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AXIOMATIZING THE SHAPLEY VALUE
WITHOUT LINEARITY

H.P. Young

This paper shows that the Shapley value can be uniquely characterized by a simple monotonicity condition without resorting to either the linearity or dummy assumptions originally used by Shapley [4].

Let N be a fixed finite set. By a *game* in characteristic function form is meant a function v which assigns to every subset S of N a real number v(S), called the value of S, such that $v(\phi) = 0$, and for all disjoint S and T

$$v(S \cup T) \geqq v(S) + v(T) \quad .$$

A *value* is a function $\varphi : V_N \rightarrow \mathbb{R}^N$ where V_N is the set of all games on N. Following Shapley we say that φ is *symmetric* if for every permutation of N we have

$$\varphi_i(v) = \varphi_{\pi(i)}(\pi v) \quad ,$$

where πv is the game defined by $(\pi v)(S) = v(\pi S)$ for all S. The value φ is *efficient* if $\sum_N \varphi_i(v) = v(N)$ for all v.

Monotonicity compares a player's inherent claims in different games on the same set N. One formulation of this property, due to Megiddo [3], is that if two games v and w differ only in that $v(N) \geq w(N)$, then we should have $\varphi_i(v) \geq \varphi_i(w)$ for all i. In other words, if in passing from w to v the claims of all proper coalitions stay the same while the total amount to be distributed increases, then no player's allocation should decrease. The analogous concept also arises in apportionment [1] and bargaining theory [2]. Megiddo shows by example that the nucleolus is not monotonic in this sense.

The following stronger formulation allows a comparison of players' allocations under more general changes in the structure of the game. For any game v and player i the *derivative of* v *with respect to* i is the function $v^i(S)$ defined for all $S \subseteq N$ such that

$$v^i(S) = \begin{cases} v(S) - v(S-i) & \text{if } i \in S \\ v(S+i) - v(S) & \text{if } i \notin S \end{cases} .$$

The value φ is *strongly monotonic* if whenever $v^i(S) \geqq w^i(S)$ for all S then $\varphi_i(v) \geqq \varphi_i(w)$. In other words, if in passing from w to v, i's marginal contribution to *every* subset increases or stays the same, then i's allocation must not decrease. In particular strong monotonicity implies monotonicity in Megiddo's sense.

It is clear that the Shapley value is strongly monotonic, since it may be written

$$\varphi_i(v) = \sum_{S: i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} v^i(S)$$

Theorem. The Shapley value is the unique symmetric and efficient value that is strongly monotonic.

Proof. First note that strong monotonicity means that for any two games $v, w \in V_N$,

(1) $$v^i(S) = w^i(S) \ \forall S \text{ implies } \varphi_i(v) = \varphi_i(w) .$$

Next consider the symmetric game w on N which is identically zero on all coalitions, so that $w^i(S) = 0$ for all i,S. By symmetry $\varphi_i(w) = \varphi_j(w)$ for all $i \neq j$ and by efficiency $\sum_N \varphi_i(w) = 0$, hence for all i, $\varphi_i(w) = 0$. By (1) it follows that for any game v on N and any $i \in N$,

(2) $$v^i(S) = 0 \ \forall S \text{ implies } \varphi_i(v) = 0 \quad .$$

That is, dummy players get nothing.

We now exploit the fact noted by Shapley that every game v can be expressed as a sum of *primitive games*

(3) $$v = \sum_{\phi \neq R \subseteq N} c_R v_R \quad ,$$

where

$$c_R v_R(S) = \begin{cases} c_R & \text{if } R \subseteq S \\ 0 & \text{if } R \nsubseteq S \end{cases}$$

The Shapley value can be expressed $\varphi_i(v) = \sum_{\phi \neq R \subseteq N} \varphi_i(c_R v_R) = \sum_{R: i \in R} c_R / |R|$. A game u is *symmetric* if for all $i \neq j$ there is a permutation π taking i to j such that $u(\pi S) = u(S)$ for all S. Letting $c_r = \max_{R: |R| = r} c_R$ for $1 \leq r \leq n$, $\tilde{c}_R = c_{|R|} - c_R$, and $u = \sum_{\phi \neq R \subseteq N} c_{|R|} v_R$, v can be rewritten in the form

(4) $$v = u - \sum_{\phi \neq R \subseteq N} \tilde{c}_R v_R$$

where $\tilde{c}_R \geq 0 \ \forall R$ and u is symmetric. Define the *index* I of v to be the minimum number of terms with $\tilde{c}_R > 0$ in some expression for v of form (4). The theorem is proved by induction on I.

If I = 0, v = u is symmetric so $\varphi_i(v) = \varphi_j(v)$ for all $i \neq j$, whence by efficiency $\varphi_i(v) = v(N)/n$, which is the Shapley value.

If I = 1, $v = u - \tilde{c}_R v_R$ for some $R \subseteq N$. For $i \notin R$, $v^i(S) = u^i(S)$ for all S, hence by monotonicity $\varphi_i(v) = u(N)/n$. By symmetry $\varphi_i(v) = \varphi_j(v)$ for all i, $j \in R$; combined with efficiency this says that

$$\varphi_i(v) = \begin{cases} u(N)/n - \tilde{c}_R/|R| & \text{for } i \in R \\ u(N)/n & \text{for } i \notin R \end{cases}$$

which is the Shapley value for v.

Assume now that $\varphi(v)$ is the Shapley value whenever the index of v is at most I. In particular this means that if $v = u - \sum_{k=1}^{I} \tilde{c}_{R_k} v_{R_k}$ then

$$(4) \qquad \varphi_i(v) = \varphi_i(u) - \sum_{k=1}^{I} \varphi_i(\tilde{c}_{R_k} v_{R_k}) = u(N)/n - \sum_{k:i\in R_k} \tilde{c}_{R_k}/|R_k|$$

Let v have index I+1 with expression

$$v = u - \sum_{k=1}^{I+1} \tilde{c}_{R_k} v_{R_k}, \quad \tilde{c}_{R_k} > 0 \quad .$$

Let $R = \bigcap_{k=1}^{I+1} R_k$ and suppose $i \notin R$. Define the game

$$w = u - \sum_{k:i\in R_k} \tilde{c}_{R_k} v_{R_k} \quad .$$

and note that w is superadditive. The index of w is at most I and $w^i(S) = v^i(S)$ for all S, so using induction it follows that

$$\varphi_i(v) = \varphi_i(w) = \varphi_i(u) - \sum_{k:i\in R_k} \varphi_i(\tilde{c}_{R_k} v_{R_k}) = \varphi_i(u) - \sum_{k=1}^{I+1} \varphi_i(\tilde{c}_{R_k} v_{R_k}) \quad ,$$

the latter since $\varphi_i(\tilde{c}_{R_k} v_{R_k}) = 0$ whenever $i \notin R_k$, by (2). But this is just the Shapley value for i.

It remains to show that $\varphi_i(v)$ is the Shapley value when $i \in R = \bigcap_{k=1}^{I+1} R_k$, i.e., that $\varphi_i(v) = \varphi_i(u) - \sum_{k=1}^{I+1} \tilde{c}_{R_k}$. Since v is symmetric on R it suffices to show that $\sum_R (\varphi_i(u) - \varphi_i(v)) = |R| \sum_{k=1}^{I+1} \tilde{c}_{R_k}$. This follows by observing that

$$\sum_{N-R} (\varphi_i(u) - \varphi_i(v)) = \sum_{k=1}^{I+1} \sum_{i\in N-R} \varphi_i(\tilde{c}_{R_k} v_{R_k})$$

$$= \sum_{k=1}^{I+1} \tilde{c}_{R_k} \frac{|R_k - R|}{|R_k|} = \sum_{k=1}^{I+1} \tilde{c}_{R_k} \frac{\left(|R_k| - |R|\right)}{|R_k|}$$

the latter since $R \subseteq R_k$ for all k.

Thus by efficiency

$$\sum_{i\in R} (\varphi_i(u) - \varphi_i(v)) = \sum_{k=1}^{I+1} \tilde{c}_{R_k} - \sum_{k=1}^{I+1} \tilde{c}_{R_k} \frac{\left(|R_k| - |R|\right)}{|R_k|} = |R| \sum_{k=1}^{I+1} \tilde{c}_{R_k} \ . \ \square$$

Observe that the proof only requires the assumption that a player's value depends just on the vector of his marginal contributions. This condition is also implicit in Shapley's axiom scheme, which requires that dummies get nothing: if $v^i(S) = w^i(S)$ for all S then i is a dummy in (v-w) so $\varphi_i(v-w) = 0$; combined with linearity it follows that $\varphi_i(v) = \varphi_i(w)$. On the other hand, the condition that $v^i(S) = w^i(S)$ implies $\varphi_i(v) = \varphi_i(u)$ is much weaker than the dummy and linearity axioms, indeed seems only slightly stronger than the dummy axiom itself. What we have shown is that by taking full advantage of efficiency and symmetry we can use it to deduce linearity and effectively characterize the Shapley value.

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