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Markowich, P.

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

PREFACE

This paper deals with the convergence of stable and consistent one-step approximations for linear parabolic initial-boundary-value problems with non-smooth solutions. The proofs given may be extended to semilinear parabolic problems using H.B. Keller's stability concept. Finally an extension to Lax's convergence theorem is given.

NUMERICAL SOLUTION OF PARABOLIC PROBLEMS WITH NON-SMOOTH SOLUTIONS

P. Markowich

In this paper we consider the problem:

I) $U_t = a(x,t)U_{xx} + b(x,t)U_x + c(x,t)U + f(x,t)$,

$(x,t) \in (0,1) \times (0,T)$

II) $U(x,0) = U_o(x)$, $x \in [0,1]$, $T > 0$

III) $U(0,t) = \gamma_0(t)$, $U(1,t) = \gamma_1(t)$, $t \in (0,T]$.

(I) is called a linear inhomogenous parabolic differential equation in one space variable x, (II) the initial condition and (III) the boundary conditions.

For the following we make the assumptions:

(A) $a, b, c, f \in C^r([0,1] \times [0,T])$, r sufficiently large
(B) $a(x,t) \geqslant k > 0$, $(x,t) \in [0,1] \times [0,T]$. stability condition
(C) $U_o(0) = \gamma_o(0)$, $U_o(1) = \gamma_1(0)$ continuity of initial and boundary functions.

We know that the initial and boundary functions determine the differentiability (smoothness) of the solution U in the points (0,0) and (1,0), which is important for the smallness of the local error of a consistent numerical procedure.

If U_o, γ_o and γ_1 are continuous functions then a unique solution U exists, which is continuous on $[0,1] \times [0,T]$ and therefore bounded in the closed set $[0,1] \times [0,T]$, and if $U_o \in C^3([0,1])$; $\gamma_0, \gamma_1 \in C^2([0,T])$ and $\dot{\gamma}_0(0)\,(\dot{\gamma}_1(0))$, $U_o''(0)$, $U_o'(0)$, $U_o(0)$ $(U_o''(1), U_o'(1), U_o(1))$, set for U_t, U_{xx}, U_x, U into the differential equation I), fulfill I), then U, U_t, U_x, U_{xx} are continuous and bounded on $[0,1] \times [0,T]$. See [1] and [2].

We gain a numerical procedure by choosing numbers N and M, and by forming the step sizes n = 1./N in x- direction and k = 1./M in t- direction, and by substituting appropriate difference approximations for U_t, U_x, U_{xx} in the net-points (x_i, t_n) with $x_i = ih$ and $t_n = nk$. So we can write our procedure in the following form assuming that $h = h(k)$ with $\lim_{k \to 0} h(k) = 0$

$$(*)\quad \begin{cases} \frac{1}{k}[B_0(k,t_n)U^n - B_1(k,t_{n-1})U^{n-1} - R(k,t_n)] - \hat{f}(t_n) = 0 \\ n = 1(1)M \text{ with} \\ U^o = U_o = (U_0(X_1), \ldots , U_0(X_{N-1}))^T \end{cases} .$$

If $B_0(k,t_n) \equiv I$, we call the scheme explicit, otherwise implicit. The U^n's are (N-1)- vectors with the approximate solutions on the n-th time level, $R(k,t_n)$ is the (N-1)- vector with worked-in boundary-conditions on the n-th time level, $\hat{f}(t_n)$ is the vector with the approximations for $f(x_i,t_n)$, i = 1(1)N-1, i.e., $\| \hat{f}(t_n) - (f(x_1,t_n), \ldots , f(x_{N-1},t_n))^T \| \to 0$ for $k \to 0$ with some appropriate norm, and $B_o(k,t_n)$, $B_1(k,t_{n-1})$ are (N-1)- square matrices derived from the difference approximations for the derivatives.

We define the local error of the procedure (*) for the parabolic problem I), II), III) in the solution U as the sequence of vectors.

$$L^n(U,k) = \frac{1}{k}[B_0(k,t_n)U(t_n) - B_1(k,t_{n-1})U(t_{n-1}) - R(k,t_n)] - \hat{f}(t_n),\ n = 1(1)M$$

where $U(t_i)$ are the vectors containing the solution U evaluated in the net-points of the i-th-time-level. Further we say that (*) is consistent with I), II), III) in U of order l if $\|L^n(u,k)\| \leqslant C(U)k^l$, where C(U) is bounded and independent of n. We can show by Taylor's expansion that C(U) is a finitelinear combination of bounds of partial derivatives of U on the rectangle $[0,1] \times [0,T]$, if $\|\cdot\|$ is the maximum norm. The second important concept concerned with difference approximations is stability. We call the difference scheme (*) stable, if $B_0(k,t_n)$ is invertible for $k \leqslant k_o$ and for all $n \leqslant N$ and if $\|B_0^{-1}(k,t_n)\| \leqslant P$ for $k \leqslant k_o$ and $n \leqslant N$ where P is independent of k and n and if

$$\|\prod_{i=n}^{m} B_0^{-1}(k,t_i)B_1(k,t_{i-1})\| \leqslant L \text{ for } k \leqslant k_o,\ t_n = nk \in (0,T]$$

with $1 \leqslant m \leqslant n$, where L is independent of n, m and k. Further we say that (*) is convergent to U, if for $t = t_n = nk$ fixed, $\lim_{\substack{k\to 0\\ n\to\infty}} \|U^n(k) - U(t_n)\| = 0$ uniformly in $t\,(U^n(k) = U^n)$.

The sequence of vectors $E^n(k) = U^n(k) - U(t_n)$ is called global error. We easily conclude convergence from stability and consistency. By solving the recursive relation (*) for $U^n = U^n(k)$ we find: $\|U^n(k)\| \leqslant L\|U^o\| + P(TL+1) \max_{1\leqslant i\leqslant n} \|\hat{f}(t_i)\|$ presuming $\gamma_0 = \gamma_1 = 0$. That means that $U^n(k)$ depends continuously on the initial condition U^o and on the disturbance $\hat{f}$ (in the norm $\|\ \|$). For the following we set $\|X\| = \max_{i=1(1)N-1} |X_i|$ for $X = (X_1, \ldots, X_{N-1})^T \in \mathbb{R}^{N-1}$. Now we can prove:

Theorem 1: consider the parabolic problem I), II) and III) with the assumptions (A), (B) and (C). Let (*) be a finit difference approximation to I), II) and III), which is stable and consistent of the order l with problems of the form I*, II*, III with solutions in $C^m([0,1] \times [0,T])$ (problem-I), II), III) with inhomogenity in $C^{m-2}([0,1] \times [0,T])$ and changed initial function) and let U_o, γ_0, γ_1 of the given problem fulfill:

a) $\gamma_0(0) = a(0,0)U_o''(0) + b(0,0)U_o'(0) + c(0,0)U_o(0) + f(0,0)$

b) $\gamma_1(0) = a(1,0)U_o''(1) + b(1,0)U_o'(1) + c(1,0)U_o(1) + f(1,0)$

with $\gamma_0, \gamma_1 \in C^m([0,T])$, $U_o \in C^3([0,1)]$, then the numerical procedure (*) is convergent for the given problem I), II) and III) in the maximum norm.

Proof: as mentioned before there exists a unique solution U of the given problem, so that U, U_t, U_x, U_{xx} are continuous and bounded in $[0,1] \times [0,T]$. (Proof in [1]).

Now let $\varepsilon > o$ be fixed. We construct the sequence of Bernstein polynomials to U on $[0,1] \times [0,\Gamma]$

$$B_n(U,x,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n}{i}\binom{n}{j} U\left(\frac{i}{n},\frac{Tj}{n}\right)(1-x)^{n-i}x^i\left(1-\frac{t}{T}\right)^{n-j}\left(\frac{t}{T}\right)^j$$

and know that: $B_n(U,.,.) \to U$

$$\frac{\partial}{\partial t} B_n(U,.,.) \to U_t$$

$$\frac{\partial}{\partial x} B_n(U,.,.) \to U_x$$

$$\frac{\partial^2}{\partial x^2} B_n(U,.,.) \to U_{xx}$$

uniformly on $[0,1] \times [0,T]$ for $n \to \infty$.

As Butzer has shown in [3] for functions U in $C^1([0,1]^2)$, we can prove it for our case.

Now we set $U_\varepsilon = B_n(U,.,.)$ with $n > N(\varepsilon)$ fixed so that

$$\|U-U_\varepsilon\|_\infty + \|U_t-U_{\varepsilon t}\|_\infty + \|U_x-U_{\varepsilon x}\|_\infty + \|U_{xx}-U_{\varepsilon xx}\|_\infty \leqslant \varepsilon$$

and define: $v_\varepsilon = U_\varepsilon - [(1-x)(U_\varepsilon(0,t)-\gamma_0(t)) + x\ (U_\varepsilon(1,t)-\gamma_1(t))]$.

We have $\begin{cases} v_\varepsilon(0,t) = \gamma_0(t) \\ v_\varepsilon(1,t) = \gamma_1(t) \end{cases}$ and v_ε is a function

in $C^m([0,1] \times [0,T])$, because γ_0, γ_1 are in $C^m([0,T])$
$B_n(U,..) = U_\varepsilon$ is in $C^\infty([0,1] \times [0,T])$ and moreover:

$$\|U-v_\varepsilon\|_\infty + \|U_t-v_{\varepsilon t}\|_\infty + \|U_x-v_{\varepsilon x}\|_\infty + \|U_{xx}-v_{\varepsilon xx}\|_\infty \leqslant 2\varepsilon+2\varepsilon+2\varepsilon+\varepsilon = 7\varepsilon \quad .$$

That means, that we have constructed a function v_ε in $C^m([0,1] \times \times [0,T])$ which has the boundary values as U and which approximates U, U_t, U_x and U_{xx} uniformly on the closed rectangle $[0,1] \times [0,T]$.

We consider the neighboring problem:

$$\text{I*)}\quad v_t = a(x,t)v_{xx} + b(x,t)v_x + c(x,t)v + f(x,t) +$$

$$+ (v_{\varepsilon t}-a(x,t)v_{\varepsilon xx}-b(x,t)v_{\varepsilon x}-c(x,t)v_\varepsilon-f(x,t))$$

$$(x,t) \in (0,1] \times (0,T]$$

$$\text{II*)}\quad v(x,0) = v_\varepsilon(x,0),\ x \in [0,1]$$

$$\text{III)}\quad v(0,t) = \gamma_0(t),\ v(1,t) = \gamma_1(t),\ t \in [0,T] \quad [\text{III*} = \text{III}]$$

which has the unique solution $v = v_\varepsilon$.

We set: $Z_\varepsilon = v_{\varepsilon t}-a(x,t)v_{\varepsilon xx}-b(x,t)v_{\varepsilon x}-c(x,t)v_\varepsilon-f(x,t)$,

$$Z_\varepsilon \in C^{m-2}([0,1] \times [0,1]),$$

and conclude

$$\|Z_\varepsilon\|_\infty \leqslant \|U_t-a(x,t)U_{xx}-b(x,t)U_x-c(x,t)U-f(x,t)\| +$$

$$+\|U_t-v_{\varepsilon t}-a(x,t)(U_{xx}-v_{\varepsilon xx})-b(x,t)(U_x-v_{\varepsilon x})-c(x,t)(U-v_\varepsilon)\| \leqslant$$

$$\leqslant 0+(1+\|a\|_\infty+\|b\|_\infty+\|c\|_\infty)\varepsilon = C_1\varepsilon,\ C_1 \in \mathbb{R} \quad .$$

The numerical procedure for I*), II*), III) has the form

$$(\bar{v})\begin{cases}\frac{1}{k}[B_0(k,t_n)V_\varepsilon^n - B_1(k,t_{n-1})V_\varepsilon^{n-1} - R(k,t_n)] = \hat{f}(t_n) + \hat{Z}_\varepsilon(t_n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , n = 1(1)M \\ V_\varepsilon^o = (v_\varepsilon(x_1,0), - , v_\varepsilon(x_{N-1},0))^T \end{cases}$$

and converges to v_ε of order l, that means:
$\| V_\varepsilon^n(k) - V_\varepsilon(t_n)\| \leqslant C(\varepsilon)k^l$, because the order of convergence is the same as the order of consistency in the case of smooth solutions.

The procedure for I), II), III) is:

$$(\bar{v}\bar{v})\begin{cases}\frac{1}{k}[B_0(k,t_n)U^n - B_1(k,t_{n-1})U^{n-1} - R(k,t_n)] = \hat{f}(t_n) \\ \\ U^o = U_o = (U_o(x_1), \ldots, U_o(x_{N-1}))^T \end{cases} .$$

We subtract $(\bar{v})$ from $(\bar{v}\bar{v})$ and get:

$$\frac{1}{k}[B_0(k,t_n)(U^n-V_\varepsilon^n) - B_1(k,t_n)(U^{n-1}-V_\varepsilon^{n-1})] = -\hat{Z}_\varepsilon(t_n)$$

$$U^o-V_\varepsilon^o = (U_o(x_1) - v_\varepsilon(x_1,0), \ldots, U_o(x_{N-1}) - v_\varepsilon(x_{N-1},0))^T$$

We use that the solution of a difference equation of this form depends continuously on the initial condition and on the disturbance, if the boundary conditions are homogenous:

$$\| U^n-V_\varepsilon^n\| \leqslant L\| U^o-V_\varepsilon^o\| + P(LT+1)\max_{1\leqslant i\leqslant h}\|\hat{Z}_\varepsilon(t_n)\| \leqslant (7L+P(LT+1)C_1)\varepsilon = C_2\varepsilon \quad .$$

We get for t = nk fixed in (0,T]:

$$\| U(t)-U^n(k)\| \leqslant \| U(t)-V_\varepsilon(t)\| + \| V_\varepsilon(t)-V_\varepsilon^n(k)\| + \| V_\varepsilon^n(k)-U^n(k)\| \leqslant$$

$$\leqslant 7\varepsilon + C(\varepsilon)k^l + C_2\varepsilon = (7+C_2)\varepsilon + C(\varepsilon)k^l \quad .$$

For $k<\left(\frac{\varepsilon}{C(\varepsilon)}\right)^{\frac{1}{l}}$ we get $\|U(t)-U^n(k)\|\leq(8+C_2)\varepsilon$, where C_2 is independent of n, ε and k. If we start the proof with $\frac{\varepsilon}{8+C_2}$ convergence follows.

Our second step is to neglect the conditions a) and b) in Theorem 1. So we prove:

Theorem 2: consider the numerical procedure (*) for I), II) and III) under the same assumptions as in Theorem 1. Let (A), (B) and (C) be valid. If $U_o \varepsilon C([0,1])$ and $\gamma_0,\gamma_1 \varepsilon C^m([0,T])$, then the numerical procedure (*) is convergent to the unique solution of I), II) and III).

Proof: Let $\varepsilon>0$ be fixed. Then we choose a function $\bar{U}_o^\varepsilon$ in $C^\infty[(0,1])$, so that $\|U_o-\bar{U}_o^\varepsilon\|_\infty<\varepsilon$. The existence of $\bar{U}_o^\varepsilon$ is a consequence of the approximation theorem of Weierstrass. We define:

$$U_o^\varepsilon = \bar{U}_o^\varepsilon-[x(\gamma_1(0)-\bar{U}_o^\varepsilon(1)) + (1-x)(\gamma_0(0)-\bar{U}_o^\varepsilon(0))]$$

We get: $U_o^\varepsilon(0) = \gamma_0(0)$ and $U_o^\varepsilon(1) = \gamma_1(0)$ and

$$\|U_o-U_o^\varepsilon\|\leq\|U_o-\bar{U}_o^\varepsilon\|+|x|\varepsilon+|1-x|\varepsilon\leq 2\varepsilon \quad .$$

Now we choose a function $y^\varepsilon(x)\ \varepsilon\ C^3([0,1])$ fulfilling $y^\varepsilon(0) = = y^\varepsilon(1) = 0$ and $\|y^\varepsilon\|_\infty\leq\varepsilon$ and form $V_o^\varepsilon = U_o^\varepsilon + y^\varepsilon$. The function V_o^ε shall satisfy:

$$1)\ \gamma_0(0) = a(0,0)V_o^{\varepsilon''}(0) + b(0,0)V_o^{\varepsilon'}(0) + c(0,0)V_o^\varepsilon(0) + f(0,0)$$

$$2)\ \gamma_1(0) = a(1,0)V_o^{\varepsilon''}(1) + b(1,0)V_o^{\varepsilon'}(1) + c(1,0)V_o^\varepsilon(1) + f(1,0)$$

That means:

$$1a)\ \gamma_0(0) - [f(0,0) + a(0,0)U_o^{\varepsilon''}(0) + b(0,0)U_o^{\varepsilon'}(0) + c(0,0)U_o^\varepsilon(0)] = a(0,0)y^{\varepsilon''}(0) + b(0,0)y^{\varepsilon'}(0)$$

$$1b)\ \gamma_1(0) - [f(1,0) + a(1,0)U_o^{\varepsilon''}(1) + b(1,0)U_o^{\varepsilon'}(1) + c(1,0)U_o^\varepsilon(1)] = a(1,0)y^{\varepsilon''}(1) + b(1,0)y^{\varepsilon'}(1) \quad .$$

We choose $y^{\varepsilon'}(0) = y^{\varepsilon'}(1) = 0$ and compute $y^{\varepsilon''}(0) = y_1$ and $y^{\varepsilon''}(1) = y_2$ from the equations 1a) and 2a) and construct:

$$y^{\varepsilon}(x) = \left\{ \begin{array}{ll} \frac{y_1}{2t_1^2} x^2 (x-t_1)^{"} & 0 \leqslant x \leqslant t_1 \\ 0 & t_1 \leqslant x \leqslant t_2 \\ \frac{y_2}{2t_2^2}(x-1)^2(x-t_2)^4 & t_2 \leqslant x \leqslant 1 \end{array} \right\} \varepsilon C^3([0,1])$$

with $0 < t_1 < \min\left(\frac{1}{2}, \sqrt[4]{\frac{729\varepsilon}{8|y_1|}}\right)$ for $y_1 \neq 0$ and

$$0 < t_2 < \min\left(\frac{1}{2}, \sqrt[4]{\frac{729\varepsilon}{8|y_2|}}\right) \text{ for } y_1 \neq 0 \quad .$$

Otherwise there is no restriction on t_1 resp t_2 (only $0<t_1<t_2<1$).

Now we consider:

$$(\Delta) \begin{cases} V_t = a(x,t)V_{xx}+b(x,t)V_x+c(x,t)V+f(x,t) \quad , & (x,t)\varepsilon(0,1]\times(0,T] \\ V(x,0) = V_o^{\varepsilon}(x) & x\varepsilon[0,1] \\ V(0,t) = \gamma_0(t) & t\varepsilon(0,T] \\ V(1,t) = \gamma_1(t) & \end{cases}$$

We have: $V_o^{\varepsilon}\varepsilon C^3([0,1])$, γ_0, $\gamma_0 \varepsilon C^m([0,T])$, $V_o^{\varepsilon}(0) = \gamma_0(0)$, $V_o^{\varepsilon}(1) = \gamma_1(0)$ and V_o^{ε}, γ_0, γ_1 fulfill the condition a) and b) in theorem 1. So we can conclude, that this problem has a unique solution V_ε, so that V_ε, $V_{\varepsilon t}$, $V_{\varepsilon x}$, $V_{\varepsilon xx}$ are continuous in $[0,1]\times[0,T]$. Also we can conclude that $Z = U-V_\varepsilon$ is the unique solution of

$$(\Delta\Delta) \begin{cases} Z_t = a(x,t)Z_{xx} + b(x,t)Z_x + c(x,t)Z \\ Z(x,0) = U_o(x) - V_o^{\varepsilon}(x) \\ Z(0,t) = Z(1,t) \equiv 0 \end{cases}$$

(U is the unique solution of the given problem).
We know that the solution Z depends continuously on the initial data Z(x,0), so we have:

$$\|Z\|_\infty = \|U-V_\varepsilon\| \leq C.\|U_o - V_o^\varepsilon\| \leq C\varepsilon \quad .$$

The numerical procedure to the given problem has the form:

$$\frac{1}{k}[B_0(k,t_n)U^n_{(k)} - B_1(k,t_{n-1})U^{n-1}_{(k)} - R(k,t_n)] = \hat{f}(t_n)$$

and to (Δ)

$$\frac{1}{k}[B_0(k,t_n)V_\varepsilon^n(k) - B_1(k,t_{n-1})V_\varepsilon^{n-1}(k) - R(k,t_n)] = \hat{f}(t_n)$$

$$V_\varepsilon^o = V_o^\varepsilon$$

We conclude by subtracting:

$$\frac{1}{k}[B_0(k,t_n)(U^n(k)-V_\varepsilon^n(k)) - B_1(k,t_{n-1})(U^{n-1}(k)-V_\varepsilon^{n-1}(k))] = 0$$

$$U^o - V_\varepsilon^o = U_o - V_o^\varepsilon$$

We get by stability: $\|U^n(k)-V_\varepsilon^n(k)\| \leq L\|U_o-V_o^\varepsilon\| \leq 3L\varepsilon$.
Applying theorem 1 we conclude, that there is a $k_o(\varepsilon)>0$ so that for all $k<k_o(\varepsilon)$, $\|V_\varepsilon(t) - V_\varepsilon^n(k)\| \leq \varepsilon$ for $t = nk$ fixed in $[0,T]$.
So,

$$\|U(t)-U^n(k)\| \leq \|U(t)-V_\varepsilon(t)\| + \|V_\varepsilon(t)-V_\varepsilon^n(t)\| + \|V_\varepsilon^n(t)-U^n(k)\| \leq$$

$$\leq C\varepsilon + \varepsilon + 3L\varepsilon = (C+1+3L)\varepsilon \quad .$$

And that means convergence.

Putting the used proof-methods on a more formal level we can derive an extension to Lax's convergence theorem for stable approximations to linear operator equations which are consistent for data in a dense set. Consider the linear and invertible operator $F : (A,\|\ \|_A) \to (B,\|\ \|_B$ where A, B are appropriate linear spaces and let $\|F^{-1}\|_B$ be bounded by k_1. That means that the solution U of the equation FU = g depends continuously on the data g. For the numerical computation of U we use approximations $F_hU_h = g_h$ with the following properties:

1) $F_h: (A_h, \| \|_{A_h}) \to (B_h, \| \|_{B_h})$ for $0<h\leqslant h_o$ (step-size, grid parameter), where A_h, B_h are appropriate linear spaces.

2) F_h is linear and invertible and $\|F_h^{-1}\|_{B_h} \leqslant k_2$ for all $h \leqslant h_o$.

The last property of F_h is called stability:

3) There exist linear and uniformly bounded operators,

$\Delta_h^A; (A, \| \|_A) \to (A_h, \| \|_{A_h})$

$\Delta_h^B; (B, \| \|_B) \to (B_h, \| \|_{B_h})$

4) $\| \Delta_h^B(g) - g_h \|_{B_h} = o(1)$ for $h \to o$.

5) The scheme $F_hU_h = g_h$ is consistent with FU = g for all $g \varepsilon X C B$, where X is dense in B, i.e.,

$\| F_h(\Delta_h^A U) - g_h \|_{B_h} = o(1)$ for $h \to 0$

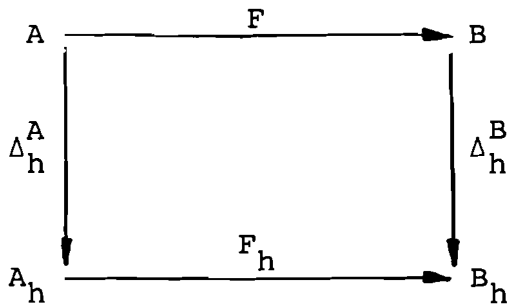
where U is the solution of FU = g.

We can conclude:

Theorem 3: under the given assumptions on F and F_h the procedure $F_hU_h = g_h$ is convergent to the solution U of the equation FU = g, for all $g \varepsilon B$, i.e.,

$$\| \Delta_h^A(U) - U_h \|_{A_h} = o(1) \text{ for } h \to 0.$$

Proof: We have the following situation:



Let ε fixed be greater o.
For solving $FU = g$ we consider the scheme $F_hU_h = g_h$. Because X is dense in B we can choose $g_\varepsilon \varepsilon X$ so that $\|g-g_\varepsilon\|_B \leqslant \varepsilon$. Instead of $FU = g$ we now solve $FU_\varepsilon = g_\varepsilon$. We conclude $\|U-U_\varepsilon\|_A \leqslant \|F^{-1}\| \; \|g-g_\varepsilon\|_B$ that means:

A) $\|U-U_\varepsilon\|_A \leqslant k_1 \varepsilon$.

Now we consider $F_hU_{\varepsilon h} = g_{\varepsilon h}$ and we easily prove the convergence of $U_{\varepsilon h}$ to U_ε for $h \to o$ and fixed $\varepsilon > o$ by the usual consistency - stability method:

$$F_hU_{\varepsilon h} = g_{\varepsilon h}$$

$$\underline{F_h\Delta_h^AU_\varepsilon = g_{\varepsilon h} + C_\varepsilon(h)} \qquad \|C_\varepsilon(h)\|_{B_h} = o(1) \text{ for } h \to o \text{ and fixed } \varepsilon > o \text{ because } g_\varepsilon \varepsilon X.$$

$$F_h(U_{\varepsilon h}-\Delta_h^AU_\varepsilon) = -C_\varepsilon(h) \Rightarrow$$

B) $\|U_{\varepsilon h}-\Delta_h^AU_\varepsilon\|_{A_h} \leqslant \|F_h^{-1}\| \; \|C_\varepsilon(h)\|_{B_h} \leqslant k_2 c_\varepsilon(h)$

$c_\varepsilon(h) \to 0$ for $h \to o$
ε <u>fixed</u> greater than o.

Now we want to find a bound for $U_{\varepsilon h} - U_h$:

$$U_{\varepsilon h} - U_h = F_h^{-1}(g_{\varepsilon h}-g_h) = F_h^{-1}(g_{\varepsilon h}-\Delta_h^Bg_\varepsilon+\Delta_h^Bg_\varepsilon-\Delta_h^Bg+\Delta_h^Bg-g_h)$$

C) $\|U_{\varepsilon h}-U_h\|_{A_h} \leqslant k_2(\|g_{\varepsilon h}-\Delta_h^Bg_\varepsilon\|_{B_h}+\|\Delta_h^Bg_\varepsilon-\Delta_h^Bg\|+\|\Delta_h^B-g_h\|)$

$$= k_2d_\varepsilon(h) \to 0 \text{ for } h \to 0 \text{ and } \varepsilon > 0 \text{ fixed}$$

because of the assumptions 2), 3) and 4). So we can conclude from (A), (B) and (C):

$$\| \Delta_h^A U - U_h \|_{A_h} \leqslant \| \Delta_h^A U - \Delta_h^A U_\varepsilon \|_{A_h} + \| \Delta_h^A U_\varepsilon - U_{\varepsilon h} \|_{A_h} + \| U_{\varepsilon h} - U_h \|_{A_h} \leqslant \| \Delta_h^A \| \cdot$$

$$\cdot \; k_1 \varepsilon + k_2 c_\varepsilon (h) + k_2 d_\varepsilon (h) \quad .$$

We can find for every $\varepsilon > o$ a $h < h(\varepsilon)$ so that $\| \Delta_h^A U - U_h \|_{A_h} \leqslant C\varepsilon$ where C is independent of ε, h and that means convergence.

It is easy to extend Theorem 3 to cases where the difference scheme F_h is uniformly continuous in h (stable) in some components of the data-vector g, but not in all. The methods for doing this are the same as used in Theorem 2, because stability of one step difference - approximation means that the solutions $U^n(k)$ depend uniformly continuous (in the grid-parameter k) on the initial data and on the disturbance but not on the boundary values.

Remark

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