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ON EVOLUTION EQUATIONS IN ESTIMATION PROBLEMS FOR SYSTEMS WITH UNCERTAINTY*

A.B. Kurzanskii

The paper deals with problems of estimating the state of a multistage linear system on the basis of available measurement parameters [1,2]. It is assumed that the disturbances in the system inputs and in the measurement are uncertain. They are taken to be unknown in advance with respective information being restricted to only a set-membership description of their values [2-4]. The total dynamic estimation process will then be described by the evolution of certain informational domains that are consistent with the results of measurement and with the constraints given in advance [3-8]. The description of these domains may be achieved within the framework of Lagrangian techniques in convex analysis [6,8]. Approximate solutions for the problems have also been considered [5,7,8].

One approach to the problem different from those mentioned above is given in this paper. Namely, a procedure that leads to imbedding of the primary problem into an auxiliary problem of linear-quadratic Gaussian estimation (Kalman filtering [1]) for

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a system with additional stochastic disturbances whose covariance matrices are given but whose mean values are uncertain. By a variation of the covariance matrices in the auxiliary problems it turns possible to approximate the primary solution with any degree of accuracy. A unified approach to the solution of both stochastic Kalman filtering problems and deterministic estimation problems under set-membership uncertainty with non-quadratic constraints as considered in this paper is therefore established.

## 1. Systems With Uncertainty. Basic Description

A system with uncertainty is understood here as a discrete-time multistage process, described by an n-dimensional equation

$$x(k + 1) = A(k)x(k) + C(k)v(k) \tag{1.1}$$

where $A(k)$, $C(k)$, $k = 0,\ldots,N$ are given matrices. The input $v(k)$, and the initial stage $x^0$ are vectors of finite-dimensional spaces $E^p$ and $E^n$ respectively. They are assumed to be unknown being restricted in advance by instantaneous "geometric" constraints

$$x(0) = x^0 \in X^0 \ , \ v(k) \in P(k) \tag{1.2}$$

where $X^0$, $P(k)$ are given convex and compact sets. It is further assumed that direct measurements of the state $x(k)$ are impossible, the available information on the process dynamics being generated by the equation

$$y(k) = G(k)x(k) + \xi(k) \quad ; \qquad k = 1,\ldots,N \tag{1.3}$$

with measurement vector $y(k) \in E^m$ and matrix $G(k)$ given. The disturbances $\xi(k)$ are unknown and restricted by

$$\xi(k) \in Q(k) \tag{1.4}$$

with the convex compact set $Q(k) \in E^q$ given in advance.

Further, the symbol $y[k,1] = \{y(k),\ldots,y(1)\}$ will denote a sequence of measurements achieved due to equation (1.3) throughout the stages whose numbers vary from k to l. Similarly, the symbol

$$h[r\ ,\ s] = \{h(r),\ldots,h(s)\}$$

denotes a sequence of vectors $h(i)$ where $i = r,\ldots,s$, while

$$R[r\ ,\ s] = \{R(r),\ldots,R(s)\}$$

stands for the sequence of sets $R(i)$ with same numbers so that

$$h[r\ ,\ s] \in R[r\ ,\ s]$$

denotes a sequence of inclusions

$$h(i) \in R(i)\ , \qquad (i = r,\ldots,s)$$

Further, if for example, $h(i) \in E^q$, then we will assume $h[i\ ,\ s] \in E_1^q \times\ldots\times E_s^q = E^{q,s}$ where $E_s^q = E^q$ for all $i = 1,\ldots,s$. Therefore, we take $R[r\ ,\ s] \subseteq E^{q,s-r}$. The symbol $x(k\ ,v[0\ ,k\ -\ 1]\ ,x^0)$ will denote the end of the trajectory $x(j)$ for system (1.1) formed for $[0\ ,k]$ with $v[0\ ,k\ -\ 1]\ ,x^0$ given.

Now assume that after s stages of system operation there appeared a measurement sequence $y[1\ ,\ s]$, generated due to relations (1.1)-(1.4).

The knowledge of $y[1\ ,\ s]$ allows us to construct an *informational domain* $X[s] = X(1\ ,s\ ,X^0)$ that consists of the ends $x(s\ ,v[0\ ,s\ -\ 1]\ ,x^0)$ of all those trajectories $x(j)$ formed for the interval $j \in [0\ ,\ s]$ that could generate the measured sequence $y[1\ ,\ s]$ under constraints (1.2)-(1.4), see, for example, (8, 9). The dynamics of the total system (1.1)-(1.3) will now be determined by the evolution of sets $X[s]$.*

*In order to simplify some further notations we will generally start the process at stage $k_0 = 0$ instead of arbitrary $k_0 = k^*$, although the basic system is *nonstationary*.

Assume $y[k+1, l], (k+1 \leq l)$ to be given, $F$ - to be a set in $E^n$; $X(k, l, F)$ to be the set of the ends $x(l, v[k, l], x^*)$ of the trajectories $x(j)$ of system (1.1) that start at stage $k$ from state $x(k) = x^*$ and are formed for the interval $j \in [k, l]$ being consistent with the realization $y[k+1, l]$ due to equation (1.3) and with constraints

$$x^* \in F , \quad v[k, l-1] \in P[k, l-1] ,$$

$$\xi[k+1, l] \in Q[k+1, l]$$

Following the scheme taken for continuous systems in [8], it is possible to verify the following assertions, see also [9].

*Lemma 1.1* Assume $F, P(k), Q(k)$ to be convex compact sets in the spaces $E^n$, $E^p$, $E^m$ respectively. Then the sets $X(l, s, F)$ are convex and compact.

*Lemma 1.2* Whatever is the set $F \subseteq E^n$, the following equality is true

$$X(k, s, F) = X(l, s, X(k, l, F)) \tag{1.5}$$

In particular $X[s] = X(k, s, X[k])$.

Condition (1.5) indicates that the transformation $X(l, s, F)$ possesses a semigroup property that allows to define a certain generalized dynamic system in the space of convex compact subsets of $E^n$. The generalized system will then absorb all the informational and dynamic properties of the total process. We also note that the sets $X[s]$ possess a sort of Markovian property: each $X[s]$ contains all the pre-history of the process and the process evolution for $r > s$ will depend only upon $X[s]$ but not on the previous $X[i]$, $i < s$.

The estimation problem will now consist in determining the projection $[\alpha(l), \beta(l)\}$ of the set $X[s]$ on any pre-assigned direction $l$.

Here

$$\alpha(l) = \inf \{(l , x) | x \in X[s]\}$$

$$\beta(l) = \sup \{(l , x) | x \in X[s]\}$$

and $(l , x)$ stands for the scalar product in the respective space $E^n$.

*Example.* Consider the system

$$x_1(k + 1) = x_1(k) + x_2(k) + v(k) ,$$

$$x_2(k + 1) = x_2(k) ,$$

$$y(k) = x_1(k) + \xi(k) ,$$

where

$$x_j(k) , v , y , \xi \in E^2 , \quad (\text{for } h \in E^2 , h = \{h^{(1)} , h^{(2)}\})$$

and

$$P = \{v : |v^{(i)}| \leq \nu\} , |\xi^{(i)}| \leq \sigma ,$$

$$x_2^0 \in X_2[0] , \quad x_1^0(0) = x_1^0$$

$$X_2[0] = \{x_2 : |x_2^{(i)} - a^{(i)}| \leq r ; \quad i = 1 , 2\}$$

It is not difficult to observe that for $\sigma = 0$ we have

$$x_2(1) = x_2^0 \in \{y(1) - x_1(0) + P\} \cap X_2[0] = Y_2[1]$$

and further on

$$X_2[k + 1] = \{y(k + 1) - y(k) + P\} \cap X_2[k]$$

Therefore

$$X[s] = \{x_1, x_2 : x_1 = y[s], x_2 \in X_2[s]\}$$

$$X_2[k+1] = \{\bigcap_{i=1}^{k} \{y(i+1) - y(i) + P\}\} \cap X_2[0] \quad ,$$

$$y(0) = x_1^0$$

is the intersection of the initial set $X_2[0]$ with k rectangles $y(i+1) - y(i) + P = M(i+1)$.

Therefore, each new measurement $y(i+1)$ generates a new set $M(i+1)$ and thus introduces an innovation into the estimation process in the form of an intersection of

$$X[i] = X[0] \cap \{\bigcap_{k=1}^{i} M(k)\}$$

with $M(i+1)$.

Although in the given example the solution is obvious the general description of $X[s]$ requires a rather cumbersome procedure. The situation therefore justifies the consideration of an approximation technique based on solving some auxiliary stochastic estimation problems. In order to explain the procedure we will start with an elementary one-stage solution.

## 2. The One-Stage Problem

Consider the system

$$z = Ax + Cv \quad , \qquad y = Gz + \xi \tag{2.1}$$

where

$$x, z \in E^n \quad , \qquad v \in E^p \quad , \qquad \xi \in E^m \quad ,$$

and the matrices A, C, G are given. Knowing the constraints

$$x \in X \ , \ v \in P \ , \ \xi \in Q \ , \tag{2.2}$$

where X, P, Q are convex and compact subsets of the spaces $E^n$, $E^p$, $E^q$ respectively and knowing the value y, one has to determine the set Z of the vectors z consistent with equations (2.1) and with the inclusions (2.2).

Denote

$$Z_s = AX + C\mathcal{P} \ ,$$

$$Z_y = \{z : y - Gz \in Q\}$$

Then obviously $Z = Z_s \cap Z_y$. Standard considerations of convex analysis (10) yield a relation for the support function

$$\rho(l|Z) = \max \{(l , z)|z \in Z\} \tag{2.3}$$

*Lemma 2.1* The equality $\rho(l|Z) = \psi(l)$ is true where

$$\psi(l) = \inf \{\Phi(l , p)|p \in E^m\} \tag{2.4}$$

$$\Phi(l , p) = \rho(A'l - A'G'p|X) + \rho(C'l - C'G'p|\mathcal{P})$$

$$+ \rho(-p|Q) + (p , y) \ ,$$

and where the prime stands for the transpose.

The problem (2.4) may be presented in another form, namely, whatever the vectors l, p, $l \neq 0$ are, it is possible to represent $p = Ml = p[l , M]$ where matrix M is of dimension $m \times n$. Condition (2.4) will then attain the following form

$$\psi(l) = \inf \{\Phi(l , p[l , M])|M \in E^{m,m}\} \tag{2.5}$$

The latter relation allows to form the inclusion

$$Z \subseteq (I_n - M'G)(AX + C\mathcal{P}) + M'(y - Q) = R(M) \qquad (2.5')$$

which is true for any matrix M. The problem (2.5) will be called as the dual problem for (2.3). (Here $I_n$ is an $n \times n$ unit matrix.)

Equality (2.5) yields

*Lemma 2.2* The following equality is true

$$Z = \{\cap R(M) | M\}$$

over all $(m \times n)$ - matrices M.

The necessity of solving (2.5) gives rise to the question of whether it is possible to calculate $\rho(l|Z)$ by a variation of the relations for some kind of a stochastic problem.

In fact it is possible to obtain an inclusion that would combine the properties of both (2.5') and of conventional relations for the linear-quadratic Gaussian estimation problem.

Having fixed a certain triplet $h = \{x^0, v, \xi\}$ that satisfies (2.2) (the set of all such triplets will be further denoted as $\mathcal{H}$), consider the system

$$w = A(x + q) + Cv \ , \quad y = Gw + \xi + \eta \ , \qquad (2.6)$$

where q, $\xi$ are independent Gaussian stochastic vectors with zero means

$$Eq = 0 \ , \qquad E\eta = 0 \ ,$$

and with covariance matrices

$$Eqq' = L \qquad E\eta\eta' = N$$

where L, N are positive definite. Assume that after one random event for the triplet h the vector y has appeared due to system (2.6). Then for the conditional variance $E(w|y)$ determined for example by means of a Bayesian procedure or by a least-square method of calculation we have

$$E(w|y) = Ax + APA'G'N^{-1}(y - GAx - GCv - \xi) + Cv \quad ,$$

$$P^{-1} = L^{-1} + A'G'N^{-1}GA \tag{2.7}$$

or in accordance with the conventional matrix transformation (11, 12)

$$P = L - LA'G'K^{-1}GAL \quad ,$$

$$K = N + GALA'G' \quad , \tag{2.8}$$

an equivalent condition

$$\bar{w}_y = E(w|y) = Ax + ALA'G'K^{-1}(y - GAx - (GCv + \xi)) + Cv \tag{2.9}$$

Note that the conditional variance

$$E((w - \bar{w}_y)\ (w - \bar{w}_y)'|y) = APA' \tag{2.10}$$

does not depend upon k and is determined only by pair

$$\Lambda = \{L , N\}$$

where $L > 0$, $N > 0$. (In the latter case further we will write $\Lambda > 0$.)

Therefore one may consider the set of all conditional mean values

$$W(\Lambda) = \{\cup \bar{w}_y | h \in H\}$$

that correspond to all possible $h \in H$. Here

$$W(\Lambda) = (I_n - ALA'G'K^{-1}G)(AX^0 + CP) + ALA'G'K^{-1}(y - Q)$$

Having denoted

$$2\Psi(\Lambda) = K^{-1}GALA$$

we find:

*Lemma 2.3* The set $W(\Lambda)$ is convex and compact. The equality is true

$$\rho(l|W(\Lambda)) = \Phi(l, p(l, \Lambda)) \qquad (2.11)$$

where

$$p(l, \Lambda) = \Psi(\Lambda)l$$

We may now observe that the relation $\Phi(l, p(l, \Lambda))$ differs from $\Phi(l, p)$ used in (2.4) by a mere substitution of $p(l, \Lambda)$ by $p$. Comparing (2.11) and (2.4) we conclude

*Lemma 2.4* Whatever the pair $\Lambda > 0$ is the inclusion

$$Z \subseteq W(\Lambda) \qquad (2.12)$$

is true.

A condition similar to (2.12) was given in paper [9]. However, by varying $\Lambda$ in (2.10) it is possible to achieve an exact description of set Z. In order to prove the respective assertion some standard assumptions are required.

*Assumption 2.1:* The matrix GA is of rank m.

We shall also make use of the following relation:

*Lemma 2.5* Under assumption 2.1 take $\Lambda = \Lambda(1, \alpha) = \{I_n, \alpha I_m\}$. Then $\Psi(\Lambda(1, \alpha))G' \to I_m$ with $\alpha \to \infty$.

The given relation follows from equality $\Psi(\Lambda(1, \alpha))G' = (\alpha I_m + D)^{-1}D$ where matrix $D = GALAG'$ is nonsingular, $L = I_n$.

*Theorem 2.1* The inclusion $z \in Z$ is true if and only if for any $l \in E^n$, $\Lambda > 0$ we have

$$(l , z) \leq \rho \ (l|W(\Lambda)) = f(l , \Lambda) \qquad (2.13)$$

Inequality (2.13) follows immediately from the inclusion $z \in Z$ due to Lemma 2.4. Therefore it sufficies to show that (2.13) yields $z \in Z$. Suppose that for a certain $z^*$ the relation (2.13) is fulfilled, however $z^* \bar{\in} Z = Z_s \cap Z_y$. First assume that $z^* \bar{\in} Z_y$. Then there exists an $\varepsilon > 0$ and a vector $p^*$ such that

$$(-p^* , y) + (p^*G , z^*) > \rho(-p^*|Q) + \varepsilon \qquad (2.14)$$

Now we will show that it is possible to select a pair of values $l^*$, $\Lambda^*$ that depend upon $p^*$ and are such that

$$(l^* , z^*) > \rho(l^*|W(\Lambda^*)) = f(l^* , \Lambda^*) \qquad (2.15)$$

Indeed, taking $l^* = G'p^*$, $\Lambda(1 , \alpha) = \{I_n , \alpha I_m\}$ we have

$$f(l^* , \Lambda(1 , \alpha)) = \Phi(l , \Lambda(1 , \alpha)) \pm ((p^* , y) + \rho(-p^*|Q))$$

$$(2.16)$$

From Lemma 2.5 and condition

$$p(l^* , \Lambda(1 , \alpha)) = K^{-1}(\alpha)GAI_nA'G'p^* , \ K(\alpha) = \alpha I_m + D$$

it follows that

$$p(l^* , \Lambda(1 , \alpha)) \to p^* , \alpha \to 0 \qquad (2.17)$$

But then from condition (2.17), from Lemma 2.4 and from the properties of function $f(l , \Lambda)$ it also follows that for any $\varepsilon > 0$ there exists an $\alpha_0(\varepsilon)$ such that for $\alpha \leq \alpha_0(\varepsilon)$ the inequality

$$|f(l^* , \Lambda(1 , \alpha)) - ((p^* , y) + \rho(-p^*|Q))| \leq \varepsilon/2$$

is true.

Comparing (2.14), (2.16), (2.18) we observe that for $\alpha \leq \alpha_0(\varepsilon)$

$$(l^*, z^*) = (p^* G, z^*) \geq f(l, \Lambda(1, \alpha)) + \varepsilon/2 \quad .$$

Therefore, with $\Lambda^* = \Lambda(1, \alpha^*)$, $0 < \alpha^* < \alpha_0(\varepsilon)$ the pair $\{l^*, \Lambda^*\}$ yields the inequality (2.15).

Now assume $z^* \bar{\in} Z_s$. Then there exists a vector $l^0$ for which

$$(l^0, z^*) \geq \zeta(l^0) + \sigma \quad , \qquad \sigma > 0 \quad ,$$

where

$$\zeta(l) = \rho(A'l|X) + \rho(C'l|P)$$

Taking $l = l^0$, $\Lambda = \Lambda(1, \alpha)$ we find:

$$\Psi(\Lambda(1, \alpha)) \to 0 \quad , \qquad \alpha \to \infty \quad .$$

But then for any $\sigma \to 0$ there exists a number $\alpha^0(\sigma)$ such that

$$|f(l^0, \Lambda(1, \alpha)) - \zeta(l^0)| \leq \sigma/2$$

provided $\alpha \leq \alpha^0(\sigma)$. Hence, for $\alpha \leq \alpha^0(\sigma)$ we have

$$(l^0, z^*) \geq f(l^0, \Lambda(1, \alpha)) + \sigma/2$$

contrary to (2.13). The theorem is thus proved.

From the given proof it follows that Theorem 2.1 remains true if we restrict ourselves to the one parametrical class

$$\Lambda^{(1)} = \{\Lambda(1, \alpha)\} \quad , \qquad \Lambda(1, \alpha) = \{I_n, \alpha I_m\}$$

Therefore, the theorem yields:

*Corollary 2.1* Under the conditions of Theorem 2.1 the inclusion $z \in Z$ is true if and only if for any $l \in E^n$ we have

$$(l\ ,\ z) \leq f(l) \quad , \tag{2.19}$$

where

$$f_1(l) = \inf \{f(l\ ,\ \Lambda(1\ ,\ \alpha))\,|\,\alpha > 0\}$$

Being positively homogeneous, the function $f_1(l)$ may, however, turn out to be nonconvex, its lower convex bound being the second conjugate $f_1^{**}(l)$ where

$$g^*(q) = \sup \{(l\ ,\ q) - g(l)\} \quad , \quad g^{**}(l) = (g^*)^*(l) \qquad [10]$$

In other words, we come to:

*Corollary 2.2* Under the conditions of Theorem 2.1, we have

$$\rho(l\,|\,Z) = f_1^{**}(l) \leq f_1(l) \tag{2.20}$$

However, if we move on to a broader class $\Lambda^{(2)} = \{L\ ,\ N\}$ where $L > 0$ and $N > 0$ depend together on m independent parameters it is possible to achieve a direct equality

$$\rho(l\,|\,Z) = f_2(l) \tag{2.21}$$

where

$$f_2(l) = \inf \{f(l\ ,\ \Lambda)\,|\,\Lambda \subseteq \Lambda^{(2)}\} = f_1^{**}(l) \quad , \tag{2.22}$$

The problem (2.22) will be called *the stochastically dual* problem for (2.4). The following assertion is true.

*Theorem 2.2* Under assumption 2.1 the conditions (2.21), (2.22) are true, where the infimum is taken over all $L > 0$, $N > 0$.

The proof of Theorem 2.2 is given in paper [17], where it is also shown that in (2.22) it suffices for $\Lambda = \{L\ ,\ N\}$ again to be in the class $\Lambda^{(2)}$.

The stochastic dual problem (2.22) may therefore replace (2.5).

## 3. Multi-Stage Systems

Returning to system (1.1)-(1.4) let us seek for $X[s] = X(1, s|X^0)$. We further introduce notations

$$Y(k) = \{x : y(k) - G(k)x(k) \in Q(k)\}$$

and $X^*(j, s|F)$ is the solution $X(s)$ of the equation

$$X(k+1) = A(k)X(k) + C(k)P(k) \quad , \qquad j \leq k < s - 1 \tag{3.1}$$

with $X(j) = F$. Then it is possible to verify the following recurrent equation similar to (2.3), see also [9].

*Lemma 3.1* Assume $y[1, k]$ to be the realization for the measurement vector $y$ of system (1.3), (1.1). Then the following condition is true

$$X[k] = X(1, k, X^0) = X^*(k-1, k|X[k-1]) \cap Y(k) \tag{3.2}$$

Formula (3.2) indicates that the innovation introduced by the k-th measurement $Y(k)$ appears in the form of an intersection (3.2). Therefore $X^*(k-1, k|X[k-1])$ is the estimate for the state of the system on stage k before the arrival of the k-th measurement while $X[k]$ is the estimate obtained after its arrival.

From suggestions similar to those of Theorem 2.1, there follows a procedure for describing the sets $X[k]$. Together with (1.1), (1.3) consider the system

$$w(k+1) = A(k)w(k) + C(k)v(k) + D(k)u(k) \tag{3.3}$$

$$k = 0, 1, \ldots, N-1 \quad ; \qquad w(0) = x^0 + w^0 \quad ,$$

$$z(k) = G(k)w(k) + \xi(k) + \eta(k) \quad , \tag{3.4}$$

where $x^0$, $v(k)$, $\xi(k)$ are deterministic, subjected "instantaneous" constraints

$$x^0 \in X^0 \quad , \quad v(k) \in P(k) \quad , \quad \xi(k) \in Q(k) \tag{3.4'}$$

while $w^0$, $u(k)$, $\eta(k)$ are independent stochastic Gaussian vectors with

$$\begin{aligned}
&\overline{w}^0 = Ew^0 = 0 \quad , \qquad \overline{u}^{(k)} = Eu(k) = 0 \quad , \\
&\overline{\eta}(k) = E\eta(k) = 0 \quad , \qquad Ew^0w^{0\prime} = P^0 \quad , \\
&Eu(k)u'(k) = L(k) \quad , \qquad E\eta(k)E\eta(k) = N(k) \quad ,
\end{aligned} \tag{3.5}$$

where L, N are positive definite.

Suppose that after k stages for system (3.3), (3.4) there arrived a measurement $z[1\,,k] \in E^{m,k}$. Having fixed the triplet

$$\zeta[0\,,k] = \{x^0\,, v[0\,, k-1]\,,\ \xi[1\,,k]\}$$

and having denoted $\omega[k] = \{v(k-1), \xi(k)\}$, $\mathcal{D}(k) = \{P(k-1), Q(k)\}$ we may find the conditional mean value

$$\overline{w}[k+1] = E\{w(k+1)\,|\,\omega(k)\ ,\ w(k)\ ,\ z(k+1)\}$$

where

$$\overline{w}(k) = Ew(k).$$

Denote

$$\begin{aligned}
W[k+1\,,F] &= W(k+1\,, L(k)\,, N(k+1)\,, F) \\
&= \cup\ \{\overline{w}[k+1]\,|\,\omega(k) \in \mathcal{D}(k)\ ,\ \overline{w}(k) \in F\}
\end{aligned}$$

From Lemma 2.4 and Theorems 2.1, 2.2 it follows:

*Theorem 3.1* Suppose assumption 2.1 holds for $A = A(k)$, $G = G(k+1)$; $k = 0,1\ldots,s$ and the sequence of observations $y[1,s]$, $z[1,s]$ for system (1.1), (1.3) and (3.3), (3.4) coincide: $y[1,s] = z[1,s]$. Then the following relation is true

$$X[s] = \{\cap\, \mathcal{W}(s, L, N, X[s-1]) \mid \Lambda \in \Lambda^{(1)}\} \quad , \qquad s > 0 \quad ,$$
$$X[0] = X^0 \quad , \qquad \Lambda = \{L, N\} \quad , \qquad P^0 = 0 \quad , \tag{3.6}$$

moreover, with $P^0 = 0$

$$\rho(l \mid X[s]) = \inf \{\rho(l \mid \mathcal{W}(s, L, N, X[s-1])\}$$

over all $(L, N) = \Lambda \subset \Lambda^{(2)}$.

*Theorem 3.2* Whatever the positive matrices $\{L(k-1), N(k)\} = \Lambda[k]$ are the following inclusions are true

$$X[k+1] \subseteq \mathcal{W}(k+1, L(k-1), N(k), X[k])$$
$$= R(k+1, \Lambda(k), X[k]) \quad , \qquad k \geq 0 \quad , \tag{3.7}$$

where

$$R(k+1, \Lambda(k), X[k]) = (I_n - H(k+1)G(k+1))(A(k)X[k]$$
$$+ H(k+1)(y(k+1) - Q(k+1)) \quad ,$$

$$X[0] = X^0 \quad , \tag{3.7'}$$

$$H(k+1) = D(k)L(k)D'(k)G'(k+1)K^{-1}(k+1) \quad ,$$

$$K(k+1) = N(k+1)G(k+1)D(k)L(k)D'(k)G'(k+1) \quad ,$$

The recurrent relations (3.7) thus allow a complete description of $X[s]$ due to equality (3.6). Solving the system

$$W(k+1) = R(k+1\,,\Lambda[k]\,,W(k)) \quad ,$$

$$W(0) = X^0$$

we find

$$X[k+1] \subseteq W(k+1)$$

where

$$\rho(l|X[k+1]) = \inf\{\rho(l|W(k+1))|\Lambda[j+1] \;;$$

$$j = 0,\ldots,k \;;\; P^0 = 0\}$$

with each pair $\Lambda[j+1] = \{L(j)\,,N(j+1)\}$ belonging to the class $\Lambda^{(2)}$. The total number of parameters over which the minimum is to be sought for does not exceed km.

The given procedure is similar to the one given in (2.7). It is justified if the sets $X[k]$ are to be known for each $k > 0$. Note that in any way with arbitrary $L(j)\,,N(j+1)$, $j = 0,\ldots,k-1$, the sets $W(k)$ always include $X[k]$.

Now assume that the desired estimate is to be found for only a fixed stage $s > 1$. Taking $z[1\,,s]$ to be known and triplet $\zeta[0\,,s]$ for system (3.3), (3.4), (3.4') to be fixed we may find the conditional mean values

$$\bar{w}(k) = E\{w(k)|z[1\,,k]\,,\,\zeta[0\,,k]\}$$

and the conditional covariance

$$P[k] = E\{w(k) - \bar{w}(k))(w(k) - \bar{w}(k))'|z[1\,,k]\}$$

where

$$Ew(0) = x^0 \quad , \qquad P(0) = P^0$$

Denoting

$$\bar{w}[j, k, F] = E\{w(k) \mid z[j+1, k], v[j, k-1],$$

$$\xi[j+1, k], \bar{w}(j)\},$$

$$\bar{W}[j, k, F] = \cup E\{w(k) \mid z[j+1, k], v[j, k-1] \in P[j, k-1]$$

$$\xi[j+1, k] \in Q[j+1, k], \bar{w}(j) \in F\},$$

$$\bar{W}[0, k, X^0] = \bar{W}(k),$$

and having in view the Markovian property for the process (3.3), (3.4) it is possible to conclude the following:

*Lemma 3.2* The equality

$$\bar{W}(k) = \bar{W}(j, k, \bar{W}(j)) \tag{3.8}$$

holds for any j, k, $j \leq k$.

The corresponding formulae that generalize (2.7), (2.9) have the form

$$\bar{W}(k+1) = (E - S(k+1)G(k+1))(A(k)\bar{W}(k) + C(k)\mathcal{P})$$

$$+ S(k+1)(z(k+1) - G(k+1)\mathcal{Q}),$$

$$S(k+1) = P(k+1)G'(k+1)N^{-1}(k+1), \tag{3.9}$$

$$P(k+1) = B(k) - B(k)G'(k+1)K^{-1}(k+1)G(k+1)B(k),$$

$$B(k) = A(k)\mathcal{P}(k)A'(k) + L(k)$$

$$K(k+1) = N(k+1) + G(k+1)B(k)G'(k+1)$$

$$P(k_0) = L,$$

If we again suppose $z[1, s] = y[1, s]$, then due to the inclusions

$$\bar{W}(k+1) \supseteq \bar{W}(k, k+1, X[k]) \quad , \; k > 0 \qquad (3.9')$$

that follow from Lemma 2.4 and to the monotonicity property

$$\bar{W}(k, k+1, F_1) \subseteq \bar{W}(k, k+1, F_2) \quad , \qquad F_1 \subseteq F_2 \quad ,$$

that follows from (3.9) we obtain in view of (3.8)

$$X[k] \subseteq \bar{W}(k) \quad , \qquad \text{for } k > 1 \quad , \qquad (3.10)$$

Consider the following condition:

*Assumption 3.1*

The system (1.1), (1.3) $v[0, s-1] = 0$, $\xi[1, s] = 0$ is completely controllable on $[0, s]$.

The given property is defined for example in [15].

In the latter case the following proposition is true:

*Theorem 3.3* Under the conditions of Theorem 3.1 and assumption 3.1 assume $y[1, s] = z[1, s]$. Then the equality

$$X[s] = \{\cap \bar{W}(s) \mid P^0 \; , \; N(k) \; , \; L(k) \equiv 0 \; , \; k = 1,\ldots,s\} \qquad (3.11)$$

is true for any $P^0 > 0$ and any diagonal $N(k) > 0$. Moreover, for the given class of matrices we have

$$\rho(l \mid X[s]) = \inf \{\rho(l \mid \bar{W}(s)) \mid P^0 \; , \; \bar{N}(k) > 0 \; , \; L \equiv 0 \quad ,$$

$$k = 1,\ldots,s\} \qquad (3.12)$$

Therefore, the precise estimate is attained here through a minimization procedure over a number z of parameters, $z \leq ms + n^2$. The proof of this assertion follows a scheme that generalizes the one for Theorems 2.1, 2.2, (see also reference [17]).

*Remark 3.1* The relations (3.9), (3.10) may therefore be treated as follows:

(a) In the case of a set-membership description of uncertainty as in (3.4') with $u(k) = 0$, $\eta(k) \equiv 0$ equations (3.9), (3.10) contain complete information on $X[k+1]$ as stated in Theorem (3.3).
(b) In the case of both set-membership and stochastic uncertainty as in (3.3)-(3.5) equation (3.9) describes the evolution of the set of the mean values of the estimates.
(c) In the case of pure stochastic uncertainty with sets $X^0$, $P(k)$, $Q(k)$ consisting of one element $(x^0, p(k), q(k))$ each, the relation (3.9) turns out to be an equality which coincides with the conventional equations of Kalman's filtering theory.

*Remark 3.2* Following the scheme of Theorem 2.1 it is possible to show that relation (3.11) holds for $P^0$, $N(k)$ selected as follows:

$$P^0 = \beta I_n \quad , \quad N(k) = \alpha(k) I_m \quad ,$$

where

$$\beta > 0 \quad , \qquad \alpha(k) > 0 \quad , \qquad k = 1,\ldots,s \quad .$$

The given procedure has a very simple interpretation for particular cases. Indeed, if we apply (3.7) for a one-stage procedure due to system

$$x_1(k+1) = x_1(k) + hx_2(k) \quad ,$$

$$x_2(k+1) = x_2(k) - hx_2(k) + v(k)h \quad ,$$

with observation

$$y(k) = x_2(k)$$

and constraints

$$x(0) \in X_0 = \{x_1^0, x_2^0 : x_1^0 \in X_1^0 \ , \ x_2^0 = c\} \quad , \quad v(0) \in P \quad ,$$

$$X_1^0 = \{x_1 : \alpha \le x_1 \le \beta\} \ , \qquad P = \{v : |v| \le \mu\} \ ,$$

then the role of L, N will be attributed to $l_1 > 0$, $l_2 > 0$, $n > 0$. According to relations (3.9) that coincide for one stage with (3.7) and to (3.10) we will have

$$w_1(1) = (1 - ph)x_1^0 - ph((c - y(1))h^{-1} + v(0)) \quad ,$$

where the parameter

$$ph = (-h^2 l_1 + h^2 l_2)(h^2 l_1 + l_2 + n)^{-1}$$

varies in the range $-1 \le ph \le h^2$. Restricting ourselves to the set $-1 \le ph \le 0$ and passing to set $\overline{W}_1(1)$ we find, assuming $ph = -\alpha$,

$$\overline{W}_1(1) = (1 - \alpha)X_1^0 + \alpha F \, , \, F = (c - y(1))h^{-1} + P$$

whence

$$X_1^* = \{\cap \overline{W}_1(1) \,|\, 0 \le \alpha \le 1\} = X_1^0 \cap F$$

Moreover,

$$\inf \{\rho(l|\overline{W}_1(1)) \,|\, 0 \le \alpha \le 1\} = \rho(l|X_1^0 \cap F)$$

It is not difficult to observe that the exact solution

$$X[1] = \{x_1 , x_2 : x_1 \in X_1^0 \cap F \, , \, x_2 = y(1)\}$$

## 4. A Particular Case. Additional Information

Assume that in system (1.1), (1.3) we have $P(k) = \{0\}$, $A(k) \equiv \{0\}$. Then $x(k) \equiv x$ and due to measurement

$$y(k) = G(k)x + \xi(k) \quad , \qquad k > 1$$

we are to identify the vector x under constraints

$$x \in X^0 \quad , \qquad \xi(k) \in Q \quad , \qquad k > 1 \tag{4.1}$$

given in advance. Assume that some additional information on vectors $\xi(k)$ is available, namely, $\xi(k)$ satisfy:

*Assumption 4.1*

(a) All the $\xi(k)$, $k \geq 1$ are independent random vectors equally distributed with continuous density $p_\xi(z)$ independent of k, its support being the set Q. (The function $p_\xi(z)$ itself may be unknown).

(b) The matrix $G[1, n] = \{G'(1), \dots G'(n)\}$ is of rank n.

(c) The function $G(k)$, $k \geq 0$ is periodic of periodic n.

Consider the sets $X[s]$ consistent with measurement $y[1, s]$ and constraints (4.1).

Note that the sequence $\{y(1), \dots, y(s), \dots\}$ of measurements is now a random sequence governed by a stochastic mechanism which under assumption 4.1 actually possesses some ergodic properties.

*Lemma 4.1* Assumption 4.1 being fulfilled, with probability 1 there will appear a sequence of measurements $\{y(1), \dots y(s)\}$ such that

$$X[s] \to \{x^0\} \quad , \qquad s \to \infty \quad ,$$

in the Hausdorff metric [16]. Here $\{x^0\}$ is an one-element set that coincides with $x^0$.

Therefore, the arrival of a minor additional information on the statistical properties of $\xi(k)$ yields an asymptotical convergence of $X[s]$ to the vector $x^0$, allowing thus to obtain an exact solution.

REFERENCES


1. Kalman, R.E. (1960) A New Approach to Linear Filtering and Prediction Problems. Trans.ASME.Ser.D, Vol. 82.

2. Krasovskii, N.N. (1964) On the Theory of Controllability and Observability of Linear Dynamic Systems. Prikl.Mat. Mech., t.28, No. 1.

3. Witsenhausen, H.S. (1968) Sets of Possible States of Linear Systems Given Perturbed Observations. IEEE Trans. Automat.Control, Vol. AC-3, February.

4. Kurzanskii, A.B. (1970) On the Duality of the Problems of Control and Observation. Prikl.Mat.Mech., t.34, No. 3.

5. Bertsekas, D., Rhodes, J. (1971) Recursive State Estimation for a Setmembership Description of Uncertainty. IEEE Trans. Automat.Control, Vol. AC-16, No. 2.

6. Kurzanskii, A.B. (1972) Differential Games of Observation. Dokl.Akad.Nauk SSR, t.207, No. 3.

7. Schlaepfer, F.M., Schweppe, F.C. (1972) Continuous-Time State Estimation Under Disturbances Bounded by Convex Sets. IEEE Trans.Automat.Control, Vol. AC-17, No. 2.

8. Kurzanskii, A.B. (1977) Control and Observation Under Conditions of Uncertainty. Moscow.

9. Katz, I.J., Kurzanskii, A.B. ((1978) Minimax multistage filtering in statistically uncertain situations. Automation and Remote Control, No. 11.

10. Rockafellar, R.T. (1970) Convex Analysis. Princeton University Press.

11. Albert, A. (1972) Regression and the Moor-Penrose Pseudo-inverse. Academic Press.

12. Bryson, A.E., Ho, Y.C. (1969) Applied Optimal Control Optimization, Estimation and Control. Blaisdell Publishing Company.

13. Cox, D.R., Hinkley, D.V. (1974) Theoretical Statistics. Chapman and Hall, London.

14. De Groot, M.H. (1970) Optimal Statistical Decisions. McGraw Hill, New York, p. 489.

15. Eykhoff, P. (1974) System Identification. Parameter and State Estimation. J. Wiley & Sons.

16. Kuratowski, K. (1968) Topology. Vol. 2, Acad. Press.

17. Koscheev, A.S., Kurzanskii, A.B. (Forthcoming) Adaptive Estimation of Multistage Systems Under Uncertainty. Izv. Akad.Nauk SSR, Tech. Cybernetics.