}^i(t)$.

14. Clearly, $\hat{S}^i_{\underline{a}\,\underline{a}}$ is semi-positive definite, as a generalized positive combination of semi-positive definite matrices $x^{i\tau}(t)x^i(t)$ and $u^{i\tau}(t)u^i(t)$.

15. It should be stressed that the methodology presented in this section is not new: it has been developed by Wierzbicki (1969) and generalized considerably for models described by Banach space equations by Wierzbicki and Dontchev (1977); a full description can also be found in Wierzbicki (1977). However, this methodology has not, as yet, been applied to economic problems. It goes without saying that the possible applications extend far beyond the the topic of this paper.

16. The expression (75) is one of the solutions of a quadratic equation; the other solution (with the sign before the square root in (75) changed to plus) is eliminated because only (75) satisfies $u^{fo}(t)=\hat{u}(t,\underline{a})$ at $\beta=0$, $x^{fo}(t)=\hat{x}(t,\underline{a})$.

17. We perform this linearization by subtracting $\dot{\hat{x}}(t,\underline{a})=-C(\hat{u}(t,\underline{a})-u_n$ from both sides of (76), subdividing the results by β and letting $\beta\to 0$. Equivalently, we can employ the theorem on the differentiable dependence of the solutions of differential equations on parameters.

18. This follows from the second Lapunov theorem on the stability of nonlinear differential equations when applied to the equation (76). Observe that even if $\beta=0$, but $x^{fo}(0)\neq x_0$ (for example, due to an error in estimating initial inflationary expectations) we obtain the unstable variational equation $\dot{\bar{x}}^{fo}(t)=C\,\dfrac{\hat{u}(t,\underline{a})\hat{x}(t,\underline{a})}{v(\hat{x},\hat{u},\underline{a})}\,\bar{x}^{fo}(t)$ for the propagation of the variation $\bar{x}^{fo}(0)$ of the initial state.

19. This does not mean that the benefit-to-cost maximizing policies might not be desirable in other applications, in particular when $\hat{\zeta}_t(t,\underline{a})$ is negligible. See Wierzbicki (1977) for examples of such applications in non-economic context.

20. Such a differentiability has been proved by Dontchev (1980) for a broader class of problems, including the problem considered here.

21. See footnote 6; an estimate of an econometric lag in u would have to be, most probably, a higher number.

REFERENCES

Alchian, A.A. 1970. Information Costs, Pricing and Resource Unemployment. In: Phelps et al. 27-52.

Althans, M., and P.L. Falb. 1966. Optimal Control. New York: Academic Press.

Barro, R.J. Rational Expectations and the Role of Monetary Policy. January 1976, 2(1), 1-32.

Brunner, K. and A.H. Meltzer, eds. 1976. The Economics of Price and Wage Controls. Amsterdam: North Holland.

Buiter, W. Crowding Out and the Effectiveness of Fiscal Policy. Journal of Public Economics. June 1977, 7(3), 309-28.

Darby, M.R. Three and a Half-Million US Employees Have Been Mislaid: Or, An Explanation of Unemployment, 1934-1941. Journal of Political Economy. February 1976, 84(1), 1-16.

De Canio, S. Rational Expectations and Learning From Experience. Quarterly Journal of Economics. February 1979, 47-57.

Dontschev, A. 1980. Perturbations, Approximations and Sensitivity Analysis of Optimal Control. Ac. Sci. of Bulgaria, mimeograph.

Dornbusch, R. and S. Fischer. 1978. Macro-economics. New York: McGraw Hill. 391-433.

Feldbaum, A.A. 1966. Foundations of the Theory of Optimal Control Systems. (In Russian) Nauka, Moscow.

Flanagan, The US Phillips Curve and International Unemployment Rate Differentials. American Economic Review. March 1973, 63(1), 114-31.

Friedman, B. Optimal Expectations and the Extreme Information Assumptions of Rational Expectations Macromodels. Journal of Monetary Economies. 5 January 1979, 23-41.

Friedman, M. The Role of Monetary Policy. American Economic Review. March 1968, 58, 1-17.

Gordon, R.J. Wage-Price Controls and the Shifting Phillips Curve Curve. Brookings Papers on Economic Activity, 1972, 3(2), 385-421.

Gronau, R. Information and Frictional Unemployment. American Economic Review. June 1971, 61(3), 290-301.

Lahiri, K. Inflationary Expectations: Their Formation and Interest Rate Effects. American Economic Review. 1976, 66, 124-31.

Lawson, T. Adaptive Expectations and Uncertainty. Review of Economic Studies. January 1980, 47(2), 305-20.

Lucas, R.E. Expectations and the Neutrality of Money. Journal of Economic Theory. April 1972, 4(2), 103-24.

Lucas, R.E. Some International Evidence in Output Inflation Tradeoffs. American Economic Review. June 1973, 63(3), 326-34.

Lucas, R.E. and E.C. Prescott. Equilibrium Search and Unemployment. Journal of Economic Theory. February 1974, 7(2), 188-209.

Lucas, R.E. and L.A. Rapping. Price Expectations and the Phillips Curve. American Economic Review. June 1969, 59(3), 342-50.

Mackay, D.I. and R.A. Hart. Wage Inflation and the Phillips Relationship. Manchester Sch.Econ.Soc.Studies. June 1974, 42(2), 136-61.

Markus, L. and E.B. Lee. 1967. Foundations of Optimal Control Theory. New York: Wiley.

McCall, T.T. Economic of Information and Job Search. Quarterly Journal of Economics. February 1970, 84(1), 113-26.

McCallum, B.T. and T.K. Whitaker. The Effectiveness of Fiscal Feedback Rules and Automatic Stabilizes Under Rational Expectations. Journal of Monetary Economics, 5 April 1979, 171-86.

Mortensen, D.T. Job Search, the Duration of Unemployment and the Phillips Curve. American Economic Review. December 1970, 60(5), 847-61.

Parkin, M., M. Summer and R. Ward. 1976. The Effects of Excess Demand Generalized Expectations and Wage-Price Controls on Inflation in the UK: 1956-71. In: Brunner and Meltzer, 193-221.

Parkin, M. The Short-Run and Long-Run Trade-Offs Between Inflation and Unemployment in Australia. Australian Economic Papers. December 1973, 12(21), 127-44.

Parsons, D.Q. Quiet Rates Over Time: A Search - Information Approach. American Economic Review. June 1973, 63(3), 390-401.

Phelps, E. 1970. Money Wage Dynamics and Labor Market Equilibrium. In: Phelps et al. 124-66.

---------, ed. 1970. Microeconomic Foundations of Employment and Inflation Theory. New York: Norton.

Salop, S.C. Systematic Job Search and Unemployment. Review of Economic Studies. April 1973, 40(2).

Sargent, T.T. 1973. Rational Expectations, the Real Rate of Interest and the Natural Rate of Unemployment. Brookings Papers on Economic Activity. 2, 429-79.

---------, A Classical Macroeconometric Model for the United States. Journal of Political Economy. April 1976, 84, 207-37.

Sargent, T.T. and N. Wallace. Rational Expectations, the Optimal Monetary Instrument and the Optimal Money Supply Rule. Journal of Political Economy. 1975, 83, 241-55.

---------, Rational Expectations and the Theory of Economic Policy. Journal of Monetary Economics. 2 April 1976, 169-184.

Shiller, R.J. Rational Expectations and the Dynamic Structure of Macroeconomic Models: A Critical Review. Journal of Monetary Economics. 4 January 1978, 1-44.

Siven, C.H. Consumption, Supply of Labor and Search of Activity in an Intemporal Perspective. Swedish Journal of Economics. March 1974, 7(1), 44-61.

Snower, D.J. 1981. Rational Expectations, Nonlinearities and the Effectiveness of Monetary Policy. Mimeograph.

Taylor, J.B. Monetary Policy During a Transition to Rational Expectations. Journal of Political Economy. October 1975, 83, 1009-21.

Tobin, J. and W. Buiter. Fiscal and Monetary Policies, Capital Formation and Economic Activity. Forthcoming in G.v. Fürstenberg: The Government and Capital Formation.

Turnovsky, S.J. The Expectations Hypothesis and the Aggregate Wage Equation: Some Empirical Evidence for Canada. Economica. February 1972, 39(153), 1-17.

-----, Empirical Evidence of the Formation of Price Expectations. Journal of the American Statistical Association. 1970, 65, 1441-54.

Vanderkamp, J. Wage Adjustment, Productivity and Price Change Expectations. Review of Economic Studies. January 1972, 39(1), 61-72.

Vanderkamp, J. Inflation: A Simple Friedman Theory with a Phillips Twist. Journal of Monetary Economics. 1975, 117-122.

Wierzbicki, A.P. 1969. Unified Approach to the Sensitivity Analysis of Optimal Control Systems. IVth Congress of IFAC, Warsaw.

Wierzbicki, A.P. Maximum Principle for Processes with Multiple Control Delays. (In Russian) Automatics and Remote Control. October 1970, 13-20.

Wierzbicki, A.P. and A.L. Dontchev. Basic Relations in Performance Sensitivity Analysis of Optimal Control Systems. Control and Cybernetics. 1974, 3, 119-139.

Wierzbicki, A.P. 1977. Models and Sensitivity of Control Systems. (In Polish) WNT, Warsaw.

Wierzbicki, A.P. 1980a. A Mathematical Basis for Satisfying Decision-Making. International Institute for Applied Systems Analysis, Laxenburg, Austria: WP-80-90.

Wierzbicki, A.P. 1980b. Multiobjective Trajectory Optimization and Model Semiregularization. International Institute for Applied Systems Analysis, Laxenburg, Austria: WP-80-181.