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ESTIMATION OF PARAMETERS OF DYNAMIC INPUT-OUTPUT MODELS WITH LIMITED INFORMATION

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1. INTRODUCTION

Dynamic I-O models ought to be closed. This means that a greater deal of what is commonly treated as final uses of internal production ought to be determined by the model. Many are the ways in which the standard open static I-O model (A in Fig. 1) can be closed. Usually the investment is the final demand component which is made endogenous by the acceleration principle. But this is nothing but a convention by which the standard static I-O model is transformed into a dynamic one (B in Fig. 1).

It is doubtful that dynamic models are more useful for regional growth analysis than static models. Closed and dynamic models can however be obtained also if we make consumption and/or the other final demand components endogenous (Δ and Γ in Fig. 1) with time lags.

More often the closing of the model is obtained simply defining new coefficients for consumption and investment as is done for the intermediate demands. Only of few cases endogenous consumption, investment and exports with behavioral equations are known. Examples of this approach are the models of Almon (1966, 1974), Morishima (1965) and all the models pooling an input-output system with a macroeconometric model of the demand side of the economy (e.g. Johansen, 1959).

The implementation of such closed models requires an enormous amount of data, which usually is not available at a regional level (see E in Fig. 1).

At this level it is then necessary to implement closed and dynamic I-O models. Both are usually obtained introducing time lags from the demand side* of the model. If the model is closed we are forced furthermore, to give up sophisticated behavioral specifications in favour of fixed coefficients between final demand and production (B_1C in Fig. 1) in order to keep the equations linear.

A crucial point is then how to estimate these final demand coefficients with the limited statistical information which is generally available at the regional level.

Before going further let's then discuss briefly the following classification scheme:

Fig. 1: Input-output models classification

<table>
<tr><td colspan="2" rowspan="2"></td><td rowspan="2">STATIC</td><td colspan="2">DYNAMIC
WITH TIME LAGS ON THE:</td></tr>
<tr><td>DEMAND SIDE</td><td>SUPPLY SIDE</td></tr>
<tr><td colspan="2">COMPLETELY OPEN</td><td>A</td><td></td><td>Z</td></tr>
<tr><td rowspan="3">MORE OR LESS CLOSED</td><td>FIXED COEFFICIENTS FOR INVESTMENT</td><td></td><td>B</td><td></td></tr>
<tr><td>FIXED COEFFICIENTS FOR CONSUMPTION AND OTHER FINAL USERS</td><td>Δ</td><td>Γ</td><td></td></tr>
<tr><td>BEHAVIORAL EQUATIONS</td><td></td><td>E</td><td></td></tr>
</table>

*Dynamic models can however be obtained with time lag from the production side without closing them (Z in Fig. 1).

In the first row we find input-output models which are completely open models because they do not explain the final demand components (A). This is the most common way to implement the input-output analysis of an economic system. These models are in most cases static and they do not allow us to simulate the growth path of that system. But there is no reason to give up this possibility because one can resort to the less used open models with time lags entering from the supply side. This kind of model (see Z in the Fig. 1) indeed allows us to obtain a dynamic description of an economic system without going into the complexities of explaining some of the final demand components.

However if we are prepared to meet these complexities we can move to the models of the second row of the table. This family of models is rather comprehensive because the feedback of production activity to the final activities of the economy may be obtained directly by fixed coefficients against production or indirectly by more complicated behavioral equations. The leading example of the first kind are the dynamic models closed with respect to investment (see B). But one can design static or dynamic closed models with respect to consumption as well (Δ and Γ in Fig. 1).

Finally the letter E refers to those models where the closing involves a more sophisticated feedback of production on final demand. It usually involves the use of price elasticities and of time lags which means that these kinds of model are non-linear dynamic.

Static input-output models (A) are implemented and used not only in almost every country, but even in many regions. As a comparison the most powerful dynamic counterpart (B) is still an unusual experiment in quantitative economics even in the most advanced countries. The reason being the alleged difficulty of implementation of the stock or capital coefficients matrix which is the cornerstone of every dynamic I-O model.

Beside the direct, but difficult, estimation of the stock coefficients one can try to implement a dynamic model with limited statistical information.

This paper deals with the problems of the estimation of capital coefficients of dynamic (closed with respect to investment) input-output models of kind B when the information available is limited.

2. A SIMPLE WAY TO DYNAMIC INPUT-OUTPUT SYSTEMS IMPLEMENTATION

2.1. Problem formulation

The unknowns of our problem are the stock or capital coefficients of the I-O model which are usually indicated as $B = \{b_{ij}\}$. The coefficient b_{ij} being the amount of capital goods produced in the sector i and required by the production of sector j per unit of output <u>increase</u>. It means that b_{ij} is both an average and a marginal stock coefficient. This is a very common assumption, but we will go further into the understanding of its nature.

The first crucial observation is due to Hawkins-Simon (1948) and Lange (1952).* It deals with the relation between the stock coefficients b_{ij} and the flow coefficients a_{ij} or stock and flow inputs. These two must stand in a relation, which can be called the turnover time. This is the number of units of time by which we measure the economic durability of good i as capital for production j. The turnover time not only depends on technological characteristics of the good i and of the production process of sector j, but also on the price system faced by the production sector j.

The A flow coefficient and the B stock coefficient matrices are then related by an exact relation of this type:

$$b_{ij} = t_{ij}a_{ij} \quad . \tag{1}$$

Unfortunately t_{ij} is not easily measurable and then knowing a_{ij} is not sufficient to know b_{ij}.

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*See also Brody (1970) and Lange (1965).

With Brody (1970) we can however make the simplyfying hypothesis that the turnover time is equal alonq rows or that a capital i has uniform turnover time irrespective of the sector where it is actually used; then we have

$$b_{ij} = t_i a_{ij} \quad . \tag{2}$$

A second crucial point about the stock coefficient b_{ij} which is worth stressing is that it is actually the product of two elementary coefficients: an intersector trade coefficient and a capital-output sectoral coefficient.

If x_j is the production level of sector j the total amount of capital needed by this sector k_j is

$$k_j = \beta_j x_j \quad , \tag{3}$$

if β_j is the gross capital-output ratio of sector j.

It is a parameter completely determined by current technology which can be thought of as very stable.

If we now define by k_{ij} the amount of capital supplied by sector i for sector j, then

$$k_{ij} = b_{ij} x_j \quad . \tag{4}$$

We can finally relate k_{ij} with k_j if we define a new coefficient τ_{ij} which is the percentage of the total amount of capital required by sector j bought from sector i:

$$k_{ij} = \tau_{ij} k_j \quad . \tag{5}$$

This is an intersectoral trade coefficient relating to capital goods and determined not only by the technology but also by the price system.

From (3), (4) and (5) follows immediately that:

$$b_{ij} = \tau_{ij} \beta_j \quad . \tag{6}$$

But relations (2) [or (1)] and (6) holds simultaneously. Then we can write:

$$b_{ij} = t_i a_{ij} = \tau_{ij}\beta_j \quad . \tag{7}$$

Relation (7) is important for the practical determination of the stock coefficient b_{ij} we are suggesting.

If we assume the flow coefficients a_{ij} to be known the relation (7) allows us to split every one of them into three elementary factors with which it will be possible to calculate the b_{ij} coefficients.

2.2. Solution

The problem is how to split the a_{ij} coefficients. The solution we suggest requires only limited information on the a_{ij} coefficients and the totals τ_i, for i = 1,n. The τ_i coefficients sum to one:

$$\sum_i \tau_i = 1 \quad ,$$

as do the τ_{ij} coefficients , $\sum_i \tau_{ij}$ = s, because they are percentages.

If we then consider the T matrix and the i unity vector we have

$$T'i = i \quad ,$$

$$Ti = \tau \quad ,$$

and (8)

$$i'\tau = 1 \quad .$$

If we assume the vector sum τ to be known, which is the sectoral percentage distribution of new capital equipment from the supply side, we can use relations (8) to constrain the unknown T matrix.

We can now see how to solve our problem simply rewriting (7) as follows:

$$\tau_{ij} = t_i a_{ij} \frac{1}{\beta_j} \quad , \tag{9}$$

or, in matrix notation:

$$T = \hat{t} \; A \; \hat{\beta}^{-1} \quad , \tag{10}$$

where $\hat{t}$ is the diagonal turnover time matrix and $\hat{\beta}^{-1}$ is the diagonal "output per unit of capital" matrix.

It is worth noting that the general term T_{ij} of this matrix T, as given by (9), can be rewritten as:

$$T_{ij} = \frac{k_{ij}}{x_j} \cdot \frac{x_j}{k_j} = \frac{k_{ij}}{k_j} \quad ,$$

[due to substitution of (4) and (3)] and interpreted as a ratio between two capital/output ratios or what is the same as a ratio between two amounts of capital. [That supplied by sector i to sector j and that globally needed by sector j: which is the percentage of k_j supplied by sector i $(0 \leq \tau_{ij} \leq 1)$].

Substituting then (10) into (8) one gets the following system:

$$\begin{cases} (\hat{t} \; A \; \hat{\beta}^{-1}) i = \tau \\ \\ (\hat{\beta}^{-1} \; A' \hat{t}) i = i \end{cases} \quad . \tag{11}$$

The unknowns are the two vectors t and β^{-1} exactly equal in number to that of the relations.

It comes out that our problem is nothing but a biproportional constraint adjustment of A into T where $\hat{t}$ is the row or "substitution effect" and the $\hat{\beta}^{-1}$ is the column or "fabrication one".

We can then use the well known RAS iterative procedure [see Stone (1963)] to estimate the T matrix starting from the

A flow matrix in the nonlinear system (11). We can afterwards estimate the B matrix in (2) or in (6).

As a conclusion we can say that if an estimate of the flow coefficients and an estimate of the regional percent distribution of the investment in the selling sector are available we can apply a biproportional adjusting technique to relation (10) (e.g. the well known RAS technique) to get the capital trade coefficients as well as the turnover time and the capital-output ratio of each sector. Starting from what can be reasonably considered a minimal amount of information we can make a very useful decomposition of a_{ij} into three parameters with a clear economic meaning, and get an estimate of stock B coefficients. This gives us an operational advantage because one can, in the following, simulate the effects on these stock coefficients and consequently on investment and production of a change in the economic life span of the stock of capital, or of a change in technology (via the A matrix and/or the β vector). This means that the RAS of system (11) must be rerun with some of the elements of A, β and t held fixed.

3. THE ESTIMATION OF CAPITAL COEFFICIENTS WHEN MORE INFORMATION IS AVAILABLE

If we cannot make a survey on the investment of every sector of the region in order to estimate directly the b_{ij} coefficients we must use some short cut method to estimate the investment or capital coefficients of systems closed with respect to investment. The common starting point of all the solutions proposed in this paper is the relation called turnover time set out as (2). This means that the use of every one of these implies the availability of the matrix of flow coefficients a_{ij}.

We assume however the availability of some other kind of statistical information. Depending on the amount of it, and on the approach used, several possibilities are open. If the supplementary information takes the form of the percent distribution of investment in the selling sector, we can follow the solution proposed in the preceding pages, but if it takes the form of a set of constraints on the stock coefficient b_{ij}, rather than on the capital trade coefficients τ_{ij}, we can use the solutions of sections 3 and 5 below.

It is worth paying a little attention to the balance between the number of the unknowns, which are the $(n \cdot n)$ coefficients b_{ij}, and the number of degrees of freedom will determine the nature of the solution adopted. In the solution just proposed we have $n(n-2)$ degrees of freedom which forced us to use a stochastic approach (the RAS technique) which implies--as is well known--a particular distance criterion.* In the present chapter (paragraph 3.2) however we will meet a situation in which these will be a kind of overdetermination. To handle this situation a stochastic approach is still needed. In the following paragraph we give a concise description of the deterministic approach that can be used when the available statistical information fits into an exactly determined mathematical system.

3.1. The estimation of capital coefficients with a deterministic approach

This approach has been proposed by Batten (Andersson-Batten, 1979) as a solution for the case faced when besides the flow coefficients a_{ij}, the vector of productions $\bar{x}_i(t)$ as well as their growth paths $\bar{\lambda}_i(t)$ are supposed to be known. An n^2 equations system can then be defined and solved to calculate the n^2 unknowns stock coefficients b_{ij}:

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*Another implication of a number of degrees of freedom can be easily seen if we confront the resulting τ_{ij} with the corresponding a_{ij}. Both are positive and at most equal to unity, but $\tau_{ij} \gtrless a_{ij}$. If it comes out that $\tau_{ij} > a_{ij}$ it means that the trade between sector i and sector j is oriented to capital goods. This may be due to a particularly high turnover time of capital goods produced by the sector i or by the particularly high productivity of overall capital in sector j or by a particularly high productivity of capital i in sector j associated to a particular high turnover time of capital good i in production j. It is apparent in (10) however that this third component is set to zero in our approach.

$$\begin{cases} b_{ij}/b_{ik} = \bar{a}_{ij}/\bar{a}_{ik} \quad , \quad i \text{ and } k = 1,n \quad , \\ \\ \sum_{j=1}^{n} b_{ij}\bar{\lambda}(t)\bar{x}_j(t) = \bar{x}_i(t) - \sum_{j=1}^{n} \bar{a}_{ij}\bar{x}_j(t) \quad , \quad i = 1,n \ . \end{cases} \tag{12}$$

The first relation follows from (2) and gives n(n-1) equations. The second, which can be rewritten as:

$$\bar{x}_i(t) = \sum_{j=1}^{n} \bar{a}_{ij}\bar{x}_j(t) + \sum_{j=1}^{n} b_{ij}\bar{\lambda}_j(t)x_j(t) \quad , \tag{13}$$

is a set of n supplementary constraints for the n^2 unknowns we want to estimate.

Here the capital input-output coefficients are fixed: the only time-dependent variables being the observed growth rate of production in the different sectors and of course, the production level. A kind of simulation analysis we can do is then that of b_{ij} with respect to variations of a_{ij}.

3.2. The stochastic approach to the estimation of the capital coefficients

The available statistical information at a regional level may frequently be more abundant than that utilized in the procedure expanded in the previous paragraph too. The deterministic approach involved in the solution of the system (12) would then underutilize all the available information. It is then useful to find a method apt to handle that case which is--typically--overdetermined because the number of the unknowns falls short of that of the constraints imposed upon them.

If this is the case an exact solution doesn't exist and we have only a feasible set in which a solution must be nonetheless found according to an optimality criterion.

Before entering into this aspect let us examine the new supplementary information with which we have to deal.

If we define the capital stock of kind k--inventories included--demanded by the economic system as a whole in period t as $K_k(t)$ it must then be:

$$\sum_j b_{kj}\bar{x}_j(t) = K_k(t) \leq K_k^*(t) \quad , \tag{14}$$

with $K \in I$, I being the set of fixed capital goods producing sectors. $K_k(t)$ is the demand for the fixed capital k, not necessarily the capacity $K_k^*(t)$, unless the capital stock is fully employed.

As in (3), $K_j(t)$ is the stock of capital needed by the sector j in the economy. Relation (3) can then be written in the form of a constraint on the column sum of the matrix B of stock or capital coefficients:

$$\sum_i b_{ij} \leq K_j(t)/\bar{x}_j(t) \quad . \tag{15}$$

The matrix B can--in other words--be constrained from the supply side of the economy with a maximum of capacity imposed (14) on the totals of its rows. But because the capacity actually used may fall short of the maximum $K_k^*(t)$ this kind of constraint can be thought of as inactive during many economic phases.

The same matrix B can however also be constrained from the input side with a--supposedly known--average capital/output required ratio. This ratio is used as a necessary condition for the sum of unknown elementary capital-output ratios of every sector j (15).

Up to now we have assumed to have information on the flow coefficients $\bar{a}_{ij}$, on the output $\bar{x}_j(t)$ and on its growth rate $\lambda_j(t)$, and on the capital stock $\bar{K}_j(t)$ required by every sector j.

But we should have realized that there is a difference between these pieces of information. The fundamental relation set up in the first row of system (12) [and in the (2)] gives us after all a rough guess on the proportions between the stock coefficients b_{ij} if we make the constancy assumption of the turnover time as depicted in the first relation of system (2).

This is only an assumption, however, which allows us to use the existing proportions between flow coefficients as starting estimates for the proportions between the n(n-1) stock coefficients and the remaining n taken as numeraire while all the other pieces of information are used to build up constraints on some linear functions of the capital input-output coefficients.

4. A DIGRESSION ON INFORMATION AND ENTROPY

If we conceive a I-O matrix $x = \{x_{ij}\}$ as a spatial system which can assume different states according to the way in which it is arranged, we can distinguish [see: Snickars-Weibull (1977) page 138] a macrostate from a microstate.* The former deals with counting while the latter deals with labeling of the $N = \sum_i \sum_j x_{ij}$ "objects" in x's cells; a basic assumption being that, however defined,** x_{ij} is integer--the matrix x is itself a macrostate--one of the many that the system can assume. Associated with every macrostate there are a number of microstates, each of them is a vector of N elements, $m = \{m_k\}$ with $k = 1,N$, where m_k is the number of the cell of x in which the object k is placed.

We want to determine x_{ij} for $\forall_{i,j}$ subject to some prior information which we use to constrain the x_{ij} unknowns. We then make two assumptions: the first is that every microstate compatible with the constraints has uniform probability while the others have zero probability. The second one is that the probability of every macrostate (obviously compatible with the constraints) is simply proportional to the number of microstates subsumed by that macrostate. This number is defined by the following combinatorial formula:

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*See also: Willekens, Por, Raquillet (1979).

**If x_{ij} is an input-output coefficients it must be rounded and multiplied by 10^{ε}, ε being the number of its decimals. This assumption is not actually necessary because x_{ij} can be thought as probability (when $\sum_i \sum_j x_{ij} = 1$), but it is useful to understand the concept of state and to give meaning to the expression $x_{ij}!$ used in the following.

$$W(x) = \frac{x!}{\Pi x_{ij}!} \quad . \tag{16}$$

If we want to choose the most probable macrostate of the system--which in a sense is an optimal use of our prior information--then we must maximize W(x) subject to the constraints on x_{ij}. The solution of this programme does not necessarily give us the actual state of the system (the actual input-output table in our case), but simply the most probable according to available prior information.

Maximizing W(x) we maximize the entropy (or the "disorder" of the system). Usually (16) is manipulated to obtain a more tractable expression for the entropy. As W(x) is invariant with respect to logorithmic transformation, and as $\ln x_{ij}!$ may be substituted by its Stirling's approximation $\ln x_{ij}! = x_{ij}\ln x_{ij} - x_{ij}$ we can write:

$$W(x) \simeq \ln x..! - \sum_i \sum_j (x_{ij}\ln x_{ij} - x_{ij}) \quad , \tag{17}$$

and then

$$W(x) \simeq \ln x..! + x.. - \sum_i \sum_j (x_{ij}\ln x_{ij}) \quad . \tag{18}$$

Because we have to maximize W(x) with respect to x_{ij} we can exclude the constant terms and write our program as a minimization constrained to find the x matrix:

$$\text{Min} \sum_i \sum_j x_{ij}\ln x_{ij} \quad . \tag{19}$$

Its solution has the maximum of the entropy which means that we have chosen the most equidistributed x_{ij} among those compatible with the economically derived constraints.

It follows that every successive message or actual measurement of x will give us a higher "information gain" with respect to our solution. If we indicate the new message with x^*_{ij} this information gain of x^*_{ij} over x_{ij} is log x_{ij}/x^*_{ij} or-log x^*_{ij}/x_{ij}. Thus the expected total information gain is:

$$\sum_i \sum_j x^*_{ij} \ln x_{ij}/x^*_{ij}$$

or (20)

$$- \sum_i \sum_j x^*_{ij} \ln x^*_{ij}/x_{ij} \quad .$$

It is interesting to note that it will automatically be greatest if x_{ij} is estimated by (19).

Now let us change perspective and imagine that we want to obtain x^*_{ij} as a new constrained estimate of x_{ij} starting from a prior information on x_{ij} indicated with $\bar{x}_{ij}$. x^*_{ij} is then the posterior distribution we want to obtain from the prior estimate $\bar{x}_{ij}$ which is different from that obtainable from relation (19).

The criterion will be the maximization of expected total information gain obtained with our estimation procedure, that is the maximization of the expected information gain of x^*_{ij} with respect to $\bar{x}_{ij}$ which means that we:

$$\text{Max} - \sum_i \sum_j x^*_{ij} \ln x^*_{ij}/\bar{x}_{ij} \quad , \tag{21}$$

But this can be rewritten as:

$$\text{Min} \sum_i \sum_j x^*_{ij} \ln x^*_{ij}/\bar{x}_{ij} \quad , \tag{22}$$

which means that we minimize a kind of distance† from x^*_{ij} to $\bar{x}_{ij}$. But this is exactly what we are looking for because we trust $\bar{x}_{ij}$ to be a good approximation of the effective, but un-

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†Other distance criteria can be used; see e.g.: Bacharach (1970).

known, distribution of the x matrix. In other words we are looking for an estimate x^*_{ij} compatible with the constraints such that x^*_{ij} is as close as possible to $\bar{x}_{ij}$--the prior information on the actual distribution--and not to the equidistribution given by (19) (which involves the maximal entropy of the system). As a conclusion we can say that (19) and (20) give the same result which is the shortest distance from the prior information: $\bar{x}_{ij}$ in the first case, and 1 in the second case. In the second case actually we do not have prior information at all and the best guess we can make is equidistribution or uniformity in $\bar{x}$ or $x_{ij} = 1$.

5. THE ESTIMATION OF CAPITAL COEFFICIENTS AS A CONSTRAINED MINIMUM-INFORMATION PROBLEM WITH CONSTRAINTS

If we consider the flow proportions, as in the first relation of system (12), as the prior most probable proportions for the stock coefficients as well, our estimation criterion will be such that our ex post or constrained stock coefficients $\hat{b}_{ij}$ estimates show proportions which are maximally close to the starting proportions.

In other words, we do not want to be surprised by the estimated $\hat{b}_{ij}/\hat{b}_{ik}$ because we trust a_{ij}/a_{ik} to be close to the real b_{ij}/b_{ik}, subject to the constraint on $\hat{b}_{ij}$.

This is an application of the criterion (22) where

$$x^*_{ij} = \hat{b}_{ij}/\hat{b}_{ik} \quad , \tag{23}$$

and

$$\bar{x}_{ij} = \bar{a}_{ij}/\bar{a}_{ik} \quad .$$

We must then find $\hat{b}_{ij}$ such that

$$\operatorname{Min} \sum_{i,j} \left(\frac{\hat{b}_{ij}}{\hat{b}_{ik}}\right) \ln \left(\frac{\hat{b}_{ij}}{\hat{b}_{ik}}\right) \Big/ \left(\frac{\bar{a}_{ij}}{\bar{a}_{ik}}\right) \quad , \tag{24}$$

subject to some constraints on the coefficients.

The task now is to add as many constraints as possible from economic analysis to reduce as much as possible the number of states within which we choose those closest to the prior distribution of the capital coefficients.

Up to now (see section 3) we have program (24) subject to (13) and (15). Other constraints are possible. If we, for example, put the relation (14) in a dynamic form we obtain a constraint of the investment flow of period t for every sector i:

$$\sum_j b_{ij}\bar{\lambda}_j(t)\bar{x}_j(t) \leq c_i(t) \quad . \tag{25}$$

We then change a stock balance into a flow balance where the new demanded capital i cannot exceed the capacity of sector i, $c_i(t)$. What is capacity in this context? It is the production not used for consumption at time t if there are no imports or if these are not possible. In this case (25) coincides with relation (13).

If this is the case we can solve (24) subject to (13) or (25) and (15) with respect to b_{ij} which must be $b_{ij} \geq 0$. We then have the following nonlinear program:

$$\begin{cases} \text{MIN} \sum_{ij} \left(\dfrac{b_{ij}}{b_{ik}}\right) \ln \left(\dfrac{b_{ij}}{b_{ik}}\right) \Big/ \left(\dfrac{a_{ij}}{a_{ik}}\right) \quad , & \forall_{ij} \\ \text{s.t.} & \\ x_i(t) - \sum_j a_{ij}x_j(t) - \sum_j b_{ij}\lambda_j(t)x_j(t) = 0 \quad , & \forall_i \\ \sum_i b_{ij} - K_j(t)/x_j(t) = 0 \quad , & \forall_j \\ b_{ij} \geq 0 \quad . & \forall_{ij} \end{cases} \tag{26}$$

A straightforward approach to the solution of this non-linear program would be to use the method of Lagrangan multipliers. Forming the Lagrange function and then differentiating it with respect to the b_{ij} coefficients one gets:

$$\begin{cases} -\sum_{l=1}^{n} \frac{b_{il}}{b_{ik}^2}\left(\ln\frac{b_{il}}{b_{ik}} / \frac{a_{il}}{a_{ik}} + 1\right) - \zeta_i\lambda_j(t)x_j(t) + \theta_j & \forall_i \quad , \quad j = k \\ \frac{1}{b_{ik}}\left(\ln\frac{b_{ij}}{b_{ik}} / \frac{a_{ij}}{a_{ik}} + 1\right) - \zeta_i x_j(t)x_j(t) + \theta_j & \forall_i, \forall_j \quad , \quad j \neq k \end{cases} \tag{27}$$

where ζ_i and θ_j are the multipliers associated with the constraints. Differentiating also with respect to those a non-linear system of $n^2 + 2n$ unknowns and equations is obtained. However, it brings us computational problems rather more difficult than those usually encountered in problems of kind (22).

Our problem is complicated indeed by its dimension because n is the number of sectors in the I-O table.

The solution via direct approach is then not viable. We must resort to a solution algorithm which is that recently developed by Murtagh-Saunders (1978). It allows us to solve large-scale non-linear programs with linear constraints using a reduced gradient approach. It has already proved to be very efficient in problems very similar to ours.

6. THE ESTIMATION OF CAPITAL COEFFICIENTS AS A CONSTRAINED MINIMUM-INFORMATION PROBLEM WITH CONSTRAINTS UPON THE CAPITAL-TRADE COEFFICIENTS.

In this section we return to the assumptions of section 2. We consider the flow coefficients as the prior most probable estimates for the capital-trade coefficients τ_{ij}. We can then apply the criterion given by (22) to obtain an estimate of τ_{ij}

maximally closed to a_{ij}, but belonging to the feasible area defined by the constraints on the row and column sums of matrix τ.

We can then use (22) assuming that

$$x^*_{ij} = \tau_{ij} \quad ,$$

and (28)

$$\bar{x}_{ij} = a_{ij} \quad ,$$

with

$$\tau_{ij} \text{ subject to (8)} \quad .$$

Thus we obtain the following program:

$$\begin{cases} \operatorname{Min} \sum_i \sum_j \tau_{ij} \ln \tau_{ij}/a_{ij} \quad , \qquad \text{s.t.} \\ \sum_i \tau_{ij} = 1 \quad , \\ \sum_j \tau_{ij} = \tau \quad . \end{cases} \tag{29}$$

It is straightforward to form its Lagrangean L and to differentiate with respect to the coefficients and to the multipliers associated with the constraints.

$$L = \sum_i \sum_j \tau_{ij} \ln \tau_{ij}/a_{ij} - \sum_j \eta_i (\sum_j \tau_{ij} - 1) - \sum_i \theta_i (\sum_j \tau_{ij} - \tau) \quad ,$$

$$\frac{\partial L}{\partial \tau_{ij}} = \frac{\tau_{ij}}{a_{ij}} - e^{\eta_j + \theta i} = 0 \quad , \qquad \forall_{ij}$$

$$\frac{\partial L}{\partial \eta_j} = \sum_i \tau_{ij} - 1 = 0 \quad , \qquad \forall_j \tag{30}$$

$$\frac{\partial L}{\partial \theta_i} = \sum_j \tau_{ij} - \tau = 0 \quad , \qquad \forall_i \quad .$$

It is now useful to write this system in matrix form:

$$\begin{cases} \tau = \hat{\theta}\, A\, \hat{\eta} \ , \\ \tau' i = 1 \ , \\ \tau i = \tau \ . \end{cases}$$

and then

$$\begin{cases} (\hat{\theta}\, A\, \hat{\eta}) i = \tau \ , \\ (\hat{\eta}\, A' \hat{\theta}) i = i \ . \end{cases} \tag{31}$$

It comes out that this is nothing but the system (11) obtained when we solved with the RAS technique in section 2.

We conclude then that the RAS estimation of capital-trade coefficients implies the use of a constrained minimum-information principle with the RAS multipliers $\hat{t}$ and $\hat{\beta}^{-1}$ being, at the same time, the Lagrange parameters $\hat{\theta}$ and $\hat{\eta}$ and meaningful economic parameters.

As a conclusion we can stress that the interpretation of the RAS multipliers as Lagrange parameters allows us to make a sensitivity analysis. These parameters indeed show how sensitive the optimal value of the objective function (the "surprise" obtained with the estimate of τ_{ij}) is to changes in the constraints (the information given by vector τ which has probably to be estimated on a priori grounds).

Changing this vector constraint τ we change the T matrix and the "shadow" turnover-time and output-per-unity-of-capital vectors. This is an alternative way of simulating the B matrix with respect to those sketched in section 2, where the "shadow" vectors have been supposed to be--in some of their elements--fixed on a priori grounds.

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