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PREFACE

Evolution and growth of natural and manmade processes have impressed human beings from the very beginning. What is evolution? Is it the passage from an initial to a higher stage? What does "higher" mean in a world of many objectives? Is "higher" bound to the existence of monotonous indicators like entropy, or is it "gambling" within a predetermined combinatoric multifold of possibilities?

Questions of this kind arise from the phenomena in our environment, from the spring-off of new species, but also from processes in our manmade technological world. How is the transition of basic innovation to technology and use of the corresponding products by society, what forecast can be made from increasing CO_2 in the atmosphere on the impact on climate, from features of sëismologic waves on future events etc. That means there is a strong connection between evolution processes and the emphasis of systems analysis as a help for strategic actions.

This paper deals with general considerations about possible growth mechanisms as a base for creating valid growth models. But the main goal is to show how the parameters in growth models can be estimated using on one hand a fuzzy approach together with vector optimization and on the other hand a Bayesian approach. It can be seen that both approaches are useful and applicable and we get informations from one approach which the other one cannot give us. We studied already the growth of cracks in materials, processes well described in [10]. Preliminary results are contained in [13].

Research will be continued to identify the superposition of driving forces and of coupled systems in which oscillations can arise because of time delays between their driving-force pulses.

DYNAMIC PROBLEMS OF EVOLUTION

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1. SAUSAGE MODEL AND DRIVING FORCES

1.1 Basic Notions of Growth Theory

We assume that the growth of any system is connected with increasing values of one or more corresponding state variables, as for example the number of individuals in a population, the GNP in an economy, the number of cells in an organ, or the biomass in a plant. Thus we demand the existence of a monotonous indicator of growth. Every growth has on the one hand autonomous features manifesting driving forces from inside of a system; on the other hand a growth process reflects environmental features arising from exogenous influences.

We consider as a first approximation a growing system within a uniform environment. The environment supplies the system with resources and takes off the "garbage" from the system (heat, excreta, outputs in the form of products, etc.). It makes no difference if we include the restricted resources within the system and thus consider the whole system to be autonomous. However, we obtain a more fruitful insight into the interaction with the environment if we also consider the environment as a growing system and try to consider evolution processes in two coupled systems in which one of them is dominant. More complicated evolution processes occur if we consider a network of coupled systems. The general demands on the behavior of such networks are formulated in Section 4. The most important properties of growing systems depend on the interaction of stochastic and deterministic influences (growth under uncertainty).

We assume that the whole phenomenon of growth can be decomposed into a deterministic trend (using a reference model for the trend description) and a stochastic influence. The decomposition is the inverse to the interaction of both components; therefore in general we need an interaction model. In this paper we assume an additive superposition depending on the unknown parameters of the reference model. In general the interaction should be described with the help of an aggregation rule from fuzzy set theory. How this can be done we show for the example of generating driving forces for the trend. (/8/,/9/)

The driving force is generally understood as the complex of all physical reasons leading to the "observed" growth rates of the determinstic trend. In our case we always describe the trend by an ordinary differential equation of first order, the right side of which is considered as a model of the physical driving force. This differential equation shows us a qualitative behavior in the phase space of the differential equation. We believe that important features of the growth, especially bifurcation phenomena, where our trajectory can split up into some different trajectories, can be well understood by the corresponding qualitative behavior of the differential equation. But in general it might also be the case that stochastic influences essentially influence the bifurcation behavior; then it would be necessary to consider the branching of stochastic processes.

These questions are connected with the problem of modeling well the continuous and discontinuous phenomena of growth processes, which are very important for a better understanding. Sometimes discontinuities are produced by the changing character of the driving forces. This is often the case when growth is produced by introducing basic innovations into the use of society.

The difference between driving forces is an expression of the use of quite different technologies.

Consideration of experiences with the evolution of real systems leads to the hypothetical Sausage Model of Evolution.

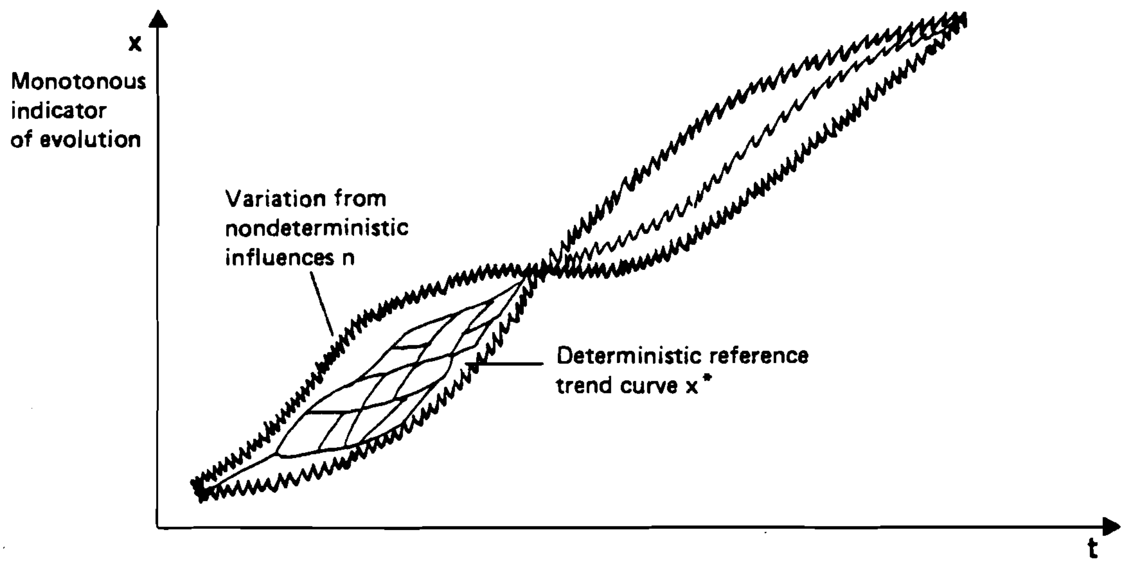

FIGURE 1: A SCHEMATIC REPRESENTATION OF THE SAUSAGE MODEL OF EVOLUTION

Sources of the nondeterministic influences:

- autonomous stochastic variations from internal processes;
- variations from internal control processes (internal feedbacks) to stabilize the motion between two equilibrium stages (steady states);
- from the area of trajectories of local bifurcations;
- stochastic influences from the environment;
- constraints from the environment.

General features of the dynamic evolution process:

- The motion between two steady states is in general a nonequilibrium motion with large exchange of resources (matter, energy, etc.) with the environment. Resources constraints are important with a moderate influence of stochastic factors from the environment.

- A more or less reliable decomposition of the motion into a deterministic trend

$$x^* = f(t,p) \qquad p = p_1, p_2, \ldots, p_k$$

and a stochastic disturbance n seems to be possible.

Very often an additive decomposition is assumed

$$x = x^* + n \quad .$$

In general we should use an appropriate model of the interaction between x^* and n. Dynamic models of evolution thus consist of:

- a deterministic trend model f(t,p);
- a model for the stochastic influences n as a stochastic process n = n(t,q);
- a model of interaction between the trend and the stochastic influence.

The parameters p of the trend and q of the stochastic process must be identified from measurements with the help of an efficient fitting procedure.

The trend between two steady states passes through the following three stages:

(1) Internal growth; organization within the system for exploring all environmental resources, which seem to be unrestricted.

(2) Acceleration of the use of all possibilities; the growth process manifests itself in increasing growth rates. The system streams into the space of possibilities like a compressed gas into an empty volume.

(3) Saturation; the constraints from external resources are felt more and more. The growth rates are decreasing and the system approaches a steady state. In this phase the system tries to find new and qualitatively different possibilities for a new evolution shift in the future.

Very often the equilibrium reached is unstable in the following sense. The future evolution can split into a finite number of quite different trajectories (bifurcation point). From a deterministic model f(t,p), under favorable conditions the different possibilities can be foreseen, but a mechanism for the choice of the future trajectory is unknown. The external stochastic influence now plays an important role and in fact determines what is going to occur in the next future.

Thus any growth process has a phase of continuous evolution followed by a discontinuous switching, a phase of revolution. If we want to model the switching process, we need a model of the part of the environment engaged in the interaction with the system considered.

What are the realistic possibilities of forecasting? Every forecasting procedure assumes that the following condition is fulfilled: the internal law of growth must implicitly be expressed in the measurements. Any procedure to find the law with the help of which the forecast is done can only amplify the constrast between the law and the nonimportant secondary influences.

For this contrast, enough information in the form of consecutive measurements must be given. Therefore we can contrast the trend f(t,p) against noise n(t,q) and vice versa to use this information during the duration of one transfer for forecasting, but we cannot contrast the law of switching without observing the environment in detail.

The problem of finding models for driving forces.

We concentrate on finding and "explaining" the trend trajectory $x^* = f(t,p)$ of an evolution process. We assume that the trend is generated by an ordinary differential equation

$$\frac{dx^*}{dt} = F(x^*,y) \quad .$$

x^* (one-dimensional or multivariate) is the growth indicator (a state variable).

$\frac{1}{x^*} \frac{dx^*}{dt}$ is the growth rate and

$F(x^*,y)$ is the driving force (a production function) for the stimulating and inhibiting influences on the growth.

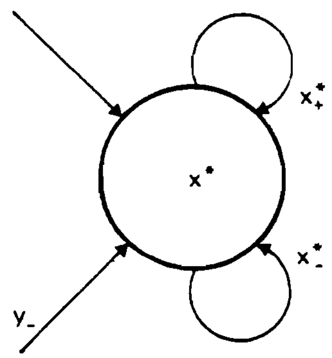

FIGURE 2: A GROWING SYSTEM AS A NODE UNDER THE INFLUENCE OF STIMULATING AND INHIBITING AUTONOMOUS AND EXTERNAL DRIVING FORCES

With increasing y_+, x_+^* the state x^* increases, and with increasing y_-, x_-^* the state x^* decreases.

The problem is how to find a relevant model for the production function of an existing growth process. We are convinced that for the solution of this problem the fuzzy set theory can make a valuable contribution.

1.2 Generation of production functions with the help of fuzzy sets

A production function is a static relationship between an output variable u and some input variables $u_1, u_2, \ldots, u_k$:

$$u = F(u_1, u_2, \ldots, u_k)$$

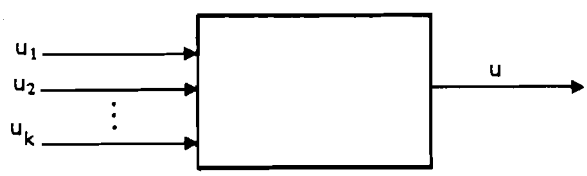

FIGURE 3: PRODUCTION FUNCTION AS A STATIC INPUT-OUTPUT RELATIONSHIP

We suppose something is known or reasonably assumed about the individual influence of input u_j on the output u. This 'knowledge' is modeled by a scalarizing function

$$\rho_j(u_j - u_{jr})$$

with the following properties: u_{jr} is a reference level of u_j corresponding to maximal effect on u (stimulating or inhibiting).

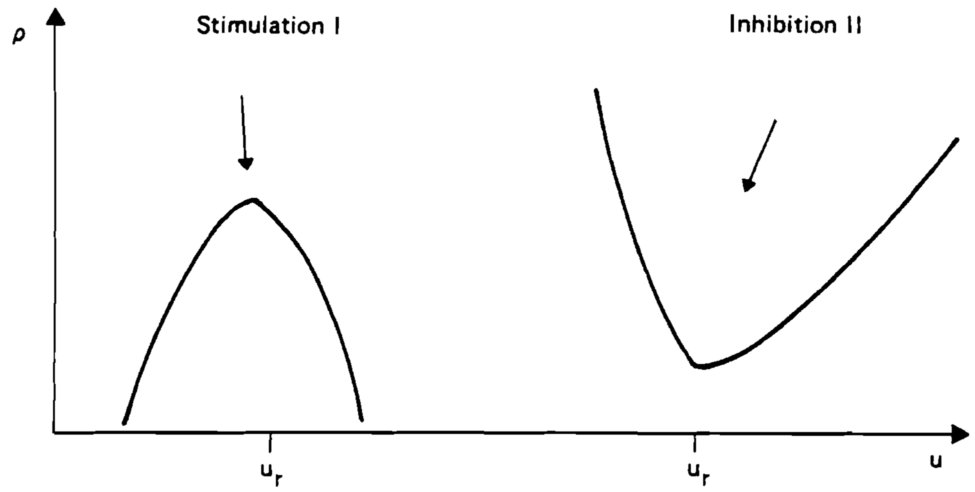

FIGURE 4: FUZZY DESCRIPTION OF A STIMULATION AND AN INHIBITION

We interpret the scalarizing function $\rho_j(u_j - u_{jr})$ as a membership function of u_{jr} which is considered to be a fuzzy set.

In case I we meet the fuzzy set u_{jr} and in case II the fuzzy complement $\bar{u}_{jr}$ of the reference level u_{jr}. Then $\rho_j(u_j - u_{jr})$ is a measure of the degree to which the concrete value u_j belongs to the corresponding fuzzy set u_{jr} or $\bar{u}_{jr}$. The output u is produced by the cooperation of $u_1, u_2, \dots, u_k$.

The fuzzy set u_{ref}, the favorable output, must then be the conjunction of all fuzzy sets u_{jr} as components

$$u_{ref} = \wedge u^*_{jr} \qquad u^*_{jr} = \begin{cases} u_{jr} & \text{stim.} \\ \bar{u}_{jr} & \text{inhib.} \end{cases}$$

In the language of membership functions, the membership function $\rho(u_{ref})$ is then

$$\rho(u_{ref}) = \text{conj}\ [\rho_1(u_{1r}),\rho_2(u_{2r}),\ldots,\rho_k(u_{kr})] \quad .$$

The production function is a monotonous increasing function of $\rho(u_{ref})$:

$$F(u_1,u_2,\ldots,u_k) = g[\rho(u_{ref})] \quad .$$

In the simple case, $g(\rho) = \rho$, we obtain the following production function model:

$$F(u_1,u_2,\ldots,u_k) = \wedge\, \rho_j(u_j - u_{jr}) \quad .$$

Very often the component functions ρ_j depend on parameters p_j which must be adjusted or which are used with exponents $e_j > 0$ (relative weights of the different influences). Sometimes it is convenient to use a threshold function for the generation of the individual membership functions $\rho_j(u_j)$. Let $u_{j\ell}$ be a lower level, and u_{ju} an upper level of u_j. Both levels are unwanted, i.e. should be described by complementary fuzzy sets:

$$\rho_{j\ell}(u_j) \qquad \text{or} \qquad \rho_{ju}(u_j)$$

$$\rho_j(u_j) = \rho_{j\ell}(u_j - u_{j\ell}) \wedge \rho_{ju}(u_{ju} - u_j) \quad .$$

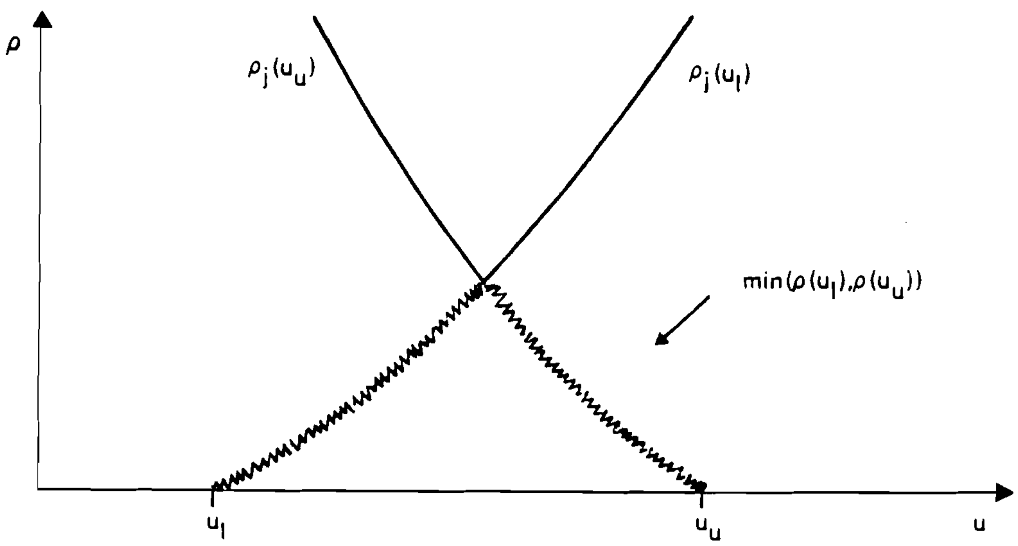

FIGURE 5: FUZZY THRESHOLDS

Some possible realizations of fuzzy conjunctions:

$$\text{conj}\ (\rho_1,\rho_2,\ldots,\rho_k) = \prod_{(i_1,i_2,\ldots,i_s)} \min\ (\rho i_1,\rho i_2,\ldots,\rho i_s)$$

$$\text{conj}\ (\rho_1,\rho_2,\ldots,\rho_k) = \Psi^{-1}[\prod_i \Psi(\rho_i)]$$

where $\Psi(u)$ is any monotonous function.

Remark: The form $F(u_1,u_2,\ldots,u_k)$ must be consistent with the measurement procedure and the estimation process for the components' membership functions.

2. DRIVING FORCES OF POWERFUNCTION PRODUCT TYPE

2.1 Hyperbolic and Parabolic Growth Laws

Special case of univariate autonomous growth

$$\frac{dx}{dt} = F(x) = K\rho(x - x_\ell) \wedge \rho(x_u - x)$$

$$g(u) = ku\ !$$

$$\rho(x - x_\ell) = (x - x_\ell)^k$$

$$\rho(x_u - x) = (x_u - x)^\ell$$

$$\wedge \Longleftrightarrow \cdot$$

We get

$$\frac{dx}{dt} = K(x - x_\ell)^k (x_u - x)^\ell \quad .$$

Stimulated growth

$$\frac{dx}{dt} = K(x - x_\ell)^k$$

$$\xi = x - x_\ell \qquad \frac{d\xi}{dt} = K\xi^k$$

$$\xi = \begin{cases} \kappa/(t_g - t)^{1/(k-1)} & \text{for } k > 1 \quad \text{hyperbolic} \\ \xi_0\, e^{kt} & \text{for } k = 1 \quad \text{exponential} \\ \kappa\ (t_g + t)^{1/(1-k)} & \text{for } k < 1 \quad \text{parabolic} \end{cases}$$

$$\kappa = 1/(|k-1|K)^{1/(k-1)} \quad t_g = 1/(\xi_0^{k-1}|k-1|K) \;;\; \xi_0 = x_0 - x_\ell .$$

Hyperbolic and parabolic growth differ remarkably because hyperbolic growth approaches infinity in a finite time t_g. Both modes of behavior are separated by the exponential growth law.

Saturated growth

$$\frac{dx}{dt} = K(x_u - x)^\ell$$

$$\xi = x_u - x \qquad \frac{d\xi}{dt} = -K\xi^\ell \qquad \xi_0 = x_u - x_0$$

$$\xi = \begin{cases} \kappa/(t_g + t)^{1/(\ell-1)} & \text{for } \ell > 1 \quad \text{hyperbolic} \\ \xi_0\, e^{-kt} & \text{for } \ell = 1 \quad \text{exponential} \\ \kappa(t_g - t)^{1/(1-\ell)} & \text{for } \ell < 1 \quad \text{parabolic} \end{cases}$$

Hyperbolic and parabolic saturation differ remarkably because parabolic saturation reaches the steady state in a finite time t_j. Both modes of behavior are separated by the exponential saturation law.

Exponential growth combined with exponential saturation - the logistic growth law

$$\frac{dx}{dt} = K(x - x_\ell)(x_u - x)$$

$$\xi = \frac{x - x_\ell}{x_u - x_\ell}$$

$$\frac{d\xi}{dt} = \tilde{K}\xi(1 - \xi) \qquad \tilde{K} = K(x_u - x_\ell)$$

$$\xi = \frac{\xi_0\, e^{kt}}{(1 - \xi_0) + \xi_0\, e^{kt}}$$

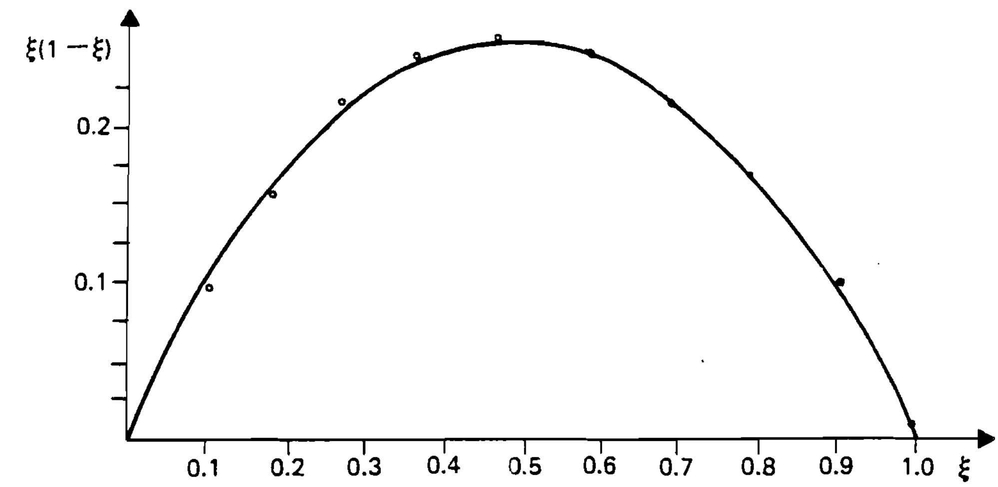

FIGURE 6: THE DRIVING FORCE OF THE LOGISTIC GROWTH CURVE

If the exponents k and ℓ have physical significance, and we are convinced they have, then the exponential growth law is unstable and separates into two stable modes: the hyperbolic and parabolic modes.

2.2 Growth Behavior of Chain Structures

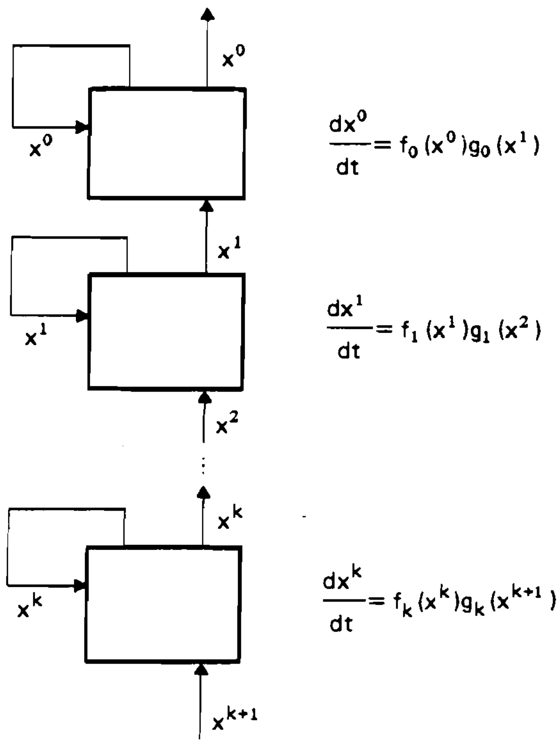

FIGURE 7: CHAIN-COUPLED SYSTEMS

Assumptions

(1) $f_i(x^i) \geqslant 0 \Rightarrow F_i(x) = \int\limits_{x_0}^{x} \frac{1}{f_i(u)}\,du$

is monotonously increasing.

(2) $g_i(x^{i+1}) \geqslant 0$

is monotonously increasing.

(3) $f_{i+1}(u) \leqslant f_i(u)$

where $f_i(u)$ converges against $f(u)$.

(4) $g_{i+1}(u) \leqslant g_i(u)$

where $g_i(u)$ converges against $g(u)$.

(5) For the initial conditions, x_0^i on the different levels must hold

$F_{i+1}(x_0^{i+1}) \leqslant F_i(x_0^i) \Rightarrow x_0^{i+1} \leqslant x_0^i$

we demand that x_0^i converges against x_0.

Then we can conclude:

$$x^i(t) = F_i^{-1}\left[F_i(x_0^i) + \int\limits_{t_0}^{t} g_i(x^{i+1})\,dt\right] .$$

From $x^{i+1}(t) \leqslant x^i(t)$, it follows that $x^i(t) \leqslant x^{i-1}(t)$. Let us stop the chain on level n putting $x^n(t) \equiv x_0^n$ and denoting the corresponding state variables by $x^{in}(t)$. Then we have

$$x^{n-1^n}(t) = F_{n-1}^{-1}(F_{n-1}(x_0^n) + g_{n-1}(x_0^n)(t - t_0)) \geqslant x_0^n$$

and therefore

$$x_0^n \leqslant x^{n-1^n}(t) \leqslant \ldots \leqslant x^{1^n}(t) \leqslant x^{0^n}(t) .$$

Let us now stop on the next higher level $n+1$ and compare $x^{i\,n+1}(t)$ with $x^{i\,n}(t)$. Now we have

$$x^{i\,n}(t) = F_i^{-1}\left[F_i(x_0^i) + \int_{t_0}^{t} g_i(x^{i+1\,n})\,dt\right]$$

$$x^{i\,n+1}(t) = F_i^{-1}\left[F_i(x_0^i) + \int_{t_0}^{t} g_i(x^{i+1\,n+1})\,dt\right] .$$

From $x^{i+1\,n+1} \geqslant x^{i+1\,n}$ it follows that

$$x^{i\,n+1} \geqslant x^{i\,n} .$$

What is occurring on level n?

$$x^{n\,u+1} = F_n^{-1}\left[F_n(x_0^n) + \int_{t_0} g_n(x_0^{n+1})\,dt\right]$$

$$> x_0^n = x^{n\,n} .$$

With a growing stop-level index n, $x^{i\,n}(t)$ can only increase. This means that on every level i, $x^{i\,n}(t)$ is a nondecreasing sequence of functions

$$\varphi^n(t) = x^{i\,n}(t)$$

$$\varphi^{n+1} \geqslant \varphi^n \geqslant \varphi^{n-1} \ldots$$

As a result we get the following diagram:

$$\begin{array}{ccc} x^{i\,n}(t) & \leqslant & x^{i\,u+1}(t) \\ \wedge\!\!\!\wedge & & \wedge\!\!\!\wedge \\ x^{i-1\,n}(t) & \leqslant & x^{i-1\,n+1}(t) \end{array} .$$

For very large n we have

$$x^{i\,n}(t) = F^{-1}\left[F(x_0) + \int_{t_0}^{t} g(x^{i+1\,n})\,dt\right]$$

or

$$F\,x^{i\,n}(t) - F\,x^{i+1\,n}(t) = \int_{t_0}^{t} [g(x^{i+1\,n}) - g(x^{i+2\,n})]dt \ .$$

If the "convergency" of $x^{i\,n}(t)$ after index n follows from this condition, then the limit follows the equation

$$x(t) = F^{-1}\ F(x_0) + \int_{t_0}^{t} g(x)dt$$

or

$$\frac{dx}{dt} = f(x)g(x) \quad . \qquad *$$

Thus consecutive systems decouple, and expose a behavior described by the equation *. Consequently, a chain of coupled exponential systems:

$$\frac{dx^i}{dt} = K\,x^i\,x^{i+1}$$

for $n \to \infty$, approaches the behavior of

$$\frac{dx^i}{dt} = K\,x^{i^2}$$

on higher levels i. Thus hyperbolic growth arises out of exponential growth. If we combine such a chain with an exponential system

$$\frac{dx^0}{dt} = K_0 x^0 x^1 \qquad K_0 \neq K$$

we get arbitrary hyperbolic growth.

3. GROWTH IN THE LONG RUN AND COUPLED GROWTH PROCESSES

In the long run we have to expect the reference structure of the kind shown in the figure below taking into account bifurcation.

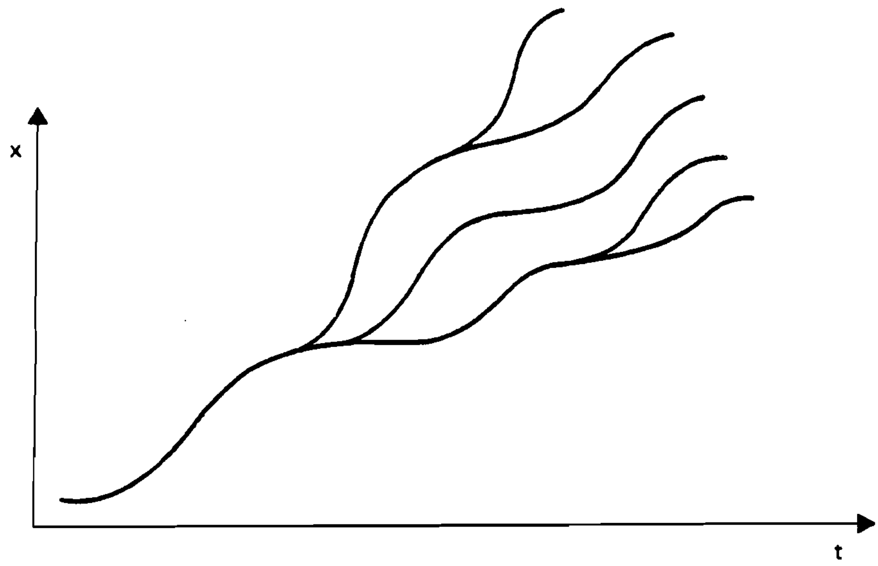

FIGURE 8: SCHEMATIC REPRESENTATION OF BIFURCATION PHENOMENA IN EVOLUTION PROCESSES

The different trajectories can be considered as "middle curves" of quite different clusters of future behavior.

A posteriori we observe one of these possible trajectories or several if we have a population of a large number of similar systems. The next figure shows one such trajectory together with the corresponding curve for the driving force.

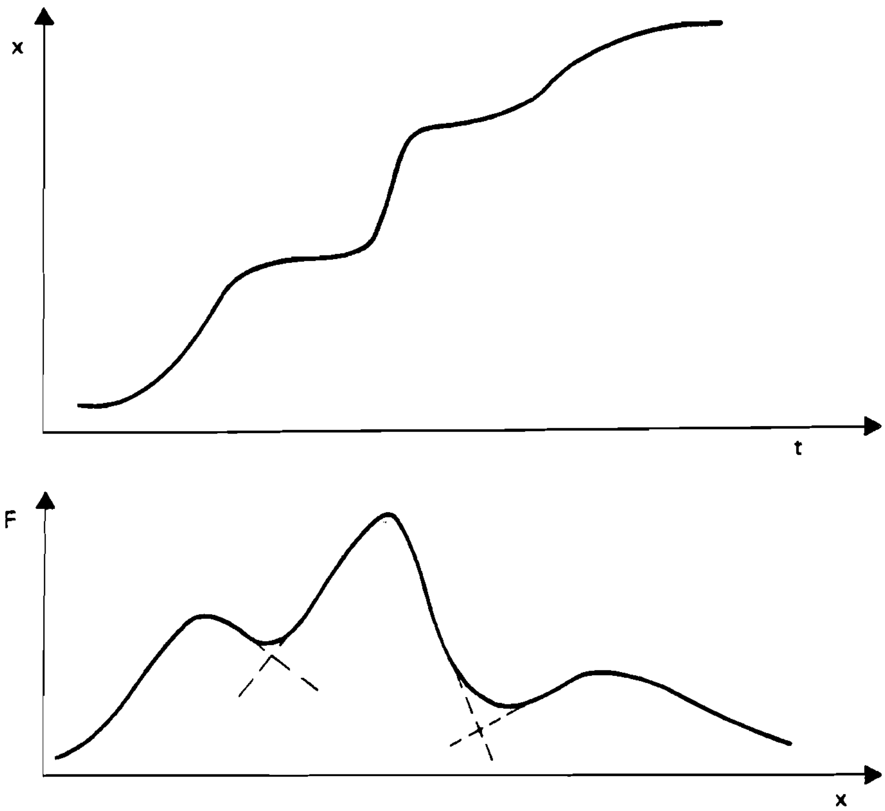

FIGURE 9: A SEQUENCE OF GROWTH PUSHES TOGETHER WITH THE CORRESPONDING SUPERPOSITION OF DRIVING FORCES

For the determination of the "best" reference curve we have to solve the wellknown "peak-resolution" problem.

Obviously the driving force F(x) is an aggregation of the driving forces $\varphi_i(x)$ of the different shifts. In general, the following questions arise.

(1) What kind of driving force determines a single growth period?

We are convinced that in many cases generalized logistic curves

$$K(x - x_\ell)^k (x_u - x)^\ell$$

are of importance. In many cases of existing software, especially that coming from statistics, Gaussian driving forces are assumed:

$$K \exp\left[-(x - a)^2 \kappa\right] \quad . \qquad /3/,/12/$$

From previous experience in fuzzy clustering, the following pulse form can be recommended

$$\frac{K}{a + \kappa(x - x_0)^c} \quad .$$

Driving forces that are often used in agriculture but also in economics are

$$K\, x^b\, e^{-cx} \quad .$$

(2) What kind of aggregation rule should be applied to combine the individual driving forces $\varphi_i(x)$ with the overall driving force F(x)?

The relevant aggregation rule should reflect in the correct manner the physical interaction between consecutive phases of the evolution process.

If we interpret $\varphi_i(x)$ as a membership function of the "fuzzy set" optimum individual driving force, we should use an appropriate disjunction rule:

$$F(x) = \bigvee_i \varphi_i(x) \quad .$$

very often indicated by + or max:

$$F(x) = \sum \varphi_i(x)$$

$$F(x) = \max_i \varphi_i(x) \quad .$$

Very frequently it is assumed that the growth in adjoining phases is qualitatively of the same kind. Then we should use for $\varphi_i(x)$ a standard form specialized only by a set of parameters:

$$F(x,p) = \sum A_i \varphi(x,p_i)$$

or

$$F(x,p) = \max_i A_i \varphi(x,p_i)$$

$$p = \{A_i, p_i\} \quad .$$

(3) How is a reference model in the long run fitted to the set of measurements?

We follow two different routes which are described in detail in Section 5.

(a) Fuzzy approach with vector optimization;

(b) Bayesian approach.

Up to now we have considered only univariate growth. Now we continue with a more complex system consisting of two coupled growing nodes.

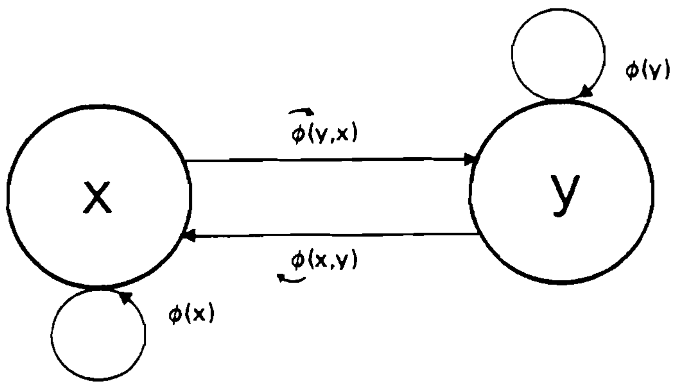

FIGURE 10: INTERACTION OF TWO COUPLED GROWING SYSTEMS

How must the autonomous and interactive driving forces be combined to get the acting driving forces of the nodes? Obviously we can apply conjunction, taking into account the fact that the forces work simultaneously, but we can also apply disjunction if we think of a superposition of the corresponding forces. It is appropriate at this time to study the behavior of the following reference system:

$$\frac{dx}{dt} = K_1 (x - x_{\ell 1})^{k_{11}} (x_{u1} - x)^{\ell_{11}} - L_1 (y - y_{\ell 2})^{k_{12}} (y_{u2} - y)^{\ell_{12}}$$

$$\frac{dy}{dt} = K_2 (y - y_{\ell 1})^{k_{22}} (y_{u1} - y)^{\ell_{22}} - L_2 (x - x_{\ell 2})^{k_{21}} (x_{u2} - x)^{\ell_{21}}$$

Expectations of the results of our research:

(1) The system should expose a bifurcation structure of possible trajectories in the phase space.

(2) Under certain conditions for the parameters every single node should show a characteristic long term run of a growing system (sequence of s-formed transfers).

(3) The long term runs of the two nodes should show us a certain delay time.

(4) Under certain conditions for the parameters the evolution process of the whole system should show oscillations (comparable with the Kondrjatev cycle in economics).

Possible examples of coupled evolution processes:

(1) production system and social system in macroeconomy;

(2) in an ecosystem the interaction between a useful population and pest system;

(3) the interaction of different processes influencing a heavy disease for example, coronary heart disease;

(4) interaction between the growth of cracks in a material and the accompaying acoustic emission.

4. DEMANDS OF A SOFTWARE INSTRUMENT FOR THE IDENTIFICATION, SIMULATION AND ANALYSIS OF COMPLEX SYSTEMS EVOLUTION

-- Law for autonomous evolution of nodes.
-- Mechanism of how the environment of a node is prepared for interaction with other nodes.
-- Interaction between nodes and the formation of cluster structures (virtually) under the action of the dialectics of affinity and aversion.
-- Stabilizing of some clusters as new particles (entities) if certain reactivity conditions are fulfilled.
-- Destabilizing mechanism contra-acting an increasing complexity.
-- Occurrence of different types of particles on a given level of aggregation because of bifurcation phenomena.
-- By iteration of this process, the generation of aggregated particles of different levels.
-- The trajectory of the whole system in every of its aggregation levels can also be considered as an evolution process.
-- Study and balance of the dynamic equilibrium on every aggregation level.

5. IDENTIFICATION OF THE PARAMETERS IN EVOLUTION MODELS

We deal first with a special case of this general problem. A trend of the form

$$\frac{dx}{dt} = F(x,p)$$

is taken as a reference. If the measurements are the growth velocities (or the growth rates) $\left(\frac{dx}{dt}\right)_i$ at points x_i, we use an additive reference model

$$y_i = F(x_i,p) + n_i$$

with noise variables n_i.

If the measurements are sampes of trajectories $y_i = x(t_i)$, we have to integrate the differential equation. The generally unknown initial condition should be included in the set of unknown parameters p. Then we use the following reference model:

$$y_i = x(t_i) + n_i \quad .$$

In general we allow that we have at every point x_i or t_i some information about the distribution of the corresponding noise n_i. If this is not the case we should combine consecutive measurements or apply moment methods.

Thus the information is given in the form shown in the following figures.

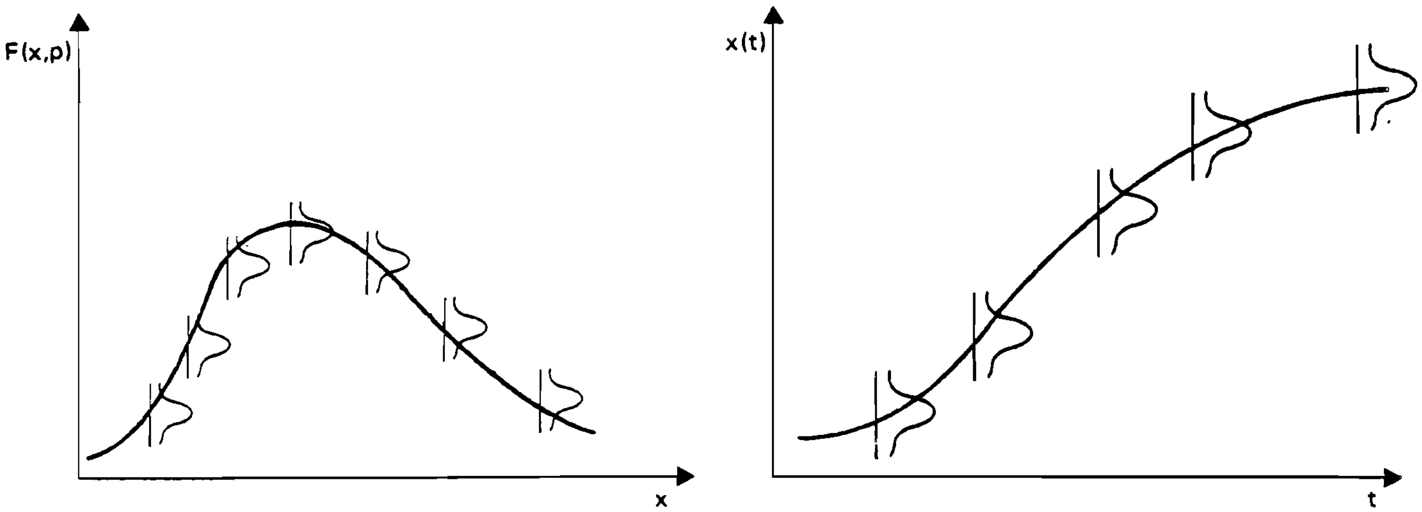

FIGURE 11: CHARACTER OF A PRIORI INFORMATION FOR GROWTH-RATE RESP. TIME TRAJECTORY MEASUREMENTS

In this paper we assume that the different noise variables n_i are "independent" of each other. Because this assumption leads to some problems we will eliminate it further on.

5.1 Fuzzy Identification Approach

(Peschel, Voigt, /1/,/2/,/3/)

We interpret the a priori information to each sample point as an elementary membership function $\varphi_{\Delta i}(n;n_i)$ belonging to the noise variables n_i. $\varphi_{\Delta i}(n;n_i)$ is a measure of the degree to which the concrete value is expected to occur. It is a relative measure and therefore only the ratios

$$\varphi_{\Delta i}(n;n_i)/\varphi_{\Delta i}(n';n_i)$$

are of interest.

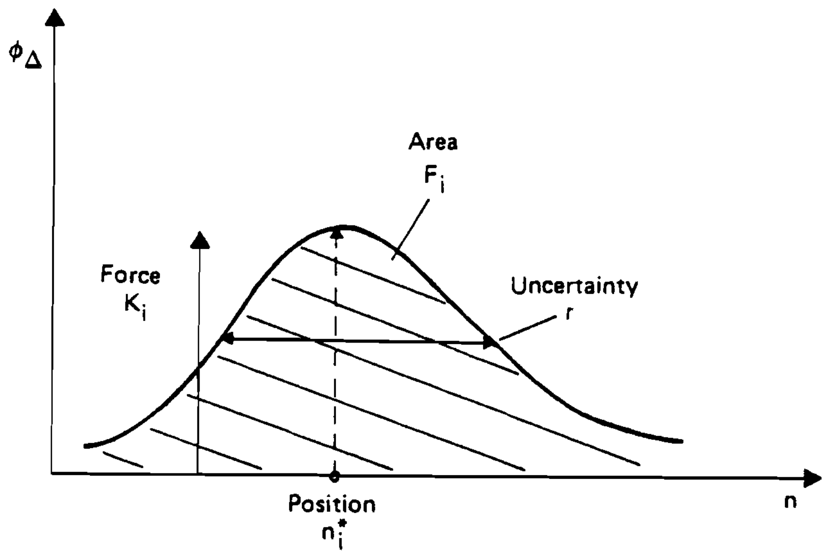

FIGURE 12: FUZZY DESCRIPTION OF A NOISY VARIABLE

The position n_i^* is a substitute for the deterministic value; the uncertainty r is a measure of fuzziness; the force K_i and the area F_i are both reliability measures often occurring in combination with each other.

We assume a priori knowledge about the reliability of our measurements and transform first all elementary membership functions in such a way that they reflect this a priori knowledge

We prefer to use a standard concept for the elementary membership function with a set of adjustable parameters:

$$\varphi_{\Delta i} = a_i \varphi(y - n_i^*, q_i) \quad .$$

a_i are given and reflect the reliability; n_i^*, q_i are given or estimated. Taking into account the additive reference then

$$\varphi_{\Delta i} = a_i \varphi\{n_i - [n_i^* - F(x_i, p)], q_i\} \quad .$$

Now we consider all n_i to be comparable with each other and replace them by a common variable $n_i \approx n_j$, i.e., we consider them as different descriptions of the same fuzzy variable n.

Gathering all the information contained in these different descriptions of the same fuzzy variable n using the disjunction rule of fuzzy sets we obtain the membership function of the fuzzy model-error estimation:

$$\varphi^*(n) = \bigvee_i a_i \varphi\{n - [n_i^* - F(x_i, p)], q_i\} \quad .$$

This is not the best error model because it still depends on the adjustable parameters

$$\{n_i^*, q_i, p\} = P$$

$$\varphi^*(n) = \Psi(n, P)$$

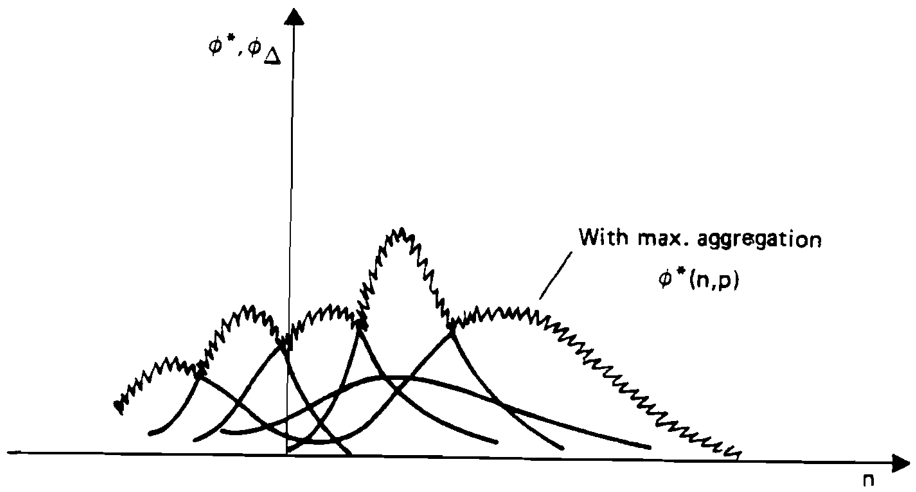

FIGURE 13: AGGREGATION OF ELEMENTARY MEMBERSHIP FUNCTIONS TO THE MODEL-ERROR MODEL

Now we formulate some reasonable demands on a "good" error model in the language of objectives (/4/,/5/,/6/,/7/).

(1) The asymmetry of the model error relative to $n = 0$ should be very small: Q_1.

(2) The breadth of the model error, the resulting uncertainty of the model, should be very small: Q_2.

(3) The steepness of the slope of the model error pulse should be very high: Q_3.

(4) The top of the model error pulse should be equally flat as well as possible: Q_4 etc.

Having agreed on the corresponding criteria we have to initiate a seeking procedure after the set of unknown parameters P to arrive at a set of efficient solutions in the sense of Pareto optimality; we have to solve a vector optimization task

$$Q_i(P) \rightarrow \text{extremum} .$$

For this concept we have elaborated a first version of a software package /11/, the first modules of which have been successfully checked and applied, but at the moment only for the case of our s-form evolution with a power-product driving force:

$$F(x,p) = K(x - x_\ell)^k (x_u - x)^\ell$$

5.2. Bayesian Identification Approach

5.2.1 General Approach

For the reference points $x_i (1 \leqslant i \leqslant N)$ we assume the measurements $y_{ij} (1 \leqslant j \leqslant m_j)$. These are used to fit a deterministic trend $f(x_i, B^T) = f_i(B^T)$ with unknown parameter vector $B^T = (b_1, b_2, \ldots, b_q)$. An additive reference between measurements and error samples $\tilde{n}_{ij}$ is assumed:

$$y_{i,j} = f_i(B^T) + \tilde{n}_{ij} \qquad 1 \leqslant i \leqslant N; \quad 1 \leqslant j \leqslant m_i \quad .$$

We suppose that the errors $\tilde{n}_{ij}$ are statistically independent, and

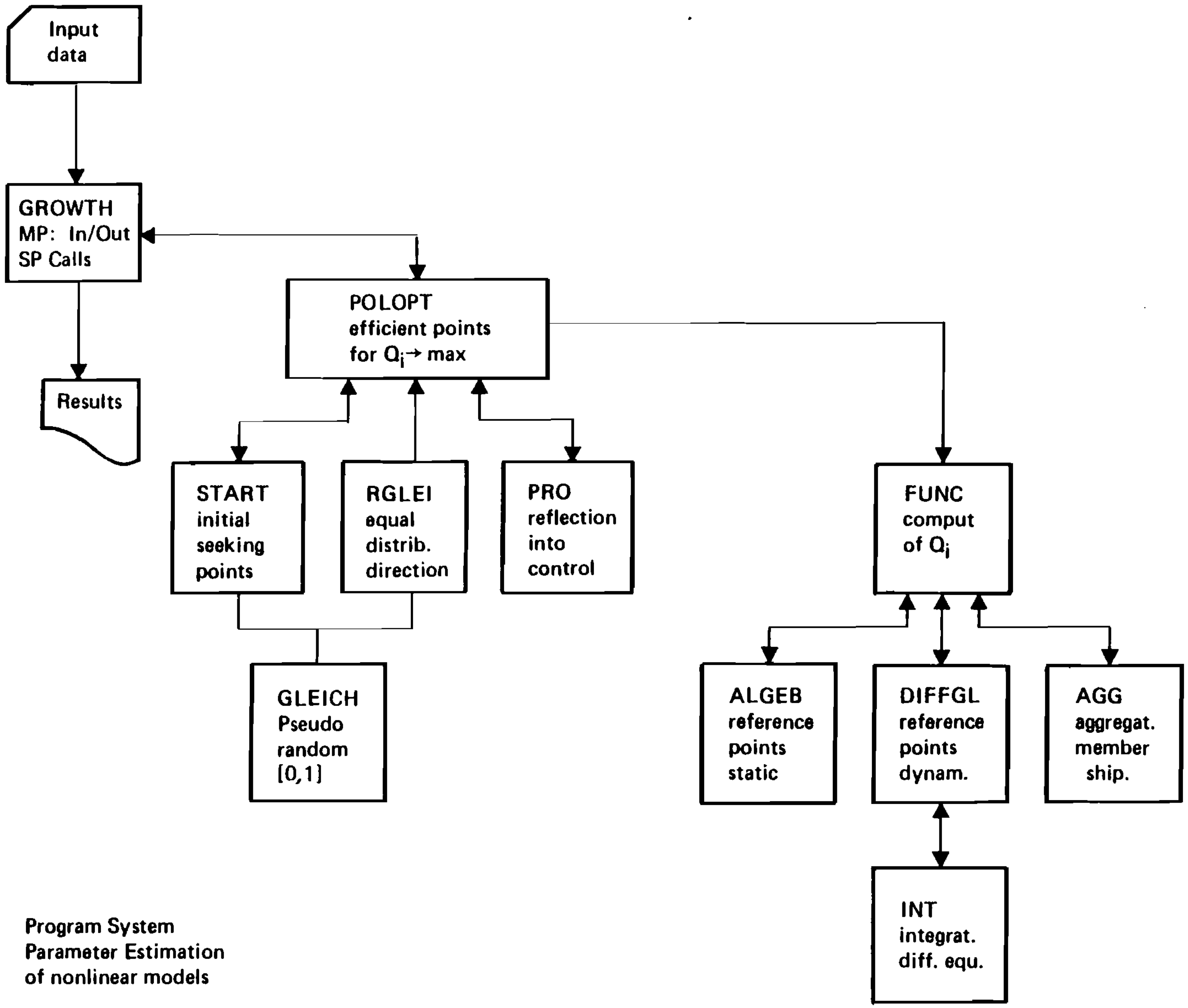

FIGURE 14: STRUCTURE OF THE SOFTWARE-PACKAGE FOR FUZZY MODEL PARAMETER ESTIMATION

that they have a common but unknown expectation n_E and unknown non-stationary variances R_{ij}^{-1}.

To simplify the estimation problem for the R_{ij}, we assume that they are constant in intervals:

$$I_n(i,j) = \{i \in [k_n + 1, k_n + s_n] \; ; \; j \in [1, m_i]\}$$

$$1 \leq n \leq M < N$$

$$< \tilde{n}_{i,j} > = n_E \qquad \forall\, i,j$$

$$< (\tilde{n}_{ij} - n_E)(\tilde{n}_{i'j'} - n_E) > = R_n^{-1} \delta_{ii'} \delta_{jj'} \qquad \forall\, i,j \in I_n$$

5.2.2 Description of the Method

In the Bayesian approach, the a posteriori probability density function of the unknown parameters is determined on the basis of an assumed a priori probability density of these parameters and the common density of all measurements. The measurement errors $\tilde{n}_{ij}$ are supposed to be independent Gaussian variables with a density

$$p(\tilde{n}_{ij}) = (2\pi R_n^{-1})^{-\frac{1}{2}} \exp\left[-\frac{R_n}{2}(\tilde{n}_{ij} - n_E)^2\right]$$

$$\forall\, i,j \in I_n(i,j)$$

n_E and R_n, with $n = 1,2,\ldots,M$ are to be estimated.

The unknown parameters B^T, n_E, $R_1,\ldots,R_M$ are assumed to be independent stochastic variables, i.e.

$$P_{pr}(B^T, n_E, R_1, \ldots, R_M) = P_{pr}(B^T) P_{pr}(n_E) \prod_{u=1}^{M} P_{pr}(R_n)$$

with the following concepts for the a priori densities

$$P_{pr}(n_E) = (2\pi K_E)^{-1} \exp\left(-R_E^2/2K_E\right)$$

$$P_{pr}(R_n) = \begin{cases} \left(\frac{\ell_{0n}}{2R_{0n}}\right)^{\frac{\ell_{0n}}{2}+1} R_n^{\ell_{0n}/2} \exp -\left(\frac{\ell_{0n}}{2}\frac{R_n}{R_{0n}}\right) & \text{for } R_n \geq 0 \\ 0 & \text{else} \end{cases}$$

Making use of the Bayesian rule, the a posteriori density of all unknown parameters is given by

$$P_{pos}(B^T,n_E,R_1,\ldots,R_M) = p\left(B^T,n_E,R_1,\ldots,R_M \Big/ \{y_{ij}\}_{j=1\ldots m_i}^{1\leq i\leq N}\right)$$

$$= C\, P_{pr}(B^T) P_{pr}(n_E) \prod_{u=1}^{M} P_{pr}(R_n) \prod_{i,j\in I_n} p(\tilde{n}_{ij})$$

with $\tilde{n}_{ij} = y_{ij} - f_i(B^T)$ as the reference signals.

First we concentrate on estimating the trend parameters B^T and the common bias n_E

$$P_{pos}(B^T,n_E) = \int_0^\infty dR_1,\ldots,\int_0^\infty dR_M P_{pos}(B^T,n_E,R_1,\ldots,R_M) \quad .$$

This leads to

$$P_{pos}(B^T,n_E) = C\, P_{pr}(B^T)\, P_{pr}(n_E) \prod_{u=1}^{M} f_n\left(\frac{x_n}{\sqrt{\gamma_n}}\right)$$

with

$$x_n^2 = <(\bar{y}_{ij} - f_i - n_E)^2>_n$$

$$f_n\left(\frac{x_n}{\sqrt{\gamma_n}}\right) = \left[1 + \left(\frac{x_n}{\sqrt{\gamma_n}}\right)^2\right]^{-\ell_n/2}$$

$$\ell_n = \ell_{0n} + M_n + 2$$

$$\gamma_n = <(y_{ij} - \bar{y}_i)^2>_n + \frac{\ell_{0n}}{R_{0n}M_n}$$

$$M_n = \sum_{i=K_n+1}^{K_n+S_n} m_i$$

where we have introduced the notations

$$\bar{k}_i = \bar{k}_{ij} = \frac{1}{m_i} \sum_{i=1}^{m_i} k_{ij}$$

$$<k_i>_n = \frac{1}{M_n} \sum_{i=K_n+1}^{K_n+S_n} m_i k_i$$

We can interpret the functions f_n as group membership functions by comparison with the fuzzy set approach. They correspond to the frequently used concept

$$\varphi_n(x) = \frac{1}{1 + b_n |x|^{\mu_n}} \quad .$$

If we put $\mu_n = \ell_n$ and $b_n = \gamma_n^{-\ell_n/2}$, both types of membership functions have the same properties: same amplitude in $x = 0$, nearly the same half lifetime and the same slope for $x \to \pm \infty$. From the above derived expression for

$$p(B^T, n_E) = \exp\left[-\frac{1}{2} Q(B^T, n_E)\right]$$

we get the cost function

$$Q(B^T, n_E) = -2\ell n P_{pr}(B^T) + \frac{n_E^2}{K_E} + \sum_{u=1}^{M} \ell_n \ell n \left[1 + \frac{<(\bar{y}_i - f_i - n_E)^2>_n}{\gamma_n}\right]$$

<u>Comparison of the polyoptimization approach with the cost function concept</u>

We approximate

$$\ell n \left[1 + \frac{<(\bar{y}_i - f_i - n_E)^2>_n}{\gamma_n}\right] \approx \frac{<(\bar{y}_i - f_i - n_E)^2>_n}{\gamma_n}$$

which corresponds to

$$f_n\left(\frac{x_n}{\sqrt{\gamma_n}}\right) = \left[1 + \left(\frac{x_n}{\sqrt{\gamma_n}}\right)^2\right]^{-\ell_n/2} \approx \tilde{f}_n\left(\frac{x_n}{\sqrt{\gamma_n}}\right) = \exp\left(-\frac{\ell_n}{2} x_n^2\right) .$$

Thus we obtain the following cost function

$$Q^*(B^T, n_E) = -2\ln P_{pr}(B^T) + \frac{n_E^2}{K_E} + \sum \alpha_n \langle (\bar{y}_i - f_i - n_E)^2 \rangle_n$$

with $\alpha_n = \ell_n/\gamma_n$.

Changing the notation a little, the cost function can be written in the following way:

$$S = \sum_{u=1}^{M} \alpha_n \qquad p_n = \frac{\alpha_n}{S}$$

$$Q_1(B^T) = n_E^{*2} = \left[\sum p_n \langle (\bar{y}_i - f_i) \rangle_n\right]^2$$

$$Q_2(B^T) = \sum_{u=1}^{M} p_n \left[\langle \overline{(y_{ij} - f_i - n_E^*)^2} + G_n\right]$$

$$G_n = \langle \overline{(y_{ij} - \bar{y}_i)^2} \rangle_n .$$

Q_1 and Q_2 are the measures of asymmetry and breadth respectively, of the error model in the polyoptimization concept.

$$Q^*(B^T, n_E) = \frac{S}{\mu}\left[-\frac{2\mu}{S} \ln P_{pr}(B^T) + \mu Q_2 + (1-\mu) Q_1 + \left(n_E - \frac{SK_E}{1 + SK_E} n_E^*\right)^2\right]$$

with

$$\mu = \frac{1 + SK_E}{2 + SK_E} .$$

The minimum with respect to n_E gives the estimation

$$n_E = \hat{n}_E(B^T) = \frac{SK_E}{1 + SK_E} n_E^*(B^T) .$$

In comparison with the polyoptimization method, the Bayesian approach gives us by variation of the a priori parameters that part of the efficient set defined by $\mu \in [\frac{1}{2}, 1]$.

However, the simplification applied holds only in the case where

$$\frac{X_n^2}{\gamma_n} = \frac{\langle(\bar{y}_i - f_i - n_E)^2\rangle_n}{\gamma_n} \quad \text{for all } n = 1,2,\ldots,M$$

is sufficiently small. This is the case only when the fluctuations of all y_{ij} around the trend are small. For the given values of B^T, n_E and X_n^2, the cost function accepts its minimum by variation of the coefficients α_n in the point

$$\bar{\alpha}_n = \frac{\ell n \; S + \sum X_n^{-2}}{\sum \ell n} \left(1 - \frac{X_n^{-2}}{S + \sum X_n^{-2}} \frac{\sum \ell n}{\ell n}\right) .$$

For

$$\frac{X_n^{-2}}{S + \sum X_n^{-2}} \frac{\sum \ell n}{\ell n} << 1$$

which is very often the case,

$$\bar{\alpha}_n \approx \ell n .$$

This means that the complete Bayesian objective contains a tendency to a uniform weighting which in the polyoptimization approach can only be realized by introducing an additional objective.

5.2.3 The Case of Partial Linear Trends -- Superposition of Driving Forces

Supposing

$$B^T = (a^T, b^T)$$

$$f_i(B^T) = a^T \phi_i(b^T) \quad .$$

Having already estimated $\hat{a}_n(b^T, n_E)$ from the partial interval

$$\hat{a}_n(b^T, n_E) = K_n^{-1}(L_n - n_E \bar{\phi}_n)$$

with

$$K_n = < \phi_i \phi_i^T > n$$

$$L_n = < \phi_i \bar{y}_i > n$$

$$\bar{\phi}_n = < \phi_i > n$$

we can write

$$x_n^2 = <(\bar{y}_i - f_i - n_E)^2> n = <(\bar{y}_{ij} - \hat{a}_n^T \phi_i - n_E)^2 > n$$

$$+ (a - \hat{a}_n)^T K_n (a - \hat{a}_n) \quad .$$

For the complete Bayesian objective we now get

$$Q(a^T, b^T, n_E) = -2\ell n \, P_{pr}(B^T) + \frac{n_E^2}{K_E}$$

$$+ \sum_{n=1}^{M} \ell_n \ell n \left[1 + \frac{(a-\hat{a}_n)^T K_n (a-\hat{a}_n)}{\gamma_n^*} \right] + \sum_{n=1}^{M} \ell_n \ell n \gamma_n^*$$

with

$$\gamma_n^*(b^T, n_E) = <\overline{(y_{ij} - \hat{a}_n^T(b^T, n_E)\phi_i(b^T) - n_E)^2} > n \quad .$$

We must determine the minimum of the cost function Q after b^T with an appropriate seeking procedure leading to an estimate $\hat{b}$. The other parameters can be determined analytically from

$$\hat{a} = \left[\sum_{n=1}^{M} K_n(b^T) p_n^*(b) \right]^{-1} \sum p_n^*(b) K_n(b^T) \hat{a}_n(b^T, n_E^*(b^T))$$

$$\hat{n}_E = \frac{s^*(\hat{b}) K_E \gamma(\hat{b})}{1 + s^*(\hat{b}) \gamma(\hat{b}) K_E} \, n_E^*(\hat{b}^T) \quad .$$

It should be mentioned that this result was obtained after the following simplification. We substituted in the cost function $\ell n(1+x^2)$ by x^2 and obtained the expression

$$Q^*(a^T,b^T,n_E) = -2\ell nP_{pr}(a^T,b^T)$$

$$+\left[S^* \frac{n_E^2}{S^*K_E} + \sum_{n=1}^{M} p_n^*(a-\hat{a})^T K_n (a-\hat{a})\right] - \sum_{n=1}^{M} \ell_n \ell np_n^*$$

$$p_n^* = \frac{\alpha_n^*}{S^*} \qquad S^* = \sum_{n=1}^{M} \alpha_n^* \qquad \alpha_n^* = \alpha^*(b^T,n_E) = \frac{\ell_n}{\gamma_n(b^T,n_E)} .$$

Assuming equally distributed a^T a priori and optimizing after a^T and n_E we get

$$Q^*(b^T) = \frac{S^*}{\mu}[\mu Q_2^*(b^T) + (1-\mu)Q_1^*(b^T)] + Q_3^*(b^T) - 2\ell nP_{pr}(b^T)$$

with the following objectives:

$$Q_1^*(b^T) = n_E^{*2}(b^T) \qquad n_E^*(b^T) = \frac{\sum p_n^* S_n^T K_n Y_n}{\sum p_n^* S_n^T K_n S_n}$$

$$Q_2^*(b^T) = \sum p_n^*(Y_n - n_E^* S_n)^T K_n (Y - n_E^* S_n)$$

$$Q_3^*(b^T) = -\sum \ell n \; \ell np_n^*$$

$$\mu(b^T,K_E) = \frac{\gamma(1+\gamma S^* K_E)}{1+\gamma(1+\gamma S^* K_E)}$$

$$\gamma = \gamma(b^T) = \sum p_n^* S_n^T K_n S_n$$

$$Y_n(b^T) = [\sum p_n^* K_n]^{-1} \sum p_n^* < \bar{y}_i \phi_i > n - K_n^{-1} < \bar{y}_i \phi_i > n$$

$$S_n(b^T) = [\sum p_n^* K_n]^{-1} \sum p_n^* < \phi_i > n - K_n^{-1} < \phi_i > n \quad .$$

The weighting coefficients will be determined at the point $n_E = n_E^*$

$$\alpha_n^*(b^T) = \frac{\ell n}{\gamma_n^*(b^T, K_E)} \qquad p_n^* = \frac{\alpha_n^*(b^T)}{S^*} \qquad S^* = \sum \alpha_n^* \quad .$$

5.3 Checking the Two Approaches

5.3.1 Checking the Fitting Procedure (Fuzzy)

We generated "measurements" for an ideal system

$$\frac{dx}{dt} = K(x - x_\ell)^k (x_u - x)^\ell$$

with the ideal parameter values

$$K = 1 \qquad x = 20 \qquad \ell = 1 \qquad x_\ell = 1 \qquad k = 1$$

and determined a set of efficient solutions by vector optimization. Two of these are represented on Figure 15

$|\mu| \to \min$ (bias) $K = 0{,}36$ $x_u = 21.0$ $\ell = 1.07$ $x_\ell = 1.22$ $x_\ell =$

$\delta \to \min$ (variance) $K = 1.09$ $x_u = 19.9$ $\ell = 0.98$ $x_\ell = 1.03$ $x_\ell =$

Figure 16 analyzes the error between the measurements and the adapted trend curve. It shows that we can not be sure to get a uniformly distributed error signal for all efficient solutions. We see that the case $\mu \to \min$ is especially bad in comparison to $\sigma \to \min$. This effect is clearly shown in Figure 17, $\mu \to \min$ and Figure 18 $\sigma \to \min$, where we have drawn the accumulated mean error

$$Q_{1i} = \frac{1}{n_i} \left| \sum_{j=1}^{n_i} f(x_j) - y_j \right| \qquad -\Delta-$$

and the mean quadratic error

$$Q_{2i} = \frac{1}{n_i - 1} \sum_{i=1}^{n_i} [f(x_j) - y_j]^2 \qquad -\Box-$$

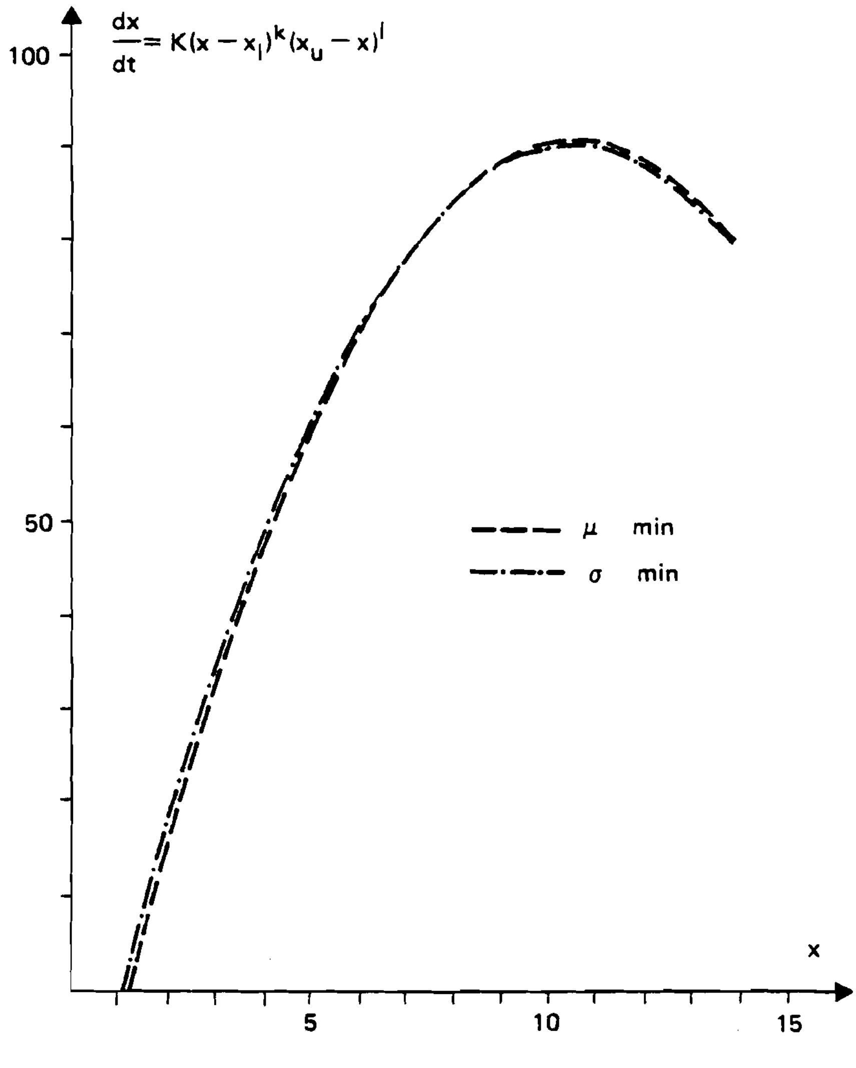

FIGURE 15: THE CORRECT CURVE TOGETHER WITH TWO APPROXIMATELY EFFICIENT SOLUTIONS

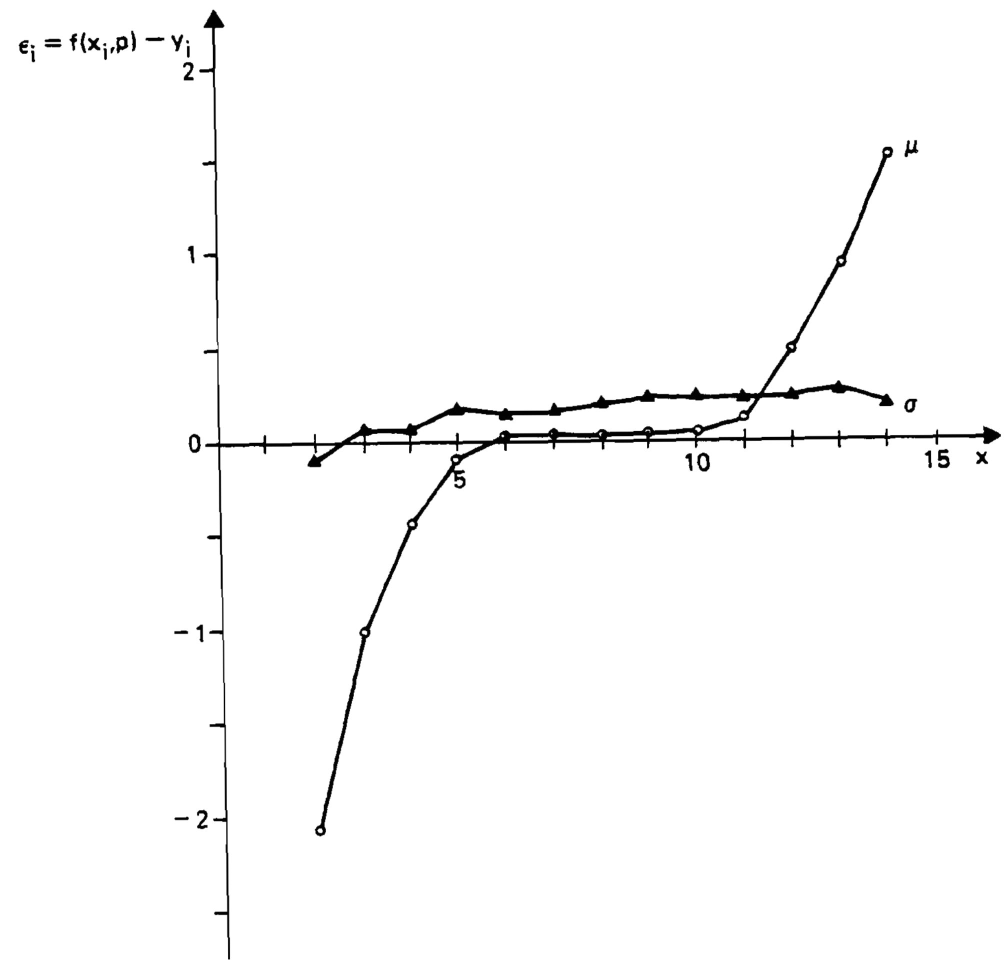

FIGURE 16: . THE ERROR DISTRIBUTION ALONG TIME-AXIS FOR TWO EFFICIENT SOLUTIONS

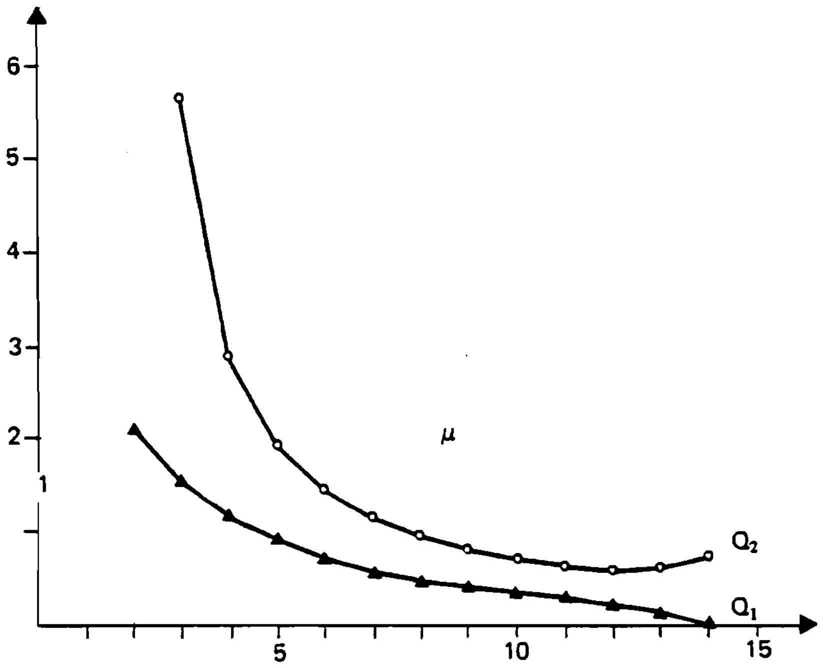

FIGURE 17: ACCUMULATED ERROR SIGNALS FOR THE SOLUTION μ → min

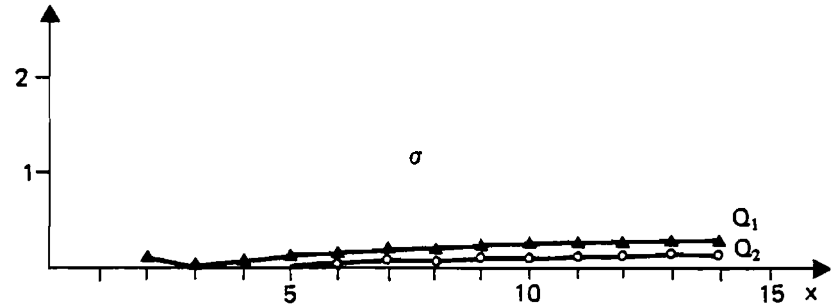

FIGURE 18: ACCUMULATED ERROR SIGNALS FOR THE SOLUTION σ → min

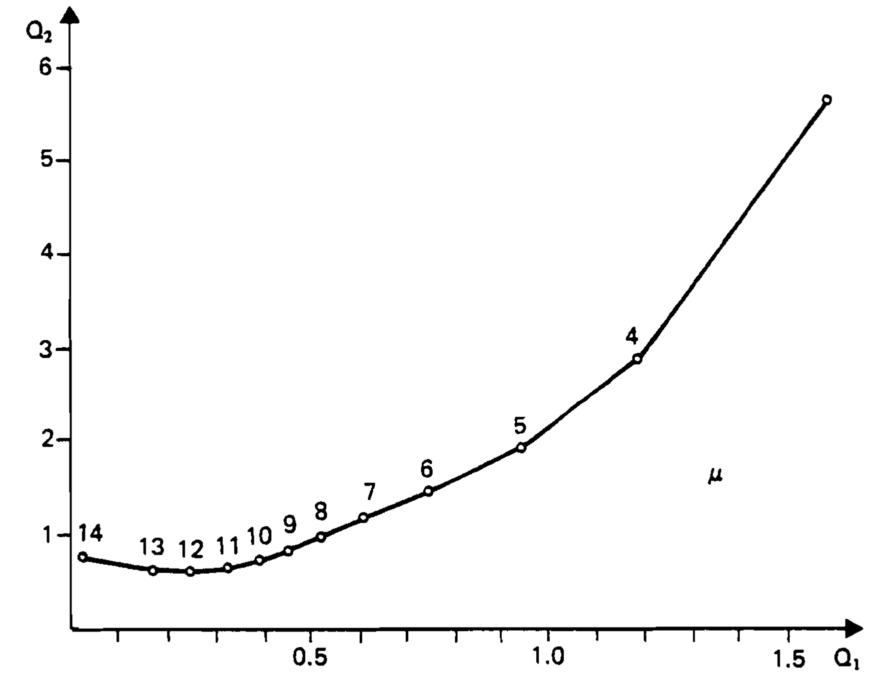

FIGURE 19: THE RELATIONSHIP BETWEEN THE TWO ACCUMULATED ERRORS FOR $\mu \rightarrow \min$

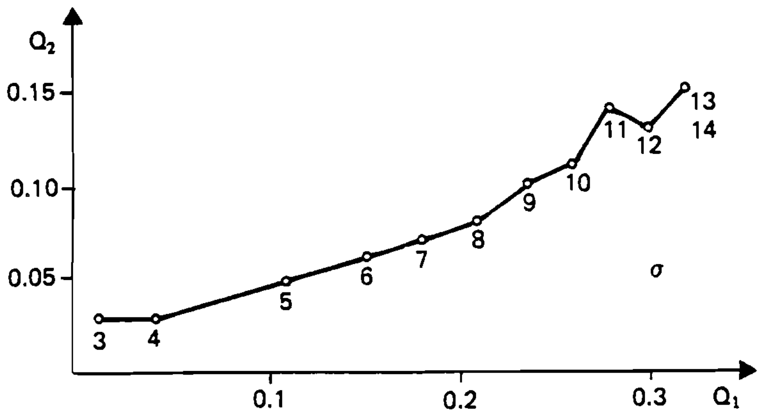

FIGURE 20: THE RELATIONSHIP BETWEEN THE TWO ACCUMULATED ERRORS FOR $\sigma \rightarrow \min$

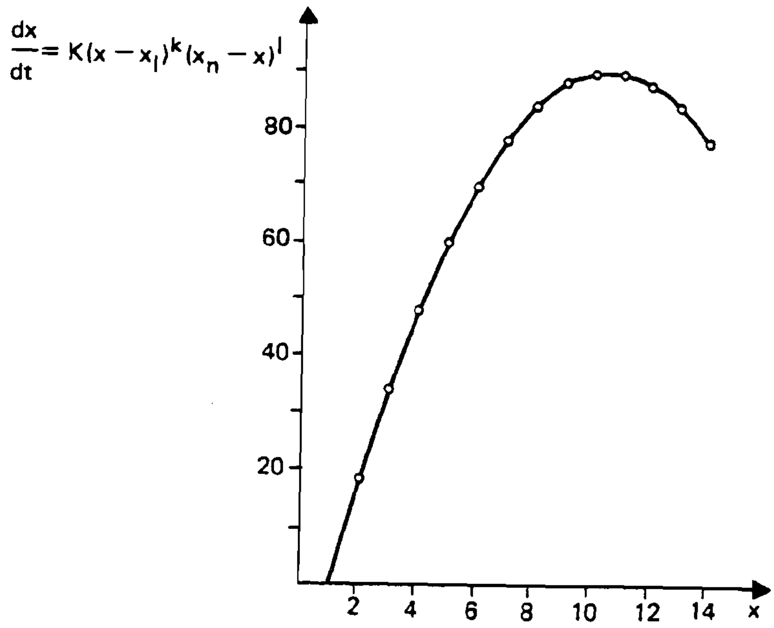

FIGURE 21: FITTING BY BAYESIAN APPROACH

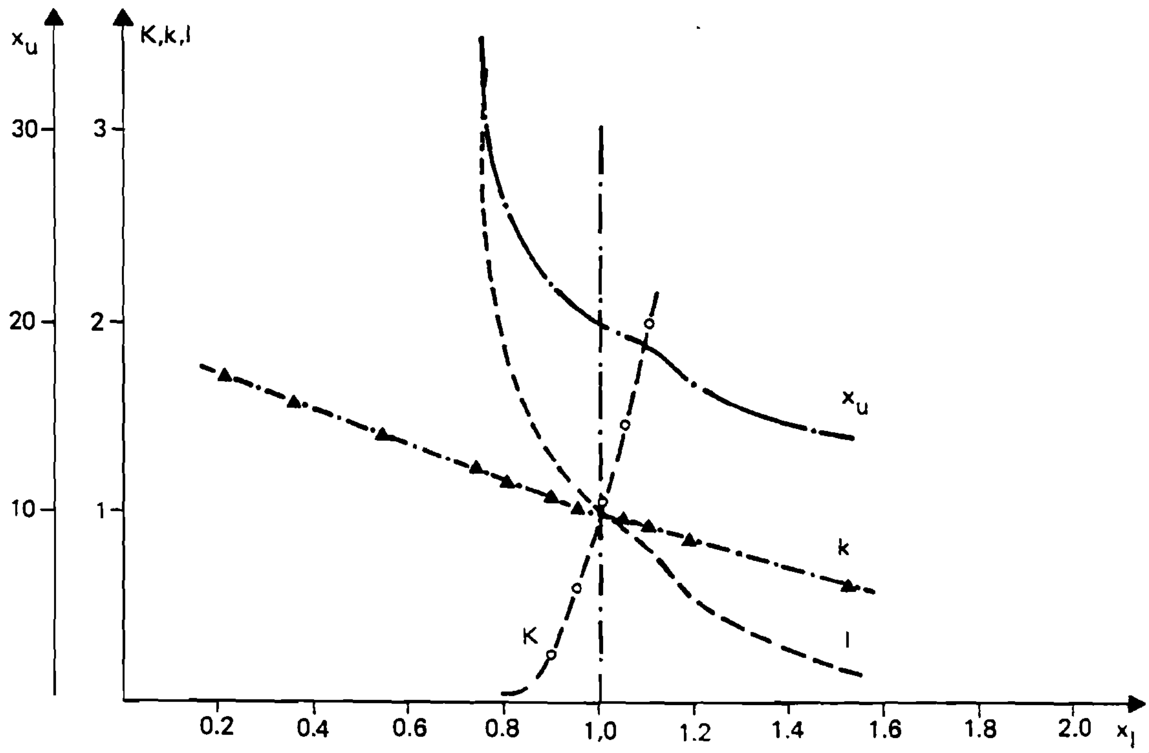

FIGURE 22: THE PARAMETER SENSITIVITY AS FUNCTION OF THE LOWER THRESHOLD

The case $\sigma \to$ min is very smooth compared with $\mu \to$ min. Figure 19, $\mu \to$ min, and Figure 20, $\sigma \to$ min, prove the same effect by exposing the relationships

$$Q_{2i} = f(Q_{1i}) \quad .$$

We can draw the conclusion that in the polyoptimization approach it is necessary to introduce an additional objective which measures the uniformity of the model error distribution.

5.3.2 Checking the Fitting Procedure (Bayesian)

We generated "measurements" for an ideal system

$$\ell n \frac{dx}{dt} = \ell n\, K + k\, \ell n(x - x_\ell) + \ell\, \ell n(x_u - x) \quad .$$

The "linear" coefficients $\ell nK, k, \ell$ were determined by linear regression on the basis of assumed values x_ℓ, x_u for the "nonlinear" coefficients. x_ℓ, x_u were iteratively determined using a one-dimensional extremum seeking procedure for each. Thus in this case we were only concerned with a two-dimensional seeking problem.

We generated "measurements" for a system with the parameter set

$$K = 1 \qquad x_u = 20 \qquad \ell = 1 \qquad x_\ell = 1 \qquad k = 1$$

and obtained the following result:

$$K = 1 \qquad x_u = 20.01 \qquad \ell = 1.0015 \qquad x_\ell = 1.0 \qquad k = 1.0001.$$

So by eye the identified reference curve cannot be distinguished from the assumed correct curve. (Figure 21). Figure 22 shows how the Bayesian optimum parameter values vary for every assumed value of x_ℓ. One critical point of this identification problem follows from it: the parameters K and ℓ are relatively sensitive, but the parameters k and x_u can be considered to be robust. This is reasonable because a variation of x_ℓ can be compensated by a corresponding change of K in a wide range, and ℓ takes information only from measurements at the end of the motion.

6. SOME CONCLUDING REMARKS

The nonlinear model

$$\frac{dx}{dt} = K(x - x_\ell)^k (x_u - x)^\ell$$

with a single growth push can be reliably identified not only by a fuzzy approach and vector optimization, but also by the Bayesian approach. The following problem arises: if we already have measurements at the points where the reference driving force is still zero, we have to set the reference to zero. In such a case the step of taking the logarithm of the driving force is forbidden and we have to pass to a higher dimensional search; this is also true in the Bayesian approach.

In our example we had a two-dimensional seeking space in the Bayesian approach, but a five-dimensional seeking space in the fuzzy approach.

The parameter identification process is a socalled inverse problem. We had already observed irregularities in the fuzzy case in the identification of K, x_u and ℓ. In the future we must introduce additional regularization measures; in the Bayesian case this was not so, since the linear regression for K, k, ℓ already has a regularization impact.

Depending on the agreed set of objectives in the polyoptimization approach we can meet valleys for some of the criteria, for example, for $Q_2 = \sigma$ in the (μ, σ) - approach and we have to apply ravine steps.

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