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A COMPUTER PROGRAM FOR MULTIOBJECTIVE DECISION MAKING BY THE INTERACTIVE SEQUENTIAL PROXY OPTIMIZATION TECHNIQUE

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PREFACE

Methodologies for decision making with conflicting multiple objectives have attracted increasing attention since the early period of IIASA activity. In the System and Decision Sciences area of IIASA, decision making processes with conflicting objectives as well as multiobjective optimization are one of the main projects and many techniques have been developed. This paper intends to provide a modest approach to such a research direction for decision sciences.

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ABSTRACT

A new interactive multiobjective decision making technique, which is called the sequential proxy optimization technique (SPOT), has been proposed by the author. Using this technique, the preferred solution for the decision maker can be derived efficiently from among a Pareto optimal solution set by assessing his marginal rates of substitution and maximizing the local proxy preference functions sequentially.

In this paper, based on the algorithm of SPOT, a computer program for multiobjective decision making with interactive procedures is presented and called ISPOT. The program is especially designed to facilitate the interactive processes for computer-aided decision making. After a brief description of the theoretical framework of SPOT, the computer program ISPOT is presented. The commands in this program and major prompt messages are also explained. An illustrative numerical example for the interactive processes is demonstrated and numerous insights are obtained.

A COMPUTER PROGRAM FOR MULTIOBJECTIVE DECISION MAKING BY THE INTERACTIVE SEQUENTIAL PROXY OPTIMIZATION TECHNIQUE

M. Sakawa

1. INTRODUCTION

The analysis of multiobjective optimization problems has evolved rapidly during the last few years. There have been more than 100 papers, dealing with multiobjective optimization problems and at least 20 different solution techniques have been proposed. The excellent survey paper of Cohn and Marks (1979) and, more recently, that of Wierzbicki (1979) are devoted to a comparative evaluation of existing techniques. Multiobjective optimization problems are concerned with decision making problems in which there are several conflicting objectives. The main aim of decision making under multiple conflicting objectives is to select as the preferred solution the best compromise among Pareto optimal solutions.

The development of decision making methodologies under multiple conflicting objectives has been one of the most active areas of research in recent years. Several techniques have been developed; among them two rival methods, namely, the multiattribute utility function (MUF) method (Keeney and Raiffa, 1976) and the surrogate worth trade-off (SWT) method (Haimes et al., 1975, and Haimes, 1977) use global and local utility (preference) modelling respectively.

The MUF method developed by Keeney et al., global utility function modelling, uses two assumptions of preference independence and utility independence to limit the utility function to specialized forms--additive or multiplicative. These global functions are mathematically simple and convenient, but they have disadvantages. Their assumptions are reasonable locally, but when assumed globally, they are very restrictive and may force the decision maker (DM) to fit a function not truly representing his or her preferences.

The SWT method developed by Haimes et al., local utility function modelling, provides an alternative approach that avoids restrictive assumptions. Instead of specifying the utility function globally, their procedures construct a sequence of local preference approximations of it.

The SWT method uses the ε-constraint problem as a means of generating Pareto optimal solutions. Objective trade-offs, whose values can be easily obtained from the values of some strictly positive Lagrange multipliers are used as the information carrier and the DM responds by expressing his degree of preference over the prescribed trade-offs by assigning numerical values to each surrogate worth function. However, the original version of the SWT method is noninteractive and some improvement, particularly in the way the information from the DM is utilized, must be made.

Recently, Chankong and Haimes (1977, 1979) and Simizu et al. (1978) independently proposed an interactive version of the SWT method on the basis of the SWT method. Their methods follow all the steps of the SWT method up to the point where all the surrogate worth values corresponding to the Pareto optimal solution are obtained from the DM. An interactive on-line scheme was constructed in such a way that the values of either the surrogate worth function or the MRS are used to determine the direction in which the utility function, although unknown, increases most rapidly. In their method, however, the DM must assess his preference at each trial solution in order to determine the step size. Such a requirement is very difficult for the DM, since he does not know the explicit form of his utility function.

On the other hand, in 1978, Oppenheimer proposed a proxy approach to multiobjective decision making. In his procedure the local proxy preference function is updated at each iteration by assessing a new MRS vector. Then the proxy is maximized to find a better point. Unfortunately, this method does not guarantee the generated solution in each iteration to be Pareto optimal. Furthermore, the systematic procedure to maximize the proxies is not mentioned, so it seems to be very difficult to do so in practice.

In order to overcome the drawbacks of the conventional methods, Sakawa (1980) has proposed a new interactive multiobjective decision making technique, which was called the sequential proxy optimization technique (SPOT), by incorporating the desirable features of the conventional multiobjective decision making methods. In his interactive on-line scheme, after solving the ε-constraint problem, the values of MRS assessed by the DM are used to determine the direction in which the utility function increases most rapidly and the local proxy preference function is updated to determine the optimal step size and Pareto optimality of the generated solution is guaranteed.

In this paper, based on the algorithm of SPOT, a computer program for multiobjective decision making by the interactive sequential proxy optimization technique, which we call ISPOT, is designed to facilitate the interactive processes for computer-aided decision making. Section 2 summarizes the theoretical development of SPOT on which the computer program ISPOT is based. A description of ISPOT is presented in Section 3. ISPOT utilizes the generalized reduced gradient (GRG) method (Lasdon et al., 1974, 1975) in order to solve the ε-constraint problems. The main part of interactive processes together with major commands and prompt messages are explained. In Section 4, the interaction processes of ISPOT are demonstrated by means of an illustrative example under the assumption of an ideal DM (i.e. consistent, rational with a well-defined structure of preference as represented by a utility function.) Several initial values of epsilons are selected and the corresponding computer outputs, which are obtained by adopting not only the sum-of-exponentials proxy but also two other types of proxy are listed in the appendices.

2. MULTIOBJECTIVE DECISION MAKING PROBLEM

The Multiobjective Optimization Problem (MOP) is represented as MOP

$$\min_{x} \ (f_1(x), f_2(x), \ldots, f_n(x)) \triangleq f(x) \tag{1}$$

subject to

$$x \in X \ = \ \{x \mid x \in E^N \ , \ g_i(x) \leqq 0 \ , \ i = 1, 2, \ldots, m\} \tag{2}$$

where x is an N-dimensional vector of decision variables, $f_1, \ldots, f_n$ are n district objective functions of the decision vector x, $g_1, \ldots, g_m$ are a set of inequality constraints and X is the constrained set of feasible decisions. Fundamental to the MOP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MOP is one where any improvement of one objective function can be achieved only at the expense of another.

Usually, Pareto optimal solutions consist of an infinite number of points, and some kinds of subjective judgment should be added to the quantitative analyses by the DM. The DM must select his preferred solution from among Pareto optimal solutions.

The multiobjective decision making problem (MDMP) we wish to solve

MDMP

$$\max_{x} \ U(f_1(x), f_2(x), \ldots, f_n(x)) \tag{3}$$

subject to

$$x \in X^P \tag{4}$$

where X^P is the set of Pareto optimal solutions of the MOP and $U(\cdot)$ is the DM's overall utility function defined on $F \triangleq \{f(x) \mid x \in E^N\}$ and is assumed to exist and is known only implicitly to the DM.

One way of obtaining Pareto optimal solutions to the MOP is to solve ε-constraint problem $P_k(\varepsilon_{-k})$ (Wierzbicki, 1979, and Keeney and Raiffa, 1976).

<u>$P_k(\varepsilon_{-k})$</u>

$$\min\ f_k(x) \tag{5}$$

subject to

$$x \in X \cap X_k(\varepsilon_{-k}) \tag{6}$$

$$\varepsilon_{-k} \in E_k \tag{7}$$

where

$$\varepsilon_{-k} \triangleq (\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n) \tag{8}$$

$$X_k(\varepsilon_{-k}) \triangleq \{x \mid f_j(x) \leqq \varepsilon_j\ ,\ j=1,\ldots,n, j\neq k\} \tag{9}$$

$$E_k \triangleq \{\varepsilon_{-k} \mid X_k(\varepsilon_{-k}) \neq \phi\} \tag{10}$$

Let us assume that $x^*(\varepsilon_{-k})$, an optimal solution to the $P_k(\varepsilon_{-k})$, be unique for the given ε_{-k}. And let AE_k be a set of ε_{-k} such that all the ε-constraint (9) is active, that is

$$AE_k \triangleq \{\varepsilon_{-k} \mid \varepsilon_{-k} \in E_k\ ,\ f_j(x^*(\varepsilon_{-k})) = \varepsilon_j\ ,\ j=1,\ldots,n\ ,\ j\neq k\}. \tag{11}$$

If the Kuhn-Tucker condition for problem $P_k(\varepsilon_{-k})$ is satisfied, the Lagrange multiplier $\lambda_{kj}(\varepsilon_{-k})$ associated with the jth active constraint can be represented as follows:

$$\lambda_{kj} = -\{\partial f_k(\varepsilon_{-k})\}/\{\partial f_j(\varepsilon_{-j})\} \quad j = 1,\ldots,n,\ j \neq k\ . \tag{12}$$

When all the ε-constraints are active, substituting the optimal solutions of $P_k(\varepsilon_{-k})$, $x^*(\varepsilon_{-k})$, given desired levels of the secondary objectives, ε_j, $j=1,\ldots,n$, $j \neq k$, the MODM can be restated as follows:

$$\max_{\varepsilon_{-k}}\ U(\varepsilon_1,\ldots,\varepsilon_{k-1},f_k[x^*(\varepsilon_{-k})],\varepsilon_{k+1},\ldots,\varepsilon_n)\ . \tag{13}$$

Throughout this paper we do the following.

Assumption 1: $U : F \to R$ exists and is known only implicitly to the DM. Moreover, it is assumed to be concave, a strictly decreasing and continuously differentiable function on F.

Assumption 2: All f_i, $i = 1,\ldots,n$ and all g_j, $j = 1,\ldots,m$ are convex and twice continuously differentiable in their respective domains and constraint set X is compact.

Assumption 3: For every feasible $\varepsilon_{-k} \in AE_k$ the solution to $AP_k(\varepsilon_{-k})$ exists and is finite.

Under Assumptions 1-3, the following theorem holds (Haimes and Chankong, 1979).

Theorem 1. Under Assumptions 1-3, the utility function $U(\varepsilon_1,\ldots,\varepsilon_{k-1},f_k[x^*(\varepsilon_{-k})],\varepsilon_{k+1},\ldots,\varepsilon_n)$ is concave with respect to $\varepsilon_{-k} \in AE_k$.

Now, before formulating the gradient, $\partial U(\cdot)/\partial\varepsilon$, of utility function U, we introduce the concept of marginal rates of substitution (MRS) of the DM.

Definition 1. At any f, the amount of f_i that the DM is willing to sacrifice to acquire an additional unit of f_j is called the MRS. Mathematically, the MRS is the negative slope of the indifference curve at f:

$$m_{ij}(f) = [\partial U(f)/\partial f_j]/[\partial U(F)/\partial f_i] = -df_i/df_j\Big|_{dU=0,df_r=0,r\neq i,j} \tag{14}$$

where each indifference curve is a locus of points among which the DM is indifferent.

The decision analyst assesses MRS by presenting the following prospects to the DM

$$f = (f_1,\ldots,f_i,\ldots,f_j,\ldots,f_n), \quad f' = (f_1,\ldots,f_i-\Delta f_i,\ldots,f_j+\Delta f_j,\ldots,f_n)$$

for a small fixed Δf_j, small enough so the indifference curve is approximately linear but large enough so the increment is meaningful, the analyst varies Δf_i until the DM is indifferent between f and f'. At this level, $m_{ij}(f) \simeq \Delta f_i/\Delta f_j$.

Now, we can formulate the gradient $\partial U(\cdot)/\partial \varepsilon_j$ of utility function $U(\cdot)$. Applying the chain rule

$$\partial U(\cdot)/\partial \varepsilon_j = \partial U(\cdot)/\partial \varepsilon_j + [\partial U(\cdot)/\partial f_k][\partial f_k/\partial \varepsilon_j] \quad j = 1,\ldots,n,\ j \neq k . \tag{15}$$

Using the relations (12) and (14), we have the following

$$\partial U(\cdot)/\partial \varepsilon_j = [\partial U(\cdot)/\partial f_k](m_{kj} - \lambda_{kj}) \quad j = 1,\ldots,n,\ j \neq k . \tag{16}$$

From the strict monotonicity of U with respect to f_k, $k = 1,\ldots,n$, $\partial U(\cdot)/\partial f_k$ is always negative. Therefore $-(m_{kj} - \lambda_{kj})$ $(j = 1,\ldots,n,$ $j \neq k)$ decide a direction improving the values of $U(\cdot)$ at a current point.

Under the assumptions 1-3, the optimality conditions for a maximization point ε_{-k} are $\partial U(\cdot)/\partial \varepsilon_{-k} = 0$, that is

$$m_{kj} = \lambda_{kj} \quad j = 1,\ldots,n,\ j \neq k . \tag{17}$$

This is a well known result that at the optimum of MRS of the DM must be equal to the trade-off rate.

If the optimality condition (17) is not satisfied at the ℓth iteration, the optimal direction of search s_j^ℓ and the corresponding direction of Δf_k^ℓ are given by:

$$s_j^\ell = -(m_{kj}^\ell - \lambda_{kj}^\ell) = \Delta \varepsilon_j^\ell \quad j = 1,\ldots,n,\ j \neq k \tag{18}$$

$$\Delta f_k^\ell = [\partial f_k(\varepsilon_{-k}^\ell)/\partial \varepsilon_{-k}^\ell]\Delta \varepsilon_{-k}^\ell = -\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda^\ell \Delta \varepsilon^\ell . \tag{19}$$

Then, we must determine the optimal step size α which maximizes $U(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell}, f_k + \alpha\Delta f_k^{\ell})$ along the direction $\Delta f^{\ell} = (\Delta\varepsilon_1^{\ell},\ldots,\Delta\varepsilon_{k-1}^{\ell},\Delta f_k^{\ell},\Delta\varepsilon_{k+1}^{\ell},\ldots,\Delta\varepsilon_n^{\ell}) \triangleq (\Delta\varepsilon_{-k}^{\ell},\Delta f_k^{\ell})$.

To solve this linear search problem, the following two problems arise.

Problem 1. The DM must assess his preference at each trial solution $(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell}, f_k^{\ell} + \alpha\Delta f_k^{\ell})$ for several values of α, in order to determine the best step size. Such requirement is very difficult for the DM, since he does not know the explicit form of his utility function.

Problem 2. Even if it is possible for the DM to assess the utility value, there remains a problem. The new trial point $f^{\ell} + \alpha\Delta f^{\ell}$, where Δf^{ℓ} is a direction vector, may be neither a Pareto optimal solution nor infeasible.

In order to resolve Problem 2, we adopt $(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell}, f_k(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell}))$ as a trial point in the process of linear search instead of $(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell}, f_k^{\ell} + \alpha\Delta f_k^{\ell})$.

Concerning Problem 1, it is necessary to construct some kind of utility (preference) function, therefore we introduce the following three types of local proxy preference functions like Oppenheimer's method (1978) in order to determine the optimal step size.

(1) sum-of-exponentials

If

$$[-\partial m_{ij}(f)/\partial f_j]/m_{ij}(f) = \omega_j$$

then

$$P(f) = -\sum a_i \exp(-\omega_i f_i) \quad . \tag{20}$$

(2) sum-of-powers ($\alpha_j \neq 0$)

If

$$[-\partial m_{ij}(f)/\partial f_j]/m_{ij}(f) = (1+\alpha_j)/f_j$$

then

$$P(f) = -\sum a_i f_i^{\alpha_i} \quad . \tag{21}$$

(3) sum-of-logarithms

If

$$[-\partial m_{ij}(f)/\partial f_j]/m_{ij}(f) = 1/(M-f_j)$$

then

$$P(f) = \sum a_i \ell_n (M - f_i) \quad , \tag{22}$$

where M is a sufficiently large positive number.

SPOT requires the MRS of the DM, but it is a question whether the DM can respond precise and consistent values of MRS through the whole searching process. So two types of consistency tests are employed in our technique following Oppenheimer (1978); the first testing MRS consistency at a single point, and the second testing consistency at successive points.

The single point test requires a second set of assessments at each point and checks whether the MRS of the DM satisfies the chain rule, i.e. $m_{kj} = m_{ki} m_{ij}$ $i,j = 1,\ldots,n$, $i \neq k$, $k \neq i$, $k \neq j$. Since only n-1 unique MRS among the objectives exist at any point, the second set can be used to measure the discrepancy E:

$$E = [(\Delta f_k/\Delta f_j) - (\Delta f_k/\Delta f_i)(\Delta f_i/\Delta f_j)]/(\Delta f_k/\Delta f_j) \quad (\%) \tag{23}$$

We set a reasonable tolerance level and if the discrepancy exceeds the tolerance, the analyst should explain the inconsistency to the DM and reassess the MRS until the discrepancy is resolved.

The second test checks for decreasing marginal rates of substitution of the proxy, which is based on the following theorem.

Theorem 2.

(1) The sum-of-exponentials proxy P(f) is strictly decreasing and concave if and only if all the parameters a_i and ω_i are strictly positive, i.e.,

$$a_i > 0 \text{ and } \omega_i > 0, \quad i = 1,\ldots,n \tag{24}$$

(2) The sum-of-powers proxy P(f) is strictly decreasing and and concave if and only if

$$a_i > 0; \quad \alpha_i > 1 \qquad i = 1,\ldots,n \quad . \tag{25}$$

(3) The sum-of-logarithms proxy P(f) is strictly decreasing and concave if and only if

$$a_i > 0 \qquad i = 1,\ldots,n \quad . \tag{26}$$

Following the above discussions, we can now describe the algorithm of the sequential proxy optimization technique (SPOT) in order to obtain the preferred solution of the DM for the MDMP.

Step 1 Choose initial point $\varepsilon^{\ell}_{-k} \in E_k$ and set $\ell = 1$.

Step 2 Set $\varepsilon_{-k} = \varepsilon^{\ell}_{-k}$, solve an ε-constraint problem $P_k(\varepsilon^{\ell}_{-k})$ for ε^{ℓ}_{-k} and obtain a Pareto optimal solution $x^*(\varepsilon^{\ell}_{-k})$, a Pareto optimal value $f^{\ell} = (\varepsilon^{\ell}_{-k}, f^{\ell}_k[x^*(\varepsilon^{\ell}_{-k})])$ and corresponding Lagrange multiplier λ^{ℓ}_{kj} $(j = 1,\ldots,n, j \neq k)$.

Step 3 If all the ε-constraints are active, go to the next step. Otherwise, change ε^{ℓ}_{-k} for inactive constraints until all the ε-constraints become active and obtain the corresponding Lagrange multipliers.

Step 4 Assess the MRS of the DM at f^{ℓ}, where Δf_j $(j = 1,\ldots,n, j \neq k)$ must be fixed small enough that the indifference curve is approximately linear but large enough that the increment is meaningful.

Step 5 For MRS at f^{ℓ}, evaluate discrepancy E. If $E < \delta_2$ go to Step 6, where the tolerance δ_2 is a prescribed sufficiency small positive number. If E exceeds the tolerance, the DM reassesses the MRS until the tolerance condition is satisfied.

Step 6 If $|m_{kj}^{\ell} - \lambda_{kj}^{\ell}| < \delta_1$ for $j = 1,\ldots,n$, $j \neq k$, stop, where the tolerance δ_1 is a prescribed sufficiency small positive number. Then a Pareto optimal solution $(\varepsilon_{-k}^{\ell}, f_k^{\ell}[x^*(\varepsilon_{-k}^{\ell})])$ is the preferred solution of the DM. Otherwise, determine the direction vector $\Delta\varepsilon_{-k}^{\ell}$ by

$$s_j^{\ell} = -(m_{kj}^{\ell} - \lambda_{kj}^{\ell}) = \Delta\varepsilon_j^{\ell} \quad (j = 1,\ldots,n\ , j \neq k)$$

Step 7 For the prescribed initial step size α_0, change the step size to be α_0 and $2\alpha_0$ and obtain the corresponding two Pareto optimal points ${}^1f^{\ell}$ and ${}^2f^{\ell}$ in the neighborhood of f^{ℓ} and assess n-1 MRS m_{kj}^{ℓ} at a point ${}^1f^{\ell}$ plus a single MRS at a third point ${}^2f^{\ell}$. If the consistency check at Step 5 is passed, select the form of the proxy function that will be used at each iteration by the measure about MRS variation. If the parameter value conditions of Theorem 2 are passed go to the next step. Otherwise, the DM reassesses the MRS until the parameter value conditions are satisfied.

Step 8 Determine the step size α which maximizes the proxy preference function $P(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell}, f_k^{\ell}[x^*(\varepsilon_{-k}^{\ell} + \alpha\Delta\varepsilon_{-k}^{\ell})]) \underline{\underline{\Delta}} P(\alpha)$ as follows. Change the step size, obtain corresponding Pareto optimal values and search for three α values α_A, α_B and α_C which satisfy

$$\alpha_A < \alpha_B < \alpha_C$$

$$P(\alpha_A) < P(\alpha_B) > P(\alpha_C)$$

This step operates either doubling or halfing the step size until the maximum is bracketed. If the maximum is not bracketed change the initial step size.

Then a local maximum of $P(\alpha)$ is in the neighborhood of $\alpha = \alpha_B$. Ask the DM whether $U(f^{\ell+1}) > U(f^{\ell})$ or not where $f^{\ell+1} = (\varepsilon_{-k}^{\ell} + \alpha_B^{\ell}\Delta\varepsilon_{-k}^{\ell}, f_k^{\ell}[x^*(\varepsilon_{-k}^{\ell} + \alpha_B^{\ell}\Delta\varepsilon_{-k}^{\ell})])$, set

$\ell = \ell+1$ and return to Step 2. Otherwise reduce α_B to be $\frac{1}{2}$, $\frac{1}{4}$ ... until improvement is achieved.

3. A COMPUTER PROGRAM FOR MULTIOBJECTIVE DECISION MAKING: ISPOT

Our computer program ISPOT is composed of a main program and a number of subroutines, which are arranged in a hierarchical structure. Here, we give a brief explanation of the current version of ISPOT. At present, some of the subroutines in ISPOT may be rather crude which will be revised in the near future.

ISPOT has three COMMANDS, i.e., GRG, DECOMP, and SPOT, and the user can select one of them in accordance with his purposes. The functions of each COMMAND are:

(i) GRG: solves nonlinear programming problems with a single objective function using the generalized reduced gradient (GRG) method proposed by Lasdon et al. (1974, 1975).

(ii) DECOMP: solves the nonlinear programming problems of a block angular structure in a two-level scheme using the dual decomposition method (Lasdon, 1970).

(iii) SPOT: solves the multiobjective decision making problems interactively by our proposed method, SPOT (Sakawa, 1980).

In the following, we explain the major subroutines which appear when SPOT is selected as a COMMAND.

Subroutine SMAIN

This subroutine is called when the command word SPOT is specified. The user can choose whether to use the dual decomposition method or not in order to solve the ε-constraint problems. The prompt message

```
WHICH DO YOU SELECT?
 1 SPOT
 2 SPOT BY DECOMP
```

is shown and the user must input either 1 or 2 according to his choice. It follows that when 1 is input to select SPOT in order to solve the multiobjective decision making problem interactively, the main part of the interaction processes is explained with the major prompt messages.

(1) DO YOU USE DEFAULT VALUES IN GRG?

GRG contains a number of tolerance parameters which must be specified when certain iterative processes should stop or when certain quantities are zero. If the user wishes to set all of them to default values, he must input YES. Otherwise, after inputting NO, desirable values for tolerance parameters are input.

(2) DO YOU USE IDEAL DM?

The values of the MRS of an ideal DM can be simulated by the explicit form of the global form of the DM. If the user wishes to test the feasibility and efficiency of the iteration processes of ISPOT under the assumption of an ideal DM, YES is input. In this case, calling the subroutine UTILITY the values of MRS are simulated by equation (14). In the case of NO, the real DM must assess his MRS by responding the amount of Δf_i that he is willing to sacrifice to acquire Δf_1 for the prescribed value of Δf_1.

(3) CORRECT VALUES OF EPSILONS

Determine the direction vector at each iteration, if at least one of the Lagrange multipliers for the corresponding ε-constraint problem becomes zero, and change the corresponding ε values in order to get the nonzero Lagrange multipliers.

(4) INPUT TOLERANCE DELTA1

If $|m_{1j} - \lambda_{1i}|$ < DELTA1 for all i = 2,...,n, the preferred solution of the DM as well as the necessary informations are listed, then the program terminates. Otherwise, go to the next iteration.

(5) INPUT INITIAL STEP SIZE

Choose an initial step size α_0 along the optimal direction of seach. Then the Pareto optimal solutions corresponding to $\alpha = 0$, α_0 and $2\alpha_0$ are calculated by GRG. The values $f_i (i = 2,\ldots,n)$, which are calculated by substituting the optimal values of the decision variable x, are adopted as Pareto optimal values instead of the values of epsilon. For that purpose subroutine SUBG is called. To list the values of f_i $(i = 1,\ldots,n)$, ε_i $(i = 2,\ldots,n)$, λ_{1i} $(i = 2,\ldots,n)$, subroutine LAGS is also called.

(6) SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWING

1 SUM OF EXPONENTIALS

2 SUM OF POWERS-1

3 SUM OF POWERS-2

As a local proxy preference function, the user must select one of the three types of proxies. Then, the parameter values are determined by calling the subroutine PARAM1, PARAM2 or PARAM3 respectively.

In the case where the sum-of-exponentials proxy is selected, if at least one of the values of parameters a_i, w_i $(i=1,\ldots,n)$ becomes nonpositive, the program displays the following prompt message:

A(I) OR W(I) IS NEGATIVE

1 GO ON

2 CHANGE INITIAL STEP IN ORDER TO FIT PROXY

3 YOUR MRS IS INCONSISTENT WITH DMR.

INPUT AGAIN YOUR MRS!

In the case of an ideal DM, the third message does not appear. The DM must select whether to reassess his MRS or to change the initial step size or to continue.

For the other two types of proxies, similar prompt messages are prepared.

(7) DO YOU FIT QUADRATIC INTERPOLATION?

For the three values A, B, and C which satisfy both $A < B < C$ and $P(A) < P(B) > P(C)$, ask the DM whether to fit quadratic interpolation in order to obtain a more precise point or not. If YES is input, by fitting quadratic interpolation, obtain the maximization point for $P(f)$ and calculate the corresponding Pareto optimal solution. In the case of NO, adopt a step size corresponding to a point B as a near optimal step size.

(8) ADOPT PREVIOUS POINT

If $P_1(\varepsilon_{-1})$ becomes infeasible in the search of the optimal step size, the program adopts the previous point as the starting point of the next iteration.

4. AN ILLUSTRATIVE EXAMPLE

We now demonstrate the interaction processes of the ISPOT by means of an illustrative example which is designed to test ISPOT under the assumption of an ideal DM.

Consider the following multiobjective decision making problem.

$$\min_{x} f(x) = (f_1(x), f_2(x), f_3(x)) \tag{27}$$

subject to

$$x \in X = \{x \mid x_1^2 + x_2^2 + x_3^2 \leqq 100,\ 0 \leqq x_1, x_2, x_3 \leqq 10\} \tag{28}$$

where

$$f_1(x) = x_1^2 + (x_2+5)^2 + (x_3-60)^2 \tag{29}$$

$$f_2(x) = (x_1+40)^2 + (x_2-224)^2 + (x_3+40)^2 \tag{30}$$

$$f_3(x) = (x_1-224)^2 + (x_2+40)^2 + (x_3+40)^2 \tag{31}$$

For illustrative purposes, we shall assume that the DM's structure of preference can be accurately represented by the utility function $U(f_1, f_2, f_3)$ where

$$U(f) = -101700f_1 - (f_2-40000)^2 - (f_3-45000)^2 \ . \qquad (32)$$

However, it should be stressed that the explicit form of utility function as in (35) is used in this example purely for simulating values of MRS. To be more specific, m_{kj} will be obtained through the following expression:

$$m_{kj}(f) = [\partial U(f)/\partial f_j]/[\partial U(f)/\partial f_k] \quad j=1,2,3, \ j \neq k \qquad (33)$$

m_{kj} obtained this way are as if they had been obtained from the ideal DM directly.

Let us now choose $f_1(x)$ as our primary objective and formulate the corresponding ε-constraint problem $P_1(\varepsilon_{-1})$.

$\underline{P_1(\varepsilon_{-1})}$

$$\min_x \ f_1(x) \qquad (34)$$

subject to

$$x \in X \cap X_1(\varepsilon_{-1}) \qquad (35)$$

where

$$X_1(\varepsilon_{-1}) = \{x \mid \varepsilon_j - f_j(x) \geqq 0 \ , \ j = 2,3\} \ . \qquad (36)$$

In this example, we set the values if the initial step size to be 1000, and the values of the tolerance parameters to be 1. Starting the initial values of x = (7,7,0), the optimal values of x corresponding to the previous ε are set automatically hereafter.

In the following, the case where the initial values of $\varepsilon'_{-1} = (\varepsilon'_2, \varepsilon'_3) = (52000,52000)$ are selected and the sum-of-exponentials are adopted as a proxy are explained especially for iteration 1 with some of the computer outputs.

The following serial numbers correspond with those in the output of Appendix 1.

In the case where two other types of proxies are adopted with the same initial value ε, the corresponding computer outputs are also listed in Appendices 2 and 3 and the interactive processes may be understood similarly.

(1) Select SPOT as a command and initiate the interactive multiobjective optimization processes.

(2) Input 3 as a number of objective functions (in this example).

(3) Input (52000,52000) as initial values of $\underline{\varepsilon}'_1$.

(4) Utilizing SPOT without the dual decomposition method, 1 is input.

(5) To solve the ε-constraint problem from phase 1 of GRG ICOUNT = 0 is input.

(6) Set the initial values of $x = (x_1,x_2,x_3,x_4,x_5,x_6) = (7,7,0,0,0,0)$ including the slack variables x_4, x_5 and x_6 corresponding to the ε-constraints because GRG is started from phase 1.

(7) To use the default values in GRG, YES is input.

(8) Upper bound constraint 100 is shown whereas ε constraints have no upper bound so $1.0 \cdot 10^{30}$ is set as $+\infty$.

(9) In GRG there are two optimality tests, i.e.:

(i) to satisfy the Kuhn-Tucker optimality conditions

(ii) to satisfy the fractional change, which means if the condition

$$|FM - OBJTST| < EPSTOPX|OBJTST|$$

is satisfied for NSTOP consecutive iterations where FM is the current objective value and OBJTST is the objective value at the start of the previous one dimensional search. NSTOP has a default value of 3. In this example it is shown that the Kuhn-Tucker optimality conditions are satisfied.

(10) To test the iteration processes using an ideal DM, YES is input.

(11) For ε'_{-1} = (52000,52000), the calculation results from GRG are shown by calling subroutine LAGS. The values of F(1), F(2) and F(3) are the obtained values of objectives and the values of EP(2) and EP(3) are selected ε values.

The values of F(2) and F(3) coincide with the values of EP(2) and EP(3) which means the ε-constraints become active so the corresponding values of Lagrange multipliers are also shown.

The Pareto optimal solution is (f_1,f_2,f_3) = (3006.5,52000, 52000).

(12) The values of tolerance parameter δ_1 are input. In this example 0.001 is set for δ_1; the preferred solution is obtained if the conditions $|\lambda_{1j} - m_{1j}| < 0.001$ $(j = 2,3)$ are satisfied.

These conditions are not satisfied, ITERATION 1 is begun.

(13) Direction vector, $S_j = \lambda_{1j} - m_{1j}$ $(j = 2,3)$ to update ε is shown, which also means the stopping criteria are not satisfied.

(14) It is requested to input the initial step size. Here, 1000 is input.

(15) When the ε values are updated to be $\varepsilon = (52000 + 1000 \cdot S_2, 52000 + 1000 \cdot S_3)$ by the direction vector and initial step size, the corresponding ε-constraint problem is solved by GRG and the results are shown.

(16) The results for the ε-constraint problem with the doubling initial step size is shown.

(17) The values of MRS of an ideal DM for three points corresponding to the Pareto optimal solutions for the step size 0, 1000 and 2000 are shown which are calculated by calling subroutine UTILITY.

(18) In order to determine the local proxy it is required to select the form of proxies. In this example 1 is input to adopt the sum-of-exponentials.

(19) The parameter values for the sum-of-exponentials proxy are calculated and listed.

(20) It is required to input the admissible maximum step size while determining the optimal step size and 100000 is set.

(21) For α = 0, 1000 and 2000, it is shown that the values of proxy P(f) become larger.

(22) The results for further doubled step size, i.e., $\varepsilon = (52000 + 4000 \cdot S_2, 52000 + 4000 \cdot S_3)$ are shown.

(23) P(f) becomes larger, the step size is further doubled.

(24) The results for $\varepsilon = (52000 + 8000 \cdot S_2, 52000 + 8000 \cdot S_3)$ are shown.

(25) The step size is further doubled, and the corresponding results for $\varepsilon = (52000 + 16000 \cdot S_2, 52000 + 16000 \cdot S_3)$ are shown.

(26) Since the values of P(f) at the point in (25) become smaller than that of (24), select whether to fit quadratic interpolation or not. In this example, in order not to fit quadratic interpolation NO is input. Then the point in (24) is adopted as a maximization point of P(f) for the direction vector in (13), i.e., the optimal step size becomes 1600.

(27) The ε-constraint problem with $\varepsilon = (52000 + 8000 \cdot S_2, 52000 + 8000 \cdot S_3)$ is solved using the saved values of x in (24) and the results are shown.

(28) Test whether the obtained trial point at ITERATION 1 is optimal or not.

(29) The optimality condition is not satisfied, the direction vector is determined, and the ITERATION 2 begins.

The same procedure continues in this manner. In this example, at the 3rd iteration the optimality condition is satisfied and the preferred values of objectives and decision variables as well as the direction vector are shown.

All the iteration processes are listed in Appendix 1.

The obtained results compare favorably with the results obtained by solving directly $\max_{x \in X} U(f_1,f_2,f_3)$ using GRG, which is

$$(f_1,f_2,f_3) = (2960 \cdot 5487, 51586 \cdot 845, 52783 \cdot 616)$$

$$(x_1,x_2,x_3) = (3 \cdot 870271, 6 \cdot 136885, 6 \cdot 881835) \quad .$$

In Appendices 2 and 3, it is also listed in the case where the other two types of proxies are selected with the same initial ε value.

Appendix 4 summarizes the obtained results for three types of proxy functions with several ε values. Although the number of iterations is different depending on the initial ε values, the obtained preferred solutions compare favorably with the true optima.

Concerning the computational study in this example, we can conclude that ISPOT will always converge to the preferred solution of the DM under the assumption that he is consistent, rational and has a well-defined structure of preference.

5. CONCLUSION

In this paper a computer program for multiobjective decision making by the interactive sequential proxy optimization technique, which we call ISPOT, is designed to facilitate interactive processes for computer-aided decision making.

The interaction processes are demonstrated by means of an illustrative example under the assumption of an ideal decision maker. In the hypothetical numerical example of this paper, the assessments of the MRS are simulated by an ideal decision maker, so the consistency check of the MRS becomes unnecessary.

It is necessary to apply our computer program ISPOT to real-world case studies by a real-world decision maker by incorporating consistency checks of his MRS assessment. From such experiences ISPOT must be revised.

An attempt to apply ISPOT to real-world environmental problems is now under consideration and will be reported elsewhere.

Furthermore, extensions of ISPOT to the nonconvex and/or non-smooth Pareto surface cases will be done in the near future.

APPENDIX 1: OUTPUT LIST USING THE SUM-OF-EXPONENTIALS PROXY WITH ε = (52000,52000)

```
COMMAND?
= SPOT                                                                  (1)
INPUT NUMBER OF OBJECTIVES
= 3                                                                     (2)
INPUT INITIAL VALUES OF EPSIRONS
= 52000.                                                                (3)
= 52000.

WHICH DO YOU SELECT ?
1   SPOT
2   SPOT BY DECOMP                                                      (4)
= 1
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0                                                                     (5)
X IS
 7.000000E+00 7.000000E+00 0.          0.          0.          0.       (6)

DO YOU USE DEFAULT VALUES IN GRG ? (YES OR NO)
= YES                                                                   (7)

UPPER BOUNDS ON INEQUALITY CONSTRAINTS ARE
 (   1)  1.000000E+02      (   2)  1.000000E+30      (   3)  1.000000E+30  (8)

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03                  (9)
DO YOU USE IDEAL DM ? (YES OR NO) :                                     (10)
= YES

          1     F(1)  =  0.30064934E+04
          2     F(2)  =  0.52000000E+05
                EP(2) =  0.52000000E+05
                LAGRANGIAN MULTIPLIER =  0.22011975E+00                 (11)
          3     F(3)  =  0.52000000E+05
                EP(3) =  0.52000000E+05
                LAGRANGIAN MULTIPLIER =  0.20118035E+00

INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001                                                                 (12)

ITERATION= 1
     S( 2) =      -0.15868455E-01                                       (13)
     S( 3) =       0.63520568E-01
INPUT INITIAL STEP SIZE
= 1000.                                                                 (14)
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)  =  0.29973553E+04
          2     F(2)  =  0.51984133E+05
                EP(2) =  0.51984133E+05
                LAGRANGIAN MULTIPLIER =  0.21900488E+00                 (15)
          3     F(3)  =  0.52063518E+05
                EP(3) =  0.52063518E+05
                LAGRANGIAN MULTIPLIER =  0.19627351E+00

INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0
```

```
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29885076E+04
          2      F(2)   =  0.51968261E+05
                 EP(2)  =  0.51968261E+05
                 LAGRANGIAN MULTIPLIER =  0.21794420E+00           (16)
          3      F(3)   =  0.52127042E+05
                 EP(3)  =  0.52127042E+05
                 LAGRANGIAN MULTIPLIER =  0.19146771E+00
POINT= 1
        M(1,2)=       0.23598820E+00
        M(1,3)=       0.13765978E+00
POINT= 2
        M(1,2)=       0.23567613E+00                               (17)
        M(1,3)=       0.13890895E+00
POINT= 3
        M(1,2)=       0.23536407E+00
        M(1,3)=       0.14015813E+00


SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 1                                                                (18)

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
      - 0.10000000E+01*EXP( 0.60379048E-05*F(1))                   (19)
      - 0.18243257E-03*EXP( 0.86865489E-04*F(2))
      - 0.38483759E-05*EXP( 0.14134425E-03*F(3))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)                               (20)
= 100000.                                                          (21)
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29716648E+04
          2      F(2)   =  0.51936527E+05
                 EP(2)  =  0.51936527E+05
                 LAGRANGIAN MULTIPLIER =  0.21597532E+00           (22)
          3      F(3)   =  0.52254080E+05
                 EP(3)  =  0.52254080E+05
                 LAGRANGIAN MULTIPLIER =  0.18213264E+00
FA<FB                                                              (23)
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29412617E+04
          2      F(2)   =  0.51873054E+05
                 EP(2)  =  0.51873054E+05
                 LAGRANGIAN MULTIPLIER =  0.21258549E+00           (24)
          3      F(3)   =  0.52508165E+05
                 EP(3)  =  0.52508165E+05
                 LAGRANGIAN MULTIPLIER =  0.16440352E+00
```

```
FA<FB

INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)   =  0.28928113E+04
           2      F(2)   =  0.51746104E+05
                  EP(2)  =  0.51746104E+05
                  LAGRANGIAN MULTIPLIER =  0.20763425E+00                (25)
           3      F(3)   =  0.53016330E+05
                  EP(3)  =  0.53016330E+05
                  LAGRANGIAN MULTIPLIER =  0.13165521E+00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO                                                                     (26)
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)   =  0.29412617E+04
           2      F(2)   =  0.51873054E+05
                  EP(2)  =  0.51873054E+05
                  LAGRANGIAN MULTIPLIER =  0.21258549E+00                (27)

           3      F(3)   =  0.52508165E+05
                  EP(3)  =  0.52508165E+05
                  LAGRANGIAN MULTIPLIER =  0.16440352E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001                                                                  (28)

ITERATION= 2
     S( 2) =       -0.20906200E-01                                       (29)
     S( 3) =        0.16750337E-01
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)   =  0.29429695E+04
           2      F(2)   =  0.51852144E+05
                  EP(2)  =  0.51852144E+05
                  LAGRANGIAN MULTIPLIER =  0.21376491E+00
           3      F(3)   =  0.52524912E+05
                  EP(3)  =  0.52524912E+05

                  LAGRANGIAN MULTIPLIER =  0.16381678E+00
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)   =  0.29447117E+04
           2      F(2)   =  0.51831239E+05
                  EP(2)  =  0.51831239E+05
                  LAGRANGIAN MULTIPLIER =  0.21495126E+00
           3      F(3)   =  0.52541665E+05
                  EP(3)  =  0.52541665E+05
                  LAGRANGIAN MULTIPLIER =  0.16323213E+00
```

```
POINT= 1
        M(1,2)=        0.23349169E+00
        M(1,3)=        0.14765319E+00
POINT= 2
        M(1,2)=        0.23308055E+00
        M(1,3)=        0.14798259E+00
POINT= 3
        M(1,2)=        0.23266941E+00
        M(1,3)=        0.14831200E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 1

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     - 0.10000000E+01*EXP( 0.90381286E-04*F(1))
     - 0.66224747E-02*EXP( 0.76915507E-04*F(2))
     - 0.69778974E-04*EXP( 0.14225485E-03*F(3))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)   =  0.29483003E+04
          2     F(2)   =  0.51789429E+05
                EP(2)  =  0.51789429E+05
                LAGRANGIAN MULTIPLIER =  0.21734549E+00

          3     F(3)   =  0.52575165E+05
                EP(3)  =  0.52575165E+05
                LAGRANGIAN MULTIPLIER =  0.16206903E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)   =  0.29558979E+04
          2     F(2)   =  0.51705804E+05
                EP(2)  =  0.51705804E+05
                LAGRANGIAN MULTIPLIER =  0.22222500E+00
          3     F(3)   =  0.52642165E+05
                EP(3)  =  0.52642165E+05
                LAGRANGIAN MULTIPLIER =  0.15976647E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)   =  0.29728013E+04

          2     F(2)   =  0.51538554E+05
                EP(2)  =  0.51538554E+05
                LAGRANGIAN MULTIPLIER =  0.23238844E+00
          3     F(3)   =  0.52776171E+05
                EP(3)  =  0.52776171E+05
                LAGRANGIAN MULTIPLIER =  0.15524592E+00
```

```
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.30137632E+04
          2      F(2)  =  0.51204055E+05
                 EP(2) =  0.51204055E+05
                 LAGRANGIAN MULTIPLIER =  0.25473020E+00
          3      F(3)  =  0.53044177E+05
                 EP(3) =  0.53044177E+05
                 LAGRANGIAN MULTIPLIER =  0.14647168E+00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29728013E+04
          2      F(2)  =  0.51538554E+05
                 EP(2) =  0.51538554E+05
                 LAGRANGIAN MULTIPLIER =  0.23238844E+00
          3      F(3)  =  0.52776171E+05
                 EP(3) =  0.52776171E+05
                 LAGRANGIAN MULTIPLIER =  0.15524592E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

ITERATION= 3
     S( 2) =       0.54749158E-02
     S( 3) =       0.23222167E-02
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03


          1      F(1)  =  0.29711703E+04
          2      F(2)  =  0.51544028E+05
                 EP(2) =  0.51544028E+05
                 LAGRANGIAN MULTIPLIER =  0.23181952E+00
          3      F(3)  =  0.52778490E+05
                 EP(3) =  0.52778490E+05
                 LAGRANGIAN MULTIPLIER =  0.15488516E+00
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0
```

```
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)  =  0.29695441E+04
          2     F(2)  =  0.51549501E+05
                EP(2) =  0.51549501E+05
                LAGRANGIAN MULTIPLIER =  0.23125249E+00
          3     F(3)  =  0.52780815E+05
                EP(3) =  0.52780815E+05
                LAGRANGIAN MULTIPLIER =  0.15452559E+00
POINT= 1
        M(1,2)=        0.22691353E+00
        M(1,3)=        0.15292369E+00

POINT= 2
        M(1,2)=        0.22702120E+00
        M(1,3)=        0.15296937E+00
POINT= 3
        M(1,2)=        0.22712887E+00
        M(1,3)=        0.15301503E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 1

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     - 0.10000000E+01*EXP( 0.47989222E-04*F(1))
     - 0.41699363E-02*EXP( 0.72349655E-04*F(2))
     - 0.59680942E-03*EXP( 0.94875566E-04*F(3))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)  =  0.29663021E+04
          2     F(2)  =  0.51560453E+05
                EP(2) =  0.51560453E+05
                LAGRANGIAN MULTIPLIER =  0.23012344E+00
          3     F(3)  =  0.52785459E+05
                EP(3) =  0.52785459E+05
                LAGRANGIAN MULTIPLIER =  0.15380956E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)  =  0.29598650E+04
          2     F(2)  =  0.51582351E+05
                EP(2) =  0.51582351E+05
                LAGRANGIAN MULTIPLIER =  0.22788587E+00
          3     F(3)  =  0.52794746E+05
                EP(3) =  0.52794746E+05
                LAGRANGIAN MULTIPLIER =  0.15239030E+00
```

```
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29471747E+04
          2      F(2)   =  0.51626152E+05
                 EP(2)  =  0.51626152E+05
                 LAGRANGIAN MULTIPLIER =  0.22348971E+00
          3      F(3)   =  0.52813327E+05
                 EP(3)  =  0.52813327E+05
                 LAGRANGIAN MULTIPLIER =  0.14960095E+00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29598650E+04
          2      F(2)   =  0.51582351E+05
                 EP(2)  =  0.51582351E+05

                 LAGRANGIAN MULTIPLIER =  0.22788587E+00
          3      F(3)   =  0.52794746E+05
                 EP(3)  =  0.52794746E+05
                 LAGRANGIAN MULTIPLIER =  0.15239030E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

THE FOLLOWING VALUES ARE YOUR PREFERRED SOLUTION

PREFERRED VALUES OF OBJECTIVES :

F(1)=            0.29598650E+04

F(2)=            0.51582351E+05

F(3)=            0.52794746E+05

PREFERRED VALUES OF VARIABLES :

X( 1)=            0.38478086E+01

X( 2)=            0.61440117E+01


X( 3)=            0.68880640E+01

DIRECTION VECTOR IS :

S(2)=            0.11100094E-03

S(3)=           -0.89873817E-03

ABSOLUTE VALUES OF WHICH ARE LESS THAN TOLERANCE DELTA1 =    0.00100
```

APPENDIX 2: OUTPUT LIST USING THE
SUM-OF-POWERS PROXY WITH ε = (52000,52000)

```
COMMAND?
= SPOT
INPUT NUMBER OF OBJECTIVES
= 3
INPUT INITIAL VALUES OF EPSIRONS
= 52000.
= 52000.

WHICH DO YOU SELECT ?
1    SPOT
2    SPOT BY DECOMP
= 1
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0
X IS
 7.000000E+00 7.000000E+00 0.           0.           0.           0.

DO YOU USE DEFAULT VALUES IN GRG ? (YES OR NO)
= YES

UPPER BOUNDS ON INEQUALITY CONSTRAINTS ARE
 (   1)  1.000000E+02       (   2)  1.000000E+30       (   3)  1.000000E+30
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
DO YOU USE IDEAL DM ? (YES OR NO) :
= YES

          1      F(1)   =  0.30064934E+04
          2      F(2)   =  0.52000000E+05
                 EP(2)  =  0.52000000E+05
                 LAGRANGIAN MULTIPLIER =  0.22011975E+00
          3      F(3)   =  0.52000000E+05
                 EP(3)  =  0.52000000E+05
                 LAGRANGIAN MULTIPLIER =  0.20118035E+00

INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

ITERATION= 1
     S( 2) =      -0.15868455E-01
     S( 3) =       0.63520568E-01
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29973553E+04
          2      F(2)   =  0.51984133E+05
                 EP(2)  =  0.51984133E+05
                 LAGRANGIAN MULTIPLIER =  0.21900488E+00
          3      F(3)   =  0.52063518E+05
                 EP(3)  =  0.52063518E+05
                 LAGRANGIAN MULTIPLIER =  0.19627351E+00

INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0
```

```
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)   =   0.29885076E+04
           2      F(2)   =   0.51968261E+05
                  EP(2)  =   0.51968261E+05
                  LAGRANGIAN MULTIPLIER =  0.21794420E+00
           3      F(3)   =   0.52127042E+05
                  EP(3)  =   0.52127042E+05
                  LAGRANGIAN MULTIPLIER =  0.19146771E+00
POINT= 1
        M(1,2)=        0.23598820E+00
        M(1,3)=        0.13765978E+00
POINT= 2
        M(1,2)=        0.23567613E+00
        M(1,3)=        0.13890895E+00
POINT= 3
        M(1,2)=        0.23536407E+00
        M(1,3)=        0.14015813E+00


SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 2

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     - 0.10000000E+01*(F(1)**( 0.10151939E+01))
     - 0.34029880E-22*(F(2)**( 0.54870799E+01))
     - 0.36150294E-36*(F(3)**( 0.83616983E+01))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)   =   0.29716648E+04
           2      F(2)   =   0.51936527E+05
                  EP(2)  =   0.51936527E+05
                  LAGRANGIAN MULTIPLIER =  0.21597532E+00
           3      F(3)   =   0.52254080E+05
                  EP(3)  =   0.52254080E+05
                  LAGRANGIAN MULTIPLIER =  0.18213264E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
```

```
           1      F(1)  =  0.29412617E+04
           2      F(2)  =  0.51873054E+05
                  EP(2) =  0.51873054E+05
                  LAGRANGIAN MULTIPLIER =  0.21258549E+00
           3      F(3)  =  0.52508165E+05
                  EP(3) =  0.52508165E+05
                  LAGRANGIAN MULTIPLIER =  0.16440352E+00
FA<FB

INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)  =  0.28928113E+04
           2      F(2)  =  0.51746104E+05
                  EP(2) =  0.51746104E+05
                  LAGRANGIAN MULTIPLIER =  0.20763425E+00
           3      F(3)  =  0.53016330E+05
                  EP(3) =  0.53016330E+05
                  LAGRANGIAN MULTIPLIER =  0.13165521E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)  =  0.28421184E+04
           2      F(2)  =  0.51492208E+05
                  EP(2) =  0.51492208E+05
                  LAGRANGIAN MULTIPLIER =  0.20350984E+00
           3      F(3)  =  0.54032660E+05

                  EP(3) =  0.54032660E+05
                  LAGRANGIAN MULTIPLIER =  0.70925890E-01
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)  =  0.28928113E+04
           2      F(2)  =  0.51746104E+05
                  EP(2) =  0.51746104E+05
                  LAGRANGIAN MULTIPLIER =  0.20763425E+00
           3      F(3)  =  0.53016330E+05
                  EP(3) =  0.53016330E+05
                  LAGRANGIAN MULTIPLIER =  0.13165521E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

ITERATION= 2
     S( 2) =      -0.23360922E-01
     S( 3) =      -0.25991379E-01
```

```
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29011456E+04
          2      F(2)   =  0.51722741E+05
                 EP(2) =  0.51722741E+05
                 LAGRANGIAN MULTIPLIER =  0.21021423E+00
          3      F(3)   =  0.52990336E+05
                 EP(3) =  0.52990336E+05
                 LAGRANGIAN MULTIPLIER =  0.13409652E+00
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =  0.29096040E+04
          2      F(2)   =  0.51699384E+05
                 EP(2) =  0.51699384E+05
                 LAGRANGIAN MULTIPLIER =  0.21283314E+00

          3      F(3)   =  0.52964346E+05
                 EP(3) =  0.52964346E+05
                 LAGRANGIAN MULTIPLIER =  0.13656845E+00
POINT= 1
        M(1,2)=      0.23099517E+00
        M(1,3)=      0.15764659E+00
POINT= 2
        M(1,2)=      0.23053576E+00
        M(1,3)=      0.15713545E+00
POINT= 3
        M(1,2)=      0.23007636E+00
        M(1,3)=      0.15662431E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 2

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     - 0.10000000E+01*(F(1)**( 0.10927865E+01))
     - 0.11082917E-18*(F(2)**( 0.48176245E+01))

     - 0.97958036E-30*(F(3)**( 0.70783073E+01))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
```

```
          1      F(1)   =   0.29269029E+04
          2      F(2)   =   0.51652659E+05
                 EP(2) =   0.51652659E+05
                 LAGRANGIAN MULTIPLIER =  0.21819542E+00
          3      F(3)   =   0.52912363E+05
                 EP(3) =   0.52912363E+05
                 LAGRANGIAN MULTIPLIER =  0.14161044E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =   0.29630772E+04

          2      F(2)   =   0.51559219E+05
                 EP(2) =   0.51559219E+05
                 LAGRANGIAN MULTIPLIER =  0.22947213E+00
          3      F(3)   =   0.52808396E+05
                 EP(3) =   0.52808396E+05
                 LAGRANGIAN MULTIPLIER =  0.15213292E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =   0.30422567E+04
          2      F(2)   =   0.51372328E+05
                 EP(2) =   0.51372328E+05
                 LAGRANGIAN MULTIPLIER =  0.25478762E+00
          3      F(3)   =   0.52600468E+05
                 EP(3) =   0.52600468E+05
                 LAGRANGIAN MULTIPLIER =  0.17539427E+00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)   =   0.29630772E+04
          2      F(2)   =   0.51559219E+05
                 EP(2) =   0.51559219E+05
                 LAGRANGIAN MULTIPLIER =  0.22947213E+00
          3      F(3)   =   0.52808396E+05
                 EP(3) =   0.52808396E+05
                 LAGRANGIAN MULTIPLIER =  0.15213292E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

ITERATION= 3
     S( 2) =       0.21522267E-02
     S( 3) =      -0.14245544E-02
```

```
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29628003E+04
          2      F(2)  =  0.51561369E+05
                 EP(2) =  0.51561369E+05
                 LAGRANGIAN MULTIPLIER =  0.22932912E+00
          3      F(3)  =  0.52806973E+05
                 EP(3) =  0.52806973E+05
                 LAGRANGIAN MULTIPLIER =  0.15216832E+00
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29625236E+04
          2      F(2)  =  0.51563520E+05
                 EP(2) =  0.51563520E+05
                 LAGRANGIAN MULTIPLIER =  0.22918623E+00
          3      F(3)  =  0.52805550E+05
                 EP(3) =  0.52805550E+05
                 LAGRANGIAN MULTIPLIER =  0.15220374E+00
POINT= 1
        M(1,2)=       0.22731990E+00
        M(1,3)=       0.15355748E+00

POINT= 2
        M(1,2)=       0.22736223E+00
        M(1,3)=       0.15352946E+00
POINT= 3
        M(1,2)=       0.22740455E+00
        M(1,3)=       0.15350145E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 2

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     - 0.10000000E+01*(F(1)**( 0.12632024E+01))
     - 0.27818301E-18*(F(2)**( 0.48707072E+01))
     - 0.10329376E-36*(F(3)**( 0.86755082E+01))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
```

```
          1        F(1)   =  0.29619715E+04
          2        F(2)   =  0.51567825E+05
                   EP(2) =  0.51567825E+05
                   LAGRANGIAN MULTIPLIER =  0.22890074E+00
          3        F(3)   =  0.52802698E+05
                   EP(3) =  0.52802698E+05
                   LAGRANGIAN MULTIPLIER =  0.15227460E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1        F(1)   =  0.29608714E+04
          2        F(2)   =  0.51576432E+05
                   EP(2) =  0.51576437E+05
                   LAGRANGIAN MULTIPLIER =  0.22833106E+00
          3        F(3)   =  0.52796999E+05
                   EP(3) =  0.52796999E+05
                   LAGRANGIAN MULTIPLIER =  0.15241656E+00

FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1        F(1)   =  0.29586884E+04
          2        F(2)   =  0.51593651E+05
                   EP(2) =  0.51593651E+05
                   LAGRANGIAN MULTIPLIER =  0.22719674E+00
          3        F(3)   =  0.52785607E+05
                   EP(3) =  0.52785607E+05
                   LAGRANGIAN MULTIPLIER =  0.15270131E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1        F(1)   =  0.29543904E+04
          2        F(2)   =  0.51628088E+05
                   EP(2) =  0.51628088E+05
                   LAGRANGIAN MULTIPLIER =  0.22494791E+00

          3        F(3)   =  0.52762813E+05
                   EP(3) =  0.52762813E+05
                   LAGRANGIAN MULTIPLIER =  0.15327422E+00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
```

```
          1       F(1)  =  0.29586884E+04
          2       F(2)  =  0.51593651E+05
                  EP(2) =  0.51593651E+05
                  LAGRANGIAN MULTIPLIER =  0.22719674E+00
          3       F(3)  =  0.52785607E+05
                  EP(3) =  0.52785607E+05
                  LAGRANGIAN MULTIPLIER =  0.15270131E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

THE FOLLOWING VALUES ARE YOUR PREFERRED SOLUTION


PREFERRED VALUES OF OBJECTIVES :

F(1)=           0.29586884E+04

F(2)=           0.51593651E+05

F(3)=           0.52785607E+05

PREFERRED VALUES OF VARIABLES :

X( 1)=           0.38659955E+01

X( 2)=           0.61234810E+01

X( 3)=           0.68961587E+01

DIRECTION VECTOR IS :

S(2)=          -0.80036551E-03

S(3)=          -0.40793175E-03

ABSOLUTE VALUES OF WHICH ARE LESS THAN TOLERANCE DELTA1 =   0.00100
```

APPENDIX 3: OUTPUT LIST USING THE SUM-OF-LOGARITHMS PROXY WITH $\varepsilon = (52000,52000)$

```
COMMAND?
= SPOT
INPUT NUMBER OF OBJECTIVES
= 3
INPUT INITIAL VALUES OF EPSIRONS
= 52000.
= 52000.

WHICH DO YOU SELECT ?
1   SPOT
2   SPOT BY DECOMP
= 1
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0
X IS
 7.000000E+00 7.000000E+00 0.            0.            0.            0.


DO YOU USE DEFAULT VALUES IN GRG ? (YES OR NO)
= YES

UPPER BOUNDS ON INEQUALITY CONSTRAINTS ARE
 (  1)  1.000000E+02       (  2)  1.000000E+30       (  3)  1.000000E+30

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
DO YOU USE IDEAL DM ? (YES OR NO) :
= YES

           1      F(1)  =  0.30064934E-04
           2      F(2)  =  0.52000000E+05
                  EP(2) =  0.52000000E+05
                  LAGRANGIAN MULTIPLIER =  0.20011973E+00
           3      F(3)  =  0.52000000E+05
                  EP(3) =  0.52000000E+05
                  LAGRANGIAN MULTIPLIER =  0.20118035E+00

INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS(RAMDA-MRS) < DELTA1 )
= 0.001

ITERATION= 1
     S( 2) =       -0.15868455E-01
     S( 3) =        0.63520568E-01
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 2
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

           1      F(1)  =  0.29973553E+04
           2      F(2)  =  0.51984133E+05
                  EP(2) =  0.51984133E+05
                  LAGRANGIAN MULTIPLIER =  0.21900448E+00
           3      F(3)  =  0.52063513E+05
                  EP(3) =  0.52063513E+05
                  LAGRANGIAN MULTIPLIER =  0.19627351E+00

INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 2
```

```
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

            1      F(1)   =  0.29885076E+04
            2      F(2)   =  0.51968261E+05
                   EP(2) =  0.51968261E+05
                   LAGRANGIAN MULTIPLIER =  0.21794420E+00
            3      F(3)   =  0.52127042E+05
                   EP(3) =  0.52127042E+05
                   LAGRANGIAN MULTIPLIER =  0.19146771E+00
POINT= 1
        M(1,2)=       0.23598820E+00
        M(1,3)=       0.13765978E+00
POINT= 2
        M(1,2)=       0.23567613E+00
        M(1,3)=       0.13890895E+00
POINT= 3
        M(1,2)=       0.23536407E+00
        M(1,3)=       0.14015813E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 3
INPUT VALUE OF M(I) SUCH THAT M(I)-F(I)>0
M(1)
= 10000.
M(2)
= 100000.
M(3)
= 100000.

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     + 0.10000000E+01*LOG(M(1)-F(1))
     + 0.16197072E+01*LOG(M(2)-F(2))
     + 0.94482923E+00*LOG(M(3)-F(3))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)

= 0
KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

            1      F(1)   =  0.29716648E+04
            2      F(2)   =  0.51936527E+05
                   EP(2) =  0.51936527E+05
                   LAGRANGIAN MULTIPLIER =  0.21597532E+00
            3      F(3)   =  0.52254080E+05
                   EP(3) =  0.52254080E+05
                   LAGRANGIAN MULTIPLIER =  0.18213264E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
```

```
          1      F(1)  =  0.29412617E+04

          2      F(2)  =  0.51873054E+05
                 EP(2) =  0.51873054E+05
                 LAGRANGIAN MULTIPLIER =  0.21258549E+00
          3      F(3)  =  0.52508165E+05
                 EP(3) =  0.52508165E+05
                 LAGRANGIAN MULTIPLIER =  0.16440352E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.28928113E+04
          2      F(2)  =  0.51746104E+05
                 EP(2) =  0.51746104E+05
                 LAGRANGIAN MULTIPLIER =  0.20763425E+00
          3      F(3)  =  0.53016330E+05
                 EP(3) =  0.53016330E+05
                 LAGRANGIAN MULTIPLIER =  0.13165521E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.28421184E+04
          2      F(2)  =  0.51492208E+05
                 EP(2) =  0.51492208E+05
                 LAGRANGIAN MULTIPLIER =  0.20350984E+00
          3      F(3)  =  0.54032660E+05
                 EP(3) =  0.54032660E+05
                 LAGRANGIAN MULTIPLIER =  0.70925890E-01
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.28928113E+04
          2      F(2)  =  0.51746104E+05
                 EP(2) =  0.51746104E+05
                 LAGRANGIAN MULTIPLIER =  0.20763425E+00
          3      F(3)  =  0.53016330E+05
                 EP(3) =  0.53016330E+05
                 LAGRANGIAN MULTIPLIER =  0.13165521E+00

INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS(FAMDA-MFSD) < DELTA1 )
= 0.001
```

```
ITERATION= 2
     S( 2) =       -0.23360922E-01
     S( 3) =       -0.25991379E-01
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29011456E+04
          2      F(2)  =  0.51722741E+05
                 EP(2) =  0.51722741E+05
                 LAGRANGIAN MULTIPLIER =  0.21021423E+00
          3      F(3)  =  0.52990336E+05
                 EP(3) =  0.52990336E+05
                 LAGRANGIAN MULTIPLIER =  0.13409652E+00
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29096040E+04
          2      F(2)  =  0.51699384E+05
                 EP(2) =  0.51699384E+05
                 LAGRANGIAN MULTIPLIER =  0.21283314E+00
          3      F(3)  =  0.52964346E+05
                 EP(3) =  0.52964346E+05
                 LAGRANGIAN MULTIPLIER =  0.13656845E+00
POINT= 1
        M(1,2)=      0.23099517E+00
        M(1,3)=      0.15764659E+00
POINT= 2
        M(1,2)=      0.23053576E+00
        M(1,3)=      0.15713545E+00
POINT= 3
        M(1,2)=      0.23007636E+00
        M(1,3)=      0.15662431E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS
2   SUM OF POWERS-1

3   SUM OF POWERS-2
= 3
INPUT VALUE OF M(I) SUCH THAT M(I)-F(I)>0
M(1)
= 10000.
M(2)
= 100000.
M(3)
= 100000.
```

```
LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     + 0.10000000E+01*LOG(M(1)-F(1))
     + 0.15683299E+01*LOG(M(2)-F(2))
     + 0.10421583E+01*LOG(M(3)-F(3))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03


          1      F(1)  =  0.29269029E+04
          2      F(2)  =  0.51652659E+05
                 EP(2) =  0.51652659E+05
                 LAGRANGIAN MULTIPLIER =  0.21819542E+00
          3      F(3)  =  0.52912363E+05
                 EP(3) =  0.52912363E+05
                 LAGRANGIAN MULTIPLIER =  0.14161044E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29630772E+04
          2      F(2)  =  0.51559219E+05
                 EP(2) =  0.51559219E+05
                 LAGRANGIAN MULTIPLIER =  0.22947213E+00
          3      F(3)  =  0.52808396E+05
                 EP(3) =  0.52808396E+05
                 LAGRANGIAN MULTIPLIER =  0.15213292E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0


KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.30422567E+04
          2      F(2)  =  0.51372328E+05
                 EP(2) =  0.51372328E+05
                 LAGRANGIAN MULTIPLIER =  0.25478762E-00
          3      F(3)  =  0.52600468E+05
                 EP(3) =  0.52600468E+05
                 LAGRANGIAN MULTIPLIER =  0.17539427E-00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03
```

```
          1      F(1)  =  0.27630772E+04
          2      F(2)  =  0.51559219E+05
                 EP(2) =  0.51559219E+05
                 LAGRANGIAN MULTIPLIER =  0.22947213E+00
          3      F(3)  =  0.52808396E+05
                 EP(3) =  0.52808396E+05

                 LAGRANGIAN MULTIPLIER =  0.15213292E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS[RAMDA-MRS] < DELTA1 )
= 0.001

ITERATION= 3
     S( 2) =        0.21522267E-02
     S( 3) =       -0.14245544E-02
INPUT INITIAL STEP SIZE
= 1000.
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29628003E+04
          2      F(2)  =  0.51561369E+05
                 EP(2) =  0.51561369E+05
                 LAGRANGIAN MULTIPLIER =  0.22932912E+00
          3      F(3)  =  0.52806973E+05
                 EP(3) =  0.52806973E+05
                 LAGRANGIAN MULTIPLIER =  0.15216832E+00
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)

= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1      F(1)  =  0.29625236E+04
          2      F(2)  =  0.51563520E+05
                 EP(2) =  0.51563520E+05
                 LAGRANGIAN MULTIPLIER =  0.22918623E+00
          3      F(3)  =  0.52805550E+05
                 EP(3) =  0.52805550E+05
                 LAGRANGIAN MULTIPLIER =  0.15220374E+00
POINT= 1
        M(1,2)=       0.22731990E+00
        M(1,3)=       0.15355748E+00
POINT= 2
        M(1,2)=       0.22736203E+00
        M(1,3)=       0.15352946E+00
POINT= 3
        M(1,2)=       0.22740435E+00
        M(1,3)=       0.15350145E+00

SELECT LOCAL PROXY PREFERENCE FUNCTION FROM AMONG THE FOLLOWINGS
1   SUM OF EXPONENTIALS

2   SUM OF POWERS-1
3   SUM OF POWERS-2
= 3
```

```
INPUT VALUE OF M(I) SUCH THAT M(I)-F(I)>0
M(1)
= 10000.
M(2)
= 100000.
M(3)
= 100000.

LOCAL PROXY PREFERENCE FUNCTION
P(F)=
     + 0.10000000E+01*LOG(M(1)-F(1))
     + 0.13648252E+01*LOG(M(2)-F(2))
     + 0.10298000E+01*LOG(M(3)-F(3))
INPUT THE MAXIMUM STEP SIZE (ALFMAX)
= 100000.
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03


          1     F(1)   =  0.29619715E+04
          2     F(2)   =  0.51567825E+05
                EP(2)  =  0.51567825E+05
                LAGRANGIAN MULTIPLIER =  0.22890074E+00
          3     F(3)   =  0.52802698E+05
                EP(3)  =  0.52802698E+05
                LAGRANGIAN MULTIPLIER =  0.15227460E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)   =  0.29608714E+04
          2     F(2)   =  0.51576432E+05
                EP(2)  =  0.51576437E+05
                LAGRANGIAN MULTIPLIER =  0.22833106E+00
          3     F(3)   =  0.52796999E+05
                EP(3)  =  0.52796999E+05
                LAGRANGIAN MULTIPLIER =  0.15241656E+00
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)

= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)   =  0.29586884E+04
          2     F(2)   =  0.51593651E+05
                EP(2)  =  0.51593651E+05
                LAGRANGIAN MULTIPLIER =  0.22719674E+00
          3     F(3)   =  0.52785607E+05
                EP(3)  =  0.52785607E+05
                LAGRANGIAN MULTIPLIER =  0.[illegible]E+00
```

```
FA<FB
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)  =  0.29543904E+04
          2     F(2)  =  0.51628088E+05
                EP(2) =  0.51628088E+05
                LAGRANGIAN MULTIPLIER =  0.22494791E+00
          3     F(3)  =  0.52762813E+05
                EP(3) =  0.52762813E+05

                LAGRANGIAN MULTIPLIER =  0.15327422E+00
DO YOU FIT QUADRATIC INTERPOLATION ? (YES OR NO) :
= NO
INPUT ICOUNT (= 0 OR 2 :MEANS START PHASE 1 OR 2)
= 0

KUHN-TUCKER CONDITIONS SATISFIED TO WITHIN 1.00000E-03

          1     F(1)  =  0.29586884E+04
          2     F(2)  =  0.51593651E+05
                EP(2) =  0.51593651E+05
                LAGRANGIAN MULTIPLIER =  0.22719674E+00
          3     F(3)  =  0.52785607E+05
                EP(3) =  0.52785607E+05
                LAGRANGIAN MULTIPLIER =  0.15270131E+00
INPUT TOLERANCE DELTA1
( STOPPING CRITERIA IS ABS(RAMDA-MRS) < DELTA1 )
= 0.001

THE FOLLOWING VALUES ARE YOUR PREFERRED SOLUTION

PREFERRED VALUES OF OBJECTIVES :


F(1)=           0.29586884E+04

F(2)=           0.51593651E+05

F(3)=           0.52785607E+05

PREFERRED VALUES OF VARIABLES :

X( 1)=           0.38639955E+01

X( 2)=           0.61234610E+01

X( 3)=           0.68961587E+01

DIRECTION VECTOR IS :

B(2)=          -0.80036551E-03

B(3)=          -0.40793175E-03

ABSOLUTE VALUES OF WHICH ARE LESS THAN TOLERANCE DELTA1 =   0.00100
```

APPENDIX 4: THE PREFERRED SOLUTIONS OF THE DM FOR THREE TYPES OF PROXY FUNCTIONS WITH SEVERAL STARTING POINTS.

Table A.1 Initial Value ε = (52000,52000)

	Sum-of-exponentials	Sum-of-powers	Sum-of-logarithms
f_1	2959.8650	2958.6884	2958.6884
f_2	51582.351	51593.651	51593.651
f_3	52794.746	52785.607	52785.607
x_1	3.8478086	3.8659955	3.8659955
x_2	6.1440117	6.1234810	6.1234810
x_3	6.8880640	6.8961587	6.8961587
s_2	1.1100094×10^{-4}	$-8.0036551 \times 10^{-4}$	$-8.0036551 \times 10^{-4}$
s_3	$-8.9873817 \times 10^{-4}$	$-4.0793175 \times 10^{-4}$	$-4.0793175 \times 10^{-4}$
Number of Iterations	3	3	3

Table A.2 Initial Value ε = (53000,53000)

	Sum-of-exponentials	Sum-of-powers	Sum-of-logarithms
f_1	2959.9680	2959.9680	2960.9704
f_2	51581.015	51581.015	51583.354
f_3	52796.072	52796.072	52786.017
x_1	3.8451645	3.8451645	3.8655155
x_2	6.1464048	6.1464048	6.1432847
x_3	6.8874055	6.8874055	6.8787927
S_2	2.1102292×10^{-4}	2.1102292×10^{-4}	2.9061103×10^{-4}
S_3	$-9.7955261 \times 10^{-4}$	$-9.7955261 \times 10^{-4}$	$-1.0825954 \times 10^{-4}$
Number of Iterations	3	3	4

Table A.3 Initial Value ε = (54000,54000)

	Sum-of-exponentials	Sum-of-powers	Sum-of-logarithms
f_1	2960.3544	2960.3544	2960.3544
f_2	51583.825	51583.825	51583.825
f_3	52789.340	52789.340	52789.340
x_1	3.8587421	3.8587421	3.8587421
x_2	6.1419207	6.1419207	6.1419207
x_3	6.8838118	6.8838118	6.8838118
S_2	1.3061003×10^{-4}	1.3061003×10^{-4}	1.3061003×10^{-4}
S_3	$-4.3755776 \times 10^{-4}$	$-4.3755776 \times 10^{-4}$	$-4.3755776 \times 10^{-4}$
Number of Iterations	5	5	5

Table A.4 Initial Value ε = (54000,50000)

	Sum-of-exponentials	Sum-of-powers	Sum-of-logarithms
f_1	2960.8648	2960.8648	2958.7263
f_2	51582.330	51582.330	51590.379
f_3	52788.234	52788.234	52790.236
x_1	3.8610440	3.8610440	3.8567036
x_2	6.1449553	6.1449553	6.1291596
x_3	6.8798115	6.8798115	6.8963159
S_2	3.2764161×10^{-4}	3.2764161×10^{-4}	$-6.0137369 \times 10^{-4}$
S_3	$-2.8382870 \times 10^{-4}$	$-2.8382870 \times 10^{-4}$	$-7.3670261 \times 10^{-4}$
Number of Iterations	6	6	6

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