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The Factors and Magnitude of Urbanization under Unchanged Natural Increase and Migration Patterns

Ledent, J.

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THE FACTORS AND MAGNITUDE OF URBANIZATION UNDER UNCHANGED NATURAL INCREASE AND MIGRATION PATTERNS

Jacques Ledent

November 1978

Preface

Roughly 1.6 billion people, 40 percent of the world's population, live in urban areas today. At the beginning of the last century, the urban population of the world totaled only 25 million. According to recent United Nations estimates, about 3.1 billion people, twice today's urban population, will be living in urban areas by the year 2000.

Rapid rates of urban demographic and economic growth increase the difficulties of providing a population with adequate supplies of food, energy, employment, social services and infrastructure. The investment needed just to maintain present standards in many rapidly urbanizing countries calls for a doubling or tripling of institutional plant within the next 25 years.

Scholars and policy-makers often disagree when it comes to evaluating the desirability of current rapid rates of urban growth in many parts of the globe. Some see this trend as fostering national processes of socioeconomic development, particularly in the poorer and rapidly urbanizing countries of the Third World; whereas others believe the consequences to be largely undesirable and argue that such urban growth should be slowed down.

Professor Nathan Keyfitz of Harvard University spent the month of May this year collaborating with HSS scholars in their research on migration, urbanization and development. During his stay, he formulated a model of the urbanization process that stimulated a number of us. In particular, Jacques Ledent responded by writing a series of three papers dealing with extensions of the Keyfitz model. This paper, the second of the series, focuses on the dynamics of urbanization under constant regimes of natural increase and migration.

A list of related papers in the Population, Resources and Growth Series appears at the end of this publication.

Andrei Rogers
Chairman
Human Settlements
and Services Area

November 1978

Abstract

This paper is the second of a series intended to shed some light on the urbanization phenomenon. Its main purpose is to contrast the results provided by two alternative models--the model proposed by Keyfitz (1978) and the continuous version of the multiregional model of population growth and distribution developed by Rogers (1968)--under constant regimes of natural increase and migration.

In both cases, the evolution of the magnitude of urbanization as well as that of the relative importance of natural increase and migration in accounting for urban growth are examined. A particular emphasis is placed on the time spans necessary to reach two cross-over points: the point at which natural increase starts exceeding inmigration in the urban region (cross-over point of type I) and the point at which the urban population becomes larger than the rural population (cross-over point of type II).

The contrast between the alternative models is illustrated with the help of an application to two actual rural-urban population systems presenting polar characteristics: those of the USSR and India.

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INTRODUCTION

Keyfitz, in a recent paper (1978), suggests a simple model intended to shed some light on whether cities grow through natural increase or inmigration. The appropriateness of the model to deal with such a problem was, however, questioned by Ledent (1978). Ledent argues that the Keyfitz model, which views migration as a net flow from the rural to the urban region, introduces a rather undesirable asymmetry between the rural and urban regions. Instead, he proposed to use a continuous version of the multiregional model of population growth and distribution, first developed by Rogers (1968) and whose long-term properties are well behaved: the ratio of urban to rural population tends toward a limit instead of increasing indefinitely.

The purpose of this paper is to examine the evolution of the factors and magnitude of urbanization implied by the two alternative models under constant regimes of natural increase and migration. The case of varying regimes will be dealt with in a forthcoming paper. The analysis is carried out mainly with an emphasis on the time spans necessary to reach two particular points in time (or cross-over points): the point at which natural increase starts exceeding inmigration in the urban region (cross-over point of type I) and the point at which the urban population becomes larger than the rural population (cross-over point of type II).

This paper consists of two parts which carry out the analysis of the urbanization problem from the two alternative models distinguished above: the first part is based on the Keyfitz model, while the second relies on the two-region continuous version of the Rogers model. A short contrast of the insights provided by the alternative models appears in the conclusion with the help of a comparison between the numerical results obtained by applying the alternative models to two actual rural-urban population systems presenting polar characteristics: those of the USSR and India.

I. ANALYSIS BASED ON THE KEYFITZ MODEL

Basically, Keyfitz (1978) considers a population system divided into two regions, urban and rural, which exhibit constant rates of natural increase, denoted by u and r respectively. In addition, he assumes a net outmigration rate from rural to urban taken as a constant fraction m (strictly positive) of the rural population. The mathematical analysis of the ensuing model has been extensively studied in Ledent (1978).

Analytics of the Model

The equations describing the population growth of the rural and urban regions are respectively:

$$\frac{dP_r(t)}{dt} = (r-m)\ P_r(t) \tag{1a}$$

and

$$\frac{dP_u(t)}{dt} = u\ P_u(t) + m\ P_r(t) \tag{1b}$$

in which $P_r(t)$ and $P_u(t)$ are the populations at time t of the rural and urban regions respectively.

It has been shown (Ledent, 1978) that the population system described by (1a) and (1b) evolves from an initial state characterized by a concentration of the whole population in the rural region if the parameters of the system are such that

$$r < u + m\,\frac{1+\overline{S}}{\overline{S}}$$

in which $\overline{S}$ is the ratio of urban to rural population in the period at which the system is observed. Thus to remain general, i.e., to prevent any peculiarities due to the value of $\overline{S}$, we impose here

$$r < u + m \tag{2}$$

Then, if t = 0 denotes the initial period, the solution of the system of equations (1a) and (1b) is given by

$$P_r(t) = P(0)\ e^{(r-m)t} \tag{3a}$$

and

$$P_r(t) = P(0)\ \frac{m}{u+m-r} \left[e^{ut} - e^{(r-m)t}\right] \tag{3b*}$$

in which P(0) is the initial population of the system. Note that

$$P_r(0) = P(0) \qquad \text{and} \qquad P_u(0) = 0$$

Letting S(t) denote the ratio of urban to rural population, and dividing (3b) by (3a), we obtain:

$$S(t) = \frac{m}{u+m-r} \left[e^{(u+m-r)t} - 1\right] \quad , \tag{4}$$

*Note that $u+m-r > 0$, as a consequence of the restriction on the parameters of the system adopted in (2).

a relationship which indicates that S(t) monotonically increases from zero (for t = 0) to + ∞ (for r → + ∞).

The part of the population of the whole system that is urban is given by:

$$\alpha(t) = \frac{S(t)}{1+S(t)} \quad . \tag{5}$$

It is then readily established that α(t) monotonically increases from zero (for t = 0) to one (for t → + ∞).

We now introduce $m_u(t)$, the urban net inmigration rate:

$$m_u(t) = \frac{mP_r(t)}{P_u(t)} = \frac{m}{S(t)} \quad , \tag{6}$$

an equation which permits one to conclude that $m_u(t)$ monotonically decreases from + ∞ (for t = 0) to 0 (for t → + ∞).

The ratio R(t) of urban net inmigration to urban natural increase,

$$R(t) = \frac{m_u(t)}{u} \quad , \tag{7}$$

is thus linked to S(t) by the following (Keyfitz 1978)

$$R(t) = \frac{m}{uS(t)} \quad . \tag{8}$$

This relationship suggests that, in consideration of the problem examined here, we must impose

$$u > 0 \tag{9}$$

so that R(t) monotonically decreases from $+\infty$ (for $t = 0$) to zero (for $t \to +\infty$).

Note that as a consequence of (9), $P_u(t)$ monotonically increases from zero (for $t = 0$) to become infinitely positive as $t \to +\infty$. By contrast, the direction of the variations of $P_r(t)$ which is also monotonic depends on the relative values of r and m: $P_r(t)$ increases from P(0) (for $t = 0$) to $+\infty$ (for $t \to +\infty$) if $r > m$, but decreases to zero (for $t \to +\infty$) if $r < m$.

The results of the above model defined by the system (1a) - (1b) and the restrictions (2) and (9) are summarized in Table 1.

<u>The Factors of Urbanization and the Cross-over Point of Type I</u>

Substituting expression (4) into (8) yields an analytic expression of R(t)

$$R(t) = \frac{u+m-r}{u\left(e^{(u+m-r)t} - 1\right)} \quad , \tag{10}$$

an expression which permits one to visualize the variations of R(t), appearing on Figure 1.

The variations of R(t) indicate that net inmigration is initially preponderant in accounting for the growth of the urban region but as time passes, its role diminishes so as to make natural increase the unique source of urban growth in the long-run. Keyfitz (1978) refers to the point in time at which natural increase is equal to net inmigration as the cross-over point. We call it here cross-over point of type I, denoted by T_1.

Table 1: The Keyfitz model as a model of urbanization: the variations of the main functions.

	t	0		+ ∞
$P_r(t)$	(a) $r > m$	P(0)	→	+ ∞
	(b) $r = m$	P(0)	→	P(0)
	(b) $r < m$	P(0)	→	+ ∞
$P_u(t)$		0	→	+ ∞
S(t)		0	→	+ ∞
(t)		0	→	1
$m_u(t)$		+ ∞	→	0
R(t)		+ ∞	→	0

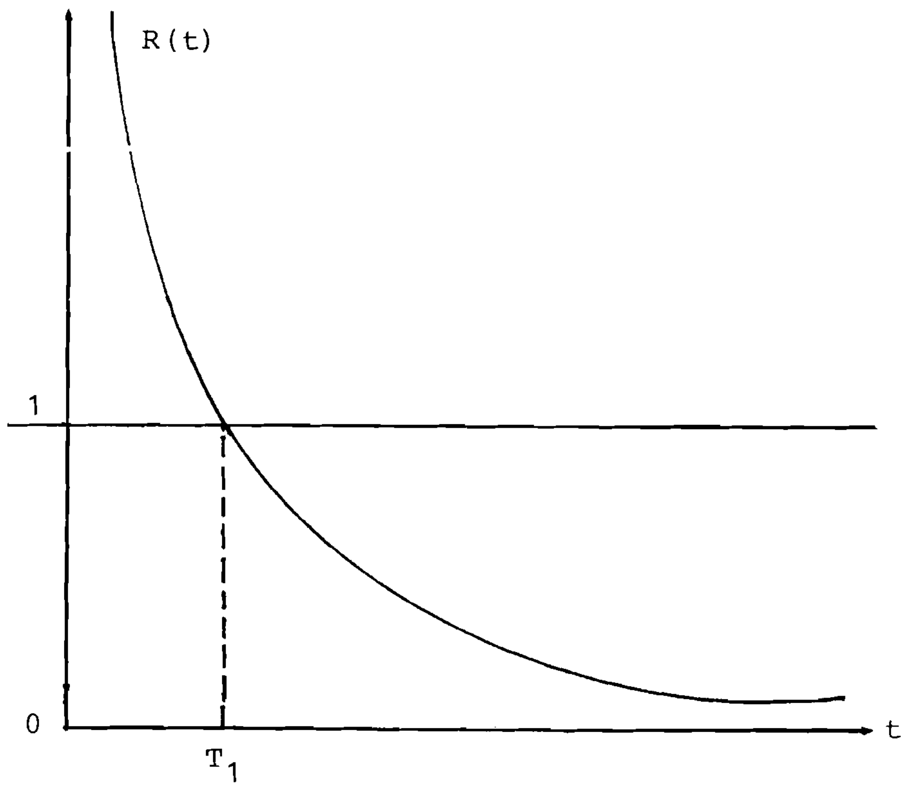

Figure 1. The Keyfitz model as a model of urbanization: the variations of R(t).

Clearly we have:

$$m_u(T_1) = u \tag{11}$$

and

$$R(T_1) = 1 \quad . \tag{12}$$

Therefore from (4) we have that:

$$S(T_1) = \frac{m}{u} \quad . \tag{13}$$

At the cross-over point T_1, the ratio of urban to rural population is simply equal to the ratio of the rural net (out) migration rate to the urban rate of natural increase. Consequently the part of the whole population which is urban at time T_1 is:

$$\alpha(T_1) = \frac{m}{u+m} \quad . \tag{14}$$

An expression of T_1 can be derived by combining (10) and (12) thus giving (Keyfitz 1978)

$$T_1 = \frac{1}{u+m-r} \ln \left(1 + \frac{u+m-r}{u}\right) \quad . \tag{15}$$

What is the impact of a parameter change on the value of T_1? Differentiating T_1 with respect to each of the three parameters yields:

$$\frac{dT_1}{du} = \frac{1}{u+m-r}\left[\frac{r-m}{u(2u+m-r)} - \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{u}\right)\right] \qquad (16)$$

$$\frac{dT_1}{dr} = \frac{1}{u+m-r}\left[-\frac{1}{2u+m-r} + \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{u}\right)\right] \qquad (17)$$

and

$$\frac{dT_1}{dm} = \frac{1}{u+m-r}\left[\frac{1}{2u+m-r} - \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{u}\right)\right] \quad . \qquad (18)$$

It is possible to show, through tedious manipulations of (16) and (17) (see Appendix 1), that, whatever the values of the parameters u, r, and m,

$$\frac{dT_1}{du} < 0 \qquad (19a)$$

and

$$\frac{dT_1}{dr} > 0 \quad . \qquad (19b)$$

The conclusion here is that the higher (smaller) the rate of natural increase in the urban (rural) region, the sooner the cross-over.* In addition, (19a) and (19b) permits one to state that, for a given u or a given r, the larger the difference r-u the later the cross-over.

*Note the contrast with the statement made by Keyfitz (1978, p.5) in the special case u=r, that "the more rapidly the population as a whole increases the sooner the cross-over."

From the comparison of (17) and (18), one finds that:

$$\frac{dT_1}{dm} = - \frac{dT_1}{dr} \tag{20}$$

and that, whatever the values of the parameters u, r, and m,

$$\frac{dT_1}{dm} < 0 \quad . \tag{21}$$

Thus we have demonstrated the suggestion made by Keyfitz (1978, p.5) that "the larger the value of m, the fraction of the countryside migrating, the sooner comes the day when natural increase exceeds migration as a factor."

Moreover, (20) suggests that, for a given u, an increase x, in the difference r-u has an impact on T_1 which has the same magnitude as a decrease x in the net outmigration rate m. Such a result is clear if one observes that m and r always intervene through their differences.

The impact of parameter changes on the part of the population which is urban at the cross-over point can also be assessed. Differentiating (14) with respect to each of the parameters yields:

$$\frac{d\alpha(T_1)}{du} = - \frac{m}{(m+u)^2} \quad ; \quad \frac{d\alpha(T_1)}{dr} = 0 \quad ; \quad \frac{d\alpha(T_1)}{dm} = \frac{u}{(m+u)^2} \quad . \tag{22}$$

Therefore the proportion of the population which is urban at the cross-over point is independent of the rural rate of natural increase. Moreover, the smaller the urban rate of natural increase and the higher the rural outmigration rate, the higher this proportion.

The Magnitude of Urbanization and the Cross-over Point of Type II

Substituting expression (4) in (5) yields an analytic expression of $\alpha(t)$

$$\alpha(t) = \frac{e^{(u+m-r)t} - 1}{e^{(u+m-r)t} + \frac{u-r}{m}} , \tag{23}$$

an expression which permits one to visualize the variations of $\alpha(t)$, appearing in Figure 2.

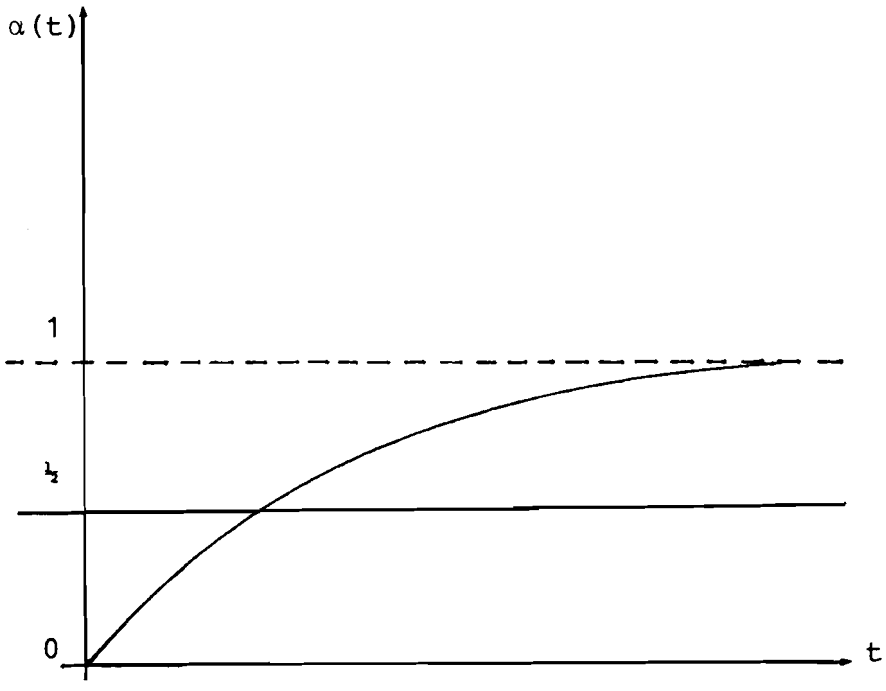

Figure 2: The Keyfitz model as a model of urbanization: the variations of $\alpha(t)$.

The variations of $\alpha(t)$ indicate that an initial totally rural population system becomes totally urban. But, how fast does the process of urbanization take place? For this purpose, following a suggestion by Andrei Rogers, we define a cross-over point, said to be of type II, as the point at which the whole population of the system is equally distributed among the rural and urban regions. Denoted by T_2, it is clearly defined by

$$\alpha(T_2) = \frac{1}{2} \tag{24}$$

i.e.,

$$S(T_2) = 1 \quad . \tag{25}$$

From (6) we have that

$$m_u(T_2) = m \tag{26}$$

and from (8) that

$$R(T_2) = \frac{m}{u} \quad . \tag{27}$$

At the cross-over T_2, the net inmigration rate of the urban region is equal to the net outmigration rate of the rural area (which one could expect since the populations in both regions are equal).

An expression of T_2 can be derived by combining (23) and (24) to obtain:

$$T_2 = \frac{1}{u+m-r} \ln \left(1 + \frac{u+m-r}{m}\right) \tag{28}$$

Note that T_2 differs from T_1 by the parameter at the denominator of the logarithmic term (m replaces u). Also, T_2 can be obtained from T_1 by simply exchanging u and m. Now, what is the impact of a parameter change on the value of T_2? We could differentiate (28) to obtain the first derivatives of T_2 with respect to each of the three parameters. However, the above remarks on the similarity of T_1 and T_2 allows one to write at once the following formulas obtained by exchanging u and m in (16) through (18):

$$\frac{dT_2}{du} = \frac{1}{u+m-r}\left[\frac{1}{u+2m-r} - \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{m}\right)\right] \tag{29}$$

$$\frac{dT_2}{dr} = \frac{1}{u+m-r}\left[-\frac{1}{u+2m-r} + \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{m}\right)\right] \tag{30}$$

$$\frac{dT_2}{dm} = \frac{1}{u+m-r}\left[\frac{r-u}{m(u+2m-r)} - \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{m}\right)\right] . \tag{31}$$

We can also conclude immediately that, since changes in u and m had impacts of the same sign on T_1, the first derivatives of T_2 have the same signs as the first derivatives of T_1, i.e.,

$$\frac{dT_2}{du} < 0 \quad ; \quad \frac{dT_2}{dr} > 0 \quad \text{and} \quad \frac{dT_2}{dm} < 0 \quad . \tag{32}$$

Then, as one would expect, the higher (smaller) the urban (rural) rate of natural increase, the sooner the cross-over. Moreover, the larger the value of m, the net outmigration rate from the rural area, the sooner also comes the day when the urban population exceeds the rural population.

Note that T_2 depends on u and r through their difference u-r. Differentiating (28) with respect to u-r, we have

$$\frac{dT_2}{d(u-r)} = \frac{1}{u+m-r}\left[\frac{1}{u+2m-r} - \frac{1}{u+m-r}\ln\left(1+\frac{u+m-r}{m}\right)\right] \qquad (33)$$

a quantity which is equal to $\frac{dT_2}{du}$ and thus negative.

Clearly the larger the difference r-u, the later the cross-over.* Subtracting (33) from (31) yields:

$$\frac{dT_2}{dm} - \frac{dT_2}{d(u-r)} = -\frac{1}{m(u+m-r)} \quad , \qquad (34)$$

a relationship from which we conclude that

$$\frac{dT_2}{d(r-u)} < -\frac{dT_2}{dm} \quad . \qquad (35)$$

In other words, an increase in the difference r-u has a smaller impact (in absolute value) on T_2 than an identical increase in the rural net outmigration rate.

*Note that in contrast to the similar statement made earlier and concerning T_1, this statement is valid regardless of the value of r (or u) permitted by the constraints (2) and (9).

Comparison of the Two Cross-over Points

Which of the two cross-over points defined above is reached first? Substracting (15) from (28) yields:

$$T_2 - T_1 = \frac{1}{u+m-r} \ln\left(\frac{1 + \frac{u+m-r}{m}}{1 + \frac{u+m-r}{u}}\right) \quad . \tag{36}$$

It follows that T_2 is greater than T_1 if the numerator of the logarithmic term is greater than its denominator, i.e., if u is larger than m. In other words, the relative values of the urban rate of natural increase and the rural net outmigration rate determine which one of the two cross-over points is reached first. This result can be obtained alternatively from the formula (27) giving the value of R(t) at the cross-over T_2. Indeed if u is larger than m, (27) indicates that $R(T_2) < 1$, i.e., that the cross-over T_1 has already been reached.

Application to Actual Population Systems

Let us suppose that in a given year, we observe an actual population system characterized by parameters u, r, and m (such that $r < u+m$) and a proportion of urban population equal to $\bar{\alpha}$.

Clearly, in the hypothetical population system (an initial totally rural population that is submitted to the constant regimes of natural increase and migration defined by u, r, and m) there is a subsequent state offering the same characteristics as the observed population system.

The time t_D, at which this correspondence occurs, is simply obtained as the root of $\alpha(t) = \bar{\alpha}$, which is unique due to the course of the evolution of $\alpha(t)$ (see Figure 2). It is readily established from (23), that

$$t_D = \frac{1}{u+m-r} \ln\left(1 + \frac{u+m-r}{m} \; \frac{1-\bar{\alpha}}{\bar{\alpha}}\right) \tag{37a}$$

or, alternatively, after noting that $\frac{1-\bar{\alpha}}{\bar{\alpha}}$ is the ratio $\bar{S}$ of the urban to rural population in the observed year,

$$t_D = \frac{1}{u+m-r} \ln\left(1 + \frac{u+m-r}{m} \bar{S}\right) \quad . \tag{37b}$$

Consequently, if around the observation period the actual population system exhibits the constant regimes of natural increase and migration defined by u, r, and m, we can simply determine whether this system has already reached or will reach the two types of cross-over points defined above.

Let T_1' and T_2' denote the time spans necessary to reach the cross-over points, of types I and II respectively, from the observed period. Indeed:

$$T_1' = T_1 - t_D \tag{38a}$$

and

$$T_2' = T_2 - t_D \tag{38b}$$

and we have

$$T_1' = \frac{1}{u+m-r} \ln\left(\frac{1 + \frac{u+m-r}{u}}{1 + \frac{u+m-r}{m} \bar{S}}\right) \tag{39}$$

$$T_2' = \frac{1}{u+m-r} \ln\left(\frac{1 + \frac{u+m-r}{m}}{1 + \frac{u+m-r}{m} \bar{S}}\right) \quad . \tag{40}$$

From these formulas it is easy to show that T_1' is positive if $\bar{S} < \frac{m}{u}$ and T_2' is positive if $\bar{S} < 1$ [results that also follow from formulas (13) and (25)].

Hence, the simple comparison of the observed ratio of population $\bar{S}$ with the quotient $\frac{m}{u}$ permits one to determine immediately whether the cross-over point of the first type has already been reached. In addition, the relative values of $\bar{S}$ and the number 1 determine whether the cross-over point of the second type has been reached or not.

By differentiating (39) and (40) with respect to the three parameters u, r, and m, we can also obtain the formulas determining the impact of a parameter change on T_1' and T_2'. However, these formulas, not shown here,* does not lead to a well defined sign for the first derivatives of T_1' and T_2', as in the case of the first derivatives of T_1 and T_2.

Note that T_1' and T_2' also depend on $\bar{S}$. One can establish that:

$$\frac{dT_1'}{d\bar{S}} = \frac{dT_2'}{d\bar{S}} = - \frac{1}{m + (u+m-r)\ \bar{S}} \quad . \tag{41}$$

---

*Recalling (38a) and (38b), one can simply obtain the first derivatives of T_1' (and T_2') by subtracting from the first derivatives of T_1 (and T_2), the first derivatives of t_D with respect to the three parameters u, r, and m:

$$\frac{dt_D}{dm} = - \frac{dt_D}{dr} = \frac{1}{u+m-r} \left[\frac{\bar{S}}{m + (u+m-r)\ \bar{S}} - \frac{1}{u+m-r} \ \ln \left(1 + \frac{u+m-r}{m} \bar{S} \right) \right] \tag{42a}$$

and

$$\frac{dt_D}{du} = \frac{1}{u+m-r} \left[\frac{\frac{r-u}{m} \bar{S}}{m + (u+m-r)\ \bar{S}} - \frac{1}{u+m-r} \ln \left(1 + \frac{u+m-r}{m} \bar{S} \right) \right] . \tag{42b}$$

Then, the higher the proportion of the observed population which is urban, the smaller the time necessary to reach the cross-over (if T'_1 or T'_2 are positive) or the longer the time elapsed since the cross-over (if T'_1 or T'_2 are negative).

Note that the impact on the various cross-over times of parameter changes can also be obtained by exogenously changing the parameters of the system and then comparing the new cross-over times calculated with the initial ones.*

Numerical Illustrations**

Rogers (1976) reports that the urban population of the USSR was growing at an annual rate of approximately 2.5 percent during the early 1970's. This rate was the sum of a rate of natural increase of 0.9 and a net migration rate of 1.6 percent. At the same time, the rural population was declining at an annual rate of 1.1 percent which was the sum of a rate of natural increase of 1.0 percent and a net migration rate of -2.1 percent. Then, in this system:

$$u = 0.009 \quad ; \quad r = 0.010 \quad ; \quad m = 0.021 \quad ; \bar{S} = \frac{0.021}{0.016} = 1.3125.$$

This observed population system corresponds to the subsequent state of a hypothetical population system, characterized as in

* In this illustration, the impact of exogenous parameter changes is obtained from the multiplier formulas derived above.
**These numerical illustrations are summarized in Table 2.

the above discussion and occurring at time t_D given by

$$t_D = \frac{1}{0.02} \ln 2.25 = 40.5 \text{ years} \quad .$$

In this system, the rural population monotonically decreases toward zero (since $r < m$) whereas the urban population increases toward $+ \infty$ (as $t \to + \infty$).

The urban net inmigration rate (= $+ \infty$ initially) monotonically decreases toward zero (as $t \to + \infty$) reaching the value 1 corresponding to the cross-over point of type I at time T_1, given by

$$T_1 = \frac{1}{0.02} \ln 3.222 = 58.5 \text{ years} \quad .$$

The time span necessary to reach this cross-over point from the observed period is then:

$$T_1' = T_1 - t_D = 18.0 \text{ years} \quad .$$

At this cross-over point, the part of the population which is urban is equal to

$$\alpha(T_1) = \frac{0.021}{0.030} = 0.70$$

i.e., 70 percent, a value which is higher than the observed 56.76 percent (an expected result since T_1' is positive).

However, $\alpha(t)$ monotonically increases toward one (for $t \to + \infty$) and takes on the value $\frac{1}{2}$ corresponding to the cross-over point of the second type at time T_2, given by

$$T_2 = \frac{1}{0.02} \ln 1.925 = 33.5 \text{ years} \quad .$$

It follows that this cross-over point has been reached 7.1 years before the observation period as $T_2' = T_2 - t_D = -7.1$ years, a result which was expected since $\bar{S} > 1$.

The impact of small parameter changes on the time spans necessary to reach the cross-over points has been calculated with reference to both the Keyfitz time frame (the one of the hypothetical system leading to the observed population system of USSR) and the actual time frame (see Table 2, under the heading "sensitivity analysis").

Therefore, a slightly less urban rate of natural increase (0.8 percent versus 0.9 percent) would delay the cross-over of type I by 4.4 years (occurring 24.4 years after the observed period) and hasten the cross-over of type II by 0.2 years (occurring 7.3 years before the observed period). Also, a slightly higher rural-urban migration rate (2.2 percent versus 2.1 percent) would delay the cross-over of type I by 0.8 years and hasten the cross-over of type II by 0.4 years.

We provide here another illustration for India observed in the late sixties, for which data can be found in Rogers and Willekens (1976):

$$u = 0.020 \quad ; \quad r = 0.022 \quad ; \quad m = 0.005 \quad ; \quad \bar{S} = 0.294 \, .$$

This observed population system appears to be identical to the

Table 2. The Keyfitz Model: Numerical Illustrations.

1 INPUT DATA

REGION	RU	BU	DU	RR	BR	DR
USSR	25.0	17.0	8.0	-11.0	19.0	9.0
INDIA	37.0	30.0	10.0	17.0	39.0	17.0

****

RU = GROWTH RATE OF URBAN POPULATION(*1000)
BU = BIRTH RATE OF URBAN POPULATION(*1000)
DU = DEATH RATE OF URBAN POPULATION(*1000)
RR = GROWTH RATE OF RURAL POPULATION(*1000)
BR = BIRTH RATE OF RURAL POPULATION(*1000)
DR = DEATH RATE OF RURAL POPULATION(*1000)

1 BASIC DATA

REGION	U	R	M	S	ALPHA
USSR	0.0090	0.0100	0.0210	1.3125	0.5676
INDIA	0.0200	0.0220	0.0050	0.2941	0.2273

****

U = NATURAL INCREASE RATE OF URBAN POPULATION
R = NATURAL INCREASE RATE OF RURAL POPULATION
M = NET OUTMIGRATION RATE OF RURAL POPULATION
S = RATIO OF URBAN TO RURAL POPULATION
ALPHA = PART OF POPULATION IN URBAN AREA

1 MAIN RESULTS

REGION	INIT	TC1	TC2	TD1	TD2	RURAL POPULATION AT				URBAN POPULATION AT				PART WHICH IS URBAN AT			
						ZERO	INIT	TC1	TC2	ZERO	INIT	TC1	TC2	ZERO	INIT	TC1	TC2
USSR	40.5	58.5	33.5	18.0	-7.1	1.00	0.64	0.53	0.69	0.00	0.84	1.23	0.69	0.00	0.57	0.70	0.50
INDIA	54.2	46.6	156.7	-7.6	102.5	1.00	2.51	2.21	14.34	0.00	0.74	0.55	14.34	0.00	0.23	0.20	0.50

****

ZERO = INITIAL PERIOD IN KEYFITZ TIME FRAME
INIT = OBSERVED PERIOD IN KEYFITZ TIME FRAME
TC1 = CROSS-OVER TYPE ONE IN KEYFITZ TIME FRAME
TC2 = CROSS-OVER TYPE TWO IN KEYFITZ TIME FRAME
TD1 = CROSS-OVER TYPE ONE IN ACTUAL TIME FRAME
TD2 = CROSS-OVER TYPE TWO IN ACTUAL TIME FRAME

1 SENSITITY ANALYSIS

REGION	CHANGE IN TC1 DUE TO CHANGE IN			CHANGE IN TC2 DUE TO CHANGE IN			CHANGE IN TD1 DUE TO CHANGE IN			CHANGE IN TD2 DUE TO CHANGE IN		
	U	R	M	U	R	M	U	R	M	U	R	M
USSR	-5.03	1.20	-1.20	-0.45	0.45	-1.61	-4.39	0.56	0.76	0.19	-0.19	0.35
INDIA	-3.21	1.04	-1.04	-10.56	10.56	-35.56	-1.82	-0.35	10.35	-9.16	9.16	-24.16

****

ALL SYMBOLS HAVE SAME MEANING AS ABOVE

THE MAGNITUDE OF THE EXOGENOUS CHANGES IN U,R AND M IS 0.001

subsequent state of a hypothetical population system, characterized as in the above discussion, and occurring at time t_D given by

$$t_D = \frac{1}{0.033} \ln 1.1764 = 54.2 \text{ years} \quad .$$

In this system, both rural and urban populations monotonically increase and become infinitely positive as $t \to +\infty$. Nevertheless, in such a system, $\alpha(t)$ monotonically increases (tending toward $+\infty$ as $t \to +\infty$) while $R(t)$ monotonically decreases (tending toward zero as $t \to \infty$). The cross-over point of type I is reached at time T_1 given by

$$T_1 = \frac{1}{0.033} \ln 1.15 = 46.6 \text{ years} \quad .$$

Thus, the urban growth in the observed Indian system is due rather to natural increase than to net migration since the cross-over point was passed 7.6 years earlier (54.2 - 46.4).

Note that, at this cross-over point, the part of the population of the whole system which is urban is equal to:

$$\alpha(T_1) = \frac{m}{m+u} = \frac{0.005}{0.025} = 0.20$$

i.e., 20 percent (compared to 22.71 percent in the observed period). On the other hand, the cross-over point of the second type is reached at time T_2 given by

$$T_2 = \frac{1}{0.003} \ln 1.6 = 156.7 \text{ years} \quad .$$

Thus, the cross-over point of type II will be reached in 102.5 years from the observed period. Again, the impact of small parameter changes on the time span necessary to reach the cross-over points has been calculated with reference to the Keyfitz and actual time frames. (See Table 2).

It appears that a slightly less urban rate of natural increase (1.9 percent versus 2.0 percent) would delay both cross-over points, the first by 1.8 years (occurring 5.8 years before the observation period) and the second by 9.2 years (occurring 111.7 years after the observation period). On the other hand, a higher rate of rural-urban migration (.6 percent versus .5 percent) would delay the cross-over of type I by 10.4 years (occurring after the observation period exactly 2.8 years later) and hasten that of type II by 24.2 years (occurring 80.1 years after the observation period).

Note that additional applications of the Keyfitz model to the world, macroregions, and regions, as defined by the United Nations (1976) for the year 1960, appear in Appendix 2. The results shown in Tables 3 and 4 suggest the following comments.

a) On the one hand, the more developed regions, as would be expected since their urban areas as a whole are more populated than their rural regions, have already passed the cross-over of type II; on the other hand, the less developed regions, on the basis of present rates, are expected to reach this cross-over in the distant future (in about 200 years for South Asia and Western Africa).

b) Surprisingly, the results relating to the cross-over of type I lead to an opposite conclusion. If present rates are unchanged, the more developed countries will reach this cross-over in the near future (in about 27 years for Western Europe). By contrast, the less developed countries have already passed this cross-over (by as much as 25 years in the case of South Asia: *their urban areas are already growing more from natural increase than from inmigration.*

There are exceptions in both groups of regions: Northern America has already reached the cross-over of type I and China has not yet reached it. This follows from the fact that in these countries, rural and urban rates of natural increase are very similar, unlike the other countries, which exhibit a higher natural increase rate in the rural region. Consequently, if the Keyfitz model is correct, the rural-urban differential in the rate of natural increase has an important impact on the time span necessary for natural increase to exceed inmigration in the urban region.

c) Finally, because the rural population decreases after the observed period in most of the developed regions considered, some doubts arise concerning the usefulness of the Keyfitz model. An alternative framework is thus needed to analyze the sources of urban growth.

II. ANALYSIS BASED ON THE ROGERS MODEL

As an alternative to the Keyfitz model, Ledent (1978) considers a continuous version of a two-region, components-of-change model (Rogers 1968). In such a model, a more symmetric hypothesis concering migration flows between the rural and urban regions is presented: constant gross migration rates out of the rural and urban regions, respectively denoted by o_r and o_u (supposed to be positive). Again, the analytics of this model have been extensively developed in Ledent (1978).

Analytics of the Model

The equations describing the population growth of the rural and urban regions are, respectively:

$$\frac{dP_r(t)}{dt} = (r-o_r)\ P_r(t) + o_u\ P_u(t) \tag{43a}$$

and

$$\frac{dP_u(t)}{dt} = o_r\ p_r(t) + (u-o_u)\ P_u(t) \quad . \tag{43b}$$

It has been shown (Ledent 1978) that the population system described by (43a) and (43b) evolves from an initial state characterized by a concentration of the whole population in the rural region if the parameters of the system are such that

$$\overline{S} < \frac{u-o_u - (r-o_r) + \sqrt{(u-o_u-r+o_r)^2 + 4o_r o_u}}{2o_u}$$

in which $\overline{S}$ is the ratio of urban to rural population in the period at which the system is observed. Thus to remain general, i.e., to prevent any peculiarities due to the value of $\overline{S}$, we impose here

$$0 < \frac{u-o_u - (r-o_r) + \sqrt{(u-o_u - r+o_r)^2 + 4o_r o_u}}{2o_u} \quad . \tag{44}$$

Then, if $t = 0$ denotes the initial period, the solution of the system consisting of equations (43a) and (43b) is given by

$$P_r(t) = A\, e^{x_1 t} - B\, e^{x_2 t} \tag{45a}$$

$$P_u(t) = C\, e^{x_1 t} - D\, e^{x_2 t} \tag{45b}$$

in which

i) x_1 and x_2 are the two real roots of:

$$x^2 - (r-o_r + u-o_u)x + (r-o_r)(u-o_u) - o_r o_u = 0 \quad . \tag{46}$$

Note that the largest root, supposed to be x_1, is necessarily positive and that the following holds:

$$x_2 \le u-o_u \le x_1 \tag{47a}$$

and

$$x_2 \le r-o_r \le x_1 \quad . \tag{47b}$$

ii) A, B, C, and D are defined by

$$A = P(0)\ \frac{x_1 - (u-o_u)}{x_1 - x_2} \tag{48a}$$

$$B = P(0)\ \frac{x_2 - (u-o_u)}{x_1 - x_2} \tag{48b}$$

$$C = P(0)\ \frac{o_r}{x_1 - x_2} \tag{48c}$$

$$D = P(0)\ \frac{o_r}{x_1 - x_2} \tag{48d}$$

where P(0) is the initial population of the system. Note that again we have:

$$P_r(0) = P(0) \qquad \text{and} \qquad P_u(0) = 0$$

One can see from (47a) that A, C, and D are positive whereas B is negative. Substituting (48a) through (48d) into (45a) and (45b) yields

$$P_r(t) = \frac{P(0)}{x_1 - x_2} \left[(x_1 - u+o_u)\ e^{x_1 t} - (x_2 - u+o_u)\ e^{x_2 t} \right] \tag{49a}$$

and

$$P_u(t) = \frac{P(0)}{x_1 - x_2} o_r \left[e^{x_1 t} - e^{x_2 t} \right] \quad . \tag{49b}$$

It can be shown that the urban population monotonically increases with t, becoming infinitely positive as $t \to +\infty$. Note that this result is identical to the one derived with the Keyfitz model, although the restriction $u > 0$ is not necessary here to obtain it.

The rural population, on the contrary, always becomes infinitely positive as $t \to +\infty$, growing at the same rate as the urban population. (Compare this result with the corresponding one obtained with the Keyfitz model). However, if $r > o_r$, the rural population increases monotonically whereas if $r < o_r$, it first decreases, reaches a minimum and then increases indefinitely.

Letting again S(t) denote the ratio of urban to rural population and dividing (45b) by (45a) we obtain

$$S(t) = \frac{C e^{x_1 t} - D e^{x_2 t}}{A e^{x_1 t} - B e^{x_2 t}} \tag{50a}$$

which after substitution of (48a) through (48d) leads to

$$S(t) = \frac{o_r \left(e^{x_1 t} - e^{x_2 t} \right)}{(x_1 - u + o_u) e^{x_1 t} - (x_2 - u + o_u) e^{x_2 t}} \quad . \tag{50b}$$

Differentiating S(t) with respect to time leads to

$$\frac{dS(t)}{dt} = \frac{o_r (x_1 - x_2)^2 e^{(x_1 - x_2)t}}{\left[(x_1 - u + o_u) e^{x_1 t} - (x_2 - u + o_u) e^{x_2 t} \right]^2} \tag{51}$$

an expression which indicates that $\frac{dS(t)}{dt}$ is positive and thus that S(t) monotonically increases: initially equal to zero, it tends toward the quotient $\frac{C}{A}$ of the coefficients of $e^{x_1 t}$ in (45a) and (45b).

This quotient can be obtained from (48a) and (48c) as

$$\frac{C}{A} = \frac{o_r}{x_1 - (u-o_u)} \qquad (52a)$$

a quantity which, after recalling (46), appears to be also equal to

$$\frac{C}{A} = \frac{x_1 - (r-o_r)}{o_u} \quad . \qquad (52b)$$

Note that in the same way it can be shown that

$$\frac{D}{B} = \frac{o_r}{x_2 - (u-o_u)} = \frac{x_2 - (r-o_r)}{o_u} \quad . \qquad (53)$$

The relationship (5) expressing the part $\alpha(t)$ of the population which is urban in terms of S(t) remains valid:

$$\alpha(t) = \frac{S(t)}{1+S(t)} \qquad (5)$$

Consequently, the variations of $\alpha(t)$ are similar to those of S(t): $\alpha(t)$ monotonically increases from zero (for t = 0) to $\frac{C}{A+C}$ (for

$t \to \infty$) where

$$\frac{C}{A+C} = \frac{o_r}{x_1 - u + o_u + o_r}$$

then, while the Keyfitz model leads in the long run to a concentration of the population in the urban area, the Rogers model leads to an equilibrium regional distribution.

The urban net inmigration rate $m_u(t)$ is now defined by

$$m_u(t) = \frac{o_r P_r(t) - o_u P_u(t)}{P_u(t)} = \frac{o_r}{S(t)} - o_u \qquad (54)$$

a relationship which permits one to conclude that $m_u(t)$ monotonically decreases from $+\infty$ (for $t = 0$) to $x_1 - u$ (for $t \to +\infty$).

The ratio $R(t)$ of urban net inmigration to urban natural increase still given by

$$R(t) = \frac{m_u(t)}{u} \qquad (7)$$

is now linked to $S(t)$ by the following

$$R(t) = \frac{o_r}{uS(t)} - \frac{o_u}{u} \qquad (55)$$

Again, we must impose here the constraint

$$u > 0 \qquad (9)$$

so that R(t) monotonically decreases from $+\infty$ (for $t = 0$) to $\frac{x_1-u}{u}$ (for $t \rightarrow +\infty$).

The results of the above model defined by the system (43a) - (43b) and the restrictions (44) and (9) are summarized in Table 3.

Table 3: The Rogers model as a model of urbanization: the variations of the model's functions.

t	0	t_r	$+\infty$
$P_r(t)$ (a) $r < o_r$	$P(0)$	↘ ↗	$+\infty$
$P_r(t)$ (b) $r \geq o_r$	$P(0)$	↗	$+\infty$
$P_u(t)$	0	↗	$+\infty$
$S(t)$	0	↗	$\frac{x_1 - (r-o_r)}{o_u}$
$\alpha(t)$	0	↗	$\frac{o_r}{x_1 - u + o_u + o_r}$
$m_u(t)$	$+\infty$	↘	$x_1 - u$
$R(t)$	$+\infty$	↘	$\frac{x_1 - u}{u}$

Finally, note that since S(t) tends toward a finite limit rather than becoming infinite in the Keyfitz model, one can expect that in some circumstances no cross-over point of type II is reached. In the same way, since R(t) tends toward $\frac{x_1-u}{u}$ rather than zero as in the Keyfitz model, this model does not necessarily lead to a cross-over point of type I either.

The Factors of Urbanization and the Cross-over Point of Type I

Substituting (50a) into (55) yields an analytic expression

$$R(t) = \frac{(o_rA - o_uC)e^{x_1t} - (o_rB - o_uD)e^{x_2t}}{u\left(C\,e^{x_1t} - D\,e^{x_2t}\right)} \tag{56}$$

Substituting (48a) through (48d) allows rewriting this expression as

$$R(t) = \frac{(x_1-u)e^{x_1t} - (x_2-u)e^{x_2t}}{u\left(e^{x_1t} - e^{x_2t}\right)} \tag{57}$$

and thus to visualize the variations of R(t), appearing in Figure 3

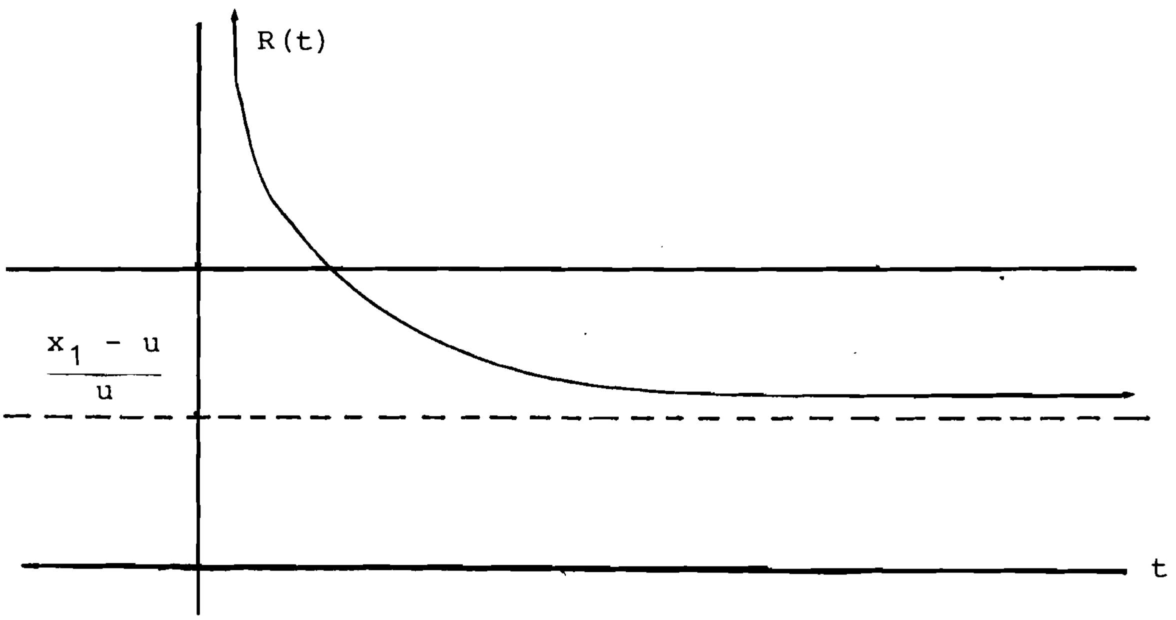

Figure 3. The Rogers model as a model of urbanization: the variations of R(t).

The variations of $R(t)$ indicate that net inmigration is initially preponderant in accounting for the growth of the urban region, but as time passes, its role diminishes and natural increase may or may not be the main source of urban growth. Clearly, Figure 3 indicates that if $\frac{x_1 - u}{u} < 1$ (i.e., $x_1 < 2u$), then $R(t)$ tends toward a limit less than one: there is a cross-over point of type I (denoted by T_1) such that for $t > T_1$ natural increase becomes preponderant. By contrast, if $x_1 > 2u$, there does not exist such a point and net inmigration accounts for the largest part of urban growth at any time t.

Again T_1 is defined by

$$m_u(T_1) = u \tag{11}$$

and

$$R(T_1) = 1 \quad . \tag{12}$$

From (54) it can be shown that

$$S(T_1) = \frac{o_r}{u+o_u} \quad . \tag{58}$$

Then, at the cross-over point T_1, the ratio of urban to rural population is simply equal to the ratio of the rural outmigration rate to the sum of the urban rates of natural increase and outmigration. Consequently, the part of the whole population which is urban at time T_1 is

$$\alpha(T_1) = \frac{o_r}{u+o_u+o_r} \quad . \tag{59}$$

An expression of T_1 can be arrived at by combining (57) and (12). We obtain

$$T_1 = \frac{1}{x_1 - x_2} \ln\left[\frac{x_2 - 2u}{x_1 - 2u}\right] \quad . \tag{60}$$

Note that T_1 is defined only if $\frac{x_2 - 2u}{x_1 - 2u}$ is greater than one, which since $x_2 < x_1$ requires $x_1 < 2u$: thus, we have obtained analytically the result previously suggested by Figure 3.

What is the impact of a parameter change on the value of T_1? It is clear that the derivation of the first derivatives of T_1 with respect to the parameters of the model is more complicated than in the case of the Keyfitz model. Since the value of T_1 depends on x_1 and x_2, it is necessary to first derive the impact of a small parameter change on x_1 and x_2 and then to determine the consequences of the changes in x_1 and x_2 on T_1. The corresponding calculations which are rather tedious (and thus are only outlined in Appendix 3) do not lead to the occurrence of definite signs for the first derivatives of T_1, as in the Keyfitz model.*

However, the impact of parameter changes on the part of the population which is urban at the cross-over point can be easily derived. Differentiating (59) with respect to each of the parameters yields:

$$\frac{d\alpha(T_1)}{du} = -\frac{o_r}{(u+o_u+o_r)^2} \quad ; \quad \frac{d\alpha(T_1)}{dr} = 0$$

$$\frac{d\alpha(T_1)}{do_u} = -\frac{o_r}{(u+o_u+o_r)^2} \quad ; \quad \frac{d\alpha(T_1)}{do_r} = \frac{u+o_u}{(u+o_u+o_r)^2} \quad . \tag{61}$$

*In practice, the derivation of the impact of parameter changes is simply obtained by comparing the new value of T_1 with the initial one.

It can be seen that the proportion of the population which is urban at the cross-over point is independent of the rural rate of natural increase. Moreover, the smaller the urban rates of natural increase and outmigration and the higher the rural outmigration rate, the higher this proportion.

The Magnitude of Urbanization and the Cross-over of Type II

Substituting expression (50b) into (5) yields:

$$\alpha(t) = \frac{o_r\left(e^{x_1 t} - e^{x_2 t}\right)}{\left(x_1 - u + o_u + o_r\right)e^{x_1 t} - \left(x_2 - u + o_u + o_r\right)e^{x_2 t}} \qquad (62)$$

an expression which permits one to visualize the variations of $\alpha(t)$ appearing in Figure 4.

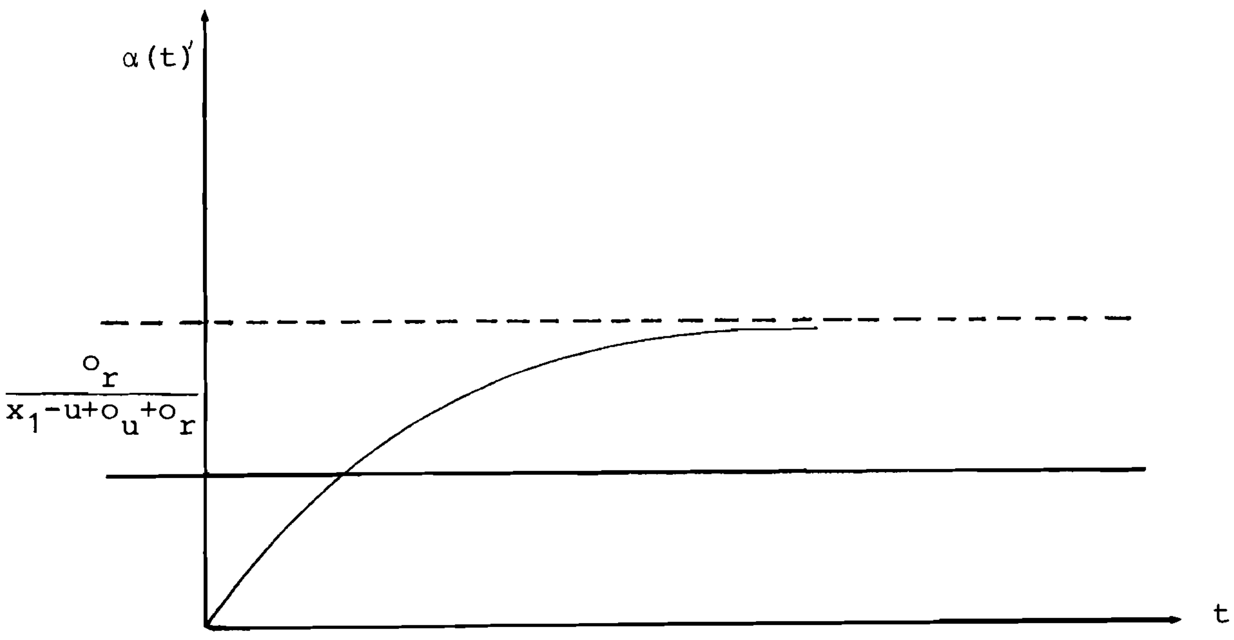

Figure 4. The Rogers model as a model of urbanization: the variations of $\alpha(t)$.

The variations of $\alpha(t)$ indicate that the population system, totally rural initially, becomes more and more urbanized but $\alpha(t)$ tends toward a limit equal to $\frac{o_r}{x_1 - u+o_u+o_r}$.

Clearly, Figure 4 indicates that $\frac{o_r}{x_1 - u+o_u+o_r} > \frac{1}{2}$ (i.e., $x_1 < u-o_u+o_r$), then $\alpha(t)$ tends toward a limit greater than $\frac{1}{2}$ and thus there exists a cross-over point of type II (denoted by T_2) defined as in Section I.

$$\alpha(T_2) = \frac{1}{2} \tag{24}$$

i.e.,

$$S(T_2) = 1 \quad . \tag{25}$$

From (55) we find that

$$R(T_2) = \frac{o_r - o_u}{u} \tag{63}$$

and from (54) that

$$m(T_2) = o_r - o_u \quad . \tag{64}$$

At the cross-over T_2 , the net inmigration rate of the urban region is equal, as one would expect, since both populations are equal, to the difference of the rural and urban outmigration rates.

An expression of T_2 can be arrived at by combining (62) and

(24). We obtain

$$T_2 = \frac{1}{x_1 - x_2} \ln \left[\frac{x_2 - u + o_u - o_r}{x_1 - u + o_u + o_r} \right] . \tag{65}$$

Note that T_2 is defined only if $\frac{x_2 - u + o_u - o_r}{x_1 - u + o_u - o_r}$ is greater than one. Since $x_2 < x_1$ requires $x_1 < u - o_u + o_r$, thus we have demonstrated analytically the result suggested by Figure 4.

What is the impact of a parameter change on the value of T_2? Again, the derivation of the first derivatives of T_2 with respect to the parameters of the model does not lead to formulas showing a definite sign as in the Keyfitz model. Such a derivation is also outlined in Appendix 3.

Comparison of the Two Cross-over Points

When both cross-over points exist, which of the two points defined above is reached first? By subtracting (6) from (65) we obtain

$$T_2 - T_1 = \frac{1}{x_1 - x_2} \ln \left[\frac{x_2 - u + o_u - o_r}{x_1 - u + o_u - o_r} \, \frac{x_1 - 2u}{x_2 - 2u} \right] . \tag{66}$$

Since $x_1 < u = o_u + o_r$ and $x_2 < 2u$, it follows that $T_2 > T_1$ if $(x_2 - u + o_u - o_r)(x_1 - 2u) > (x_1 - u + o_u - o_r)(x_2 - 2u)$, i.e., if

$$u > o_r - o_u \quad . \tag{67}$$

Application to Actual Population Systems

Let us suppose that, in a given year, we observe a population system characterized by parameters u, o_u, r and o_r and a proportion of urban population equal to $\bar{\alpha}$.

If those parameters satisfy the condition (44), there exists a hypothetical population system, initially totally rural and submitted to the constant regimes of natural increase and migration defined by u, r, and m, that at some point presents characteristics as the observed population system.

The time t_D, at which this correspondence occurs, is simply obtained as the root of $\alpha(t) = \bar{\alpha}$, which is unique due to the course of evolution of $\alpha(t)$ (see Figure 4). It is readily established from (62) that

$$t_D = \frac{1}{x_1 - x_2} \ln \left[\frac{o_r - \bar{\alpha}\ (x_2 - u + o_u + o_r)}{o_r - \bar{\alpha}\ (x_1 - u + o_u + o_r)} \right] \tag{68a}$$

or alternatively,

$$t_D = \frac{1}{x_1 - x_2} \ln \left[\frac{o_r - \bar{S}\ (x_2 - u + o_u)}{o_r - \bar{S}\ (x_1 - u + o_u)} \right] \quad . \tag{68b}$$

Consequently, if around the observation period the actual population system exhibits the constant regimes of natural increase and migration defined by u, o_u, r and o_r, we can simply determine whether this system has already reached or will reach the two types of cross-over points defined above.

Again, let T_1' and T_2' denote the time spans necessary to reach the cross-over points, of types I and II respectively, from the observed period. Using (39) we have

$$T_1' = \frac{1}{x_1 - x_2} \ln \left[\frac{x_2 - 2u}{x_1 - 2u} \; \frac{o_r - \bar{\alpha}\,(x_1 - u + o_u + o_r)}{o_r - \bar{\alpha}\,(x_2 - u + o_u + o_r)} \right] \tag{69}$$

and

$$T_2' = \frac{1}{x_1 - x_2} \ln \left[\frac{x_2 - u + o_u + o_r}{x_1 - u + o_u + o_r} \; \frac{o_r - \bar{\alpha}\,(x_1 - u + o_u + o_r)}{o_r - \bar{\alpha}\,(x_2 - u + o_u + o_r)} \right]. \tag{70}$$

From these formulas, it is easy to show that T_1' is positive if $\bar{\alpha} < \frac{o_r}{u + o_u + o_r}$ $\left(\text{or } \bar{S} < \frac{o_r}{u + o_u}\right)$ and that T_2' is positive if $\bar{\alpha} < \frac{1}{2}$ $\left(\text{or } \bar{S} < 1\right)$. Note that the knowledge of the variations of $S(t)$ and formulas (58) and (25) lead to the same results.

Hence, the simple comparison of the observed ratio of population $\bar{S}$ with the quotient $\frac{o_r}{u + o_u}$ permits one to immediately determine whether the cross-over point of the first type has already been reached. In addition, the relative values of $\bar{S}$ and the number 1 determine whether the cross-over point of the second type has been reached or not.

Again, the impact of small parameter changes on T_1' and T_2' can be obtained after tedious computations (See Appendix 3).

Numerical Illustrations*

The parameters relevant to the application of the Rogers model to the USSR case are:

$$u = 0.009; \quad o_u = 0.011; \quad r = 0.010; \quad o_r = 0.035; \quad \bar{S} = 1.3125 \ .$$

This observed population system corresponds to the subsequent state of a hypothetical population, characterized as in the above discussions and occurring at time t_D given by:

$$t_D = 30.2 \text{ years}^{**} \quad .$$

In this system, the urban population monotonically increases while the rural population first decreases, passes through a minimum and then increases. Both populations become infinitely positive, growing at the same rate $x_1 = 0.925$ percent.

The urban net migration rate (+ ∞ initially) monotonically decreases toward $x_1 - u = 0.00025$ (as $t \to \infty$) reaching the value 1 corresponding to the cross-over point of type I at time T_1 given by

$$T_1 = \frac{1}{0.0455} \ln 6.2 = 40.1 \text{ years} \quad .$$

The time span necessary to reach this cross-over point from the observed period is then

$$T_1' = t_1 - t_D = 9.9 \text{ years} \quad .$$

*These numerical illustrations are summarized in Table 4.
**The two roots of (46) are $x_1 = 0.00925$ and $x_2 = -0.03625$.

Table 4. The Rogers Model: Numerical Illustrations.

1 INPUT DATA

REGION	RU	BU	DU	OU	RR	BR	DR	OR
USSR	25.0	17.0	8.0	11.0	-11.0	19.0	9.0	35.0
INDIA	37.0	30.0	10.0	10.0	17.0	39.0	17.0	7.0

****

RU = GROWTH RATE OF URBAN REGION
BU = BIRTH RATE OF URBAN REGION
DU = DEATH RATE OF URBAN REGION
OU = OUTMIGRATION RATE OF URBAN REGION
RR = GROWTH RATE OF RURAL REGION
BR = BIRTH RATE OF RURAL REGION
DR = DEATH RATE OF RURAL REGION
OR = OUTMIGRATION RATE OF RURAL REGION

1 BASIC DATA

REGION	U	OU	R	OR	S	ALPHA
USSR	0.0090	0.0110	0.0100	0.0350	1.3125	0.5676
INDIA	0.0200	0.0100	0.0220	0.0070	0.2941	0.2273

****

U = NATURAL INCREASE RATE OF URBAN POPULATION
OU = OUTMIGRATION RATE OF URBAN POPULATION
R = NATURAL INCREASE RATE OF RURAL POPULATION
OR = OUTMIGRATION RATE OF RURAL POPULATION
S = RATIO OF URBAN TO RURAL POPULATION
ALPHA = PART OF POPULATION WHICH IS URBAN

1 MAIN RESULTS

REGION	INIT	TC1	TC2	TD1	TD2	RURAL POPULATION AT				URBAN POPULATION AT				PART WHICH IS URBAN AT				
						ZERO	INIT	TC1	TC2	ZERO	INIT	TC1	TC2	ZERO	INIT	TC1	TC2	INF
USSR	30.2	40.1	23.5	9.9	-6.7	1.00	0.58	0.53	0.63	0.00	0.76	0.93	0.63	0.00	0.57	0.64	0.50	0.76
INDIA	49.9	37.7	0.0	-12.2	0.0	1.00	2.29	1.84	0.00	0.00	0.67	0.43	0.00	0.00	0.23	0.19	0.00	0.38

****

ZERO = INITIAL PERIOD IN KEYFITZ TIME FRAME
INIT = OBSERVED PERIOD IN KEYFITZ TIME FRAME
TC1 = CROSS-OVER TYPE ONE IN KEYFITZ TIME FRAME
TC2 = CROSS-OVER TYPE TWO IN KEYFITZ TIME FRAME
TD1 = CROSS-OVER TYPE ONE IN ACTUAL TIME FRAME
TD2 = CROSS-OVER TYPE TWO IN ACTUAL TIME FRAME
INF = LONG RUN

1 SENSITIVITY ANALYSIS

REGION	CHANGE IN TD1 DUE TO CHANGE IN				CHANGE IN TD2 DUE TO CHANGE IN			
	U	OU	R	OR	U	OU	R	OR
USSR	-2.23	-1.25	0.36	0.68	0.16	-0.38	-0.17	0.29
INDIA	-0.74	-2.38	-0.69	7.71	0.00	0.00	0.00	0.00

****

ALL SYMBOLS HAVE SAME MEANING AS ABOVE
THE MAGNITUDE OF THE EXOGENOUS CHANGES IN U,OU,R AND OR IS 0.001

At this cross-over point, the part of the population which is urban is equal to

$$\alpha(T_1) = \frac{0.035}{0.055} = 0.636$$

i.e., 63.6 percent, a value higher than the observed 56.76 percent (which was expected since T_1' is positive).

However, $\alpha(t)$ monotonically increases toward a limit which appears to be 0.7568 (i.e., 75.68 percent is ultimately concentrated in the urban region). It takes on the value $\frac{1}{2}$ corresponding to the cross-over point of type II at time T_2 given by

$$T_2 = 23.5 \text{ years} \quad .$$

It follows that this cross-over point has been reached 6.7 years before the observation period.

The impact of small parameter changes on the time spans necessary to reach the cross-over points has been calculated with reference to both the Keyfitz time frame (the one of the hypothetical systems leading to the observed population system of the USSR) and the actual time frame.

It turns out that, a slightly smaller urban rate of natural increase (0.8 percent versus 0.9 percent) would delay the cross-over of type I by 2.2 years and hasten the cross-over of type II by 0.2 years. Also, a slightly higher rural outmigration rate (3.6 percent rather than 3.5 percent) would delay both cross-over points (by 0.6 and 0.3 years respectively), while a slightly higher urban outmigration rate (1.2 percent instead of 1.1 percent) would hasten both cross-over points (by 1.3 and 0.4 years respectively).

In the case of the rural-urban system of India in the late sixties, the parameters of the Rogers model are

$$u = 0.020; \quad o_u = 0.010; \quad r = 0.022; \quad o_r = 0.007; \quad \text{and} \quad \bar{S} = 0.294.$$

This observed population system appears to be identical to the subsequent state of a hypothetical population system, characterized as in the above discussion and occurring at time t_D given by

$$t_D = \frac{1}{0.0175} \ln 2.396 = 49.9 \text{ years}^{*} \quad .$$

In this system, both rural and urban populations monotonically increase, ultimately growing at a 2.125 percent rate.

$R(t)$ monotonically decreases from $+\infty$ (for $t = 0$) to $x_1 - u = 0.00125$ (as $t \to +\infty$) reaching the value 1 corresponding to the cross-over point of type I at time T_1 given by

$$T_1 = \frac{1}{0.0175} \ln 1.933 = 37.7 \text{ years} \quad .$$

Thus, the urban growth in the observed Indian system is due rather to natural increase than to net inmigration since the cross-over point has been passed 12.1 years earlier (49.8 - 37.7).

At this cross-over point, the part of population which was urban was then

$$\alpha(T_1) = \frac{0.007}{0.037} = 0.1892$$

*The two roots of (46) in this case are $x_1 = 0.02125$ and $x_2 = 0.00375$.

i.e., 18.92 percent (compared to 22.73 percent in the observation period).

On the other hand, $\alpha(t)$ monotonically increases from zero (for $t = 0$) to 38.36 percent (for $t \to + \infty$). Therefore, the observed Indian system does not admit any cross-over point of type II.

What about the effect of small parameter changes on the cross-over of type I? The results of Table 4 under the heading "Sensitivity Analysis" indicate that a slightly smaller urban rate of natural increase would delay the cross-over points by 0.74 years (for a .1 percent decrease in u). Indeed, higher out-migration rates also delay the occurrence of this cross-over (the delay is 7.7 years if the rural outmigration rate increases from .7 to .8 percent).

Note that, surprisingly, a larger rural rate of natural increase contributes to hasten even more the occurrence of the cross-over.

CONCLUSION

This paper has sought to examine the importance and the forces of urbanization under constant schemes of natural increase and migration. Following up an earlier paper (Ledent, 1978), we have pointed out the problems raised by the use of the Keyfitz model and evaluated the relevance of using an alternative framework for this purpose, namely a continuous two-region version of the Rogers model.

The main conclusion is that, if the second model removes some of the limitations of the Keyfitz model (possible vanishing rural population), it also brings its own difficulties. The existence of a limiting regional distribution, $\alpha(\infty) < 1$ (contrasting with $\alpha(\infty) = 1$ in the case of the Keyfitz model) does not ensure the existence of cross-over points, especially in the case of the cross-over of type II.

Table 5 shows the contrast between the results of the two alternative models when applied to two actual population systems.

It indicates the existence of a large discrepancy between the cross-over measures to which they lead.

Table 5. The results of the two models contrasted.

	USSR		INDIA	
	Keyfitz Model	Rogers Model	Keyfitz Model	Rogers Model
T_1	58.5	40.1	46.6	37.7
$\alpha(T_1)$	0.70	0.64	0.20	0.19
T_2	33.4	23.5	156.7	----
T_D	-40.5	-30.2	-54.2	-49.9
T_1'	18.0	9.9	7.6	-12.2
T_2'	- 7.1	- 6.7	102.5	----

Which of the two models is then the most relevant to give insights into the urbanization phenomenon? This paper has not brought any definite answer to this question; nevertheless it suggests that the Rogers framework is more appropriate because

a) its limiting regional distribution (a consequence of the Markov chain formulation of the model) is less restrictive than the vanishing of the rural population in the alternative model, and

b) a rural-urban net migration rate is much more volatile than the corresponding rural and urban outmigration rates because of the relative variations of the urban and rural populations.

In practice, however, the use of the Rogers framework might well be hindered by the fact that actual migration data for most regional systems generally consist of data on net migration only (United Nations, 1976).

Finally, since actual population systems do not exhibit constant schemes of natural increase and migration, the question of whether urban areas grow from natural increase or from inmigration, ought to be reexamined in the context of varying rates. This extension will be carried out in a forthcoming paper.

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Appendix 1

The Keyfitz model: Obtaining the signs of the first derivatives of T_1.

The first derivative of T_1 with respect to the urban rate of natural increase $\frac{dT_1}{du}$, given by (16), has the sign of

$$y = \frac{(r - m)(u + m - r)}{u(2u + m - r)} - \ln\left(1 + \frac{u + m - r}{u}\right) .$$

Because of the restrictions (2) and (9), it is obvious that y is negative if r is less than or equal to m. In the case $r > m$, we must analyze the variations of y to determine its sign.

Since the first component of y can be rewritten as

$$(u + m - r)\left[\frac{2}{2u + m - r} - \frac{1}{u}\right] ,$$

the first derivative $\frac{dy}{du}$ is obtained as

$$(u + m - r)\left[- \frac{4}{(2u + m - r)^2} + \frac{1}{u^2}\right] + \frac{r - m}{u(2u + m - r)}$$

$$- \frac{- \frac{m - r}{u^2}}{1 + \frac{u + m - r}{u}} .$$

This expression reduces to

$$\frac{(u + m - r)(m - r)(4u + m - r)}{u^2(2u + m - r)^2} + \frac{r - m}{u(2u + m - r)} - \frac{r - m}{u(2u + m - r)}$$

in which the last two terms cancel out. Thus $\frac{dy}{du}$ has the sign of $(m - r)$. Since $r > m$, $\frac{dy}{du}$ is negative and y decreases. Moreover, we note that if $u = r - m$ [the smaller value that u can take because of the constraint (2)], y takes on the value zero. We then conclude that y is also negative, whatever the values of the model parameters, in the case $r > m$. Then, in all circumstances

$$\frac{dT_1}{du} < 0 \quad ,$$

and $\frac{dT_1}{dr}$, given by (17), has the sign of

$$z = -\frac{u + m - r}{2u + m - r} + \ln\left(1 + \frac{u + m - r}{u}\right)$$

which can be determined by analyzing the variations of z. Differentiating z with respect to r yields:

$$\frac{dz}{dr} = \frac{u}{(2u + m - r)^2} - \frac{\frac{1}{u}}{1 + \frac{u + m - r}{u}} \quad ,$$

an expression which reduces to

$$\frac{dz}{dr} = - \frac{u + m - r}{(2u + m - r)^2} \quad ,$$

which is indeed negative.

Then, z monotonically decreases. Because of the constraint (2), z is always higher than the value it takes on when $r = u + m$, i.e., zero. Thus, z is always positive and

$$\frac{dT_1}{dr} > 0 \quad .$$

Finally, since $\frac{dT_1}{dm} = - \frac{dT_1}{dr}$, we have

$$\frac{dT_1}{dm} < 0 \quad .$$

Appendix 2

The Keyfitz model: Application to the world, macroregions, and regions as defined by the United Nations.

The Model proposed by Keyfitz has been applied to the world, macroregions as defined by the United Nations (1976) for the year 1960.

Table A1 shows the input data found in the UN publication and used to derive the parameters requested by the Keyfitz model. The latter are displayed in Table A2.

Table A3 shows the various time spans necessary to reach both of the cross-over points (in the Keyfitz time frame and actual time frames). Comments suggested by the results of this Table appear in the text at the end of the first section.

The impact of small parameter changes on the various cross-over measures was calculated using the formulas that give their first derivatives with respect to the model parameters. It was also determined by comparing the new cross-over measures, calculated for small parameter changes equal to .1 percent, with the reference measures. The corresponding results are shown in Table A4 and A5 respectively; they indicate the existence of small discrepancies.

Table A1. INPUT DATA

REGION	RU	BU	DU	RR	BR	DR
WORLD	33.0	27.7	11.6	12.5	39.8	19.1
*********************						
MORE DEVELOPED REG.	23.5	20.1	8.9	-2.6	23.3	9.3
LESS DEVELOPED REG.	45.5	37.9	15.4	16.5	44.1	21.7
*********************						
AFRICA	44.8	41.6	18.0	18.0	47.8	25.1
NORTHERN AMERICA	24.3	24.2	8.9	-1.2	24.8	9.3
LATIN AMERICA	44.6	35.1	10.8	12.7	44.2	12.6
EAST ASIA	48.6	29.8	12.9	8.6	36.7	19.3
SOUTH ASIA	36.7	40.0	17.2	21.2	47.1	22.9
EUROPE	17.9	17.8	10.2	-4.2	21.8	10.0
OCEANIA	26.2	22.5	8.9	13.2	36.3	13.1
USSR	34.5	20.8	6.5	-1.4	26.5	8.4
*********************						
WESTERN AFRICA	49.9	41.1	20.0	17.9	50.2	27.1
EASTERN AFRICA	49.9	44.6	18.9	20.1	46.9	24.8
NORTHERN AFRICA	42.3	43.8	17.1	18.5	47.4	22.1
MIDDLE AFRICA	58.6	47.2	20.6	13.0	44.8	27.7
SOUTHERN AFRICA	32.9	32.1	15.1	16.3	47.6	20.1
NORTHERN AMERICA	24.3	24.2	8.9	-1.2	24.8	9.3
TROPICAL SOUTH AMER.	49.6	31.1	11.2	11.7	45.0	12.8
MIDDLE AMER.(MAINL.)	47.0	42.7	11.5	21.1	47.0	13.0
TEMPERATE SOUTH AME.	30.2	24.3	9.1	-9.1	34.3	9.5
CARIBBEAN	34.2	30.8	11.3	15.1	41.9	12.9
CHINA	50.3	33.9	15.4	9.7	38.2	20.7
JAPAN	29.2	15.8	6.6	-5.9	18.5	8.6
OTHER EAST ASIA	56.2	35.8	9.0	14.9	43.3	13.6
MIDDLE SOUTH ASIA	32.6	39.6	17.9	21.1	47.2	23.9
SOUTH EAST ASIA	43.3	42.2	16.2	21.9	46.7	21.1
SOUTH WEST ASIA	46.4	38.0	15.1	18.6	48.9	19.5
WESTERN EUROPE	19.5	17.4	10.6	-6.5	20.9	11.2
SOUTHERN EUROPE	21.0	19.3	9.1	-2.2	23.0	9.4
EASTERN EUROPE	19.2	17.3	9.6	-3.8	22.6	9.3
NORTHERN EUROPE	11.2	17.4	11.0	-6.4	17.6	11.1
AUSTRALIA AND N.Z.	25.8	22.2	8.9	1.8	29.0	7.5
MELANESIA	47.9	45.8	13.8	22.4	42.7	19.8
MICRONESIA AND POL.	47.6	35.5	9.1	25.8	42.6	12.9
USSR	34.5	20.8	6.5	-1.4	26.5	8.4
*********************						

RU = GROWTH RATE OF URBAN POPULATION(*1000)
BU = BIRTH RATE OF URBAN POPULATION(*1000)
DU = DEATH RATE OF URBAN POPULATION(*1000)
RR = GROWTH RATE OF RURAL POPULATION(*1000)
BR = BIRTH RATE OF RURAL POPULATION(*1000)
DR = DEATH RATE OF RURAL POPULATION(*1000)

Table A2. BASIC DATA

REGION	U	R	M	S	ALPHA
WORLD	0,0161	0,0207	0,0082	0,4852	0,3267
*********************					
MORE DEVELOPED REG.	0,0112	0,0140	0,0166	1,3496	0,5744
LESS DEVELOPED REG.	0,0225	0,0224	0,0059	0,2565	0,2042
*********************					
AFRICA	0,0236	0,0227	0,0047	0,2217	0,1815
NORTHERN AMERICA	0,0153	0,0155	0,0167	1,8556	0,6498
LATIN AMERICA	0,0243	0,0316	0,0189	0,9310	0,4821
EAST ASIA	0,0169	0,0174	0,0088	0,2776	0,2173
SOUTH ASIA	0,0228	0,0242	0,0030	0,2158	0,1775
EUROPE	0,0076	0,0118	0,0160	1,5534	0,6084
OCEANIA	0,0136	0,0232	0,0100	0,7937	0,4425
USSR	0,0143	0,0181	0,0195	0,9653	0,4912
*********************					
WESTERN AFRICA	0,0211	0,0231	0,0052	0,1806	0,1529
EASTERN AFRICA	0,0257	0,0221	0,0020	0,0826	0,0763
NORTHERN AFRICA	0,0267	0,0253	0,0068	0,4359	0,3036
MIDDLE AFRICA	0,0266	0,0171	0,0041	0,1281	0,1136
SOUTHERN AFRICA	0,0170	0,0275	0,0112	0,7044	0,4133
NORTHERN AMERICA	0,0153	0,0155	0,0167	1,8556	0,6498
TROPICAL SOUTH AMER.	0,0199	0,0322	0,0205	0,6902	0,4084
MIDDLE AMER.(MAINL.)	0,0312	0,0340	0,0129	0,8165	0,4495
TEMPERATE SOUTH AME.	0,0152	0,0248	0,0339	2,2600	0,6933
CARIBBEAN	0,0195	0,0290	0,0139	0,9456	0,4860
CHINA	0,0185	0,0175	0,0078	0,2453	0,1970
JAPAN	0,0092	0,0099	0,0158	0,7900	0,4413
OTHER EAST ASIA	0,0268	0,0297	0,0148	0,5034	0,3348
MIDDLE SOUTH ASIA	0,0217	0,0233	0,0022	0,2018	0,1679
SOUTH EAST ASIA	0,0260	0,0256	0,0037	0,2139	0,1762
SOUTH WEST ASIA	0,0229	0,0294	0,0108	0,4596	0,3149
WESTERN EUROPE	0,0068	0,0097	0,0162	1,2756	0,5606
SOUTHERN EUROPE	0,0102	0,0136	0,0158	1,4630	0,5940
EASTERN EUROPE	0,0077	0,0133	0,0171	1,4870	0,5979
NORTHERN EUROPE	0,0064	0,0065	0,0129	2,6875	0,7288
AUSTRALIA AND N.Z.	0,0133	0,0215	0,0197	1,5760	0,6118
MELANESIA	0,0320	0,0229	0,0005	0,0314	0,0305
MICRONESIA AND POL.	0,0264	0,0297	0,0039	0,1840	0,1554
USSR	0,0143	0,0181	0,0195	0,9653	0,4912
*********************					

U = NATURAL INCREASE RATE OF URBAN POPULATION
R = NATURAL INCREASE RATE OF RURAL POPULATION
M = NET OUTMIGRATION RATE OF RURAL POPULATION
S = RATIO OF URBAN TO RURAL POPULATION
ALPHA = PART OF POPULATION IN URBAN AREA

Table A3. MAIN RESULTS

REGION	INIT	TC1	TC2	TD1	TD2	RURAL POPULATION AT ZERO	RURAL INIT	RURAL TC1	RURAL TC2	URBAN POPULATION AT ZERO	URBAN INIT	URBAN TC1	URBAN TC2	PART WHICH IS URBAN AT ZERO	PART INIT	PART TC1	PART TC2
WORLD	53.5	56.1	101.1	2.6	[illegible]	1.00	1.50	2.02	3.54	0.00	0.95	1.53	3.54	0.00	0.33	0.34	0.50
**********************																	
MORE DEVELOPED REG.	54.5	58.2	33.8	3.7	-10.7	1.00	0.87	0.86	0.89	0.00	1.17	1.27	0.89	0.00	0.77	0.60	0.50
LESS DEVELOPED REG.	38.6	39.4	116.9	0.8	78.3	1.00	1.09	1.92	6.69	0.00	0.49	0.50	6.89	0.00	0.20	0.21	0.50
**********************																	
AFRICA	41.9	38.0	140.1	-3.8	96.2	1.00	2.12	1.98	12.45	0.00	0.47	0.39	12.45	0.00	0.18	0.17	0.50
NORTHERN AMERICA	63.1	44.3	41.6	-18.8	-21.5	1.00	0.93	0.95	0.95	0.00	1.72	1.03	0.95	0.00	0.65	0.52	0.50
LATIN AMERICA	39.0	33.6	41.3	-5.3	2.3	1.00	1.64	1.53	1.69	0.00	1.53	1.19	1.69	0.00	0.48	0.44	0.50
EAST ASIA	28.0	48.1	80.8	20.1	52.0	1.00	1.27	1.51	1.99	0.00	0.35	0.79	1.99	0.00	0.22	0.34	0.50
SOUTH ASIA	68.1	42.4	267.2	-25.7	199.1	1.00	4.24	2.46	288.20	0.00	0.91	0.32	288.20	0.00	0.18	0.12	0.50
EUROPE	64.7	79.4	46.8	14.7	-17.9	1.00	0.76	0.72	0.82	0.00	1.16	1.51	0.82	0.00	0.61	0.68	0.50
OCEANIA	78.1	72.5	98.1	-5.7	19.9	1.00	2.80	2.60	3.65	0.00	2.23	1.91	3.65	0.00	0.44	0.42	0.50
USSR	36.6	47.2	37.6	10.6	1.0	1.00	0.95	0.94	0.95	0.00	0.92	1.28	0.95	0.00	0.49	0.58	0.50
**********************																	
WESTERN AFRICA	32.9	44.1	149.9	11.2	116.9	1.00	1.80	2.20	14.62	0.00	0.33	0.54	14.62	0.00	0.15	0.20	0.50
EASTERN AFRICA	37.2	35.2	238.4	-2.0	201.2	1.00	2.11	2.03	120.51	0.00	0.17	0.16	120.51	0.00	0.08	0.07	0.50
NORTHERN AFRICA	51.5	32.7	96.5	-18.9	45.0	1.00	2.59	1.83	5.96	0.00	1.13	0.47	5.96	0.00	0.30	0.20	0.50
MIDDLE AFRICA	26.0	30.4	107.5	4.3	81.5	1.00	1.40	1.48	4.05	0.00	0.18	0.23	4.05	0.00	0.11	0.13	0.50
SOUTHERN AFRICA	61.5	57.6	86.6	-3.9	25.1	1.00	2.73	2.56	4.10	0.00	1.92	1.69	4.10	0.00	0.41	0.40	0.50
NORTHERN AMERICA	63.1	44.3	41.6	-18.8	-21.5	1.00	0.95	0.95	0.95	0.00	1.72	1.03	0.95	0.00	0.65	0.52	0.50
TROPICAL SOUTH AMER.	29.7	42.1	41.0	12.3	11.3	1.00	1.42	1.64	1.62	0.00	0.98	1.69	1.62	0.00	0.41	0.51	0.50
MIDDLE AMER. (MAIN.)	48.9	27.8	57.3	-21.2	8.3	1.00	2.81	1.80	3.35	0.00	2.29	0.74	3.35	0.00	0.45	0.29	0.50
TEMPERATE SOUTH AME.	39.6	39.3	22.2	-0.3	-17.4	1.00	0.70	0.70	0.82	0.00	1.55	1.56	0.82	0.00	0.69	0.69	0.50
CARIBBEAN	59.5	46.2	62.5	-13.3	3.0	1.00	2.06	2.01	2.57	0.00	2.32	1.43	2.57	0.00	0.49	0.42	0.50
CHINA	27.6	44.2	85.8	16.5	58.1	1.00	1.31	1.54	2.30	0.00	0.52	0.65	2.30	0.00	0.20	0.30	0.50
JAPAN	37.2	64.3	44.4	27.1	7.2	1.00	0.80	0.68	0.77	0.00	0.63	1.18	0.77	0.00	0.44	0.63	0.50
OTHER EAST ASIA	28.6	32.9	49.6	2.3	21.0	1.00	1.53	1.58	2.09	0.00	0.77	0.87	2.09	0.00	0.33	0.36	0.50
MIDDLE SOUTH ASIA	89.3	45.5	401.9	-43.0	312.6	1.00	6.58	2.61	4821.59	0.00	1.53	0.25	4821.59	0.00	0.17	0.09	0.50
SOUTH EAST ASIA	51.9	35.7	181.9	-16.2	130.0	1.00	3.11	2.19	53.71	0.00	0.67	0.31	53.71	0.00	0.18	0.12	0.50
SOUTH WEST ASIA	39.1	40.0	77.9	0.9	38.9	1.00	2.47	2.11	4.26	0.00	0.95	0.99	4.26	0.00	0.31	0.32	0.50
WESTERN EUROPE	53.9	81.5	45.1	27.6	-8.9	1.00	0.70	0.59	0.75	0.00	0.90	1.40	0.75	0.00	0.56	0.70	0.50
SOUTHERN EUROPE	61.7	64.2	45.7	2.5	-16.9	1.00	0.87	0.87	0.90	0.00	1.28	1.35	0.90	0.00	0.59	0.61	0.50
EASTERN EUROPE	60.3	79.5	44.7	19.2	-15.[illegible]	1.00	0.83	0.74	0.84	0.00	1.13	1.64	0.84	0.00	0.60	0.69	0.50
NORTHERN EUROPE	101.5	85.8	53.8	-15.7	-47.7	1.00	0.92	0.56	0.71	0.00	1.40	1.16	0.71	0.00	0.73	0.67	0.50
AUSTRALIA AND N.Z.	56.7	54.2	40.0	-2.5	-16.7	1.00	1.12	1.10	1.07	0.00	1.73	1.63	1.07	0.00	0.61	0.60	0.50
MELANESIA	49.2	27.3	313.1	-21.9	263.9	1.00	3.01	1.84	1111.26	0.00	0.0[illegible]	0.03	1111.26	0.00	0.03	0.02	0.50
MICRONESIA AND POL.	46.5	37.5	238.[illegible]	-9.1	192.0	1.00	3.32	2.63	470.26	0.00	0.53	0.39	470.28	0.00	0.16	0.13	0.50
USSR	36.6	47.2	37.6	10.6	1.0	1.00	0.95	0.94	0.95	0.00	0.92	1.25	0.95	0.00	0.49	0.58	0.50
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ZERO = INITIAL PERIOD IN KEYFITZ TIME FRAME
INIT = OBSERVED PERIOD IN KEYFITZ TIME FRAME
TC1 = CROSS-OVER TYPE ONE IN KEYFITZ TIME FRAME
TC2 = CROSS-OVER TYPE TWO IN KEYFITZ TIME FRAME
TD1 = CROSS-OVER TYPE ONE IN ACTUAL TIME FRAME
TD2 = CROSS-OVER TYPE TWO IN ACTUAL TIME FRAME

Table A4. SENSITIVITY ANALYSIS

REGION	CHANGE IN TC1 DUE TO CHANGE IN			CHANGE IN TC2 DUE TO CHANGE IN			CHANGE IN TD1 DUE TO CHANGE IN			CHANGE IN TO2 DUE TO CHANGE IN		
	U	R	M	U	R	M	U	R	M	U	R	M
WORLD	-4,62	1,47	-1,47	-4,54	4,54	-14,88	-3,27	0,12	5,83	-3,19	3,19	-7,58
*********************												
MORE DEVELOPED REG.	-4,89	1,32	-1,32	-0,79	0,79	-2,77	-3,72	0,14	2,16	0,38	-0,38	0,71
LESS DEVELOPED REG.	-2,28	0,72	-0,72	-5,48	5,48	-19,73	-1,59	0,03	5,82	-4,79	4,79	-13,19
*********************												
AFRICA	-2,13	0,67	-0,67	-7,68	7,68	-28,34	-1,31	-0,14	8,08	-6,87	6,87	-19,59
NORTHERN AMERICA	-2,84	0,78	-0,78	-0,70	0,70	-2,50	-1,39	-0,67	3,02	0,75	-0,75	1,29
LATIN AMERICA	-1,65	0,50	-0,50	-0,73	0,73	-2,46	-0,99	-0,16	1,82	-0,07	0,07	-0,15
EAST ASIA	-3,37	1,02	-1,02	-2,60	2,60	-9,24	-3,00	0,65	2,19	-2,23	2,23	-6,04
SOUTH ASIA	-2,68	0,88	-0,88	-31,10	31,10	-103,56	-0,44	-1,36	22,86	-28,86	28,86	-79,82
EUROPE	-9,14	2,36	-2,36	-0,92	0,92	-3,17	-7,50	0,71	2,11	0,73	-0,73	1,31
OCEANIA	-7,85	2,60	-2,60	-4,74	4,74	-14,36	-4,83	-0,42	8,11	-1,72	1,72	-3,65
USSR	-3,21	0,88	-0,88	-0,59	0,59	-2,04	-2,66	0,32	1,10	-0,03	0,03	-0,06
*********************												
WESTERN AFRICA	-2,88	0,93	-0,93	-9,63	9,63	-32,52	-2,36	0,41	5,60	-9,11	9,11	-25,99
EASTERN AFRICA	-1,82	0,58	-0,58	-19,07	19,07	-84,86	-1,18	-0,06	16,84	-18,43	18,43	-67,44
NORTHERN AFRICA	-1,56	0,49	-0,49	-3,64	3,64	-13,44	-0,40	-0,67	6,85	-2,48	2,48	-6,10
MIDDLE AFRICA	-1,34	0,40	-0,40	-3,75	3,75	-17,53	-1,04	0,10	5,25	-3,45	3,45	-11,88
SOUTHERN AFRICA	-4,96	1,64	-1,64	-3,68	3,68	-11,18	-3,10	-0,23	5,61	-1,81	1,81	-3,93
NORTHERN AMERICA	-2,84	0,78	-0,78	-0,70	0,70	-2,50	-1,39	-0,67	3,02	0,75	-0,75	1,29
TROPICAL SOUTH AMER.	-2,58	0,79	-0,79	-0,75	0,75	-2,45	-2,17	0,38	0,90	-0,35	0,35	-0,76
MIDDLE AMER.(MAINL.)	-1,13	0,35	-0,35	-1,36	1,36	-4,73	-0,11	-0,67	3,66	-0,34	0,34	-0,72
TEMPERATE SOUTH AME.	-2,24	0,58	-0,58	-0,21	0,21	-0,72	-1,66	-0,01	0,76	0,38	-0,38	0,62
CARIBBEAN	-3,15	1,00	-1,00	-1,79	1,79	-5,72	-1,52	-0,63	4,39	-0,16	0,16	-0,32
CHINA	-2,84	0,86	-0,86	-2,91	2,91	-10,63	-2,49	0,51	2,65	-2,55	2,55	-7,12
JAPAN	-6,01	1,53	-1,53	-0,80	0,80	-2,85	-5,43	0,95	0,85	-0,22	0,22	-0,46
OTHER EAST ASIA	-1,39	0,42	-0,42	-1,02	1,02	-3,55	-1,02	0,06	1,58	-0,65	0,65	-1,55
MIDDLE SOUTH ASIA	-3,09	1,02	-1,02	-74,66	74,66	-236,99	0,83	-2,89	42,42	-70,74	70,74	-193,55
SOUTH EAST ASIA	-1,89	0,61	-0,61	-13,10	13,10	-47,75	-0,63	-0,65	13,28	-11,84	11,84	-33,86
SOUTH WEST ASIA	-2,36	0,76	-0,76	-2,72	2,72	-8,86	-1,64	0,03	3,30	-2,00	2,00	-4,80
WESTERN EUROPE	-9,70	2,39	-2,39	-0,84	0,84	-2,93	-8,54	1,23	1,15	0,32	-0,32	0,60
SOUTHERN EUROPE	-5,94	1,61	-1,61	-0,91	0,91	-3,15	-4,45	0,11	2,62	0,59	-0,59	1,07
EASTERN EUROPE	-9,14	2,38	-2,38	-0,85	0,85	-2,89	-7,68	0,92	1,62	0,61	-0,61	1,11
NORTHERN EUROPE	-10,77	2,64	-2,64	-1,17	1,17	-4,18	-7,28	-0,65	5,26	2,32	-2,32	3,71
AUSTRALIA AND N.Z.	-4,24	1,21	-1,21	-0,69	0,69	-2,32	-2,93	-0,10	2,22	0,62	-0,62	1,11
MELANESIA	-1,09	0,34	-0,34	-22,30	22,30	-220,32	-0,05	-0,70	79,13	-21,26	21,26	-140,85
MICRONESIA AND POL.	-2,10	0,70	-0,70	-27,13	27,13	-84,11	-1,03	-0,38	12,14	-26,06	26,06	-71,28
USSR	-3,21	0,88	-0,88	-0,59	0,59	-2,04	-2,66	0,32	1,10	-0,03	0,03	-0,06
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ALL SYMBOLS HAVE SAME MEANING AS ABOVE

THE MAGNITUDE OF THE EXOGENOUS CHANGES IN U,R AND M IS 0,001

Table A5. SENSITIVITY ANALYSIS TWO

REGION	CHANGE IN TD1 DUE TO CHANGE IN			CHANGE IN TD2 DUE TO CHANGE IN		
	U	R	M	U	R	M
WORLD	-2,96	0,13	5,00	-2,99	3,43	-6,53
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MORE DEVELOPED REG.	-3,37	0,15	1,99	0,37	-0,39	0,66
LESS DEVELOPED REG.	-1,48	0,03	4,89	-4,50	5,12	-11,29
**********************						
AFRICA	-1,22	-0,14	6,58	-6,40	7,42	-16,33
NORTHERN AMERICA	-1,26	-0,69	2,82	0,72	-0,78	1,22
LATIN AMERICA	-0,93	-0,16	1,70	-0,07	0,08	-0,14
EAST ASIA	-2,79	0,68	1,88	-2,14	2,34	-5,41
SOUTH ASIA	-0,37	-1,44	16,75	-24,95	34,38	-56,94
EUROPE	-6,60	0,75	1,91	0,70	-0,76	1,22
OCEANIA	-4,19	-0,45	6,91	-1,59	1,89	-3,38
USSR	-2,46	0,33	1,02	-0,03	0,03	-0,05
**********************						
WESTERN AFRICA	-2,19	0,42	4,55	-8,39	9,97	-21,25
EASTERN AFRICA	-1,10	-0,07	11,29	-16,76	20,49	-50,30
NORTHERN AFRICA	-0,36	-0,69	5,94	-2,35	2,62	-5,37
MIDDLE AFRICA	-0,98	0,10	4,24	-3,30	3,61	-10,34
SOUTHERN AFRICA	-2,76	-0,24	4,89	-1,69	1,95	-3,40
NORTHERN AMERICA	-1,26	-0,69	2,82	0,72	-0,78	1,22
TROPICAL SOUTH AMER.	-2,03	0,40	0,83	-0,34	0,36	-0,71
MIDDLE AMER.(MAINL.)	-0,09	-0,69	3,36	-0,33	0,36	-0,66
TEMPERATE SOUTH AME.	-1,55	-0,01	0,73	0,37	-0,38	0,60
CARIBBEAN	-1,37	-0,66	3,97	-0,15	0,17	-0,29
CHINA	-2,32	0,52	2,28	-2,44	2,68	-6,34
JAPAN	-4,92	0,99	0,75	-0,21	0,23	-0,44
OTHER EAST ASIA	-0,97	0,06	1,45	-0,53	0,68	-1,44
MIDDLE SOUTH ASIA	0,82	-3,11	28,22	-56,64	95,58	-119,30
SOUTH EAST ASIA	-0,57	-0,67	10,36	-10,80	13,12	-26,57
SOUTH WEST ASIA	-1,52	0,04	2,93	-1,90	2,11	-4,27
WESTERN EUROPE	-7,53	1,29	1,01	0,31	-0,33	0,56
SOUTHERN EUROPE	-3,99	0,11	2,40	0,57	-0,61	1,00
EASTERN EUROPE	-6,78	0,97	1,46	0,59	-0,64	1,04
NORTHERN EUROPE	-6,21	-0,90	4,79	2,22	-2,44	3,44
AUSTRALIA AND N.Z.	-2,66	-0,11	2,06	0,60	-0,64	1,04
MELANESIA	-0,04	-0,72	29,94	-19,62	23,21	-65,86
MICRONESIA AND POL.	-0,94	-0,39	9,38	-22,60	30,93	-51,84
USSR	-2,46	0,33	1,02	-0,03	0,03	-0,05
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ALL SYMBOLS HAVE SAME MEANING AS ABOVE

THE MAGNITUDE OF THE EXOGENOUS CHANGES IN U,R AND M IS 0,001

Appendix 3

The Rogers model: Sensitivity Analysis.

The two expressions (60) and (65), giving T_1 and T_2 respectively, can be rewritten as

$$T_i = \frac{1}{x_1 - x_2} \ln K \qquad \text{for } i = 1,2$$

so that

$$\frac{dT_i}{T_i} = \frac{dK}{K \ln K} - \frac{d(x_1 - x_2)}{x_1 - x_2} \qquad \text{for } i = 1,2 \quad .$$

Since $\ln K = (x_1 - x_2)T_i$, this can be rewritten as

$$dT_i = \frac{1}{x_1 - x_2} \left[\frac{\frac{dx_2}{du} - 2}{x_1 - 2u} - \frac{\frac{dx_1}{du} - 2}{x_1 - 2u} - T_1 \left(\frac{dx_1}{du} - \frac{dx_2}{du} \right) \right]$$

and

$$\frac{dT_2}{du} = \frac{1}{x_1 - x_2} \left[\frac{\frac{dx_2}{du} - 1}{x_2 - u + o_u - o_r} - \frac{\frac{dx_1}{du} - 1}{x_1 - u + o_u - o_r} - T_2 \left(\frac{dx_1}{du} - \frac{dx_2}{du} \right) \right]$$

in which $\frac{dx_1}{du}$ and $\frac{dx_2}{du}$, obtained by differentiating (46) with respect to u, are such that

$$\frac{dx_i}{dt} = \frac{x_i - (r - o_r)}{2x_i - (r - o_r + u - o_r)} = \frac{x_i - (r - o_r)}{x_i - x_j} \quad \text{for } i,j = 1,2 \ (j \neq i).$$

Similar formulas are obtained in the case of small variations in o_u, r and o_r.

In contrast to the case of the Keyfitz model, this model does not lead to first derivatives having a definite sign (which explains why we do not provide formal expressions of these derivatives). Recalling (38a) and (38b) also permits one to obtain the first derivatives of T_1' and T_2'. For example, in the case of a small variation in u, we have

$$\frac{dT_1'}{du} = \frac{dT_1}{du} - \frac{dt_D}{du} \qquad \text{and} \qquad \frac{dT_2'}{du} = \frac{dT_2}{du} - \frac{dt_D}{du} \quad .$$

in which $\frac{dt_D}{du}$, obtained by differentiating (68b) is equal to:

$$\frac{dt_D}{du} = \left[\frac{\bar{s}\left(1 - \frac{dx_2}{du}\right)}{o_r - \bar{s}(x_2 - u + o_u)} - \frac{\bar{s}\left(1 - \frac{dx_1}{du}\right)}{o_r - \bar{s}(x_1 - u + o_u} - t_D\left(\frac{dx_1}{du} - \frac{dx_2}{du}\right)\right] \quad .$$

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