International Institute for Applied Systems Analysis
I I A S A www.iiasa.ac.at

Some Problems of Linkage Systems

Ermoliev, Y.M.

IIASA Working Paper

WP-80-102

June 1980

SOME PROBLEMS OF LINKAGE SYSTEMS

Yuri Ermolev

June 1980
WP-80-102

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

SOME PROBLEMS OF LINKAGE SYSTEMS

Yuri Ermolev

INTRODUCTION

At present many different models have been developed which describe separate activities of a real economy. Some examples are energy, water and other resources supply models, problems of national settlement planning, industrial or agricultural production models, manpower and educational planning models, and resources allocation models. These models have exogenous variables which describe interactions between one subsystem and the other subsystems. One can consider these variables as endogenous or as decision variables when these submodels are linked with a model of the whole system.

There are different aspects of the linkage of submodels. We define here linkage as the opposite to decomposition. If in a linkage problem one tries to obtain a model of the whole system by uniting models of the subsystems, then in the decomposition problem one must try to split a model of the system into a number of small models of the subsystems. As a result, one may obtain a large-scale model. How does one solve a corresponding mathematical problem? Is it necessary to collect all submodels in one place? Are there numerical methods which allow us to use different computers for running separate submodels?

The objective of this paper is to discuss the possible ways to formulate the problems of linkage mathematically and possibilities for applying methods of optimization to these problems. For instance, in special cases of linkage the known decomposition technique can be used.

A Deterministic Case with a Common Objective

Let us suppose that each model of a subsystem (submodel) can be described in the following LP-form:

$$(a(k),x(k)) = \max \tag{1}$$

$$A(k)x(k) \leq b(k) \tag{2}$$

$$B(k)x(k) \leq y(k) \tag{3}$$

$$x(k) \geq 0 \quad . \tag{4}$$

There are N submodels $k = \overline{1,N}$. Linking the submodels is carried out by parameters $y(k)$. There is a nonempty feasible set of endogenous (linking) variables $y(k)$, $k = \overline{1,N}$, corresponding to the feasible conditions of linkage. Let us assume that these conditions of linkage are described by linear constraints

$$\sum_{k=1}^{N} D(k)y(k) \leq d \quad , \tag{5}$$

in particular,

$$\sum_{k=1}^{N} y(k) = d, \; y(k) \geq 0 \quad . \tag{5'}$$

In this case the linkage version may be a problem of allocating the vector resources among subsystems $k = \overline{1,N}$.

Sometimes equations (5) and (5') have only a single solution, for instance, if the linkage variables are bound by a strong econometrics relationship then the problem of linkage leads to the solving of equations (see Almon and Nyhus 1977; Keyzer et al. 1977). Let us assume that equation (5) has many solutions. In this case, it is natural to consider the problem of finding the best variables $y(k)$, $k = \overline{1,N}$. Denote by $x(k,y)$ the solution of the k-th problem (1) - (4) for given $y(k)$. Then the problem of linkage might be the problem of finding y such that $y = (y(1),\ldots, y(N))$, which maximizes the nonlinear objective function

$$f(y) = \sum_{k=1}^{N} [(c(k),x(k,y)) + (e(k),y(k))] \quad , \tag{6}$$

under the conditions of (5). Generally speaking, there is a set value $f(y)$ for each given y and there must be a certain sense of maximizing the mapping $f(y)$. In particular, if

$$e(k) = 0,\ c(k) = a(k),\ k = \overline{1,N} \quad ,$$

then function $f(y)$ is a nondifferentiable piecewise linear concave function and for the maximization of this function one can use the well-known finite methods of decomposition or iterative methods of nondifferentiable optimization (see, for instance, Ermolev 1978).

The number of vertices of the feasible polyhedral set for such problems is so large that finite-step methods, based on moving from one vertice to another, yield very small steps at each iteration and consequently very slow convergence. Moreover, the known finite methods are not robust versus computational errors. The nondifferentiable approach made it possible to develop easily implemented iterative decomposition schemes of the gradient type. These approaches do not use the basic solution of the linear programming problem which makes it possible to start the computational process from any point and leads to computational stability.

Let us consider an application of nondescent methods of nondifferentiable optimization--the method of a generalized gradient. Denote by $v(x,y)$ the optimal value of dual variables in problems (1) - (4), which corresponds to the constraint (3). Then a generalized gradient of function $f(y)$ defined by (6) is

$$\hat{f}_y(y) = (v(1,y) + e(1),\ldots, v(N,y) + e(N)) \quad .$$

Therefore the generalized gradient method for the considered problem of maximization (6) subject to (5) reduces to the following steps. Let

$$y^s = (y^s(1),\ldots, y^s(N)) \quad ,$$

be an approximate solution after the s-th iteration ($s = 0,1\ldots$), y^0 being arbitrary. For the given y^s the following subproblems are solved

$$(u(k),b(k)) + (v(k),y^s(k)) = \min \quad , \tag{7}$$

$$u(k)A(k) + v(k)B(k) \geq a(k) \quad , \tag{8}$$

$$u(k) \geq 0,\ v(k) \geq 0 \quad , \tag{9}$$

for $k = \overline{1,N}$. If $(u(k,y^s),v(k,y^s))$ are the solutions of these problems, then the next value of y will be calculated from

$$y^{s+1} = \pi_Y(y^s + \rho_s(v^s + e)),\ s = 0,1,\ldots \quad , \tag{10}$$

where ρ_s is a step-size multiplier; $v^s = (v(1,y^s),\ldots, v(N,y^s))$, $e = (e(1),\ldots, e(N))$; Y is the set defined by (5), and $\pi_Y(\cdot)$ is the projection operator on Y.

In problems (7) - (9) only the objective function changes with the number of iterations. Therefore the previous solution $(u(k,y^s),v(k,y^s))$ can be used for calculating the solution $(u(k,y^{s+1}),v(k,y^{s+1}))$ in the next iteration s+1. For this reason it is possible to calculate $(u(k,y^{s+1}),v(k,y^{s+1}))$ very quickly.

The projection operation could be easily implemented for constraint (5') if the number of components of d is not too large. This operation can also be simplified for constraint (5) when information about the previous iteration is used.

The convergence conditions for the procedure demand that

$$\rho_s \geq 0, \; \rho_s \to 0, \; \sum_{s=0}^{\infty} \rho_s = \infty \quad .$$

Note that according to algorithm (10) the solutions of subproblems (7) - (9) can be carried out on different computers and the information on $x(k,y^s)$ and $e(k)$ is needed for solving the problem of linking (6) and (5) only.

A More General Case with Different Objectives

In a more general case each submodel can be described in the following form: maximize

$$g^0(x,y,k) \quad ,$$

subject to

$$g^i(x,y,k) \leq 0, \; i = \overline{1,m} \quad ,$$

$$x \in X(k) \quad ,$$

where $k = 1,\ldots, N$, and the linking variable y is given. Let $x(k,y)$ be an optimal solution and $X(k,y)$ be a set of optimal solutions. Then for finding a desirable point y in the feasible set of linking variables Y one can use the set valued mapping

$$F(y) = f(x(1,y),\ldots, x(N,y),y) \quad ,$$

and a problem of linkage is the problem of maximization of the mapping $F(y)$. One way to understand this problem is the maximization of function

$$F(y) = \max f(x(1), \ldots, x(k), y) \quad , \tag{11}$$

$$x(k) \in X(k,y) \quad ,$$

$$k = \overline{1,N} \quad ,$$

in the feasible set of linking variables Y.

If $f(x(1), \ldots, x(k), y) = \sum_{k=1}^{N} g^0(x(k), y, k)$, functions $g^\nu(x,y,k)$, $\nu = 0,1,\ldots, m$, $k = \overline{1,N}$ are concave as functions of variables (x,y), then function $F(y)$ will also be concave and for solving the obtained problem there exists a numerical method similar to method (10).

If $f(x(1) \ldots, x(x), y) \neq \sum_{k=1}^{N} g^0(x(k), y, k)$, then for given y, the maximization of $f(x(1) \ldots, x(N), y)$, $x(k) \in X(k,y)$ is equivalent to maximizing

$$f(x(1), \ldots, x(N), y) \quad ,$$

under constraints

$$g^i(x(k), y, k) \leq 0, \; i = \overline{1,m} \quad ,$$

$$x(k) \in X(k), \; k = \overline{1,N} \quad ,$$

$$f(x(k), \ldots, x(N), y) \geq f(z(1), \ldots, z(N), y) \quad ,$$

for all $z(k)$ such that

$$g^i(z(k), y, k) \leq 0, \; i = \overline{1,m} \quad ,$$

$$Z(k) \in X(k), \; k = \overline{1,N} \quad .$$

This problem can be approximated by the following stochastic maximin-type problem. Maximize the function

$$f(x(1),\ldots, x(N),y) - M\int_Z \min(0,f(x(1),\ldots, x(N),y) -$$

$$- f(z,y))p(z)dz \quad ,$$

subject to

$$g^i(x(k),y,k) \quad 0,\ i = \overline{1,m} \quad ,$$

$$x(k) \in X(k),\ k = \overline{1,N},\ y \in Y \quad ,$$

where M is a large positive number, $p(z(1),\ldots, z(N))$ is an arbitrary nongenerate density over set $X = X(1)\cdots X(N)$.

$$Z = \{(x(1),\ldots, x(N)):g^i(x(k)y,k) \leq 0,\ x(k) \in X(k),$$

$$i = \overline{1,m},\ k = \overline{1,N}\} \quad .$$

In some cases problems of this type are solved by stochastic quasigradient methods (see Ermolev 1978; Ermolev and Nurminski 1980).

A Stochastic Case

The stochastic aspects of linkage systems is a very important practical extension of the above case. Below we will consider one possible stochastic formulation of the linkage problem. Instead of common constraint (5), let (5') be the constraint with only the vector of total resources being random. Denote this by $d(w) = (d_1(w),\ldots, d_r(w))$, where w is a random parameter which is supposed to be an element of some probabilistic space.

Let us assume that the distribution of resources between subsystems be carried out according to the proportions:

$$y_j(k) = h_j(k)d_j(w) \;;\; j = \overline{1,r} \quad ,$$

where for unknown $h(k) = (h_1(k), \ldots, h_r(k))$

$$\sum_{k=1}^{N} h_j(k) = 1 \quad , \quad h_j(k) \geq 0 \quad . \tag{12}$$

Consider an optimal solution $x(k,h,w)$ for the k-th subsystem which maximizes (1) subject to (2); (4) and

$$B(k)x(k) \leq H(k)d(w) \quad ,$$

where $H(k)$ is a diagonal matrix with $h_j(k)$ on it diagonal and $h_j(k)$ is fixed. It is natural to introduce the mathematical expectation of the stochastic set valued mapping

$$f(h,w) = \sum_{k=1}^{N} [(c(k)x(k,h,w)) + (e(k), H(k)d(w))] \quad ,$$

as the objective function of the whole system. If $a(k) = c(k)$, $e(k) = 0$, then function

$$F(h) = Ef(h,w) = \int f(h,w)P(dw) \quad , \tag{13}$$

will be a concave function and the solution of the problem of linkage (the maximization of function $F(h)$ subject to (12) can be obtained by a stochastic quasigradient method (see Ermolev 1978). This method is the natural extension of the generalized gradient method (10) to the stochastic case.

Let $h^s = (h^s(1), \ldots, h^s(N))$ be an approximate solution after s iterations, w^0, $w^1, \ldots, w^s, \ldots$, which result from independent draws over w. The second component $v^s(k)$ of the optimal solution $(u^s(k), v^s(k))$ of subproblem

$$(u(k), b(k)) + (v(k), H^s(u)d(w^s)) = \min \quad ,$$

$$u(k)A(k) + v(k)B(k) \geq a(k) \quad ,$$

$$u(k) \geq 0 \quad , \quad v(k) \geq 0 \quad ,$$

is obtained. Here $H^s(k)$ is the diagonal matrix with $h_j^s(k)$, $j = 1, \ldots, r$, at the main diagonal. Then the new approximation is

$$h^{s+1} = \pi_{\{h\}}[h^s + \rho_s(v^s + \theta^s)] \quad , \quad s = 0,1,\ldots \quad ,$$

where $\{h\}$ is the set of $h = (h(1), \ldots, h(N))$, which are satisfied according to (13); $\theta^s = (\theta^s(1), \ldots, \theta^s(N))$, $\theta^s(k) = (\theta_1^s(k), \ldots, \theta_r^s(k))$, $\theta_j^s(k) = e_j(k)d_j(w^s)$, $j = \overline{1,r}$.

It is not difficult to show that the conditional mathematical expectation

$$E(v^s+\theta^s/h^s) = \hat{F}_h(h^s) \quad ,$$

where $\hat{F}_h(h^s)$ is the subgradient of $F(h)$. The convergence conditions of this kind of procedure follow from the general conditions for stochastic quasigradient methods.

Dynamic Systems

There may be several possibilities for formulating linkage problems for dynamic systems. Let us consider only one of them. The behavior of $k = 1, \ldots, N$ subsystems is described by the following state equations

$$z^k(t+1) = \phi^k(t)z^k(t) + \psi^k(t)x^k(t) + \gamma^k(t)y^k(t) + g^k(t)$$

$$z(0) = z^0, \; t = 0,1,\ldots, T-1 \quad ,$$

where $z^k(t)$ are state variables, $x^k(t)$ are control variables, and $y^k(t)$ are linkage variables. There are objective functions

$$(\beta^k(T), z^k(T)) + \sum_{t=0}^{T-1} [(\alpha^k(t), x^k(t)) + (\beta^k(t), z^k(t))] \quad (14)$$

and constraints

$$A^k(t)z^k(t) + B^k(t)x^k(t) + R^k(t)y^k(t) \le b^k(t) \; , \qquad t = 0,\ldots, T-1 \tag{15}$$

$$x^k(t) \in X^k(t) \; , \; y^k(t) \in Y^k(t) \quad , \quad t = 0,\ldots, T-1 \; . \tag{16}$$

Here all matrices $\phi^k(t)$, $\psi^k(t)$, $\gamma^k(t)$, $A^k(t)$, $B^k(t)$, $R^k(t)$ and vectors are assumed to be fixed. Let $x^k(t,y)$, $t = 0,1,\ldots, T-1$ be an optimal control and $z^k(t,y)$, $t = 0,1,\ldots, T-1$ be the optimal trajectory of the k-th subsystem. The problem of linkage is to find such linkage variables $y^k(t)$, $t = 0,1,\ldots, T-1$, which maximize the set-valued mapping

$$F(y) = \sum_{k=1}^{N} \left((e^k(T), z^k(T,y)) + \sum_{t=0}^{T-1} [(c^k(t), x^k(t,y)) + (\delta^k(t), z^k(t,y))] + (e^k(t), y^k(t)) \right) \tag{17}$$

subject to

$$\sum_{k=1}^{N} D^k(t)y^k(t) \le d(t) \quad . \tag{18}$$

If $c^k(t) = \alpha^k(t)$, $\delta^k(t) = \beta^k(t)$, $e^k(t) = 0$, then the problem of maximizing the concave function $F(y)$ subject to (17) is the problem of decomposition of the dynamic systems.

This particular problem arises, for instance, in planning a dairy farm: as cattle grow, the dairy farm subsystem is linked with the crop subsystem. Similarly, a model of an agricultural region might be linked with a model of water resources management. Problems (12) - (18) are very difficult even if $c^k(t) = \alpha^k(t)$, $\delta^k(t) = \beta^k(t)$, $e^k(t) = 0$. The development of special methods which take into account the dynamic structure of the problem is needed. In addition, these special methods would allow us to use different computers for running separate submodels.

One such method was described in Ermolev (1978) and it is the extension of method (10) on a dynamic case. The remarkable feature of this method is that it is also applicable in the case when the coefficient of the original problem is subjected to random disturbances. This method, like (10), consists of solving the primal and dual problem.

Problems (13) - (16) are equivalent to the following problem--to find $x^k(t)$, $y^k(t)$ subject to constraints (17) and (18) which maximize function

$$\min_{\lambda} \Big\{ \sum_{t=0}^{T-1} [(\lambda^k(t), b^k(t)) - (p^k(t+1), g^k(t))] - (p^k(0), Z^0) + \\ + \sum_{t=0}^{T-1} (\alpha^k(t) - p^k(t+1)\psi^k(t) - \lambda^k(t) B^k(t), x^k(t)) + \\ + \sum_{t=0}^{T-1} (e^k(t) - p^k(t+1)\gamma^k(t) - \lambda^k(t) R^k(t), y^k(t)) \Big\} , \tag{19}$$

where variables $\lambda^k(t)$, $p^k(t)$ are subjected to the constraints

$$p^k(t) = p^k(t+1)\phi^k(t) + \lambda^k(t) A^k(t) - \beta^k(t) , \\ p^k(T) = \beta^k(T) , \; t = T-1, \ldots, 0 , \tag{20}$$

$$\lambda^k(t) \geq 0 , \; t = T-1, \ldots, 0 . \tag{21}$$

The method consists of the following. Let $x^k(t,s)$, $y^k(t,s)$ be the approximation of optimal control and linking variables after s iterations. Compute the corresponding trajectory $z^k(t,s)$ from (13). For given $x^k(t,s)$, $Z^k(t,s)$ to find a solution of the simple subproblems by choosing dual control $\lambda^k(t,s)$, $t = T-1,\ldots,0$, and corresponding trajectory $p^k(t,s)$, $t = T,\ldots, 0$, which minimize the linear function

$$\sum_{t=0}^{T-1} [(\lambda^k(t), b^k(t)) - (p^k(t+1), g^k(t))] - (p^k(0), z^0) +$$

$$+ \sum_{t=0}^{T-1} (\alpha^k(t) - p^k(t+1)\phi^k(t) - \lambda^k(t)B^k(t), x^k(t,s)) +$$

$$+ \sum_{t=0}^{T-1} (e^k(t) - p^k(t+1)\gamma^k(t) - \lambda^k(t)R^k(t), y^k(t,s)) \quad ,$$

under constraints (20) and (21). According to the well-known discrete Pontryagin's principle the solution of this problem is reduced to the solution of the simplest static, linear programming subproblems:

$$\min_{c \geq \lambda^k(t) \geq 0} [(\lambda^k(t), b^k(t) - A^k(t)z^k(t,s) - B^k(t,s) - R^k(t)y^k(t,s))]$$

where $z^k(t,s)$ is a trajectory corresponding to control $x^k(t,s)$ and linking variables $y^k(t,s)$ and vector c has a large number of components. Let $\lambda^k(t,s)$ be the optimal solution of this subproblem and $p^k(t,s)$ be the trajectory found from (20) for $\lambda^k(t) = \lambda^k(t,s)$. The next approximation for optimal control will be

$$x^k(t,s+1) = \Pi_{X(k)} [x^k(t,s) + \rho_s(\alpha^k(t) - p^k(t+1,s)\psi^k(t) -$$

$$- \lambda^k(t,s)B^k(t))] \quad ,$$

and for linking variables $y(t,s) = (y^1(t,s), \ldots, y^N(t,s))$

$$y(t,s+1) = \Pi_y(y(t,s) + \rho_s\mu(t,s)) \quad , \quad s = 0,1,\ldots \quad ,$$

where $\Pi_{X(k)}(\cdot)$ is the projection operator on the set $x(k)$, $\Pi_y(\cdot)$ is the projection operator on the set defined by constraints (16) and (17), and ρ_s are the step-size multipliers which should satisfy the same conditions as in procedure (10), $y(t,s) = (y^1(t,s), \ldots, y^N(t,s))$, $\mu(t,s) = (e^1(t) - p^1(t+1,s)\gamma^1(t) - \lambda^1(t,s)R^1(t), \ldots, e^N(t) - p^N(t+1,s)\gamma^N(t) - \lambda^N(t,s)R^N(t))$.

There may be a great variety of iterative methods for solving the described problems. The approach studied in this paper is based on methods of nondifferentiable optimization. These methods have some advantages which make them attractive for certain categories of users. Among these advantages are logical simplicity of algorithms which vary slightly in all the mentioned cases, low core requirements, numerical stability, and the possibility of handling a relatively big problem on several small computers.

REFERENCES

Almon, C., and D. Nyhus. 1977. The Inforum International System of Input-Output Models and Bilateral Trade Flows. Inforum Research Report No. 21.

Ermolev, Yu. 1978. Methods of Nondifferentiable and Stochastic Optimization and their Applications. WP-78-62. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Ermolev, Yu., and E. Nurminski. 1980. Stochastic Quasigradient Algorithms for Minimax Problems. Edited by M. Dempster. Proceedings of the International Conference on Stochastic Programming. London: Academic Press.

Keyzer, M.A., C. Lemarechal, and R. Mifflin. 1978. Computing Economic Equilibria Through Nonsmooth Optimization. RM-78-13. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Nurminski, E. 1979. Some Theoretical Considerations on Linkage Problems. WP-79-117. Laxenburg, Austria: International Institute for Applied Systems Analysis.