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ABSTRACT

A correspondence is observed between a class of n-person cooperative games and production functions with fixed, discrete factor inputs. This correspondence motivates a simple way of valuing the players (or factors): the players, or factor representatives, set prices on themselves in the face of a market of buyers. A noncooperative price-setting game results for which equilibrium prices always exist. Interpreted as a cooperative game it always has a core, which reduces to the core of the original game if the latter is nonempty. This concept was originally applied to the problem of determining the relative value of the players in a voting game when a market exists for their votes.

The Market Value of a Game

Consider a cooperative game, by which we mean a group of individuals whose cooperation can produce valuable results. The potential 'value' of the different possible subgroups of individuals is assumed known. A fundamental problem is how to attribute value to the individuals based on their contribution to coalitions.

It is customary to treat this problem axiomatically, regarding the game as an entity complete in itself, played in isolation from the rest of the universe [4,7]. Doubtless this is a convenient and reasonably correct assumption for many types of games. However, there is a large class of games, including for example many 'economic' types of games, for which this approach is inappropriate. These games are characterized by the feature that the players' actions frequently have value to agents *outside* the game and this value is completely transferable.

Examples of such games abound. A classic instance would be any cooperative agreement among a group of individuals to produce goods for an external market by a division of labor. A second example is the control of production by oligopolies. Yet a third is the class of *political* games, in which allocation decisions to outside interest groups are made by certain coalitions of decision makers. The latter is an example of a situation where the players' (i.e., the politicians') actions actually have no direct value, in and of themselves, to the players at all: rather, they have value only to the constituents, who are outside the game. Of course, these constituents may in turn be willing to compensate the players for their actions.

It will be shown that the existence of an external market for a game has direct implications for the way the players can be valued. The reason is that under various valuations there will be an incentive for outsiders to buy control of particular subsets of players on which profits can be made.

If the core of the game exists, and if the players act cooperatively, then they can be consistently valued by any imputation in the core, and outside agents will not be able to make any profits. If the core does not exist, however, then the players cannot obtain the whole value of the game in the face of a market. Assuming that the players cooperate, exactly enough profits will be skimmed off by entrepreneurs to allow a core to exist on what is left over. If the players act noncooperatively, then even more may be skimmed off in pure profits. In other words, the nonexistence of the core of a game means that positive profits can be made by entrepreneurs, and that structurally there is a possibility for exploitation.

These results apply not only to the class of games mentioned above, but to virtually any production function with discrete factor inputs. Further, while it would be possible to treat the external market as a part of an 'enlarged' game, a good deal of flexibility is sacrificed in so doing: the essential role of the game as production activity is obscured, and other important connections with the outside universe such as opportunity costs are also lost. A more fruitful approach is to focus on the game as the entity of interest without forgetting the market "environment" that conditions its behavior.

Let v be the characteristic function of a cooperative n-person game in which payoffs are made to the players by agents outside the game in return for valuable actions the players perform. The set of players will be denoted by $N = \{1,2,...,n\}$.

v is assumed to satisfy the following two properties:

(1) $$v(S) \geqq v(\phi) = 0 \quad \text{for all } S \subseteq N ,$$

(2) $$v(S \cup T) \geqq v(S) + v(T) \quad \text{whenever } S \cap T = \phi .$$

v may be thought of as a *production function* whose *factor inputs* are the players. In the language of production theory, condition (1) allows free disposal, and condition (2) allows joint production.

Conversely, virtually any production activity whose inputs are discrete in nature can formally be described by such a production function v. If each factor is thought of as being "represented" by a player (for example, its owner), then a cooperative game is defined. This establishes a correspondence between cooperative games and discrete production functions that has important consequences.

Example 1. (*Team Recruitment*)

An example of a team recruitment problem is the following: athletic managers are to recruit teams from a draft pool of players in a sport. Each potential team has a box office value depending on the individuals composing it. The value of two teams taken together is at least the sum of their values taken separately.

Another example would be the recruitment of performing artists by booking agents. Consider the following simple numerical example. A night-club owner wants to hire singers from a "pool" consisting of a soprano (S), alto (A), and a contralto (C). The values of the different combinations of players are

$$v(\phi) = 0$$

$$v(S) = 80 \qquad v(S,A) = 900$$

$$v(A) = 50 \qquad v(S,C) = 800$$

$$v(C) = 40 \qquad v(A,C) = 400$$

$$v(S,A,C) = 1000$$

Example 2. (*Joint Resource Use*)

Four countries A,B,C,D, border a sea that can be exploited for commercial fishing. Each may establish controls on overfishing within its own territorial waters; however, because of interdependencies, cooperation in setting controls leads to greater productivity in the total size of the catch. Let $v(\mathcal{S})$ represent the monetary value that a set $\mathcal{S}$ of countries can obtain by setting policies together:

$$v(\phi) = 0$$

$$v(A) = v(B) = v(C) = 1, \quad v(D) = 0$$

$$\begin{array}{llll} v(A,B) = 10 & & v(A,B,C) = 13 \\ v(A,C) = 9 & & v(A,B,D) = 12 \\ v(A,D) = 8 & & v(A,C,D) = 11 \\ v(B,C) = 7 & & v(B,C,D) = 7 \\ v(B,D) = 4 & & \\ v(C,D) = 5 & & v(A,B,C,D) = 15 \end{array} .$$

These examples are relatively simple; the structure of the production function v may in reality be extremely complicated combinatorially, reflecting complex substitution possibilities between the factors. Hence the relative value of the factors is not at all obvious. In fact, it will turn out that there may be a multiplicity of valuations of the factors; nevertheless definite bounds can be placed on the region within which all economically plausible valuations must fall.

Two approaches may be taken. The first allows the market participants outside the game to set prices by bidding on the factors. The second views the external market as responding to prices that are set by the players. The latter approach only makes sense of course if the factors really are represented by agents who can act to set prices; in other words, if the production function really is a "game" rather than a collection of mute factors.

It is the second approach that will be adopted in this paper. However it can be shown that a natural bidding model leads to exactly the same values as are derived here, hence the two approaches are compatible [10].

Given v, let $\underset{\sim}{p} = (p_1, p_2, \ldots, p_n) \geqq \underset{\sim}{0}$ be a hypothetical set of *prices* for the players. For any subset $S \subseteq N$ the *profit* of S relative to $\underset{\sim}{p}$ is $v(S) - \sum_S p_i$. In particular, the empty set always yields zero profit. [1)]

The external market will be treated in the sketchiest of terms. It may consist of one, several, or many agents, who respond to prices set by the players. Only one assumption is made regarding the market:

(3) *Market Postulate. For any given prices* $\underset{\sim}{p}$ *the set of players bought constitutes a maximum profit set.*

This assumption is certainly plausible if there is only one buyer. If there are several buyers, they can be thought of as arriving at the purchase window in some order, and a similar outcome obtains. Other market models supporting this hypothesis can easily be imagined.

<u>COOPERATIVE MARKET VALUE</u>

Let the players in the game propose some division of their joint proceeds. What is the maximum amount they can obtain?

Suppose for example that the trio proposes the distribution

$$p_S = \$633\ 1/3 \quad , \quad p_A = \$233\ 1/3 \quad , \quad p_C = \$133\ 1/3 \quad .$$

This distribution effectively establishes prices for the various players. A market viewing the distribution will see that each of the three duos constitutes a most profitable set (the profit being \$33 1/3 for each), whereas the trio would yield zero profit. Hence one of the duos will be hired, and some player will be excluded. But then the players will not obtain the full \$1000, so the proposed distribution is infeasible.

Of course, it might be argued, the players could act as a coalition and simply insist that they must all be bought together or not at all. Then they would receive \$1000, and could split it as proposed. Unfortunately there is a very strong incentive for such a coalition to break up, since any duo would be better off by offering to defect (for a bonus) and the market agents might well try to induce them to do so.

The conclusion is that if the trio has any hope of obtaining \$1000, they cannot split it in this way.

In general, let $x_1, x_2, \ldots, x_n$ be a proposed distribution to the coalition N. As viewed by the market, the amounts x_i constitute effective prices for the players. Hence N will only be able to obtain the amount $\sum_N x_i$ if the set of players bought contains all players i such that $x_i > 0$. Thus there must be a maximum profit set T with respect to $\underset{\sim}{x}$ such that $x_i = 0$ for $i \notin T$.

If q is the profit from T, then

$$(4) \qquad q = v(T) - \sum_T x_i \geqq v(S) - \sum_S x_i \qquad \text{for all } S \subseteq N \quad .$$

However, $x_i = 0$ for $i \notin T$ implies

$$(5) \qquad v(N) - \sum_N x_i = v(N) - \sum_T x_i \geqq v(T) - \sum_T x_i = q \quad ,$$

so equality holds in (5) and N is also a maximum profit set with respect to $\underset{\sim}{x}$.

The *maximum amount* N can obtain is therefore q*, where q* is an optimum to the linear program

$$(6) \qquad \begin{aligned} &\min q \\ &\text{subject to} \quad q + \sum_S p_i \geqq v(S) \qquad \text{for all } S \\ &\qquad\qquad\quad q + \sum_N p_i = v(N) \quad . \\ &\qquad\qquad\qquad\quad \underset{\sim}{p} \geqq \underset{\sim}{0} \end{aligned}$$

Here $\underset{\sim}{p}$ has been identified with $\underset{\sim}{x}$. Note that $q^* \geqq 0$ by virtue of the inequality with $S = \phi$.

It is easily seen that (6) always has an optimal solution, since $q = v(N)$, $\underset{\sim}{p} = \underset{\sim}{0}$ is feasible and q is clearly bounded below. Any optimal solution $\underset{\sim}{p}^*$ to (6) will be called a *cooperative market value* for v. A cooperative market value represents a distribution to the players that yields a maximum total return to the players in the presence of a market which provides the payoffs. If the game v has a core, then the minimum value of q is zero, and the cooperative market values are precisely the distributions in the core. Thus the cooperative market value concept generalizes the core in a natural way.

In Example 1 the unique cooperative market value is $p_S^* = \$600$, $p_A^* = \$200$, $p_C^* = \$100$, and the market absorbs \$100 in pure profits. In this sense the players can be exploited by a booking agent, nightclub owner, or other entrepreneur.

Of course, for the players to actually receive these amounts, all of them must be bought. But each duo also yields as much profit as the trio, \$100. What assures that the market will buy the trio instead of some duo at these prices? The answer is that the players *could* receive these amounts; moreover they can all *assure* themselves of *up to* these amounts, since if all "shade" their prices by a small amount ε, then {S,A,C} will be the *unique* most profitable set as viewed by the market.

In general for any game v, if all players with positive cooperative market values "shade" their prices by a small amount ε, then all are certain of being bought. Hence the cooperative market value can also be interpreted as a limiting distribution to the players that is independent of how the market resolves a "tie."

One can similarly ask how much any *subcoalition* $C \subseteq N$ could obtain (in the limit) by a suitable pricing of its members. For C to be able to guarantee itself an amount α, it must be true that no matter what prices are asked by the players *not* in C, C will be contained in some maximum profit set and receive the amount $\underset{\sim}{\alpha}$. More precisely, C can guarantee itself α if there is a $|C|$-vector $\underset{\sim}{p}^C \geqq \underset{\sim}{0}$, $\sum_C p_i^C = \alpha$, such that for any prices $\underset{\sim}{p}^{N-C} \geqq \underset{\sim}{0}$ quoted by the players in $N - C$, there exists a maximum profit set T relative to $\underset{\sim}{p} = (\underset{\sim}{p}^C, \underset{\sim}{p}^{N-C})$ with $\sum_{C \cap T} p_i = \alpha$.

The maximum amount C can obtain in this way is denoted by $v^*(C)$, and the n-person game v^* so defined is called the *cooperative sell-out game*. It can then be shown that

every cooperative market value for v *is in the core of* v^*.

This result, which is a special case of Theorem 2 below, says that no coalition can, by any pricing policy, guarantee itself more than it gets under the prices represented by a cooperative market value. Combined with the earlier observations about the relationship between cooperative market values and the core of v, it also implies that

the core of v^* *is always nonempty, and contains the core of* v.

It will also be shown (Theorem 2) that while the core of v^* may contain distributions other than the cooperative market values, none of these meet the test of being a "noncooperative equilibrium" (to be defined below) and therefore are not viable.

In the trio game of Example 1, consider the coalition of the soprano (S) and the alto (A). If both have positive prices, $p_S, p_A > 0$, then $\{S,A\}$ will be contained in a maximum profit set when C charges p_C only if

$$\left\{\begin{array}{r} 400 - p_C - p_A \leqq 900 - p_S - p_A \\ 800 - p_C - p_S \leqq 900 - p_S - p_A \\ 0 \leqq 900 - p_S - p_A \end{array}\right\} \text{ or } \left\{\begin{array}{r} 400 - p_C - p_A \leqq 1000 - p_S - p_A - p_C \\ 800 - p_C - p_S \leqq 1000 - p_S - p_A - p_C \\ 0 \leqq 1000 - p_S - p_A - p_C \end{array}\right\}$$

that is,

$$\left\{\begin{array}{l} p_S \leqq 500 + p_C \\ p_A \leqq 100 + p_C \\ p_S + p_A \leqq 900 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} p_S \leqq 600 \\ p_A \leqq 200 \\ p_S + p_A \leqq 1000 - p_C \end{array}\right\}$$

The $\min\limits_{p_C \geqq 0} \; \max\limits_{p_S, p_A > 0} (p_S + p_A)$ is achieved when $p_S = 600$, and $p_A = 200$. Thus $v^*(S,A) = \$800$. Now the maximum amount that the singleton set $\{S\}$ can guarantee itself is at most $v(S) = \$80$, since otherwise the others might charge so much that only the empty set yields a nonnegative profit. On the other hand, with $p_S = \$80$ none of the sets ϕ, $\{A\}$, $\{C\}$, $\{A,C\}$ can yield a higher profit than does $\{S,A,C\}$, hence $v^*(S) = \$80$. By this type of reasoning we find that

$$v^*(\phi) = 0$$

$$v^*(S) = 80 \qquad v^*(S,A) = 800$$

$$v^*(A) = 50 \qquad v^*(S,C) = 700$$

$$v^*(C) = 40 \qquad v^*(A,C) = 300$$

$$v^*(S,A,C) = 900$$

In this case the cooperative market value $\underset{\sim}{p}^* = (600,200,100)$ is the unique vector in the core of v^*.

NONCOOPERATIVE MARKET VALUE

If the core of the original game v is empty, it can easily be imagined that the players will not cooperate at all, because there is not enough economic glue to hold self-organized coalitions together. This does not mean, however, that the benefits the players can produce by joint action will be lost. It simply means that coalitions will be organized from the outside by entrepreneurs. Under this regime production is efficient -- the maximum possible value v(N) will be produced. For their services, however, the entrepreneurs extract a profit. This profit is necessarily positive if the game has no core; in fact, it may be very large. The minimum profit that will be extracted if the players cooperate in setting their prices is the optimal q^* of (6). This value will be called the *extractable value* of the game, and denoted by $q^*(v)$. It represents the minimum amount that must be skimmed off of each coalition's value for the core to come into existence.

If the players do not cooperate, then even greater profits may be extracted. However the possible prices that can hold even in this setting are quite restricted. In fact, we shall show that the attempts of the players to find their most advantageous prices relative to the others' has an equilibrium outcome.

Define the "price-setting" game as follows. For any possible prices $\underset{\sim}{p}$ let $f(\underset{\sim}{p})$ be the maximum profit set that is actually bought at these prices. Typically there is only one such set for a given $\underset{\sim}{p}$, however f serves as a tie-breaking rule if there is more than one maximum profit set. The function f is called a *market schedule*. Some specification of f is necessary for the price setting game to be well-defined, and it turns out that there always is some choice of f that yields a price equilibrium. Happily, the equilibrium prices and payoffs, if they do exist, do not depend on which particular f they come from.

Define the *noncooperative sell-out game* for a given f to be the game whose player set is N, and in which a *strategy* of player i is to name a nonnegative real number p_i (his *price*); the *payoff* to i given the strategy $\underset{\sim}{p} = (p_1,p_2,\ldots,p_n)$ is then

$$\varphi_i(\underset{\sim}{p},f) = \begin{cases} p_i & \text{if } i \in f(\underset{\sim}{p}) \;, \\ 0 & \text{if } i \notin f(\underset{\sim}{p}) \;. \end{cases}$$

A price vector $\underset{\sim}{p} \geqq \underset{\sim}{0}$ is a *strong noncooperative equilibrium* (hereafter called simply an *equilibrium*) and $(\underset{\sim}{p},f)$ is an *equilibrium pair* if there is no set of players that can change prices in such a way that *each* receives a higher payoff than before, assuming that none of the other players changes price [1]. That is, *there is no nonempty* $C \subseteq N$ *such that*

$$p_i' = p_i \text{ for all } i \notin C \quad ,$$

$$\phi_i(\underset{\sim}{p}',f) > \phi_i(\underset{\sim}{p},f) \quad \text{for all } i \in C \quad .$$

The class of all equilibria is easily characterized. Let $\mathcal{S}^0 = \{S \subseteq N : v(S) = \max\}$ be the family of maximum value sets, called *critical sets*. By assumption, N is a critical set. The *critical players* N^0 are the players contained in every critical set: $N^0 = \bigcap_{S \in \mathcal{S}^0} S$.

For any price vector $\underset{\sim}{p}$ define $\overline{\underset{\sim}{p}}$, the *normalization* of $\underset{\sim}{p}$, as follows:

$$\begin{aligned} \overline{\underset{\sim}{p}} &= p_i \quad \text{if} \quad i \in N^0 \\ &= 0 \quad \text{otherwise} \quad . \end{aligned}$$

$\underset{\sim}{p}$ is *normal* if $\underset{\sim}{p} = \overline{\underset{\sim}{p}}$.

<u>*Theorem 1*</u>. *Any game* v *has an equilibrium price vector* $\underset{\sim}{p}$; *moreover* $\underset{\sim}{p} \geqq \underset{\sim}{0}$ *is an equilibrium and* q *the corresponding extracted profit, if and only if*

(i) $q + \sum_S \overline{p}_i \geqq v(S) \quad$ for all $S \subseteq N$,

(ii) $q + \sum_T p_i = v(T) \quad$ for some critical set $T \subseteq N$,

(iii) for every player k there exists $S \subseteq N$ with $k \notin S$ and $q + \sum_S p_i = v(S)$.

These conditions say that $\underset{\sim}{p}$ is an equilibrium if the maximum profit q is realized on some critical set T and T remains a maximum profit set (with the same profit q) even when all non-critical players quote a price of zero; further, no player is in *every* maximum profit set. The latter condition is clearly necessary, else some player could raise his price further.

If the conditions are accepted for the moment as sufficient, it is easy to see why an equilibrium exists, and how to construct one. Beginning with prices $\underset{\sim}{p}^0$ and some critical player k, raise his price to the point where he is not contained in some maximum profit set, and do this successively for all of the critical players. In a finite number of steps, condition (iii) must be satisfied. At this point the constructed price vector $\underset{\sim}{p}$ satisfies $p_i = \overline{p}_i = 0$ for all noncritical i, and by construction all critical sets are maximum profit sets, so conditions (i)-(iii) are satisfied and $\underset{\sim}{p}$ is an equilibrium. (This does not imply of course that *all* equilibria may be constructed in this way.)

This theorem is a generalization of results in [8] for simple games, and the proof that the conditions are necessary and sufficient is given in the Appendix. Here we note several corollaries.

The conditions imply that in any maximum profit set S all noncritical players have zero price. In particular, no matter what f is, all noncritical players will receive zero payoff. Furthermore, if S* is the set of players sold and T the maximum profit critical set guaranteed by condition (ii), then any player in T - S* with a *positive* price could lower his price and improve his income. Hence all players in T - S* must have zero price, which implies S* is also a critical set. This shows that in equilibrium the payoff to any player i is precisely $\overline{p}_i$.

<u>*Corollary 1.1:*</u> $(\underset{\sim}{p}, f)$ *is an equilibrium pair iff* $\underset{\sim}{p}$ *satisfies (i)-(iii) and* $f(\underset{\sim}{p})$ *is a critical set.*

<u>*Corollary 1.2:*</u> *For any equilibrium price vector* $\underset{\sim}{p}$, *the equilibrium payoffs are* $\overline{\underset{\sim}{p}}$.

<u>*Corollary 1.3:*</u> *If* $\underset{\sim}{p}$ *is an equilibrium then so is* $\overline{\underset{\sim}{p}}$.

Corollaries 1.2 and 1.3 say that the payoff vector from any equilibrium set of prices is itself a normal equilibrium. Any normal equilibrium will be called a *noncooperative market value* for v.

<u>*Theorem 2*</u>. *Every cooperative market value is a noncooperative market value; in fact, the set of cooperative market values is precisely the set of noncooperative market values contained in the core of* v*.

The proof is given in the Appendix.

The fishing rights game (Example 2) has a one-point core consisting of the allocation (6,4,3,2) to players A,B,C,D respectively. This valuation of the players is both a cooperative and a noncooperative market value for the game and the corresponding extractable profit is zero. However, there are noncooperative market values not in the core. The reader may verify that each of the valuations $(8-x, 2+x, 3, x)$ where $0 \leqq x < 2$ is a noncooperative market value yielding a profit of $2-x$ units to the market.

<u>OPPORTUNITY COSTS</u>

In the sell-out game, if a player is not bought he gets nothing. This is because, by assumption, value can only be obtained through intermediary agents. Thus, if the players in a football draft pool are not hired by a team, they get nothing; if a fishing fleet does not buy a licence to fish in the territorial waters, the country gets nothing; if the singer is not contracted by the night-club owner, he gets nothing.

This assumption ignores, however, the true relation between the players in the game and the universe outside the game: in reality, each player or factor has an *opportunity cost* of being employed in the game v as opposed to doing something else. The maximum value of doing something else establishes a *floor price* $p_i^0 \geqq 0$ for each player i in the game. Thus if the football player or singer is not recruited, he can get an alternative job or collect unemployment compensation; if the country's coastal waters are not exploited for fishing they could be used for example for waste dumping. The opportunity costs $\underset{\sim}{p}^o$ are additional data of the problem not specified in the game v; however, these costs must be included in realistic applications.

The previous theory dealt with the "pure" case $\underset{\sim}{p}^o = \underset{\sim}{0}$. The results generalize in a straight-forward manner to the case of an arbitrary $\underset{\sim}{p}^o \geqq \underset{\sim}{0}$. However, direct connections with the core generally do not survive, except in the case where the opportunity cost of each player i is identified with v(i).

Given $\underset{\sim}{p}^o$, the *cooperative sell-out value* $v^*(C)$ of a coalition $C \subseteq N$ is the sum of its opportunity costs plus the maximum *additional* amount it can obtain by some distribution over and above these costs.

Now a coalition C can obtain an amount α in addition to its opportunity costs if there is a distribution $x_i \geqq 0$ for all $i \in C$ such that for $p_i^C = x_i + p_i^o$ and any feasible prices $p_j^{N-C} \geqq p_j^o$ demanded by the other players, there exists a maximum profit set T with respect to $(\underset{\sim}{p}^C, \underset{\sim}{p}^{N-C})$ with $\sum_{C \cap T} x_i = \alpha$.

For any distribution $\underset{\sim}{x} = (x_1, \ldots, x_n)$ which yields a maximum for the coalition N, $\underset{\sim}{x} + \underset{\sim}{p}^o = \underset{\sim}{p}$ will be called a *cooperative market value for* v *relative to* $\underset{\sim}{p}^o$.

A *critical set relative* to $\underset{\sim}{p}^o$ is any set which yields maximum profits when prices are $\underset{\sim}{p}^o$. The *critical players* N^o are those contained in every critical set. A price vector $\underset{\sim}{p}$ is *normal* if $\underset{\sim}{p} = \overline{\underset{\sim}{p}}$ where $\overline{\underset{\sim}{p}}$ is defined by

$$\overline{p}_i = \begin{cases} p_i & \text{if } i \in N^o \\ p_i^o & \text{if } i \notin N^o \end{cases} .$$

If $\underset{\sim}{x} + \underset{\sim}{p}^o = \underset{\sim}{p}$ is a cooperative market value then for some T

$$q = v(T) - \sum_T p_i \geqq v(S) - \sum_S p_i \quad \text{for all S and } p_i = p_i^o \text{ for all } i \notin T.$$

Hence

$$v(T) - \sum_T p_i^o - \sum_{T-S} x_i \geqq v(S) - \sum_S p_i^o \quad \text{for all } S \quad .$$

If S is critical, then equality must hold. Hence maximum profits are attained on all critical sets, and $p_i = p_i^o$ for all noncritical players.

The cooperative market values relative to $\underset{\sim}{p}^o$ are therefore the optimal solutions to the linear program

(7) $\quad \min q$

subject to

$$\underset{\sim}{p} \geqq \underset{\sim}{p}^o \text{ with } p_i = p_i^o \text{ for } i \notin N^o \quad ,$$

$$q + \sum_S p_i \geqq v(S) \quad \text{for all } S \quad ,$$

$$q + \sum_T p_i = v(T) \quad \text{for all critical sets } T \text{ with respect to } \underset{\sim}{p}^o .$$

The optimum value of (7) is called the *extractable value* of the game v given floor prices $\underset{\sim}{p}^o$. Clearly an optimum always exists.

We illustrate these ideas with the "trio" example. Suppose that each of the singers can work instead as a typist and earn \$250. The profit from the different coalitions, net of opportunity costs, is

$$w(\phi) = 0$$

w(S) = -170	w(S,A) = 400
w(A) = -200	w(S,C) = 300
w(C) = -210	w(A,C) = -100

$$w(S,A,C) = 250$$

The unique critical set is the duo {S,A}. If the soprano and the alto receive premiums of \$300 and \$100 respectively, then {S,A} yields a net profit of zero and no other set yields a positive net profit. Therefore by (7) the extractable value is zero, and (550,350,250) is a cooperative market value for v with the given floor prices. This is not the unique cooperative market value, however. The same analysis holds for any amounts x_S and x_A such that $x_S + x_A = 400$, $x_S \geqq 300$, $x_A \geqq 0$. Hence the cooperative market values form the family $\{(650 - x_A, 250 + x_A, 250): 0 \leqq x_A \leqq 100\}$.

If each player's opportunity cost is taken to be v(i), then the core of v exists if and only if the extractable value of the game is zero, and the cooperative market values constitute precisely the core of v.

For arbitrary floor prices $\underset{\sim}{p}^o \geqq \underset{\sim}{0}$ the definition of the noncooperative sell-out game is generalized as follows. The strategy space is defined to be the set of all $\underset{\sim}{p}$ with $\underset{\sim}{p} \geqq \underset{\sim}{p}^o$, and for any market schedule f, the payoff is

$$\begin{aligned} \phi_i(\underset{\sim}{p}, f) &= p_i \quad \text{if} \quad i \in f(\underset{\sim}{p}) \quad , \\ &= p_i^o \quad \text{if} \quad i \notin f(\underset{\sim}{p}) \quad . \end{aligned}$$

An *equilibrium* means a strong noncooperative equilibrium with respect to this game. Theorems 1 and 2 and their corollaries now generalize *verbatim* with the added condition that $\underset{\sim}{p} \geqq \underset{\sim}{p}^o$.

APPLICATION TO VOTING GAMES

A legislature is a natural example of a situation in which the players' actions do not yield value to the players directly, but to interest groups outside the game. The representatives of the interest groups, called *lobbyists*, view the game as a production process in which the object produced is a decision, and their object is to buy combinations of legislators that yield a desired outcome.

In the context of voting games opportunity costs also arise in a natural way. The alternative to selling out is to "stay honest," which doubtless (to most legislators) has a positive value. The fact that this value is particular to the legislator and may be nontransferable is of no importance: a floor price $p_i^o \geqq 0$ is assumed given for each player as a datum of the problem.

The voting game v is now interpreted as a production function in the following way. For a given issue that some interest group wants to have passed, every losing set S has zero monetary value, while every winning set T has a monetary value $v(T) = L$, where L is presumed to be "large" relative to the players' floor prices. If L is sufficiently large relative to the players' floor prices (a not unreasonable presumption) then, unless there is a veto player, the value of L is immaterial to the determination of either cooperative or noncooperative market value. This is because the price ceiling of a player is determined by the possibility of substituting other players for him, hence ultimately on the others' floor prices, not on the value of L.

Example 3. Consider the weighted voting game (3,1,1,1,1,1) where a weighted vote of 5 or more is required to win. Let every player have the same floor price $p^o > 0$, and let the value of every winning set to a lobbyist be some large number L. The critical sets S are those of form {3,1,1}, and player 1 is the unique critical player. By Theorem 1, the normal noncooperative equilibria are the solutions $\underset{\sim}{p}$ to the system

(0) $\underset{\sim}{p} \geqq \underset{\sim}{p}^o \underset{\sim}{1}$

(i) $q + \sum_S p_i \geqq L$ for all winning sets S

(ii) $q + \sum_S p_i = L$ for all critical sets S

(iii) $q + \sum_S p_i = L$ for some winning set S not containing player 1.

Since only player 1 is critical $p_i = p^o$ for players 2 to 6. Moreover, the only winning set *not* containing player 1 is {2,3,4,5,6}, so by conditions (ii) and (iii), $p_1 + 2p^o = 5p^o$, whence $p_1 = 3p^o$. Thus $p^* = (3p^o,p^o,p^o,p^o,p^o,p^o)$ is the unique noncooperative market value (hence it is also the unique cooperative market value) and market profits are $q^* = L - 5p^o$.

The ideas of "cooperative" and "noncooperative" market values and floor prices were first introduced [8] in the context of voting games under the names "canonical equilibrium" and "strong noncooperative equilibrium" respectively. The original motivation was to develop a measure of power with more economic content than such value concepts as the Shapley-Shubik and Banzhaf measures. In defining power it was argued that it is not what a player *asks* but what he *gets* that counts, hence the proper measure of his power is not his price but his expected payoff. In [8] this was interpreted to mean his expected bribe income under cooperative market value prices, i.e. his expected bribe income over all equilibrium pairs (p,f) when p is a cooperative market value and f a market schedule. In Example 3, there are 10 critical sets, each of which might be the set bought in equilibrium, hence the probability that any given noncritical player is bribed is 2/5 and the expected bribe incomes are $(3p^o,2p^o/5,2p^o/5,2p^o/5,2p^o/5,2p^o/5)$.

However, this interpretation ignores the fact that even if a player is not bribed he still receives an implicit payoff -- his opportunity cost. The present model includes opportunity costs in the payoff function and leads to the more satisfactory result that price and payoff are the same -- at least for all "normal" prices.

In the presence of a veto player -- that is, a player which is necessary for every winning coalition -- the total value L of the winning coalitions to the lobbyist enters explicitly, since this value is the only effective ceiling on the price of such a player. (In [8] the value of the winning sets was assumed to be infinite and equilibrium was not defined for this case.) To see the effect of a veto player consider the same example as above but with a quota of 6 required to win. Then player 1 is a veto player, the unique noncooperative market value is

$$\underset{\sim}{p} = (L - 3p^{o}, p^{o}, p^{o}, p^{o}, p^{o}, p^{o}) \quad ,$$

and market profits are zero.

This model of vote buying holds whether there are one, several or many lobbyists trying to obtain control of the voters. However, it may well happen that there are lobbyists on opposite sides of an issue -- one trying to buy votes *for*, the other *against*. In this situation the present model does not always apply, since it is predicated on the assumption of a uniform market in which all buyers perceive the *same* production function. This will be the case for two opposing lobbyists only if the winning coalitions are the same as the blocking coalitions (i.e., only if the voting game is *decisive*, like simple majority rule). Various special bidding mechanisms have been investigated for the case of two opposing lobbyists [5,6,9].

RELATION TO OTHER VALUES

The preceding argument has shown that in the face of a market, the players of a game with transferable value may not be able to distribute to themselves the whole value of the game (unless the game has a core). This distinguishes the concept of market value from a number of other value concepts. (The idea that the players should be able to distribute the whole value of the game goes back to von Neumann and Morgenstern [7].) Here we contrast the market value with several other value concepts using the "trio" example, and show how the latter fail to satisfy certain simple market tests.

The unique cooperative (and noncooperative) market value for the trio is (600,200,100) and it is assumed that all three singers are hired. Clearly no singer will do better by asking less, and if any tries to get more, only her rivals will be hired and she will go begging. Moreover, $900 is the maximum that all can get under these conditions.

Consider by contrast the Shapley value for this game: (495, 280, 225). This valuation will not stand in the presence of recruiters (i.e. night club owners) because certain subcoalitions yield more profit than the whole coalition. The unique most profitable coalition under these prices is the soprano-alto duo, with a profit of $125. In these circumstances one of two things must happen: either the prices of S and A will rise, or the price of C will fall (or both). Moreover this conclusion follows without postulating any cooperative behavior on the part of the players, so the Shapley value fails the test of noncooperative price equilibrium.

A second value concept, the generalized *Banzhaf* value (see [2,3]) is based on a probabilistic assessment by each player of his value in joining an existing coalition. If S is a coalition and $i \notin S$ then the value of i joining S is $v(S \cup \{i\}) - v(S)$. If all prior coalitions $S \subseteq (N - \{i\})$ are equally likely we obtain the *Banzhaf value* of i:

$$\beta_i = \sum_{S \subseteq N-\{i\}} [v(S \cup \{i\}) - v(S)]/2^{n-1} .$$

The sum of the values may be *more* or *less* than the "value of the whole," $v(N)$.

The Banzhaf values for Example 1 are $(\beta_S, \beta_G, \beta_O) = (572.5, 357.5, 302.5)$. These are implausible as market values for the simple reason that all nonempty sets yield a negative profit.

Another frequently used value concept is the *least core* and a specialization thereof, the *nucleolus*. The least core is an allocation $\underset{\sim}{x} \geqq \underset{\sim}{0}$ to the players such that the whole amount $v(N)$

is divided, and the excess profit possible from any subcoalition is a minimum, that is

$$\min q$$

$$\text{(i)} \quad \underset{\sim}{x} \geqq \underset{\sim}{0}$$

$$\text{(ii)} \quad q + \sum_{S} x_i \geqq v(S)$$

$$\text{(iii)} \quad \sum_{N} x_i = v(N) \quad .$$

These conditions are very close to the definition of the cooperative market value, except for the crucial assumption (iii) that the players must receive the whole value of the game. Unfortunately, this simple difference leads to a value which also fails to satisfy the test of noncooperative market equilibrium.

The least core for the trio consists of the single imputation $(p_S, p_A, p_C) = (633\ 1/3,\ 233\ 1/3,\ 133\ 1/3)$. There are three most profitable sets: {S,A}, {S,C}, and {A,C}. However, only one of them will be bought -- which one depends on additional, unspecified factors. Whether the choice is deterministic or probabilistic, however, *some* player will always be able to quote a slightly *lower* price and thus make sure that he is bribed with *certainty*, in other words to increase his *expected income*. So the players -- acting independently -- will be both motivated and able to upset this allocation: it fails the test of noncooperative equilibrium.

CONCLUSION

In summary, if the action of players in a cooperative game has value to agents outside the game, a market for the game may be created that conditions the plausible valuations that can be placed on the players. The players are assumed to place values on themselves and the market responds. If they do so noncooperatively, then strong equilibrium prices or values can be shown to exist. But even if they set their values cooperatively they will not be able to realize the whole value of the game if the core is empty. This means that, for purely structural reasons, the players may be exploitable by outside entrepreneurs.

It is always possible, of course, to enlarge the original game so as to incorporate the market. Introduce a new factor "O" (the market) and define the game $\hat{v}$ on $\{0,1,\ldots,n\}$ by $\hat{v}(S) = v(S-\{0\})$ if $0 \in S$, $\hat{v}(S) = 0$ otherwise. It will then be seen that the cooperative and noncooperative market values are particular imputations in the core of this *augmented game*. But the attempt to encompass everything within a game having larger boundaries obscures important features such as opportunity costs. The game theoretic apparatus is more useful when it is employed flexibly to bring into sharp focus certain types of interactions without forgetting the larger system in which they are embedded. Here this approach was used to focus on the players rather than the market agents; exactly the opposite approach could be taken in which price formation in the market is modelled in detail. It may be shown, however [10], that this approach also leads to precisely the value concepts described here.

APPENDIX

Proof of Theorem 1.

The existence of an equilibrium has already been noted in the text for $\underset{\sim}{p} = \underset{\sim}{0}$ and the argument is similar for general $\underset{\sim}{p}^o$.

Sufficiency of the Conditions

Let $\underset{\sim}{p} \geqq \underset{\sim}{p}^o$ satisfy conditions (i) - (iii) of Theorem 1. By (ii) there is a critical set T that is also a maximum profit set relative to $\underset{\sim}{p}$. Notice that (i) and (ii) imply $p_i = \bar{p}_i$ for all $i \in T$. Notice further that for any critical set S*

$$q \geqq v(S^*) - \sum_{S^*} \bar{p}_i = v(S^*) - \sum_{S^*} p_i^o - \sum_{S^* \cap T} (\bar{p}_i - p_i^o)$$

$$= v(T) - \sum_{T} p_i^o - \sum_{S^* \cap T} (\bar{p}_i - p_i^o)$$

$$= v(T) - \sum_{T} \bar{p}_i = q \quad .$$

Thus

(8) $$q = v(S^*) - \sum_{S^*} \bar{p}_i \quad \text{for } \textit{all} \text{ critical sets } S^* \quad .$$

Let f be *any* market schedule such that $f(\underset{\sim}{p}) = T$. We will show that $(\underset{\sim}{p}, f)$ is an equilibrium pair (thus also establishing half of Corollary 1.1).

If not, then for some $\underset{\sim}{p}' \geqq \underset{\sim}{p}^o$, differing from $\underset{\sim}{p}$ only on a nonempty coalition C, we have $\varphi_i(\underset{\sim}{p}', f) > \varphi_i(\underset{\sim}{p}, f)$ for all $i \in C$. Hence for $T' = f(\underset{\sim}{p}')$,

(9) $$C \subseteq T'$$

(10) $$p_i' > p_i = \bar{p}_i \quad \text{for all } i \in C \cap T \quad ,$$

and

(11) $$p_i' > p_i^0 = \bar{p}_i \quad \text{all } i \in C - T \quad .$$

Now

(12) $$v(T) - \sum_T p_i' \leqq v(T') - \sum_{T'} p_i'$$

and

(13) $$q = v(T) - \sum_T \bar{p}_i \geqq v(T') - \sum_{T'} \bar{p}_i \quad .$$

Subtracting (12) from (13),

(14) $$\sum_T (p_i' - \bar{p}_i) \geqq \sum_{T'} (p_i' - \bar{p}_i) \quad .$$

For $i \in T - C$, $p_i' = p_i = \bar{p}_i$, so (14) can be written

(15) $$\sum_{T \cap C} (p_i' - \bar{p}_i) \geqq \sum_{T'} (p_i' - \bar{p}_i) \quad .$$

By (10) and (11), $p_i' \geqq \bar{p}_i$ for all $i \in C$, hence for all $i \in N$. Since $T' \supseteq T \cap C$, (15) implies that $p_i' = \bar{p}_i$ for all $i \in (T' - (T \cap C))$. Thus $p_i' = \bar{p}_i$ for all $i \in C - T$, contradicting (11) unless $C - T = \phi$. Thus

(16) $$C \subseteq T \cap T' \quad \text{and} \quad p_i' = \bar{p}_i \quad \text{for all } i \in T' - C \quad .$$

By hypothesis $C \neq \phi$; choose $j \in C$. By (iii) there is a set S^* such that $j \notin S^*$ and $q = v(S^*) - \sum_{S^*} p_i$. By (i) and the fact that $\tilde{p} \geqq \tilde{\bar{p}}$, $q = v(S^*) - \sum_{S^*} \bar{p}_i$, whence $p_i = \bar{p}_i$ for all $i \in S^*$. Therefore, under prices $\tilde{p}'$ the profit of S^* is

$$v(S^*) - \sum_{S^*} p_i' = v(S^*) - \sum_{S^*} \bar{p}_i - \sum_{S^* \cap C} (p_i' - \bar{p}_i) = q - \sum_{S^* \cap C} (p_i' - \bar{p}_i) .$$

However, the profit of T' is

$$v(T') - \sum_{T'} p_i' = q - \sum_C (p_i' - \bar{p}_i) \quad ,$$

which is smaller, since $j \in C - S^*$ and $p_j' > \bar{p}_j$. This contradiction to the definition of T' establishes the sufficiency of the conditions.

Necessity of the Conditions

Let $\underset{\sim}{p} \geqq \underset{\sim}{p}^o$ be an equilibrium, that is, a strong noncooperative equilibrium relative to some specific market schedule f. Let $f(\underset{\sim}{p}) = T$, and $q = v(T) - \sum_T p_i$. We show first that T is a critical set (this will, incidentally, prove the other half of Corollary 1.1).

Suppose not. Let $\bar{T}^+ = \{i \notin T : p_i > p_i^o\}$, and define $\underset{\sim}{p}^\varepsilon$ such that

$$p_i^\varepsilon = \begin{cases} p_i & \text{if } i \notin \bar{T}^+ \\ p_i^o + \varepsilon & \text{if } i \in \bar{T}^+ \end{cases} .$$

Consider an arbitrary critical set S*.

$$v(S^*) - \sum_{S^*} p_i^o > v(T) - \sum_T p_i^o$$

implies

$$v(S^*) - \sum_{S^*} p_i^o - \sum_{S^* \cap T} (p_i - p_i^o) > v(T) - \sum_T p_i^o - \sum_T (p_i - p_i^o)$$

The two sides of this expression can be rewritten as

$$v(S^*) - \sum_{S^*} p_i^\varepsilon + \sum_{S^*-T} (p_i^\varepsilon - p_i^o) > v(T) - \sum_T p_i^\varepsilon .$$

Since $\sum_{S^*-T} (p_i^\varepsilon - p_i^o) = \varepsilon |S^* \cap \bar{T}^+|$, it follows that for all sufficiently small $\varepsilon > 0$,

(17) $$v(S^*) - \sum_{S^*} p_i^\varepsilon > v(T) - \sum_T p_i^\varepsilon .$$

Therefore T is *not* maximum profit under $\underset{\sim}{p}^{\varepsilon}$. Since T *is* maximum profit under $\underset{\sim}{p}$ and $\underset{\sim}{p}$ differs from $\underset{\sim}{p}'$ only on T', it follows that every maximum profit set S with respect to $\underset{\sim}{p}^{\varepsilon}$ has a nonempty intersection with $\overline{T}^{+}$. Let $C (C \neq \phi)$ be a *minimal* element of the following family, ordered by inclusion:

$$\mathscr{C}_{\varepsilon} = \{S \cap \overline{T}^{+} : S \text{ is maximum profit under } \underset{\sim}{p}^{\varepsilon}\} \quad ,$$

and let

$$p_i' = \begin{cases} p_i & \text{if } i \notin C \\ p_i^{o} + \varepsilon & \text{if } i \in C \end{cases} \quad ,$$

where ε is chosen so small that $p_i^{o} + \varepsilon < p_i$ for all $i \in \overline{T}^{+}$. In particular, $\underset{\sim}{p}' \geqq \underset{\sim}{p}^{\varepsilon}$ with $>$ for $i \in \overline{T}^{+} - C$. Now the maximum profit under $\underset{\sim}{p}'$ is the same as under $\underset{\sim}{p}^{\varepsilon}$; call it q'. In particular, $f(\underset{\sim}{p}') = S'$ has profit q' under both $\underset{\sim}{p}'$ and $\underset{\sim}{p}^{\varepsilon}$. By choice of ε, $S' \cap \overline{T}^{+} \subseteq C$. By definition of C, $S' \cap \overline{T}^{+} = C$. Therefore $C \subseteq f(\underset{\sim}{p}')$.

But $\underset{\sim}{p}'$ is obtained from $\underset{\sim}{p}$ when all members of C reduce their prices from p_i to $p_i^{o} + \varepsilon$. Moreover, the payoff to $i \in C$ is p_i^{o} under $\underset{\sim}{p}$ but $p_i^{o} + \varepsilon$ under $\underset{\sim}{p}'$, since $C \subseteq f(\underset{\sim}{p}')$. So $\underset{\sim}{p}$ was not a strong noncooperative equilibrium. This contradiction establishes that $T = f(\underset{\sim}{p})$ must be a critical set. In particular, (ii) of Theorem 1 holds for T.

With $\underset{\sim}{p}^{\varepsilon}$ defined as above, the argument beginning at (17) shows that for all sufficiently small $\varepsilon > 0$ we must have

$$v(S) - \sum_{S} p_i^{\varepsilon} \leqq v(T) - \sum_{T} p_i^{\varepsilon} = q \qquad \text{for all } S \subseteq N \quad .$$

The limit pf $\underset{\sim}{p}^{\varepsilon}$ as $\varepsilon \to 0$ is the vector $\underset{\sim}{p}^{T}$ defined by

$$p_i^{T} = \begin{cases} p_i & \text{if } i \in T \\ \overline{p}_i & \text{if } i \notin T \end{cases} \quad ,$$

and

$$(18) \qquad v(S) - \sum_S p_i^T \leq v(T) - \sum_T p_i^T = v(T) - \sum_T p_i = q \quad \text{for all } S \subseteq N.$$

Suppose that $p_k > \bar{p}_k = p_k^o$ for some $k \in T$. By definition of $\bar{\underset{\sim}{p}}$, k is not critical, hence there is a critical set S such that $k \notin S$. Then

$$v(S) - \sum_S p_i^o = v(T) - \sum_T p_i^o$$

implies

$$v(S) - \sum_S p_i^T = v(S) - \sum_S p_i^o - \sum_{S \cap T} (p_i - p_i^o)$$

$$= v(T) - \sum_T p_i^o - \sum_{S \cap T} (p_i - p_i^o) > v(T) - \sum_T p_i \quad ,$$

the $>$ since $p_k > \bar{p}_k = p_k^o$ and $k \in T-S$. This contradicts (18), showing that $p_i = \bar{p}_i$ for all $i \in T$, hence $\underset{\sim}{p}^T = \bar{\underset{\sim}{p}}$. But then (18) is precisely condition (i) of the theorem.

Finally, condition (iii) is clearly necessary, since if a player were contained in *every* maximum profit set, he could raise his price and do better. □

<u>Proof of Theorem 2.</u>

Let $\underset{\sim}{p}^*$ be a cooperative market value relative to some given floor prices $\underset{\sim}{p}^o$. We show first that $\underset{\sim}{p}^*$ is also a noncooperative market value.

We know that $\underset{\sim}{p}^*$ is an optimal solution to (7), hence $\underset{\sim}{p}^*$ is normal ($\underset{\sim}{p}^* = \bar{\underset{\sim}{p}}^*$) and maximum profits q^* are attained on some critical set.

By Theorem 1 it remains only to show that no player k is in *every* maximum profit set. Such a player k would have to be critical, since all critical sets are maximum profit. But then player k's price could be increased, contradicting the fact that q^* is a minimum for (7).

Next we show that $\underset{\sim}{p}^*$ is in the core of v^*. Since we have $\sum_N p_i^* = v^*(N)$ by definition of v^*, it remains only to show that

$$\sum_S p_i^* \geqq v^*(S) \text{ for all } S \subset N \quad .$$

Suppose to the contrary that $\sum_S p_i^* < v^*(S)$ for some S. By definition of $v^*(S)$ there is an $|S|$-vector $\underset{\sim}{x} \geqq 0$ such that $v^*(S) = \sum_S x_i + \sum_S p_i^o$. Consider the price vector $\underset{\sim}{p}'$ that equals $x_i + p_i^o$ on S and p_i^o on N - S. By definition of $v^*(S)$ there is a maximum profit set S' relative to $\underset{\sim}{p}'$ such that $x_i = 0$ for $i \in S - S'$. If $x_k > 0$ for some $k \in S - N^o$, then there is a critical set not containing k, and under $\underset{\sim}{p}'$ it will yield a higher profit than S', a contradiction. Hence $x_k = 0$ for all $k \in S - N^o$.

Now consider the vector $\underset{\sim}{p}''$ that equals $x_i + p_i^o$ on S and p_i^* on N - S. By the preceding,

(19) $$p_i'' = p_i^* \text{ for all } i \notin S \cap N^o \text{ and } \sum_{S\cap N^o} p_i'' > \sum_{S\cap N^o} p_i^* \quad .$$

Again by definition of $v^*(S)$ there is a maximum profit set S" relative to $\underset{\sim}{p}''$ such that

(20) $$p_i'' = p_i^o \text{ for all } i \in S - S'' \quad .$$

Thus for any critical set T,

(21) $$q^* = v(T) - \sum_T p_i^* \geqq v(S'') - \sum_{S''} p_i^* \quad ,$$

whereas

(22) $$q'' = v(S'') - \sum_{S''} p_i'' \geqq v(T) - \sum_T p_i'' \quad ,$$

whence

(23) $$\sum_T (p_i'' - p_i^*) \geqq \sum_{S''} (p_i'' - p_i^*) \quad .$$

By (19), $p_i'' = p_i^*$ for all $i \in S'' - T$, and by (20) $p_i'' - p_i^* \leqq 0$ for all $i \in T - S''$. Hence equality holds in (21)-(23). In particular, all critical sets are maximum profit sets under $\underset{\sim}{p}''$.

Thus $(q'',\underset{\sim}{p}'')$ is feasible for (7). But (19) implies $q'' < q^*$, a contradiction.

Conversely, let $\underset{\sim}{p}$ be a noncooperative market value in the core of v^*. Then $\sum_N p_i = v^*(N)$. Now $v^*(N) = \sum_N p_i^*$ for some optimal solution $\underset{\sim}{p}^*$ to the linear program (7); moreover, from the definition of v^* and the fact that $\underset{\sim}{p}^*$ is normal, $v^*(T) = \sum_T p_i^*$ for every critical set T. Since $\underset{\sim}{p}$ is assumed to be in the core of v^*,

$$(24) \qquad \sum_T p_i \geqq v^*(T) = \sum_T p_i^* \quad \text{for all critical } T \quad .$$

Since $\underset{\sim}{p}$ and $\underset{\sim}{p}^*$ are normal,

$$(25) \qquad \sum_N p_i = \sum_{N-T} p_i^o + \sum_T p_i \geqq \sum_{N-T} p_i^o + \sum_T p_i^* = \sum_N p_i^* \quad .$$

But $\sum_N p_i = \sum_N p_i^* = v^*(N)$, so all inequalities are equalities in (24) and (25). Hence $\underset{\sim}{p}$ is also an optimal solution to (7), so $\underset{\sim}{p}$ is a cooperative market value.□

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FOOTNOTE

1. By profit we mean pure profit. If the opportunity cost of alternative investment is some profit rate $\pi > 0$, then the pure profit of a set S of factors is $v(S) - (1+\pi)\sum_S p_i$. This situation is treated by simply defining the new game. $v'(S) = v(S)/(1+\pi)$ for each S and proceeding as above.