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PREFACE

The ever increasing complexity of the systems to be modeled and analyzed, taxes the existing mathematical and numerical techniques far beyond our present day capabilities. By their intrinsic nature, some problems are so difficult to solve that at best we may hope to find a solution to an approximation of the original problem. Stochastic optimization problems, except in a few special cases, are typical examples of this class.

This however raises the question of what is a valid "approximate" to the original problem. The design of the approximation must be such that (i) the solution to the approximate provides approximate solutions to the original problem and (ii) a refinement of the approximation yields a better approximate solution. The classical techniques for approximating functions are of little use in this setting. In fact very simple examples show that classical approximation techniques dramatically fail in meeting the objectives laid out above.

What is needed, at least at a theoretical level, is to design the approximates to the original problem in such a way that they satisfy an epi-convergence criterion. The convergence of the functions defining the problem is to be replaced by the convergence of the sets defined by these functions. That type of convergence has many properties but for our purpose the main one is that it implies the convergence of the (optimal) solutions.

This article is devoted to the relationship between the epi-convergence and the classical notion of pointwise-convergence. A strong semicontinuity condition is introduced and it is shown to be the link between these two types of convergences. It provides a number of useful criteria which can be used in the design of approximates to difficult problems.

CONVERGENCE OF FUNCTIONS: EQUI-SEMICONTINUITY

Given a space X, by $\overline{R}^X$ we denote the space of all functions defined on X and with values in $\overline{R}$, the extended reals. We are interested in the relationship between various notions of convergence in $\overline{R}^X$, in particular between pointwise convergence and that induced by the convergence of the epigraphs. We extend and refine the results of De Giorgi and Franzoni (1975) (collection of "equi-Lipschitzian" functions with respect to pseudonorms) and of Salinetti and Wets (1977) (sequences of convex functions on a reflexive Banach space). The range of applicability of the results is substantially enlarged, in particular the removal of the convexity, reflexivity (Salinetti and Wets 1977) and norm dependence (De Giorgi and Franzoni 1975) assumptions is significant in many applications. The work in this area was motivated by: the search for "valid" approximations to extremal statistical problems, variational inequalities and difficult optimization problems, cf., the above mentioned articles. Also by relying only on minimal properties for the topology of the domain space and for the class of functions involved, the derivation itself takes on an elementary and insightful character.

By their nature the results are asymmetric; semicontinuity is a one-sided concept. We have chosen to deal with lower semicontinuity and epigraphs rather than upper semicontinuity and hypographs. Every assertion in one setting has its obvious counterpart in the other. This choice however, does condition the addition rule for the extended reals, viz. $(+\infty) + a = +\infty$ for all $a \in \overline{R}$ and $(-\infty) + a = -\infty$ for all $a \in [-\infty, +\infty[$. Also, note that we are working with the extended reals, thus every collection of elements of $\overline{R}$ has lower and upper bounds in $\overline{R}$; all limits involving extended-real numbers must be interpreted in that sense.

I LIMIT FUNCTIONS

Let (X,τ) be a topological space and f a generic element of $\overline{R}^X$. The *effective domain* of f is

$$\text{dom } f = \{x \in X \mid f(x) < +\infty\}$$

and its *epigraph* is

$$\text{epi } f = \{(x,\eta) \in X \times R \mid f(x) \leq \eta\} \quad .$$

The function f is τ-*lower semicontinuous* (τ-*l.sc.*) if epi f is a closed subset of $X \times R$, or equivalently if

(d_0) to each $x \in \text{dom } f$ and to each $\varepsilon > 0$, there corresponds a τ-neighborhood V of x such that

$$\inf_{y \in V} f(y) \geq f(x) - \varepsilon \quad ;$$

($\sim d_0$) to each $x \notin \mathrm{dom}\, f$ and to each $a \in R$, there corresponds a τ-neighborhood V of x such that

$$\inf_{y \in V} f(y) \geq a \quad .$$

Note that if $\sigma \supset \tau$, i.e., σ is finer than τ, then f τ-l.sc. implies f σ-l.sc. .

To define limits of collection of functions, i.e., elements of $\bar{R}^X$, we adopt the following framework: N is an index space and H is a filter on N. (If τ has a local countable base at each point, it would be sufficient to consider limits in terms of sequences, unfortunately many interesting functional spaces do not have this property.) The e_τ-*limit inferior* of a filtered collection of functions $\{f_\nu, \nu \in N\}$ is denoted by $li_\tau f_\nu$, and is defined by

(1.1) $$(li_\tau f_\nu)(x) = \sup_{G \in G_\tau(x)} \sup_{H \in H} \inf_{\nu \in H} \inf_{y \in G} f_\nu(y)$$

where $G_\tau(x)$ is the family of (open) τ-neighborhoods of x. The e_τ-*limit superior* is denoted by $ls_\tau f_\nu$, and is defined similarly,

(1.2) $$(ls_\tau f_\nu)(x) = \sup_{G \in G_\tau(x)} \inf_{H \in H} \sup_{\nu \in H} \inf_{y \in G} f_\nu(y).$$

In the literature on Γ-convergence, these two functions are known respectively as the $\Gamma^-(\tau)$-limit inferior and the $\Gamma^-(\tau)$-limit superior, cf. De Giorgi and Franzoni (1975)[1]. By $\ddot{H}$ we denote the *grill* associated with the filter H, i.e. the family of subsets of N that meet every set H in H. Given any collection $\{a_\nu \in \bar{R}, \nu \in N\}$, it is easy to verify the identity

(1.3) $$\sup_{H' \in \ddot{H}} \inf_{\nu \in H'} a_\nu = \inf_{H \in H} \sup_{\nu \in H} a_\nu$$

if we observe that H is the "grill" of $\ddot{H}$, i.e. the collection of all subsets of N that meet every set in $\ddot{H}$. From this it follows that

(1.4) $$(ls_\tau f_\nu)(x) = \sup_{G\in G_\tau(x)} \sup_{H\in \ddot{H}} \inf_{\nu\in H} \inf_{y\in G} f_\nu(y) \quad .$$

Since $H \subset \ddot{H}$ it follows directly that

(1.5) $$li_\tau f_\nu \le ls_\tau f_\nu \quad .$$

The collection $\{f_\nu, \nu\in N\}$ admits an e_τ-limit, denoted by $lm_\tau f_\nu$, if

$$ls_\tau f_\nu = li_\tau f_\nu = lm_\tau f_\nu \quad ,$$

in which case the f_ν are said to *epi-converge* to $lm_\tau f_\nu$. This terminology is justified by the fact that epi $lm_\tau f_\nu$ is the limit of the epigraphs of the f_ν; this is made explicit here below.

The *limit inferior Li* C_ν and *limit superior Ls* C_ν of a filtered collection $\{C_\nu, \nu \in N\}$ of subsets of a topological space are defined by

(1.6) $$Li\ C_\nu = \cap_{H\in \ddot{H}} cl(\cup_{\nu\in H} C_\nu)$$

and

(1.7) $$Ls\ C_\nu = \cap_{H\in H} cl(\cup_{\nu\in H} C_\nu)$$

Since $H \subset \ddot{H}$ and thus we always have that

$$Li\ C_\nu \subset Ls\ C_\nu \, .$$

The filtered collection $\{C_\nu, \nu \in N\}$ is said to have a *limit, Lm* C_ν, if the limits inferior and superior coincide, i.e.,

(1.8) $$Ls\ C_\nu = Lm\ C_\nu = Li\ C_\nu$$

All these limit sets are closed as follows directly from their definitions.

Proposition 1.9. (Mosco 1969) *Suppose that* $\{f_\nu, \nu \in N\} \subset \bar{R}^X$ *is a filtered collection of functions Then*

$$(1.10) \qquad \text{epi } li_\tau f_\nu = Ls \text{ epi } f_\nu$$

and

$$(1.11) \qquad \text{epi } ls_\tau f_\nu = Li \text{ epi } f_\nu$$

Proof. We first derive (1.10). From the definition (1.7) of Ls epi f_ν, it follows that $(x,\alpha) \in Ls$ epi f_ν if and only if

$$(x,\alpha) \in \text{cl}(\cup_{\nu\in H} \text{epi } f_\nu) \qquad \text{for all } H \in \mathcal{H},$$

or equivalently--because the sets involved are epigraphs--if and only if for all $\varepsilon > 0$ and $G \in \mathcal{G}_\tau(x)$ such that

$$G \times]\alpha - \varepsilon, +\infty[\cap (\cup_{\nu\in H} \text{epi } f_\nu) \neq \emptyset \qquad \text{for all } H \in \mathcal{H}$$

or still, if and only if for to every $H \in \mathcal{H}$, $\varepsilon > 0$ and $G \in \mathcal{G}_\tau(x)$ there correspond $\nu \in H$ and $y \in G$ such that

$$f_\nu(y) \geq \alpha - \varepsilon \quad .$$

This holds, if and only if

$$\alpha \geq \sup_{G\in\mathcal{G}_\tau(x)} \sup_{H\in\mathcal{H}} \inf_{\nu\in H} \inf_{y\in G} f_\nu(y) \quad ,$$

and, as follows from (1.1), if and only if $\alpha \geq (li_\tau f_\nu)(x)$ or equivalently, if and only if $(x,\alpha) \in$ epi $li_\tau f_\nu$.

In view of (1.4), the proof of (1.11) follows from exactly the same argument with the grill $\ddot{H}$ replacing H. □

Corollary 1.12 *Given any filtered collection of functions* $\{f_\nu, \nu \in N\} \subset \bar{R}^X$, *the functions* $li_\tau f_\nu$, $ls_\tau f_\nu$, *and* $lm_\tau f_\nu$ *if it exists, are* τ*-lower semicontinuous.*

Proof. The lower semicontinuity follows directly from (1.10) and (1.11) since they imply that the epigraphs are closed. □

We shall be interested in the implications of a change in topology for X. In particular, we have the following:

Proposition 1.13. *Suppose that* σ *and* τ *are two topologies defined on* X *such that* $\sigma \supset \tau$. *Then*

(1.14) $$li_\tau f_\nu \leq li_\sigma f_\nu \quad ,$$

and

(1.15) $$ls_\tau f_\nu \leq ls_\sigma f_\nu \quad .$$

Proof. This follows from the definitions (1.1) and (1.2) and the fact that $\sigma \supset \tau$ implies that $G_\sigma(x) \supset G_\tau(x)$. □

In some applications, in particular those involving variational inequalities, it is useful to use a stronger notion of limit function. Again, let σ and τ be two topologies defined on X, the $e_{\tau,\sigma}$-*limit* of a collection of functions $\{f_\nu, \nu \in N\} \subset \bar{R}^X$, denoted by $lm_{\tau,\sigma} f_\nu$, exists if

$$\text{(1.16)} \qquad li_\tau f_\nu = lm_{\tau,\sigma} f_\nu = ls_\sigma f_\nu \ .$$

The case of interest is $\sigma \supset \tau$, this models the situation when X is a normed linear (functional) space, and σ and τ are respectively the strong and weak topologies; in this setting this limit function is called the *Mosco limit*, cf. Mosco (1969) and Attouch (1979), for example.

Proposition 1.17. *Suppose that σ and τ are two topologies defined on X such that $\sigma \supset \tau$. Moreover suppose that $lm_{\tau,\sigma} f_\nu$ exits. Then*

$$lm_\sigma f_\nu = lm_{\tau,\sigma} f_\nu = lm_\tau f_\nu$$

Proof. This follows directly from Proposition (1.13), inequality (1.5) and the definition (1.16) of $lm_{\tau,\sigma} f_\nu$. □

II τ/σ-EQUI-SEMICONTINUITY

As already indicated in Section I, we are interested in exploring the relationship between the limit functions of a collection of functions $\{f_\nu, \nu \in N\} \subset \bar{R}^X$, when X is equipped with different topologies, say σ and τ. The question of the equality between lm_τ and lm_σ was already raised in connection with the existence of the Mosco limit $lm_{\tau,\sigma}$. Recall also that for variational problems epi-convergence essentially implies the convergence of the solutions, it is thus useful to have conditions that allow us to pass from epi-convergence in a given topology to epi-convergence in a finer topology because of the stronger continuity properties of the solution of the limit problem, consult Attouch (1979), Theorem 2.1, for example. Finally, a special and extreme case is when

$\sigma = \iota$, the discrete topology. The study of the connections between lm_τ and lm_ι becomes that of the relationship between epi-convergence and pointwise-convergence. This is particularly useful in the design of approximation schemes for optimization problems. We deal with this special case of pointwise-convergence at the end of this section.

The inequalities (1.14) and (1.15), relating the e_τ-limits inferior and superior, become equalities if the family of functions $\{f_\nu, \nu \in N\}$ is τ/σ-equi-lower semicontinuous. This property, defined below, is not only sufficient (Theorem 2.3) but is also necessary (Theorem 2.10). It constitutes in fact a sort of compactness condition, this is clarified in Section IV.

Definition 2.1. A filtered collection of functions $\{f_\nu, \nu \in N\} \subset \bar{R}^X$ is τ/σ-*equi-lower semicontinuous* (τ/σ-*equi-l.sc.*) if there exists a set $D \subset X$ such that

(d) given any $x \in D$, to every $\varepsilon > 0$ and every $W \in G_\sigma(x)$ there correspond $H \in \mathcal{H}$ and $V \in G_\tau(x)$ such that for all $\nu \in H$

$$\inf_{y \in V} f_\nu(y) \geq \inf_{y \in W} f_\nu(y) - \varepsilon \quad ,$$

and

(~d) given any $x \notin D$, to every $a \in R$ there correspond $H \in \mathcal{H}$ and $V \in G_\tau(x)$ such that for all $\nu \in H$

$$\inf_{y \in V} f_\nu(y) \geq a \quad .$$

We call D the *reference set*. If $\sigma \subset \tau$, then (d) holds with $V = W$ and H arbitrary, and hence any collection is τ/σ-equi-l.sc. with $D = X$. In applications, as far as we can tell, the only case of genuine interest is when σ is finer than τ; however, the results are derived for arbitrary topologies [2].

Proposition 2.2. *Suppose that $\sigma_2 \supset \sigma_1$ and $\tau_2 \subset \tau_1$. Then for any collection of functions, τ_2/σ_2-equi-lower semi-continuity implies τ_1/σ_1-equi-lower semicontinuity.*

Proof. Follows simply from the definition (2.1) and the inclusions $G_{\sigma_2}(x) \supset G_{\sigma_1}(x)$ and $G_{\tau_2}(x) \subset G_{\tau_1}(x)$. □

Theorem 2.3. *Suppose that the filtered collection of functions $\{f_\nu, \nu \in N\} \subset \bar{R}^X$ is τ/σ-equi-l.sc. . Then*

(2.4) $$\mathrm{li}_\sigma f_\nu \le \mathrm{li}_\tau f_\nu$$

and

(2.5) $$\mathrm{ls}_\sigma f_\nu \le \mathrm{ls}_\tau f_\nu \quad .$$

Proof. We start with the proof of (2.4). Given $x \in D$ and $\varepsilon > 0$, it follows from the definition of $\mathrm{li}_\sigma f_\nu$ that there exists $G_\varepsilon \in G_\sigma(x)$ and $H_\varepsilon \in H$ such that for all $\nu \in H_\varepsilon$

$$(\mathrm{li}_\sigma f_\nu)(x) \le [\inf_{y \in G_\varepsilon} f_\nu(y)] + \varepsilon \quad .$$

In turn, (d) guarantees the existence of $V \in G_\tau(x)$ and $H' \in \mathcal{H}$ such all $\nu \in H'$

$$\inf_{y\in G_\varepsilon} f_\nu(y) \leq \inf_{y\in V} f_\nu(y) + \varepsilon ,$$

and hence for all $\nu \in H' \cap H_\varepsilon (\in \mathcal{H})$ we have that

$$(li_\sigma f_\nu)(x) \leq \inf_{\nu\in H'\cap H_\varepsilon} \inf_{y\in V} f_\nu(y) + 2\varepsilon \quad .$$

This yields

$$(li_\sigma f_\nu)(x) \leq \sup_{V\in G_\tau(x)} \sup_{H\in \mathcal{H}} \inf_{\nu\in H} \inf_{y\in V} f_\nu(y) + 2\varepsilon$$

$$= (li_\tau f_\nu)(x) + 2\varepsilon \quad .$$

Since this holds for every $\varepsilon > 0$, we have that $li_\sigma f_\nu \leq li_\tau f_\nu$ on D.

If $x \notin D$, condition (~d) implies that for every $a \in R$, there exists $V_a \in G_\tau(x)$ and $H_a \in \mathcal{H}$ such that

$$(li_\sigma f_\nu)(x) \geq \inf_{\nu\in H_a} \inf_{y\in V_a} f_\nu(y) \geq a \quad .$$

Hence $(li_\tau f_\nu)(x) = +\infty$ for every x in $X \setminus D$ and the inequality $li_\sigma f_\nu \leq li_\tau f_\nu$ is trivially satisfied.

In view of (1.4), the same argument can be used to derive (2.5) replacing simply li by ls and $\mathcal{H}$ by $\ddot{\mathcal{H}}$. □

Corollary 2.6. *Suppose that the filtered collection of functions* $\{f_\nu, \nu \in N\} \subset \bar{R}^X$ *is* τ/σ*-equi-l.sc. . Then*

$$(2.7) \qquad li_\sigma f_\nu = li_\tau f_\nu$$

and

$$(2.8) \qquad ls_\sigma f_\nu = ls_\sigma f_\nu \quad .$$

Moreover dom $li_\sigma f_\nu$ = dom $li_\tau f_\nu$ *is the smallest of all subsets* D *of* X *with respect to which both* (d) *and* (~d) *hold for the collection* $\{f_\nu, \nu \in N\}$, *i.e.,* dom $li_\sigma f_\nu$ *is the smallest possible reference set.*

Proof. The equalities follow directly from Theorem 2.5 and the Proposition (1.13). To obtain the last assertion, we note that if $C \subset D$, $li_\sigma f_\nu = +\infty$ on $D \backslash C$ and the collection $\{f_\nu, \nu \in N\}$ is τ/σ-equi-l.sc. with respect to D, it is also τ/σ-equi-l.sc. with respect to C. Clearly dom $li_\sigma f_\nu$ is the smallest such set C since for any strictly smaller set $C' \subset$ dom $li_\sigma f_\nu$, (~d) will fail on (dom $li_\sigma f_\nu$)$\backslash C'$. □

Corollary 2.9. (Convergence Theorem). *Suppose that* $\sigma \supset \tau$ *and that the filtered collection of functions* $\{f_\nu, \nu \in N\}$ *is* τ/σ*-equi-l.sc. then*

$$f = \mathrm{lm}_\tau f_\nu$$

if and only if

$$f = \mathrm{lm}_\sigma f_\nu \quad .$$

Proof. From $f = \mathrm{lm}_\tau f_\nu$ and Proposition 1.13 it follows that

$$f \le \mathrm{li}_\tau f_\nu \le \mathrm{li}_\sigma f_\nu \quad .$$

On the other hand from the Theorem, more precisely (2.5), the τ/σ-equi-l.sc. yields

$$f \ge \mathrm{ls}_\tau f_\nu \ge \mathrm{ls}_\sigma f_\nu \quad ,$$

and hence $f = \mathrm{lm}_\sigma f_\nu = \mathrm{li}_\sigma f_\nu = \mathrm{ls}_\sigma f_\nu$ as follows from (1. 5)

If $f = \mathrm{lm}_\sigma f_\nu$, then Proposition 1.13 implies that

$$f \ge \mathrm{ls}_\sigma f_\nu \ge \mathrm{li}_\tau f_\nu \quad .$$

and τ/σ-equi-lower semicontinuity yields via (2.4)

$$f \le \mathrm{li}_\sigma f_\nu \le \mathrm{li}_\tau f_\nu \quad .$$

To complete the proof we again appeal to (1. 5). □

The next Theorem shows that τ/σ-equi-semicontinuity is a *minimal* condition that allows to pass from the epi-convergence in one topology to the epi-convergence in another topology.

Theorem 2.10. *Suppose that* $\{f_\nu, \nu \in N\} \subset \bar{R}^X$ *is a filtered collection of functions such that* $-\infty < ls_\sigma f_\nu \le li_\tau f_\nu$. *Then the collection* $\{f_\nu, \nu \in N\}$ *is* τ/σ*-equi-l.sc. . Moreover if* $\sigma \supset \tau$, *then also*

$$(2.10) \qquad lm_\sigma f_\nu = lm_\tau f_\nu \quad .$$

Proof. The equality (2.10) follows from the assumptions via (1. 5) and Proposition 1.13. For brevity, let $f = li_\tau f_\nu$. To prove equi-l.sc. we argue by contradiction. First suppose that $x \notin \mathrm{dom}\, f$ and (~d) fails, i.e., there exists $a \in R$ such that for every $V \in G_\tau(x)$ and $H \in H$ there exists $\nu \in H$ and $y \in V$ with

$$f_\nu(y) < a \quad .$$

Then $f(x) = (li_\tau f_\nu)(x) \le a$, contradicting the hypothesis that $x \notin \mathrm{dom}\, f$.

If $f(x) = (li_\tau f_\nu)(x) \ge (ls_\sigma f_\nu)(x)$ is finite and (d) fails, it means that there exists $\varepsilon > 0$ and $W \in G_\sigma(x)$ such that for every $H \in H$ and $V \in G_\tau$

$$\varepsilon + \inf_{y\in V} f_\nu(y) < \inf_{y\in W} f_\nu(y)$$

for some $\nu \in H$. In particular, this must hold for some $\nu' \in H'$ with the pair (H', G_ε) constructed as follows. From the definitions of li_τ and ls_σ, it follows that

(i) there exist $G_\varepsilon \in G_\tau(x)$ and $H_\varepsilon \in H$ such that

$$(li_\tau f_\nu)(x) - \varepsilon/4 \le \inf_{\nu\in H_\varepsilon} \inf_{y\in G_\varepsilon} f_\nu(y) \quad ,$$

and

(ii) to $W \in G_\sigma(x)$, there corresponds $H_W \in H$ such that

$$(ls_\tau f_\nu)(x) + \varepsilon/4 \geq \sup_{\nu \in H_W} \inf_{y \in W} f_\nu(y) \quad .$$

Now simply define $H_\varepsilon \cap H_W = H' (\in H)$ and because (d) fails, for some $\nu' \in H'$

$$\varepsilon + \inf_{y \in G_\varepsilon} f_{\nu'}(y) < \inf_{y \in W} f_{\nu'}(y)$$

and thus

$$\varepsilon + \inf_{\nu \in H'} \inf_{y \in G_\varepsilon} f_\nu(y) < \sup_{\nu \in H'} \inf_{y \in W} f_\nu(y)$$

Hence

$$f(x) + 3\varepsilon/4 = \varepsilon + (li_\tau f_\nu)(x) - \varepsilon/4 \leq \varepsilon + \inf_{\nu \in H_\varepsilon} \inf_{y \in G_\varepsilon} f_\nu(y)$$

$$\leq \varepsilon + \inf_{\nu \in H'} \inf_{y \in G_\varepsilon} f_\nu(y) < \sup_{\nu \in H'} \inf_{y \in W} f_\nu(y)$$

$$\leq \sup_{\nu \in H_W} \inf_{y \in W} f_\nu(y) \leq (ls_\sigma f_\nu)(x) + \frac{\varepsilon}{4} \leq f(x) + \frac{\varepsilon}{4} \quad ,$$

a clear contradiction. □

The *pointwise-limit functions* of a filtered collection of functions $\{f_\nu, \nu \in N\}$ are denoted by $li\ f_\nu$ and $ls\ f_\nu$ and are defined by

(2.11) $$li\ f_\nu(x) = \sup_{H \in H} \inf_{\nu \in H} f_\nu(x)$$

and

(2.12) $\quad \text{ls } f_\nu(x) = \inf_{H\in\mathcal{H}} \sup_{\nu\in H} f_\nu(x)$

$$= \sup_{H\in\ddot{\mathcal{H}}} \inf_{\nu\in H} f_\nu(x) \quad .$$

The last equality follows from (1.3).

Let ι denote the discrete topology on X, then $\mathcal{G}_\iota(x)$ consists of all subsets of X that contain x. From this it follows that

$$\text{li } f_\nu = \text{li}_\iota f_\nu \quad \text{and} \quad \text{ls } f_\nu = \text{ls}_\iota f_\nu$$

and thus the preceding results also yield the relationship between epi-convergence and pointwise-convergence, for example, (1.14) and (1.15) become

(2.13) $\quad \text{li}_\tau f_\nu \leq \text{li } f_\nu$

and

(2.14) $\quad \text{ls}_\tau f_\nu \leq \text{ls } f_\nu \quad .$

When $\sigma = \iota$ it is possible to replace (d) by :

(d_p) given any $x \in D$, to every $\varepsilon > 0$ there corresponds $H \in \mathcal{H}$ and $V \in \mathcal{G}_\tau(x)$ such that for all $\nu \in H$

$$\inf_{y\in V} f_\nu(y) \geq f_\nu(x) - \varepsilon$$

This condition is easier to verify and is in fact equivalent to (d) as we show next. Clearly (d) implies (d_p) since $\{x\} \in \mathcal{G}_\iota(x)$. On the other hand

given $x \in D$, and any $\varepsilon > 0$ and $W \in \mathcal{G}_\iota(x)$ (any set containing x), we always have that

$$\inf_{y \in W} f_\nu(y) - \varepsilon \leq f_\nu(x) - \varepsilon$$

If (d_p) is satisfied, there then exists $H \in \mathcal{H}$ and $V \in G_\tau(x)$ such that

$$f_\nu(x) - \varepsilon \leq \inf_{y \in V} f_\nu(y)$$

for all ν H. Combining the two preceding inequalities we get (d). In this setting, Theorem 2.3 and its corollaries, and Theorem 2.10 become:

Theorem 2.15. *Suppose that* $\{f_\nu, \nu \in N\} \subset \bar{R}^X$ *is a filtered collection of functions:*

(i) *If the collection is* τ*-equi-l.sc., then*

$$li_\tau f_\nu = li\ f_\nu \quad \textit{and} \quad ls_\tau f_\nu = ls\ f_\nu \quad .$$

Also, $f = lm\ f_\nu$ *if and only if* $f = lm_\tau f_\nu$.

(ii) *If* $-\infty < f = lm\ f_\nu = lm_\tau f_\nu$, *then the collection of functions* $\{f_\nu, \nu \in N\}$ *is* τ*-equi-l.sc.* .

By means of Proposition 2.2, we obtain as corollaries to the above, a whole slough of convergence results. For example:

Corollary 2.16. *Suppose that* $\sigma \supset \tau$. *If* $f = lm\ f_\nu$ *and the filtered collection* $\{f_\nu, \nu \in N\}$ *is* τ*-equi-l.sc., then* $f = lm_{\tau,\sigma} f_\nu$. *Also, if* $f = lm_{\sigma,\tau} f_\nu$ *and the collection is* τ*-equi-l.sc. then* $f = lm\ f_\nu$.

The assertions of Theorem 2.15 remain valid with a weakened version of τ-equi-l.sc., when X is a subset of a *linear* topological space and the $\{f_\nu, \nu \in N\}$ are *convex* functions. For (~d) we substitute the following condition:

($\sim d_c$) given any $x \notin \text{cl } D$, to every $a \in R$ there corresponds $H \in \mathcal{H}$ and $V \in \mathcal{G}_\tau(x)$ such that for all $\nu \in H$

$$\inf_{y \in V} f_\nu(y) \geq a \quad .$$

Obviously (~d) implies ($\sim d_c$), the converse also holds in the "convex" case, but that needs to be argued. To start with, we need the convexity of some limit functions which we obtain as a corollary to the next proposition.

Proposition 2.17. *Suppose that* $\{C_\nu, \nu \in N\}$ *is a filtered collection of convex subsets of a linear topological space. Then* $Li\ C_\nu$ *is convex.*

Proof. From the definition (1.6) of $Li\ C_\nu$, it follows that $x \in Li\ C_\nu$ if and only if to every neighborhood V of x, there corresponds $H \in \mathcal{H}$ such that for all $\nu \in H$

$$C_\nu \cap V \neq \emptyset \quad . \tag{2.18}$$

Now take $x^0, x^1 \in Li\ C_\nu$ and for $\lambda \in [0,1]$ define

$$x^\lambda = (1-\lambda)x^0 + \lambda x^1 \quad .$$

We need to show that if V^λ is a neighborhood of x^λ, there exists $H^\lambda \in \mathcal{H}$ such that $C_\nu \cap V \neq \emptyset$ for all $\nu \in H^\lambda$. Define

$$V^0 = V^\lambda - x^\lambda + x^0$$

and

$$V^1 = V^\lambda - x^\lambda + x^1 \quad .$$

These are neighborhoods of x^0 and x^1 and thus there exist H^0 and H^1 such that (2.18) is satisfied. Let $H^\lambda = H^0 \cap H^1$. Since $\mathcal{H}$ is a filter, $H^\lambda \in \mathcal{H}$ and clearly for all $\nu \in H^\lambda$ we have that

$$V^0 \cap C_\nu \neq \emptyset \quad \text{and} \quad V^1 \cap C_\nu \neq \emptyset \quad ,$$

from which it follows that for all $\nu \in H^\lambda$

$$V^\lambda \cap C_\nu \neq \emptyset$$

because all the C_ν are convex. □

Corollary 2.19. *Suppose that* $\{f_\nu, \nu \in N\}$ *is a filtered collection of convex functions defined on the linear topological space* (X,τ). *Then* $ls_\tau f_\nu$ *is a convex function, and if they exist so are* $lm_\tau f_\nu$ *and* $lm\ f_\nu$.

Proof. Recall that a function is convex if and only if its epigraph is convex. Thus the convexity of $ls_\tau f_\nu$ follows from (1.11) and Proposition 2.17 since by assumption all the $\{\text{epi}\ f_\nu, \nu \in N\}$ are convex. The rest follows from the facts that if they exist $lm_\tau = ls_\tau$ and $lm = lm_\iota$. □

Note however that in general $li_\tau f_\nu$ is not convex, although the f_ν are convex. Consider, for example $X = R$, τ the natural (or the discrete) topology and for $k = 1,2,\ldots$ the functions

$$f_{2k}(x) = |x - 1| \quad ,$$

and

$$f_{2k-1}(x) = |x + 1| \quad .$$

Then clearly $li_\tau f_\nu$ is not convex, since

$$li_\tau f_\nu = \begin{cases} |x + 1| & \text{if } x \leq 0 \\ |x - 1| & \text{if } x \geq 0 \end{cases} \quad .$$

Proposition 2.20. *Suppose that* $\{f_\nu, \nu \in N\}$ *is a filtered collection of convex functions defined on the linear topological space* X. *Moreover suppose that either* $-\infty < \mathrm{lm}_\tau f_\nu$ *exists or that* $-\infty < \mathrm{lm}\ f_\nu$ *exists and is* τ-*l.sc.* . *Then the collection* $\{f_\nu, \nu \in N\}$ *is* τ-*equi-l.sc. if and only if it satisfies* (d_p) *and* $(\sim d_c)$, *with the same reference set* D.

Proof. Since $(\sim d)$ implies $(\sim d_c)$, the only thing to prove is the converse in the presence of (d_p), convexity and the existence of a limit function. From the proof of Theorem 2.3, with $\sigma = \iota$, we see that (d_p) implies that $\mathrm{li}\ f_\nu \leq \mathrm{li}_\tau f_\nu$ and that $\mathrm{ls}\ f_\nu \leq \mathrm{ls}_\tau f_\nu$ on D. Similarly that $(\sim d_c)$ yields the same relations on $X \backslash \mathrm{cl}\ D$. Combining these inequalities with (2.13) and (2.14), we have that (d_p) and $(\sim d_c)$ imply that

$$\mathrm{li}_\tau f_\nu = \mathrm{li}\ f_\nu \text{ and } \mathrm{ls}_\tau f_\nu = \mathrm{ls}\ f_\nu \tag{2.21}$$

on $X \backslash Q$, where $Q = \mathrm{cl}\ D \backslash D$. Moreover, in view of Corollary 2.19, $\mathrm{ls}_\tau f_\nu$ is always convex and so are $\mathrm{lm}\ f_\nu$ and $\mathrm{lm}_\tau f_\nu$ if they exist.

If $-\infty < f = \mathrm{lm}\ f_\nu$ exists and is τ-l.sc., it follows from the above that $f = \mathrm{ls}_\tau f_\nu = \mathrm{li}_\tau f_\nu$ on $X \backslash Q$. Convexity also yields the equality on Q. We argue this by contradiction. Suppose to the contrary that for some $x^1 \in Q$

$$\alpha = (\mathrm{li}_\tau f_\nu)(x^1) < f(x^1) \quad .$$

Take $x^0 \in \mathrm{dom}\ f \subset D$, and without loss of generality, assume that $f(x^0) = 0$. Given any $\varepsilon > 0$, $G \in G_\tau(x^1)$, $H \in \mathcal{H}$, the definition of li_τ yields $\nu_\varepsilon \in H$ and $y_\varepsilon \in G$ such that

$$\alpha \geq f_{\nu_\varepsilon}(y_\varepsilon) - \varepsilon \quad .$$

For $\lambda \in [0,1]$, define

$$x^\lambda_{H,G} = (1-\lambda)x^0 + \lambda y_\varepsilon \quad .$$

The convexity of the f_ν, implies that

$$f_{\nu_\varepsilon}(x^\lambda_{H,G}) \leq (1-\lambda)f_{\nu_\varepsilon}(x^0) + \lambda f_{\nu_\varepsilon}(y_\varepsilon) \leq (1-\lambda)f_{\nu_\varepsilon}(x^0) + \lambda(\alpha+\varepsilon) \quad .$$

Now note that for any fixed $\lambda \in [0,1]$, $x^\lambda = (1-\lambda)x^0 + \lambda x^1$ is a limit point of the filtered collection $\{x^\lambda_{H,G},\ (H,G) \in H \times G_\tau(x^1)\}$. Hence, we have that for every $\lambda \in [0,1[$

$$f(x^\lambda) \leq (li_\tau f_\nu)(x^\lambda) \leq (1-\lambda)f(x^0) + \lambda\alpha = \lambda\alpha \quad .$$

Let $\lambda \uparrow 1$. From the lower semicontinuity of f we get that $f(x^1) \leq \alpha$, contradicting our working hypothesis. And thus we have shown that $lm_\tau f_\nu = lm\ f_\nu =$ on X, and hence the collection is τ-equi-l.sc. as follows from Theorem 2.10, with $\sigma = \iota$.

On the other hand, if $f = lm_\tau f_\nu$ exists and the collection of convex functions $\{f_\nu, \nu \in N\}$ satisfies (d_p) and $(\sim d_c)$ with respect to D (necessarily containing dom f), it follows from (2.21) that on $X \backslash Q$,

$$lm_\tau f_\nu = li\ f_\nu = ls\ f_\nu \quad .$$

Corollary 2.6 implies that $D \supset \text{dom}\ lm_\tau f_\nu$ and thus $lm_\tau f_\nu = +\infty$ on Q. By (2.13), on all of X we have that

$$f = lm_\tau f_\nu \leq li\ f_\nu \leq ls\ f_\nu \quad ,$$

from which it follows that on Q, $f = \text{li}\, f_\nu = \text{ls}\, f_\nu = +\infty$. Thus we have shown that on all of X, $\text{lm}_\tau f_\nu = f = \text{lm}\, f_\nu$. Again with $\sigma = \iota$ Theorem (2.10) then yields the τ-equi-l.sc. of the f_ν. □

When X is a reflexive Banach space and the $\{f_\nu, \nu \in N\}$ are convex, the original definition of τ-equi-l.sc., as given in Salinetti and Wets (1977), coincides with this weakened version involving (d_p) and $(\sim d_c)$. Condition (α) of Salinetti and Wets (1977) is precisely (d_p). In general $(\sim d_c)$ implies (γ) of Salinetti and Wets (1977) and because the closed balls of a reflexive space are weakly compact (γ) implies $(\sim d_c)$. Condition (β) of Salinetti and Wets (1977) is automatically satisfied if the functions f_ν converge pointwise (Salinetti and Wets, 1977, Lemma 2.ii) and it is implied by (d_p) and $(\sim d_c)$ if the f_ν epi-converge. Thus, Theorem 1., 2. and 3. of Salinetti and Wets (1977) are special cases of Theorem 2.15 and Corollary 2.16.

III THE HYPERSPACE OF CLOSED SETS

Let (Y,η) be a topological space. In this section we have collected some facts about the *(hyper)space* of closed subsets of Y equipped with the topology of set-convergence, as defined by (1.8). This turns out to be a variant of the Vietoris finite topology, at least when (Y,η) is separated (Hausdorff) and locally compact. The results found in this section can be extracted from articles by Choquet (1947-48), and by Michael (1951) and from the book by Kuratowski (1958).

By $\mathcal{F}_Y$, or simply $\mathcal{F}$ if no confusion is possible, we denote the *hyperspace of closed subsets* of Y. The topology $\mathcal{T}$ on $\mathcal{F}$ is generated by the subbase of open sets:

$$\{F^K, K \in K\} \quad \text{and} \quad \{F_G, G \in G\}$$

where K and G are the hyperspaces of compact and open subsets of Y respectively, and for any $Q \subset Y$.

$$F^Q = \{F \in F \mid F \cap Q = \emptyset\} \quad ,$$

and

$$F_Q = \{F \in F \mid F \cap Q \neq \emptyset\} \ .$$

Proposition 3.1. *Suppose that* Y *is separated and locally compact,* $\{C_\nu, \nu \in N\}$ *is a filtered collection of subsets of* Y, *and* $C \subset Y$ *is closed. Then*

(i) $C \subset Li\ C_\nu$ *if and only if to every* $G \in G$ *such that* $C \cap G \neq \emptyset$, *there corresponds* $H_G \in H$ *such that for every* $\nu \in H_G$, $C_\nu \cap G \neq \emptyset$.

(ii) $C \supset Ls\ C_\nu$ *if and only if to every* $K \in K$ *such that* $C \cap K = \emptyset$, *there corresponds* $H_K \in H$ *such that for every* $\nu \in H_K$, $C_\nu \in K = \emptyset$.

Moreover $C = Lm\ C_\nu$ *if and only if* $C = T\text{-}\lim C_\nu$.

Proof. It will be sufficient to prove (i) and (ii) since the last assertion follows immediately from (i) and (ii) and the construction of T.

Suppose first that $x \in C$, then $C \cap G \neq \emptyset$ for all $G \in G_\eta(x)$. The "if" part of (i), implies that $C_\nu \cap G \neq \emptyset$ for all $\nu \in H_G$ with $H_G \in H$. Every H' in H meets every $H \in \ddot{H}$ and hence

$$(\cup_{\nu \in H} C_\nu) \cap G \neq \emptyset$$

for every $H \in \ddot{H}$ and $G \in G_\eta(x)$. Thus for every $H \in \ddot{H}$, $x \in \mathrm{cl}(\cup_{\nu \in H} C_\nu)$ and consequently by (1.5) $x \in Li\ C_\nu$, i.e., $C \subset Li\ C_\nu$.

If $C \subset Li\ C_\nu$, then $C \cap G \neq \emptyset$ implies that $G \cap (\cap_{H \in \ddot{H}} \text{cl}(\cup_{\nu \in H} C_\nu)) \neq \emptyset$, i.e., for every $H \in \ddot{H}$

$$(\cup_{\nu \in H} C_\nu) \cap G \neq \emptyset$$

or equivalently there exists $H_G \in H$ such that for all $\nu \in H_G, C_\nu \cap G \neq \emptyset$, again because H consists of all the subsets of N that meet every set in $\ddot{H}$. This completes the proof of (i).

Suppose that $x \in Ls\ C_\nu$, then for every $H \in H$, $x \in \text{cl}(\cup_{\nu \in H} C_\nu)$, cf. (1.6). If $x \notin C$, by local compactness of Y, there is a compact neighborhood K of x such that $K \cap C = \emptyset$. The "if" part of (ii) then implies that $K \cap (\cup_{\nu \in H_K} C_\nu) = \emptyset$ for some $H_K \in H$, i.e., $x \notin \text{cl}(\cup_{\nu \in H_K} C_\nu)$ contradicting the assumption that $x \in Ls\ C_\nu$.

Now suppose that $C \supset Ls\ C_\nu$, $C \cap K = \emptyset$, but for every $H \in \ddot{H}$ we can find ν such that $C_\nu \cap K \neq \emptyset$, i.e., there exists $H' \in H$ such that $C_\nu \cap K \neq \emptyset$ for every $\nu \in H'$. Since K is compact, it follows that the $\{C_\nu \cap K, \nu \in H'\}$ admit at least one cluster point $x \in K$. Then for every $H \in H$

$$x \in \text{cl}(\cup_{\nu \in H} C_\nu) \cap K \quad ,$$

and consequently $x \in Ls\ C_\nu \cap K$. But this contradicts the assumption that $C \supset Ls\ C_\nu$. □

Thus T is indeed the topology of set-convergence as defined in Section I. The next Proposition yields the properties of (F,T) that are needed in the sequel.

Proposition 3.2. *Suppose that* Y *is separated (Hausdorff) and locally compact. Then* (F,T) *is regular and compact.*

Proof. By construction the sets $\{\mathcal{F}_K; K \in \mathcal{K}\}$ and $\{\mathcal{F}^G; G \in \mathcal{G}\}$ are the complements of open (base) sets, and thus are closed. In particular, this implies that singletons are closed, since

$$F = (\cap_{y \in F} \mathcal{F}_{\{y\}}) \cap \mathcal{F}^G \quad ,$$

$G = Y \backslash F$ is open.

To see that $(\mathcal{F}, \mathcal{T})$ is separated, let F_1 and F_2 be two subsets of $\mathcal{F}$ such that $F_1 \neq F_2$. Then there is some y that belongs to F_1 but not to F_2 (or vice-versa). Since Y is locally compact by assumption and F_2 is closed, there exists K°, an open precompact neighborhood of y, such that $K = \text{cl } K^\circ$ is disjoint of F_2. Hence

$$F_1 \in \mathcal{F}_{K^\circ} \quad \text{and} \quad F_2 \in \mathcal{F}^K \quad .$$

The compactness of $(\mathcal{F}, \mathcal{T})$ follows from Alexander's characterization of compactness in terms of the finite intersection property of a subbase of closed (hyper)sets. Suppose that

$$(3.3) \qquad (\cap_{i \in I'} \mathcal{F}_{K_i}) \cap (\cap_{j \in J} \mathcal{F}^{G_j}) = \emptyset$$

where $K_i \in \mathcal{K}$, $G_j \in \mathcal{G}$ and, I and J are arbitrary index sets. We must show that the family of sets $\{K_i, i \in I ; G_j, j \in J\}$ contains a finite subfamily that has an empty intersection. Let $G = \cup_{j \in J} G_j$ and note that $G \in \mathcal{G}$. Now observe that (3.3) holds if and only if

$$\cap_{i \in I} (\mathcal{F}_{K_i} \cap \mathcal{F}^G) = \emptyset$$

or still, if and only if for some $i_0 \in I$, $\mathcal{F}_{K_{i_0}} \cap \mathcal{F}^G = \emptyset$, or equivalently, if and only if there exists $i_0 \in I$ such that

$$K_{i_0} \subset G$$

But K_{i_0} is compact and thus the open cover $\{G_j, j \in J\}$ contains a finite subcover $\{G_{j_1}, \ldots, G_{j_q}\}$. Hence (3.3) holds if and only if

$$F_{K_{i_0}} \cap (\cap_{i=1} F^{G_{j_i}}) = \emptyset$$

Since (F, T) is compact and separated, it is also regular. □

IV COMPACTNESS CRITERIA FOR SPACES OF SEMICONTINUOUS FUNCTIONS

The relationship between pointwise- and e_τ-limits through equi-semicontinuity suggests a number of compactness criteria for spaces of semicontinuous and continuous functions, the celebrated Arzelà-Ascoli Theorem being a special case of these. Our approach in fact provides an unconventional proof of this classical result.

Although a few of the (weaker) subsequent statements remain valid in a more general setting, we shall assume henceforth that the domain-space (X, τ) is separated and locally compact. Let $SC(X)$ be the *space of* τ*-l.sc. functions* with range $\bar{R}$ and domain X. The elements of $SC(X)$ are in one-to-one correspondence with the elements of E, *the hyperspace of epigraphs*, i.e. the closed subsets E of $Y = X \times R$ such that $(x,a) \in E$ implies that $(x,b) \in E$ for all $b \geq a$. Note that $\{\emptyset\} \in E$ and corresponds to the (continuous) function $f \equiv +\infty$. E is a subset of F_Y, the hyperspace of closed subsets of $Y = X \times R$.

Proposition 4.1. *Suppose that* (X,τ) *is separated and locally compact. Then* $\mathcal{E} \subset \mathcal{F}_Y$ *is compact with respect to the* $\mathcal{T}$ *topology. Moreover the* $\mathcal{T}$*-relative topology on* $\mathcal{E}$ *can be generated by the subbase of open sets:*

$$\{\mathcal{E}^{K,a}\ ;\ K \in \mathcal{K}_X, a \in \bar{R}\}$$

and

$$\{\mathcal{E}_{G,a^\circ}\ ; G \in \mathcal{G}_X, a \in \bar{R}\} \quad ,$$

where for any $Q \subset X$ *and* $a \in \bar{R}$

$$\mathcal{E}^{Q,a} = \{E \in \mathcal{E} \,|\, E \cap (Q \times]-\infty, a]) = \emptyset\}$$

and

$$\mathcal{E}_{Q,a^\circ} = \{E \in \mathcal{E} \,|\, E \cap (Q \times]-\infty, a[) \neq \emptyset\}$$

Proof. Suppose $F \in \mathcal{F}_Y \backslash \mathcal{E}$, then there exists $x \in X$ and $a < b$ such that $(x,a) \in F$ but $(x,b) \notin F$. The local compactness of X yields an open precompact set K° such that

$$\mathcal{F}^{K\times\{b\}} \cap \mathcal{F}_{K^\circ \times]a-\varepsilon, a+\varepsilon[} \quad ,$$

with $K = \text{cl } K^\circ$ and $0 < \varepsilon < b-a$, is an open neighbourhood of F that does not contain any epigraphs. Thus $\mathcal{F}\backslash\mathcal{E}$ is open or equivalently $\mathcal{E}$ is closed. Since $\mathcal{F}$ is compact, so is $\mathcal{E}$.

To see that the $\mathcal{T}$-relative topology on $\mathcal{E}$ can be generated the subbase described above, note that the topological properties of $Y = X \times R$ imply that the sets of the type

$$\{F^{K\times[a,b]} \; ; K \in K_X \, , a,b \in R\}$$

and

$$\{F_{G\times]a,b[} \; ; G \in G_X, a,b \in R\}$$

also are a subbase for T on F_Y. The restriction of this subbase to E, yields

$$E^{K\times[a,b]} = E^{K,a}$$

and

$$E_{G\;]a,b[} = E_{G,a^\circ} \quad . \quad \square$$

Combining Propositions 3.2 and 4.1 we get:

Corollary 4.2. *The topological space* (E,T) *is regular and compact.*

From Propositions 1.9, 3.1 and 4.1, with e_τ the topology of epi-convergence in SC(X), we also get:

Corollary 4.3. *The topological space* $(SC(X), e_\tau)$ *is regular and compact.*

The above implies that any closed subset of SC is compact. In particular, note that for any $a \in R$ and $D \subset X$, the set

$$SC^a(D) = \{f \in SC \,|\, f \leq a \text{ on } D\} = \bigcap_{x\in D} \{f \in SC \,|\, f(x) \leq a\}$$

is compact. To see this simply observe that $\{f \in SC \,|\, f(x) \leq a\}$ is closed since it corresponds in E to the T-closed set

$$\{E \in E \,|\, E \cap (\{x\}\times]-\infty,a]) \neq \emptyset\}$$

Also, for any $a \in R$ and any open $G \in X$, the set

$$SC_a(G) = \{f \in SC \mid f \geq a \text{ on } G\}$$

is closed since it corresponds in E to the T-closed set

$$\{E \in E \mid E \cap (G \times]-\infty, a[) = \emptyset\} \quad .$$

We have just shown that:

Corollary 4.4. *Any bounded collection of* τ-*l.sc. functions is a compact subset of* $(SC(X), e_\tau)$.

The topological space (SC,p) is the space of τ-l.sc. functions equipped with the topology of pointwise convergence. We already know that neither pointwise nor epigraph-convergence implies the other. However, in view of Theorem 2.15, these topologies coincide on τ-equi-l.sc. subsets of SC:

Definition 4.5. A set $A \subset SC(X)$ is *equi-l.sc.* if there exists a set $D \subset X$ such that

(d_{SC}) given any $x \in D$, to every $\varepsilon > 0$, there corresponds $V \in G_\tau(x)$ such that for every f in A

$$\inf_{y \in V} f(y) \geq f(x) - \varepsilon \quad ,$$

and

($\sim d_{SC}$) given any $x \notin D$, to every $a \in R$ there corresponds $V \in G_\tau(x)$ such that for all f in A,

$$\inf_{y \in V} f(y) \geq a \quad .$$

Theorem 4.6. *Suppose that (X,τ) is separated and locally compact. Then any τ-equi-l.sc. family of τ-l.sc. functions contains a (filtered) subfamily converging pointwise to a τ-l.sc. function. Moreover, if the family of functions is bounded, it contains a subfamily converging pointwise to a bounded τ-l.sc. function.*

Proof. As follows from Theorem (2.15), for τ-equi-l.sc. subsets of $SC(X)$, the p-closure or e_τ-closure coincide. The first statement then follows from Corollary 4.3 and the second from Corollary 4.4. □

Every property derived for $(SC(X),e_\tau)$ has its counterpart in $(-SC(X),-e_\tau)$, the space of τ-*upper semicontinuous functions* (τ-u.sc.) with the topology $-e_\tau$ of *hypo(graph)-convergence*. In particular, $(-SC(X),-e_\tau)$ is compact and any bounded subfamily is precompact. And thus, any τ-*equi-u.sc.* family of (bounded) u.sc. functions contains a subfamily converging pointwise to a (bounded) τ-u.sc. function.

Given $\{f_\nu,\nu\in N\}$ a filtered collection of functions, the $-e_\tau$-*limit inferior* is $-(ls_\tau - f_\nu)$ and the $-e_\tau$-*limit superior* is $-(li_\tau - f_\nu)$. The hypographs of these functions being precisely *Li* hypo f_ν and *Ls* hypo f_ν. We always have that

$$li_\tau f_\nu \leq li\ f_\nu = -(ls - f_\nu) \leq -(ls_\tau - f_\nu)$$

and

$$ls_\tau f_\nu \leq ls\ f_\nu = -(li - f_\nu) \leq -(li_\tau - f_\nu) \quad .$$

In each one of the preceding expressions, the first (second resp.) inequality becomes an equality if the collection is τ-equi-l.sc. (τ-equi-u.sc. resp.).

Let $\bar{C}(X) = SC(X) \cap -SC(X)$ be the space of continuous extended-real valued functions, $\pm e_\tau$ the join of the two topologies e_τ and $-e_\tau$, and again p the topology of pointwise convergence. In general $(\bar{C}(X), \pm e_\tau)$ is not compact but as we shall see, its equi-continuous subsets are precompact. A subset $A \subset \bar{C}(X)$ is *equi-continuous* if it is both τ-equi-l.sc. and τ-equi-u.sc. with the same reference set D being used in the verification of the equi-sc. conditions. (Note that necessarily D must be open.)

Proposition 4.7. *Suppose that* X *is separated and locally compact. Then* $A \subset \bar{C}(X)$ *is precompact (with respect to* $\pm\varepsilon_\tau$*) if and only if it is equi-continuous.*

Proof. If *A* is equi-continuous, it is equi-l.sc. and hence every subset of *A* contains a filtered family $\{f_\nu, \nu \in N\}$ such that $lm_\tau f_\nu = lm\ f_\nu$, but by assumption the $\{f_\nu, \nu \in N\}$ are also equi-u.sc. and thus contain a subfamily (a finer filter on N) such that

$$lm_\tau f_\nu = lm\ f_\nu = -(lm_\tau - f_\nu)$$

from it follows that *A* is precompact.

On the other hand, if *A* is not equi-continuous, then assume for example, that τ-equi-lower semicontinuity fails. This means that for some collection of functions $\{f_\nu, \nu \in N\}$ and some x, we have that

$$(lm_\tau f_\nu)(x) < (li\ f_\nu)(x) = -(ls - f_\nu)(x) \leq -(ls_\tau - f_\nu)(x) \quad .$$

Hence there is obviously no subcollection of the $\{f_\nu\}$ whose hypographs converge to $lm_\tau f_\nu$, since at x the $-e_\tau$-limit inferior of the $\{f_\nu\}$ is strictly larger than $(lm_\tau f_\nu)(x)$. Thus *A* cannot be precompact. □

Finally, we consider the space C(X) of continuous real-valued functions with the topologies $\pm e_\tau$, p and $\|\cdot\|$, the last one being the sup-norm topology induced by the pseudo-norm defined by

$$\|f\| = \sup_{x\in X} |f(a)| \quad .$$

This pseudo-norm induces a topology on C. The fundamental system of neighborhoods of an element f is defined by the sets $\{g \in C \mid \|f-g\| < a\}$ with $a > 0$. Note that if X is compact, then $\|\cdot\|$ is a norm on C(X) and the topology $\|\cdot\| \subset \pm e_\tau$ as can easily be verified. In general however these two topologies are not comparable.

Theorem 4.8. *Suppose that* X *is separated and locally compact and* $A \subset C(X)$ *is equi-continuous and bounded. Then* A *is* $\pm e_\tau$*-precompact.*

Proof. This follows from the fact that bounded subsets of SC(X) and -SC(X) are e_τ and $-e_\tau$-compact respectively, cf. Corollary 4.4. As in Proposition 4.7 equi-continuity providing the link between the limit functions. □

Corollary 4.9. (Arzelá-Ascoli) *Suppose that* X *is separated and compact. Then* A *is precompact, with respect to the* $\pm e_\tau$*-topology, and consequently with respect to the* $\|\cdot\|$ *topology, if and only if* A *is equi-continuous and bounded.*

Sufficiency follows from Theorem 4.8. The necessity of equi-continuity is argued as in Proposition 4.7. Finally, if A is unbounded, there exist $\{f_\nu, \nu \in N\}$ and $\{x_\nu, \nu \in N\}$ such that $f_\nu(x_\nu) \downarrow -\infty$ (or $\uparrow +\infty$). The compactness of X implies that the family $\{x_\nu, \nu \in N\}$ admits an accumulation point, say x. Then $(li_\tau f_\nu)(x) = -\infty$ (or $-(li\ -f\ )(x) = +\infty$) and hence the $\pm e_\tau$-closure of A can not be in C(X) if A is unbounded. □

APPENDIX

There is an intimate connection between the semicontinuity properties of multifunctions and the convergence of (filtered) families of sets. The appendix is devoted to clarifying these relations; most of this can be found in one form or another in Choquet (1947-1948) or Kuratowski (1958).

A map Γ with domain Y and whose values are subsets of X (possibly the empty set) is called a *multifunction*. The graph of Γ is

$$\text{grph } \Gamma = \{(y,x) \in Y \times X \mid x \in \Gamma(y)\} \ .$$

We recall that the *image* of $A \subset Y$ is $\Gamma A = \cup_{y \in A} \Gamma(y)$ and the *preimage* of $B \subset X$ is $\Gamma^{-1}B = \{y \in Y \mid \Gamma(y) \cap B = \emptyset\}$.

A neighborhood base $\mathcal{B}(y_0)$ of $y_0 \in Y$ is a filter base on Y. A multifunction Γ is said to be *upper semicontinuous (u.sc.)* at y_0 whenever

$$(Ls\ \Gamma)(y_0) = \cap_{W \in \mathcal{B}(y_0)} \text{ cl } \Gamma W \subset \Gamma(y_0)$$

or equivalently if to each $x^0 \notin \Gamma(y^0)$ we can associate neighborhoods Q of x^0 and W of y^0 such that $\Gamma W \cap Q = \emptyset$. Note that Γ is u.sc. (at every y) if and only if grph Γ is closed.

In the literature one can find a couple of closely connected defintions of upper semicontinuity. A multifunction Γ is said to be *K-u.sc.* at y^0, if to each closed set F disjoint of $\Gamma(y_0)$ there corresponds a neighborhood W of y_0 such that $\Gamma W \cap F = \emptyset$, or equivalently if to each open set G that includes $\Gamma(y_0)$ there corresponds a neighborhood W of y_0 such that $\Gamma W \subset G$. If X is regular, then Γ closed-valued and K-u.sc. at y_0 implies Γ u.sc. at y_0. If X is compact and Γ is closed-valued at y_0 then both notions coincide.

A multifunction is said to be *C-u.sc.* at y_0, if to each compact set K disjoint of $\Gamma(y_0)$ there corresponds V a neighborhood of y_0 such that $\Gamma V \cap K = \emptyset$. Obviously u.sc. implies C-u.sc. . The converse can be obtained with anyone of the following assumptions

(i) X is locally compact,

(ii) Γ^{-1} is K-u.sc. at every x_0 (for example, if $f = Y \to X$ is a continuous function and $\Gamma = f^{-1}$, then Γ^{-1} is K-u.sc.),

(iii) X is metizable, y_o has a countable neighborhood base and Γy_o is closed, cf. Dolecki (1980).

The proof of the last assertion proceeds as follows: Suppose that Γ is not u.sc. at y_o. Then there exists $x_0 \notin \Gamma y_o$ and neighgorhood bases $\{Q_\nu, \nu = 1,2,...\}$ of x_0 and $\{W_\nu, \nu = 1,2,...\}$ of y_o such that for all ν ,

$$\Gamma y_o \cap Q_\nu \neq \emptyset$$

because Γy_o is closed, and for all ν

$$\Gamma W_\nu \cap Q_\nu \neq \emptyset$$

because Γ is not u.sc. at y_o. For every ν, pick $x_\nu \in \Gamma W_\nu \cap Q_\nu$.

The set $K = \{x_1, x_2, \ldots, x_0\} \subset X$ is compact (every subsequence converges to x_0) and disjoint of Γy_o but meets every ΓW. This contradicts the C-u.sc. of Γ at y_o.

A multifunction is *lower semicontinuous (l.sc.)* at y_o if

$$\Gamma(y_o) \subset (Li\ \Gamma)(y_o) = \cap_{V \in \ddot{B}(y_o)} \text{cl}\ \Gamma V$$

where $\ddot{B}(y_o)$ is the grill associated to the filter base $B(y_o)$, or equivalently if $\Gamma^{-1}G$ is a neighborhood of y_o whenever G is an open set that meets $\Gamma(y_o)$.

For a given set X, we denote by $P(X)$ the power set of X, i.e., the hyperspace containing all subsets of X, by $F(X) = F$ the hyperspace of closed subsets of X, and ${}_oF = F \setminus \{\emptyset\}$. We now consider the multifunction Λ from $P(X)$ into X defined by $\Lambda Q = Q$. We have that $\Lambda^{-1}A = \{Q | Q \cap A \neq \emptyset\}$ and $(\Lambda^{-1}A)^c = \{F | F \subset A^c\}$.

We restrict Λ to F. The sets $\{\Lambda^{-1}G, G \text{ open}\}$ form a subbase for a topology on ${}_oF$ (but not for F). Similarly, the collection $\{(\Lambda^{-1}K)^c, K \text{ compact}\}$ constitutes a subbase for another topology on F. The supremum of these two topologies yields a topology T on F. It is the coarsest topology for which Λ is both l.sc. and C-u.sc. The topology V, the *Vietoris topology*, on F has a subbase consisting of the collections $\{\Lambda^{-1}G, G \text{ open}\}$ and $\{(\Lambda^{-1}F)^c, F \text{ closed}\}$. It is the coarsest topology for which the multifunction $\Lambda : F \rightrightarrows X$ is l.sc. and K-u.sc. .

NOTES

1. When convergence in the τ topology can be defined in terms of sequential convergence, the limit functions can also be obtained as follows: let $N = \{1,2,\ldots\}$, then

$$(li_\tau f_\nu)(x) = \inf_{\substack{\{\nu_\mu\}\subset N \\ \{x_\mu \to x\}}} \liminf_{\mu\in N} f_{\nu_\mu}(x_\mu)$$

and

$$(ls_\tau f_\nu)(x) = \inf_{\{x_\nu \to x\}} \limsup_{\nu} f_\nu(x_\nu),$$

where in the first expression the infinum is over all subsequences of functions $\{f_{\nu_\mu}, \mu\in N\}$ and all sequences $\{x_\mu, \mu\in N\}$ converging to x.

2. A function f from X to $\bar{R}$ is τ/σ-*l.sc.* if (d) and (~d) hold with $D = \text{dom } f$ and $f_\nu = f$ for all $\nu\in N$. If $\tau\supset\sigma$ the concept is is essentially meaningless since then any function $f\in\bar{R}^X$ is then τ/σ-l.sc. . If $\sigma\supset\tau$, then f is τ/α-*l.sc.* if and only if τ-cl(σ-cl epi f) = σ-cl epi f . In particular if $\sigma = \iota$ then τ/ι-l.sc. corresponds to the usual notion of τ-l.sc. .

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