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Young, H.P.

IIASA Research Report
June 1977

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LOBBYING AND CAMPAIGNING WITH APPLICATIONS TO THE MEASURE OF POWER

H.P. Young

RR-77-13
June 1977

This work was supported in part by the National Science Foundation under Grant MPS 75-07414 with the Graduate School of the City University of New York.

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International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

PREFACE

This paper deals with a field of study in the Systems and Decision Sciences Area that focuses on institutional structures and their role in shaping decisions. One aspect of this task is concerned with institutions that make decisions by voting: this has wide application in both governmental and nongovernmental (e.g. corporate) contexts. A particular problem addressed by this task is how to operationally define the idea of the "power" of different actors in a voting body. The result is a set of models that can be used as a normative basis for estimating the effects different institutional arrangements have on the relative power of their members.

In this paper two basic models of power are described and applications are made to a variety of examples. Some concepts and notation from game theory are used, but the style is mainly expository, relying in some cases on results proved formally in a previous IIASA Research Report by the author, *Power, Prices, and Incomes in Voting Systems*. The present paper summarizes and interprets some of these earlier results, and goes on to develop a second approach to measuring power that is applicable in somewhat different contexts.

SUMMARY

The intent of this paper is two-fold. First, given lobbying and campaigning as a fact of life in the political sphere, we ask how should a calculating lobbyist allocate his resources most effectively to achieve his goals? Two models are presented: the case of one lobbyist acting unopposed, and the case of two opposing lobbyists; each is shown to lead to a certain concept of equilibrium payments to voters. These solutions may find practical application by practitioners of lobbying and campaigning. But the models also have a theoretical interest: they provide a new approach to the problem of finding a normative measure of power in a voting system. In fact, two new measures of power are defined. While they are related in certain ways, their differences also point to the importance of considering the context of the problem in which power is to be measured.

Lobbying and Campaigning with Applications to the Measure of Power

1. INTRODUCTION

The object of representative institutions being to give a voice to the diverse interests of society, it cannot be held surprising that particular interests will use whatever means lie at their disposal to influence the votes of the representatives. The nature of the influence takes many forms: from the provision of information favorable to a particular point of view, to private vacations and gifts, to outright bribes.

The phenomenon of vote buying has been known since ancient times. Plutarch reports that Clodius, after being caught in Caesar's house (in pursuit of Caesar's wife) masquerading as a woman at a female ritual, greased the palms of a calculated majority of the judges, and was acquitted. Cicero gave testimony against Clodius, and after the acquittal, Clodius questioned whether the judges had believed Cicero's testimony. "Yes", replied Cicero, "five-and-twenty of them trusted me and condemned you, and the other thirty did not trust you, for they did not acquit you till they had got your money" [14].

In modern times, lobbying and campaigning has come to be accepted, within limits, as a normal part of representative government. For example, a so-called "stringent" code of ethics recently passed by the United States House of Representatives (but not to take effect until 1979) restricts outside income to a total of 15% of a member's salary, limits "honoraria" to $1000, and provides for the disclosure of the source of gifts exceeding certain amounts.* Campaigning for votes through large expenditures, mobilized by party organizations, has become an

*Bill passed in the U.S. House of Representatives, March 2, 1977.

integral part of the electoral process. Indeed it is sometimes said that it is the balancing of countervailing *particular* interests that often leads to a decision in the *public* interest.

On the other hand, it must also be pointed out that, frequently, only *one* interest group or lobbyist has *both* a high stake in a given issue *and* the wherewithal to affect the outcome. The problem is, how should such a lobbyist deploy his resources amongst the various voters to efficiently achieve his ends? How does his strategy differ when he encounters a lobbyist on the opposite side of the issue, from the case when he is unopposed? In the case of party organizations competing for votes, what is the most efficient allocation of campaign funds?

In this paper we describe how a calculating lobbyist (or party) should allocate his resources most efficiently--both in the presence of an opponent, and when he operates unopposed. The style of the paper is chiefly expository, and proofs of certain technical results--particularly in the case of the one lobbyist model--are given elsewhere [19]. Various applications of the models are discussed, including such diverse problems as the determination of the relative salaries of various members of government, and campaign fund allocations in U.S. Presidential elections. Finally, it is suggested that the resulting equilibrium prices lead to two interesting new value concepts for n-person simple games.

2. LOBBYING WITHOUT OPPOSITION

A *voting game* (also known as a *simple game*) is specified by a set N of *voters* together with a list $\mathcal{S}$ of all subsets S of voters (called *winning sets*) whose support is sufficient to pass a measure. Thus, S is a winning set if a measure would pass when all the voters in S vote *yes* and all the voters not in S vote *no*. The following conditions are usually assumed for any voting game $G = (N,\mathcal{S})$:

(1) and $\phi \notin \mathcal{S}$, $S \in \mathcal{S}$ and $S \subseteq T$ implies $T \in \mathcal{S}$,

that is, if S wins, then any set containing S also wins.

Given (1), it is easy to see that to describe G we need only specify the *minimal* winning sets S such that no proper subset of S is a winning set.

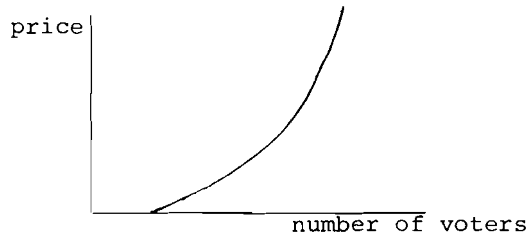
A common example of a voting game is the so-called *weighted voting game*. Here each of the players i, $1 \leqq i \leqq n$, casts a vote with *weight* w_i, and a bill passes if and only if the total weighted vote *for* the bill is at least as high as a given *quota* $q > 0$; thus the winning sets are

$$\mathcal{S} = \{S \subseteq N: \sum_{i \in S} w_i \geqq q\} ,$$

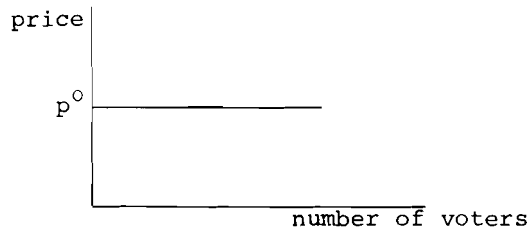
and the game has the *representation* $(q; w_1, w_2, \ldots, w_n)$.

It has been said that "every man has his price". In this paper we will be interested in how voters' prices might be determined in terms of their "value" to a lobbyist trying to buy votes. The models are simplified, and do not pretend to deal with such factors as the bargaining skill of different voters (or of the lobbyist), or the probabilities of different coalitions forming. On the other hand, while the models appear to be completely cynical (i.e., by assuming that everyone's vote can be bought), the analysis can be made just as well for the case where some voters would vote with the lobbyist anyway, and others cannot be bought at all (see Section 8 below). Indeed, this just amounts to saying that some voters have a price of zero, and others have a price of plus infinity.

We begin with the assumption that a lobbyist has a large quantity of funds at his disposal, and a bill (or several bills) that he would like to have passed. The passage of these bills is assumed to be sufficiently valuable to him that he is willing (like Clodius) to pay the price necessary to have the measure pass with certainty, rather than merely with some high probability: in other words, he is out to capture some winning set, and is assumed to have the funds requisite to do so. For a "virgin legislature", we may expect that the supply of voters at various possible prices has the classical shape; some perhaps will go along for nothing and, as the price paid goes up, increasingly many voters will be purchasable at that price or less:



The minimum prices at which the voters are willing to offer themselves will be denoted by the price vector $\underset{\sim}{p}^{o} = (p_1^o, p_2^o, \ldots, p_n^o)$. Voters may arrive at these prices, called *minimum expectations*, by calculations that take into account the risk of accepting a bribe, with the possible loss of income, prestige, or personal honor if caught. Lacking more detailed information, we may sometimes assume, for *a priori* calculations, that this "supply curve" is perfectly elastic, i.e. that all voters have the *same* minimum price:



The behavior of the lobbyist is summarized by the following axiom.

(2) *The lobbyist will bribe the least expensive collection of players sufficient to pass the measure.*

Given this assumption, and supposing that all players have equal influence (for example, they all have the same number of votes in a weighted voting game), the lobbyist's course of action seems clear: he should begin bribing at the low end of the supply curve and proceed until he has bought a coalition just sufficient to win. If some players are structurally more important than others, which is the interesting case from the theoretical point of view, then the lobbyist could find the least cost winning set by calculating costs on a *per vote* basis, in the case of a weighted voting game for example. However, in this case, it may also be possible for some players to obtain *more* than their minimum price because of their strategic positions. The problem is to determine whether an "equilibrium" set of prices exists in this case, and how to find it.

Example 1

The New York City Board of Estimates consists of eight members having weighted votes as shown in Table 1, a simple weighted majority being required to pass [10].

Table 1. Weights of the New York City Board of Estimates (1975).

Voters		Weights
1	Mayor	4
2	Controller	4
3	Council President	4
4	Brooklyn Borough President	2
5	Manhattan Borough President	2
6	Queens Borough President	2
7	Bronx Borough President	2
8	Richmond Borough President	2
	Total weight	22

A lobbyist representing certain interests in New York City might attempt to influence the Board by offering them considerations in return for their votes. Certain members of the Board might respond by indicating whether the price offered is too low (i.e., below their minimum expectations, or what they would consider worthwhile under any conditions). But if they set their prices too high, the lobbyist always has the possibility of approaching other members who may not be so demanding.

Suppose that all members have a minimum price of $1000. While the supply curve for *voters* is perfectly elastic, on a *per vote* basis, the Mayor, the Controller, and the Council President are a better buy. In fact, these three players constitute the *unique* least-cost winning set. This means in particular that some, or all of them, could ask for more and get away with it. How much more? If the Mayor raises his price to more than $2000, then any two Borough Presidents can be *substituted* for him at less cost; that is, the lobbyist can plausibly walk away from any such price demand by the Mayor. On the other hand, for any price *less* than $2000 the first three players still constitute the unique least-cost winning set. Thus, if the lobbyist walks away from a price demand by the Mayor of, say, $1900, the lobbyist may end up having to pay more to get his winning set. On the other hand, the Mayor will get nothing. A bargaining process may establish some price in between the Mayor's minimum and the price ($2000) above which other players undercut him, depending on the relative bargaining abilities of the lobbyist and the Mayor, or on other assumptions.

In this model we shall make the simplifying assumption that the lobbyist is a *price taker*, i.e., that his only ability to bargain lies in his possibility of going to some other players who will cost him less. Thus, for any given vector of prices $\underset{\sim}{p} = (p_1, p_2, \ldots, p_n)$ that are *feasible* in the sense that $\underset{\sim}{p} \geqq \underset{\sim}{p}^0$, the lobbyist chooses a winning set S such that $\sum_{i \in S} p_i$ is a minimum, and pays p_i to each $i \in S$. If there are several minimum cost sets for $\underset{\sim}{p}$, the lobbyist chooses one of them. Thus, for any feasible $\underset{\sim}{p}$ there is an associated set S of bribees $f(\underset{\sim}{p})$. The function f is called the *payment schedule* for the lobbyist.

In the terminology of game theory, the choice of a payment schedule, f, by the lobbyist defines an n-person game on the voters in which the *strategy* of a voter is to quote a price $p_i \geqq p_i^o$, and the *payoff* equals p_i if $i \in f(\underset{\sim}{p})$ (i.e., if i is bribed) and zero otherwise. Returning to our example, we see that the Mayor can charge any amount up to $2000 and get away with it. By a similar process, the Controller and the Council President can also raise their prices up to $2000 each. At the breakeven point, namely

$$(3) \qquad \underset{\sim}{p} = (2000,2000,2000,1000,1000,1000,1000,1000) ,$$

it happens that *all* minimal winning sets cost the same, $6000, so the lobbyist can choose to bribe any one of them to equal advantage. Suppose that, in fact, he chooses to bribe the first three players. (One plausible reason for such a choice is that it is the *smallest* winning set, hence there are fewer voters to deal with.) Then the price vector (3) has the following property:

> *No player or group of players is able to change to some other feasible set of prices such that each does strictly better than before.*

Such a $\underset{\sim}{p}$ is said to be a *strong equilibrium* [13].

It is a remarkable fact that, if the voting game G has no *veto player*, that is, no player whose agreement is necessary to win, then a strong equilibrium always exists for any specification of minimum prices $\underset{\sim}{p}^o$ (see (9) below).

For Example 1 above, it turns out that the price vector (3) is, essentially, the *unique* strong equilibrium, in a sense to be made precise below. Moreover, uniqueness typically holds for many real examples of voting games (and equal minimum prices). However, there are also voting games with multiple equilibria.

Example 2

Consider the weighted voting game on seven voters with quota 11 and weights (5,3,3,2,2,2,2). Let the lobbyist have a payment schedule f that always chooses a minimum cost winning set, and, moreover, if voters 1,2,3 constitute one such set among several, then this set is the one chosen. If minimum prices are \$1 each, then the price vector $\underset{\sim}{p}^1 = (3,1,1,1,1,1,1)$ is a strong equilibrium, and so is $\underset{\sim}{p}^2 = (2.50,1.50,1.50,1,1,1,1)$. The latter, it may be observed, is proportional to the weights.

Thus, several strong equilibrium prices may result. Nevertheless, there is, in general, one among the strong equilibria that is more stable than the others. To see this, it is necessary to consider the possibility of coalitions.

From the outset, let us remark that coalition formation in the context of selling votes is probably an infrequent phenomenon, since secrecy is generally at a premium. Therefore, in fact, not all possible forms of cooperative behavior will be considered. In Example 1, for instance, consider a coalition C consisting of the first three voters, and let each of them agree to charge a very high price. Since every winning set meets C, at least one of them must be bribed. C can therefore guarantee itself a very high payoff, provided the members can agree on a division of the spoils. In fact, if we postulate a large but finite budget limit β for the lobbyist, C can guarantee itself (using pure strategies) a payoff of up to $\beta/3$. In this way, the cooperative value of different coalitions could be defined; unfortunately, the core does not always exist for these games.

Moreover, notice that, typically, not all voters in such a coalition will be bribed; rather, some members receive all the payments while others receive nothing. Such a coalition seems fragile, and presumes an extraordinary degree of cooperation among the coalition members. Notice, for example, that the members who are not directly bribed do not even control enough to assure getting their own minimum expectations.

But a more fundamental problem with this approach is that if this degree of cooperation is assumed, then the nature of the game itself changes: any coalition such as C that wins and meets every other winning set could simply act as a block (that is, as a veto player) and guarantee itself a payoff of β. Unfortunately, there is then no very satisfactory way of imputing stable payoffs to the voters.*

Returning to the problem of defining an equilibrium which is stable under some limited form of cooperative behavior, it seems reasonable to require, in particular, that *if a coalition is to hold under prices* $\tilde{p}$*, then every member of the coalition must be bribed, i.e. must receive directly his minimum expectation.*

Thus, if $\tilde{p}$ is a price vector, and there exists some coalition of players which can adjust their prices so that

(4) *the whole coalition is bribed*

and

(5) *after minimum expectations are met there is a distribution of the remainder such that each member of the coalition is strictly better off than before,*

then we may presume that such a coalition could form, and that the given prices are unstable. We say that a price vector $(p_1, p_2, \ldots, p_n) = \tilde{p}$ is a *collective equilibrium* (for $\tilde{p}^o$) if there is no coalition of players who can change their prices and improve their position as in (4) and (5).**

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*For these games the core is, in most cases, empty.

**More precisely, $\tilde{p}$ is a collective equilibrium if there is no $C \subseteqq N$, $C \neq \emptyset$, and feasible prices $\tilde{p}'$, where $p_i' = p_i$ for all $i \notin C$, such that

$$C \subseteq f(\tilde{p}') ,$$

(6) $$\sum_{i \in C} (p_i' - p_i^o) \geqq \sum_{i \in C \cap f(\tilde{p})} (p_i - p_i^o) ,$$

where = is allowed only if $C \cap f(\tilde{p}) = \emptyset$ and $p_i^o > 0$ for all $i \in C$.

Every collective equilibrium is, in particular, a strong equilibrium, but a collective equilibrium has the additional property that it is stable against certain types of coalition formation with side-payments. It is therefore a considerably stronger concept of equilibrium than is usually considered--indeed than normally exists--for n-person games. For the special class of games considered here, however, a collective equilibrium almost always does exist, and normally it is unique (see (9) below). Moreover, while this equilibrium seems to depend on the payment schedule f chosen by the lobbyist, we shall see that a description of the equilibrium can be given independently of f.

Returning to Example 2, consider the prices $\underset{\sim}{p}^1 = (3,1,1,1,1,1,1)$. The first three players, if they choose to cooperate, can collectively do better by adopting the prices \$2.50,\$1.50, \$1.50 instead of \$3,\$1,\$1. All three are assured of being bribed under the given payment schedule f; (alternatively they could shave off ε from these prices and be assured of being bribed for any choice of f). Voter 1 gives up \$.50, so the price of his cooperation must be that the other two will more than compensate him for his losses. They can do this and still retain more than \$1 each for themselves. The price vector $\underset{\sim}{p}^1$ can therefore be upset by a coalition satisfying (4) and (5), so it is not a collective equilibrium. On the other hand, it may be verified that $\underset{\sim}{p}^2 = (2.50,1.50,1.50,1,1,1,1)$ *is* a collective equilibrium.

Are there other collective equilibria for this example? In a trivial sense, there are: for example another one is

$$\underset{\sim}{p}^3 = (2.50,1.50,1.50,1,1,1,5) \quad .$$

However, $\underset{\sim}{p}^3$ differs from $\underset{\sim}{p}^2$ in an uninteresting way, because *the members who count*, i.e., the ones who are actually bribed, receive the same amount in both cases, since $f(\underset{\sim}{p}^3) = f(\underset{\sim}{p}^2)$ = voters {1,2,3}. For given $\underset{\sim}{p}^o$ we define a *canonical equilibrium* $\underset{\sim}{p}$ for f, also called an *equilibrium without opposition*, to be a collective equilibrium for f in which the members who are not bribed quote their minimum expectations, that is,

(7) $$p_i = p_i^o \quad \text{for} \quad i \notin f(\underset{\sim}{p}) \quad .$$

The restriction to canonical equilibria is justified by the following fact [19].

(8) *Every collective equilibrium* $\underset{\sim}{p}$ *for* f *differs from a canonical equilibrium* $\hat{\underset{\sim}{p}}$ *only in that some players who are not bribed (i.e., not in* $f(\underset{\sim}{p})$ *or in* $f(\hat{\underset{\sim}{p}})$*) have raised their prices.*

(9) *Theorem* [19]. *If* G *has no veto players then, for some* f, *a canonical equilibrium exists and normally it is unique.*

Notice that, if G has a veto player j, then this player is in a position to demand an arbitrarily high price. Hence, relative to the others, his price is in some sense infinite, and we should not expect an equilibrium to exist in this case. (However, we might compute an equilibrium for the remaining voters by setting the price of j equal to ∞, and computing the prices for the G^j that results by "removing" j: $G^j = (N, \mathcal{S}^j)$ where $\mathcal{S}^j = \left\{S \subset N : S \cup \{j\} \in \mathcal{S}\right\}$.)

It may be shown (see (12) below) that $\underset{\sim}{p}^2$ above is actually the *unique* equilibrium without opposition for the given minima in Example 2; similarly, (3) is the unique solution for Example 1. In both cases the solutions are proportional to the members' weights. This is in contrast to the Shapley-Shubik values [16,17], which for Example 1 are in the proportions (2.2:2.2:2.2:1:1:1:1:1), and to the Banzhaf values [1], which are in the proportions (2.17:2.17:2.17:1:1:1:1:1).

Any canonical equilibrium for a given set of minimum expectations may be computed, *independently of* f, by means of a certain linear program, the form of which gives insight into its structure. For given minima $(p_1^o, p_2^o, \ldots, p_n^o) = \underset{\sim}{p}^o$, let S^o be the family of all winning sets that are minimum cost *relative to* $\underset{\sim}{p}^o$,

$$S^o = \{S \in S: \underset{\sim}{p}^o(S) = \min\} \quad .$$

The members of S will be called *critical sets*.

Further, let N^o, the set of *critical voters*, be the voters contained in every critical set:

$$N^o = \bigcap_{S \in S^o} S \quad .$$

Then [19]

(10) *in any canonical equilibrium* $\underset{\sim}{p}$ *only the critical voters can have prices higher than their minimum expectations;*

further,

(11) *if* $\underset{\sim}{p}$ *is a canonical equilibrium for* f, *then* $f(\underset{\sim}{p}) \in S^o$; *moreover* $\underset{\sim}{p}$ *is also a canonical equilibrium for any payment schedule* g *such that* $g(\underset{\sim}{p}) \in S^o$.

In view of these results, we shall sometimes refer to a canonical equilibrium (i.e., an equilibrium without opposition) without reference to any particular f.

In Example 1 the set S^o = {Mayor, Controller, Council President} is the unique critical set (and these the critical voters) when the minimum expectations of all members are the same. But the price vector $\underset{\sim}{p}$ of (3) is only in equilibrium for an f satisfying $f(\underset{\sim}{p}) = S^o$. Why should the lobbyist choose this set instead of some other equal-cost winning set, such as the Mayor plus any four of the Borough Presidents? First, the lobbyist has an interest in achieving a stable (i.e. equilibrium) solution, which the latter is not. But more important, since, by (10), only the critical players are above their minima, they can lower their prices by a hair and be sure that they are in *every* minimum cost winning set, so that the lobbyist is *certain* to bribe them. Viewed in this light, we may say that the canonical equilibrium $\underset{\sim}{p}$ and the corresponding f, where $f(\underset{\sim}{p}) \in \mathcal{S}^o$, represent a kind of *limiting behavior* on the part of both the voters and the lobbyist.

The principal result for equilibria without opposition is the following [19].

(12) *Theorem. For given minimum expectations* $\underset{\sim}{p}^o$ *and* N^o, $\mathcal{S}^o$ *as above,* $\underset{\sim}{p}$ *is an equilibrium without opposition if and only if* $\underset{\sim}{p}$ *is optimum for the linear program*

$$\max\ \underset{\sim}{p}(N^o)$$

(13) *subject to* $\underset{\sim}{p}(S) \geqq \underset{\sim}{p}(S^o)$ *for all* $S \in \mathcal{S}$ *and all* $S^o \in \mathcal{S}^o$

$$p_i \geqq p_i^o \quad \text{for all } i \in N^o$$

$$p_i = p_i^o \quad \text{for all } i \notin N^o \quad .$$

What this means is that, in equilibrium, the critical voters maximize their take, while making sure that they cannot be undercut by some winning set not containing all of them.

(14) *This solution has a natural interpretation in terms of "substitution". For any voter* i *let* S^i *be a smallest cardinality winning set containing* i. *The minimum number of voters that can be substituted for* i *in some choice of* S^i *and still have a winning set is denoted by* r_i.

(15) *Theorem* [19]. *If* G *is a weighted voting game in which there are no veto players, and if all voters have the same minimum expectation,* p^o, *then every canonical equilibrium* $\underset{\sim}{p}$ *satisfies*

(16) $$p^o(r_i - 1) \leqq p_i \leqq p^o r_i \quad \textit{whenever} \quad p_i > p^o \quad .$$

This theorem says that, if the minimum expectations of all voters are equal, then in equilibrium the price that a voter i may charge, if it is *more* than his minimum price p^0, must be *approximately equal to* p^0 times the number of voters who could replace him. It is precisely this possibility of substitution between voters that creates the conditions for equilibrium, since it prevents any one player from raising his price too high.

In practice, the substitution theorem (15) holds also for many voting games that are not representable as weighted voting games. Consider the voting game consisting of the members of the United States House of Representatives (R), Senate (S), Vice-President (V), and President (P). We may represent the minimal winning sets of this game (the *U.S. Federal Game*) schematically as follows:

$$\{218R, 50S, V, P\} \quad , \quad \{218R, 51S, P\} \quad , \quad \{290R, 67S\} \quad .$$

Suppose that all members have, *a priori*, the same minimum expectations, say \$1000 each. Then it may be verified that the unique equilibrium without opposition is \$88,000 for the President and \$1000 for everyone else. This result can also be arrived at by considering the substitution possibilities. Any voter (except the President) who tries to charge more than \$1000 can be replaced by someone who is willing to charge less; the President, on the other hand, can hold out for 88 times this amount since it takes 88 voters to replace him.

In equilibrium the lobbyist will, by (11), bribe some set of form {218R,50S,V,P} or {218R,51S,P}, since these constitute S^0. If we assume that he bribes any one of these with equal probability, we may compute the *expected income*, ψ_i, of each player:

$$(17) \qquad \psi_i = p_i s_i^0 / s^0 \quad ,$$

where s^0 is the number of sets in S^0, and s_i^0 is the number of sets in S^0 containing voter i. For the U.S. Federal Game this results in the following expected incomes for a Representative, a Senator, a Vice-President, and a President (which should be interpreted as relative amounts):

R	\$501.15
S	\$504.95
V	\$504.95
P	\$88,000.00 .

The expected incomes of the voters stand in somewhat different ratios to each other than do their prices. Since it is not what a voter *charges* but what he *gets* that matters in the end, we may regard the expected incomes as a way of estimating the power of various voters in terms of their ability to extract money for their votes.

The extraordinarily important position of the President with respect to passing a measure is evident. Consider now the *complementary game* $\bar{G}$ defined by

$$\bar{G} = (N, \bar{\mathcal{S}}) \quad , \quad \bar{\mathcal{S}} = \{S \subseteq N : N - S \notin \mathcal{S}\}$$

in which the lobbyist is trying to buy votes to *block* passage of a bill. Again assume that each player has a minimum price of \$1000. The complementary game is described by the minimal winning sets

$$\{146R, P\} \ , \ \{34S, P\} \ , \ \{51S\} \ , \ \{50S, V\} \ , \ \{218R\} \quad .$$

The equilibrium prices and expected incomes turn out to be as follows:

		Price	Income
(18)	R	\$1000	0
	S	\$1000	\$340
	V	\$1000	0
	P	\$17,000	\$17,000 .

For comparison with the results given by other power indices we may consider the average expected income from passing and blocking:

R	$250.58
S	$422.48
V	$252.48
P	$52,500.00

The proportions for various indices are shown below.

	Income	Shapley-Shubik	Banzhaf
R	1.000	1.000	1.000
S	1.686	4.268	2.081
V	1.008	2.732	2.081
P	209.514	168.186	26.128

It is clear that the average Senator plays a more crucial role in the blocking of a measure than his counterpart in the House; moreover the President is relatively less powerful in his ability to block than to pass. (It should also be observed that, *a priori*, it is easier, i.e., requires less funds, for a lobbyist to block a measure than to pass it.)

What, though, happens when two lobbyists compete, one on either side of the issue? This is the subject of the next section.

3. LOBBYING WITH OPPOSITION

For many types of legislation it is doubtless true that there is only one special interest group that is directly interested in the matter, and which can work more or less unopposed to attempt to get legislation passed that is particularly beneficial to itself (e.g., special tax legislation for certain qualifying groups, and monopoly or license rights).

Nevertheless, with the advent of citizens' action groups that provide a focus for representing the public interest against certain private interests, the phenomenon of competing lobbies is increasingly encountered. In these situations, the financial resources of the public interest groups are in general much less than their competitors'; this is in fact a particularly interesting case that we shall examine in detail.

But a model of vote buying by two competing forces has implications far beyond the context of lobbying. First, it has important applications to the problem of effectively allocating campaign funds in elections. In this context the Electoral College system in the United States is a particularly interesting example. Second, the model provides a new approach to the problem of setting fair "salaries" for different government positions: here one can think of the lobbyist as being in competition with the government, the government's objective being to discourage bribery attempts by setting salaries in proper balance.

Third, and most important, the model provides yet another new approach to measuring power in voting games that is different from those of Shapley-Shubik and of Banzhaf.

Let two different *lobbyists* (or *parties*) compete for votes in a voting game $G = (N, \mathcal{S})$. Let the lobbyist who wishes to buy *pro* votes be called A, and the lobbyist who wants *con* votes be called B. Further, let $a \geqq 0$ be the total financial (or equivalent) resources of A, $b \geqq 0$ the total financial (or equivalent) resources of B.

The object of each lobbyist will be to spend more than his opponent on some set of voters capable of deciding the issue.

Thus if A offers p_i to voter i and B offers q_i to voter i, then i will side with A if $p_i > q_i$, will side with B if $p_i < q_i$, and there will be a tie (with a 50-50 probability of i going either way) if $p_i = q_i$.

A *pure strategy* for A is a *price vector* $p = (p_1, \ldots, p_n)$ satisfying $\underset{\sim}{p} \geqq 0$, $\sum_i p_i \leqq a$; similarly a pure strategy for B is a price vector $\underset{\sim}{q} = (q_1, \ldots, q_n)$, $\underset{\sim}{q} \geqq 0$, $\sum_i q_i \leqq b$. We say that

$$\text{(19)} \qquad \begin{array}{l} \text{A } \textit{wins} \text{ if } \{i \in N: p_i > q_i\} \in \mathcal{S} \quad , \\ \text{B } \textit{wins} \text{ if } \{i \in N: q_i \not> p_i\} \notin \mathcal{S} \quad . \end{array}$$

This defines a 2-person, zero sum game with payoff function $\underset{\sim}{v}(\underset{\sim}{p}, \underset{\sim}{q})$ as follows.

$$\text{(20)} \qquad \begin{array}{lll} (v_A(\underset{\sim}{p},\underset{\sim}{q}), v_B(\underset{\sim}{p},\underset{\sim}{q})) & = & (1,-1) \quad \text{if A wins} \quad , \\ (v_A(\underset{\sim}{p},\underset{\sim}{q}), v_B(\underset{\sim}{p},\underset{\sim}{q})) & = & (-1,1) \quad \text{if B wins} \quad . \end{array}$$

In the case of ties, we may extend this definition by letting w_A be the number of sets $S \in \mathcal{S}$ such that $p_i \geqq q_i$ for all $i \in S$, $q_i \geqq p_i$ for all $i \notin S$, and w_B be the number of $S \notin \mathcal{S}$ such that $p_i \geqq q_i$ for all $i \in S$, $q_i \geqq p_i$ for all $i \notin S$. Then

$$\left(v_A(\underset{\sim}{p},\underset{\sim}{q}), v_B(\underset{\sim}{p},\underset{\sim}{q})\right) = \left(\frac{w_A - w_B}{w_A + w_B}, \frac{w_B - w_A}{w_A + w_B}\right) .$$

Let

$$\underset{\sim}{P} = \{\underset{\sim}{p} \geqq 0, \sum_{i=1}^{n} p_i \leqq a\} \quad ,$$

$$\underset{\sim}{Q} = \{\underset{\sim}{q} \geqq 0, \sum_{i=1}^{n} q_i \leqq b\} \quad .$$

An *equilibrium pair* of price vectors is a pair $(\underset{\sim}{p},\underset{\sim}{q})$ such that $\underset{\sim}{p} \in \underset{\sim}{P}$, $\underset{\sim}{q} \in \underset{\sim}{Q}$, and

$$\text{(21)} \qquad \begin{array}{lll} v_A(\underset{\sim}{p}',\underset{\sim}{q}) & \leqq & v_A(\underset{\sim}{p},\underset{\sim}{q}) \quad \text{for all } \underset{\sim}{p}' \in \underset{\sim}{P} \quad , \\ v_B(\underset{\sim}{p},\underset{\sim}{q}') & \leqq & v_B(\underset{\sim}{p},\underset{\sim}{q}) \quad \text{for all } \underset{\sim}{q}' \in \underset{\sim}{Q} \quad . \end{array}$$

An equilibrium pair in *pure* strategies does not usually exist, so that one is forced to consider *mixed strategies* over the infinite sets $\underset{\sim}{P}$ and $\underset{\sim}{Q}$. Specifically, a mixed strategy for

A is a measure μ defined on $\underset{\sim}{P}$ such that $\mu(\underset{\sim}{P}) = 1$ and a mixed strategy for B is a measure ν on $\underset{\sim}{Q}$ such that $\nu(\underset{\sim}{Q}) = 1$; the *expected* payoff is

$$(22) \qquad \begin{aligned} v_A(\mu,\nu) &= \int_{\underset{\sim}{P}\times\underset{\sim}{Q}} v_A(\underset{\sim}{p},\underset{\sim}{q})\, d(\mu\times\nu) \\ v_B(\mu,\nu) &= \int_{\underset{\sim}{P}\times\underset{\sim}{Q}} v_B(\underset{\sim}{p},\underset{\sim}{q})\, d(\mu\times\nu) = 1 - v_A(\mu,\nu) \quad , \end{aligned}$$

assuming these integrals exist. Equilibrium pairs are defined analogously to (21), but explicit forms for the equilibrium measures (probability distributions) μ and ν are technically very difficult to compute, and perhaps difficult also to interpret: it is not clear that a lobbyist or campaign organization would ever in fact use such complex probability distributions to determine a strategy. The type of game defined by (19) - (22) is similar to a class known as *Colonel Blotto* games, but differs in the objective function: in Colonel Blotto games a player wins a unit if he outbids his opponent on that unit, and the objective is to maximize the expected number of (weighted) units won (see for example [7,8,9,12,15]), whereas here the objective is to maximize the probability of winning (see Section 6 below).

In this paper we shall focus on a particular case of the model in which a *limiting* equilibrium in *pure* strategies exists, and which therefore yields a solution to the problem that is practical to apply. Moreover, it is conjectured that this pure strategy solution is the same as the *expected payoff* in games where the roles of the two lobbyists are symmetric, in which case it would represent quite a general value concept for voting games (see Section 4 below).

The case we shall consider is that in which one of the lobbyists, say B, has "substantially" more resources than A, $b \gg a$. The meaning of "substantially" will be made precise presently. This situation undoubtedly arises for many types of lobbying, as noted above, where opposition groups are thin and poorly financed; but it has also frequently been the case in U.S. Presidential elections, where the Republicans were often considerably better-heeled than the Democrats (see Section 6 below).

If one lobbyist (or party), B, has substantially more funds than A, then he may in fact be in a position, by judicious distribution of his resources, to *entirely prevent* A from succeeding in buying a winning set of voters. For example, suppose lobbyist A has \$6000 to try to pass a measure through the New York City Board of Estimates, but that there is an interest group B opposed to the measure having a budget of \$11,800. If B offers the amounts

$$\underset{\sim}{p} = (\$2100,\$2100,\$2100,\$1100,\$1100,\$1100,\$1100,\$1100)$$

to the eight members of the Board, respectively, then A will be unable, within his resources, to make *any* counter-offer such that the measure will pass. In fact, B's best strategy will be to spend *as little as possible* and still be certain of preventing A from winning. It is not difficult to see that this is in fact achieved by the following distribution:

$$\underset{\sim}{p} = (2000+\varepsilon,\ 2000+\varepsilon,\ 2000+\varepsilon,\ 1000+\varepsilon,\ 1000+\varepsilon,\ 1000+\varepsilon,\ 1000+\varepsilon,\ 1000+\varepsilon)$$

where ε is vanishingly small. In other words, B can spend just over \$11,000 and thwart A. We call such a distribution (in the limit, as ε goes to 0) a *defensive equilibrium* for B. Notice the interesting circumstance that the result is precisely the same as the equilibrium without opposition when the voters' initial minimum expectations are equal.

In general, let $G = (N, \mathcal{S})$ be an arbitrary voting game, and suppose $b \gg a$. B's objective is to find a $\underset{\sim}{p}$ such that $\underset{\sim}{p}(S) > a$ for all $S \in \mathcal{S}$ and $\sum_i p_i$ is a minimum with this property. In the limit, this amounts to finding a $\underset{\sim}{p}$ solving

$$\min \sum_i p_i$$

(23) $$\text{subject to } \underset{\sim}{p}(S) \geqq a \quad \text{for all} \quad S \in \mathcal{S} \quad \text{and} \quad \underset{\sim}{p} \geqq 0 \quad .$$

Any optimal solution to the linear program (23) constitutes a *defensive equilibrium* for G (given resources $a \geqq 0$ for lobbyist A). Notice that the *relative* value of an optimal $\underset{\sim}{\bar{p}}$ for (23) does not depend on the value of a (for positive a) because $(a'/a)\underset{\sim}{\bar{p}}$ is optimal for $a' > 0$ if and only if $\underset{\sim}{\bar{p}}$ is optimal for $a > 0$ ($\underset{\sim}{\bar{p}} = \underset{\sim}{0}$ if $a = 0$). The meaning of B having "substantially" more resources than A can now be made precise: if $\pi(a)$ is the optimum value of (23), then B's resources must exceed $\pi(a)$. Since (23) is always primal and dual feasible we have the following result.

(24) *For any voting game* G *a defensive* ***equilibrium*** *always exists, and normally it is unique (up to multiplication by a scalar).*

4. NEW VALUE CONCEPTS FOR VOTING GAMES

The model of a single lobbyist buying votes was shown to lead to an equilibrium set of prices in any voting game without veto players. Moreover, the expected incomes to the players associated with this equilibrium gives a natural way of thinking about measuring relative power in a voting body.

In the situation of two lobbyists competing for votes, a single set of voters' "prices"--i.e., a pure strategy solution--exists in the case where the lobbyists have substantially unequal resources (in the sense defined in the preceding section). In this situation the prices are given by solving the linear program *min* $\sum_i p_i$, *subject to* $\underset{\sim}{p} \geqq 0$ and $\underset{\sim}{p}(S) \geqq a$ for some $a > 0$ and all winning sets S. As the relative values do not depend on a, we may normalize so that $|\underset{\sim}{p}| = \sum_i p_i = 1$. Then the equilibrium is given by the compact formulation

$$\max_{\substack{\underset{\sim}{p} \geqq 0 \\ |\underset{\sim}{p}| = 1}} \; \min_{S \in \mathcal{S}} \; \underset{\sim}{p}(S) \qquad .$$

In the general case, when no restrictions are placed on the respective resources, a and b, of the two lobbyists, an equilibrium solution, *if it exists*, will be a pair of mixed strategies μ and ν. The relative values of the various voters may be defined in this case as the *expected payoffs* to each player

$$\bar{p}_i = \int_P p_i d\mu + \int_Q q_i d\nu \quad .$$

These expected payoffs provide a new *value* concept for n-person simple games.

A particularly natural case to investigate is the situation where the lobbyists have equal resources ($a = b$). While we shall not pursue this in detail here, we shall suggest what the answer is for a particular class of problems. We say that a voting game G is *decisive* (also sometimes called *proper*) if, for every subset S of voters, exactly one of S, N - S wins. An example of this situation is (weighted) simple majority rule with the total number of votes *odd*. Notice that in a decisive game G, the complement $\bar{G}$ is the same as G. Hence, for two lobbyists with equal resources, the associated 2-person game is completely symmetric in the two players. For this case, we propose that

An equilibrium for the two-lobbyist model exists, and the (normalized) expected payoffs $\bar{p}$ *to the players are the same as some pure strategy solution in the unequal resources case, namely,* $\bar{p}$ *solves*

$$\max_{\substack{p \geq 0 \\ |p| = 1}} \; \min_{S \in \mathcal{S}} \; p(S) \qquad (25)$$

The expression (25) represents a new value concept for voting games (when the equilibrium exists) that differs from both the Banzhaf and the Shapley-Shubik values. Some of its properties in the case of weighted voting games are developed in Section 7.

5. APPLICATION TO LEGISLATORS' SALARIES

A second application of the two-lobbyist model is to provide a rationale for determining the *relative salaries* of different legislators. Indeed, it can be said that one function of a legislator's salary is to protect him from the temptation of accepting bribes (assuming that accepting bribes is illegal). For, by accepting a bribe, the legislator risks losing his position, and hence his salary. One objective for setting salaries could be to find the most efficient distribution of salaries, that is, a distribution that minimizes the total cost to the state and protects against a given level of corruption. The solution is provided precisely by the defensive equilibrium computed from (23).

However, since the desire is to protect legislators from attempted bribes to either *pass* or *block* a measure, we must consider the defensive equilibrium for both the voting game G and its complement $\bar{G}$ and see which solution dominates. To illustrate, let us compute salaries for the U.S. Federal Game. Since it is really only the *relative* salaries we are interested in, the choice of a (the amount presumed to be available to a potential lobbyist) is arbitrary. Let us obtain a solution that protects against any lobbying effort of less than a = $1,000,000. It is sufficient to compute the optimum to (23) using only the *minimal* winning sets. It turns out that the program is degenerate and there are two extreme optimal solutions:

	(i)			(ii)
R	0		R	0
S	$14,925.37	and	S	$14,925.37
V	$14,925.37		V	0
P	$238,805.90		P	$253,731.30

These solutions (and any convex combination of them) give an optimal distribution of salaries needed to protect against the possibility that a lobbyist with $1,000,000 succeeds in *passing* a measure. That members of the House receive nothing in an efficient distribution reflects the peculiar circumstance that 67 out of 100 and 51 out of 100 are slightly larger majorities than 290 out of 435 and 218 out of 435, so that a dollar spent on protecting the Senate from bribery goes a little further than a dollar spent to protect the House. The "degenerate" role of the Vice-President reflects the fact that he is useful in passing a measure only if the President also concurs.

Now consider the complementary game. Since it is easier to block a measure than to pass it, the lobbyist's $1,000,000 will go further. Therefore, to prevent the lobbyist from successfully *blocking* a measure, higher salaries than those found above will be required: in fact, the unique solution is

(26)	R	$4,587.16
	S	$19,607.84
	V	$19,607.84
	P	$333,333.33

which are in the proportions 1 : 4.27 : 4.27 : 72.67. The President's salary is 17 times a Senator's, which, it should be noted, is precisely the ratio of their equilibrium prices when a lobbyist tries to block a measure unopposed (under the assumption of equal minimum expectations for all voters).

The actual salaries (in 1976) were

R	$44,600
S	$44,600
V	$65,600
P	$200,000 .

The solution (26) gives too low an *absolute* salary for a Representative, but this depended on our arbitrary choice of $a = \$1,000,000$. In fact, $44,600 may be taken as a minimum acceptable salary, and the salaries in (26) scaled up accordingly; this results in theoretical protection against a lobbyist with up to $9,722,791 in resources.

6. APPLICATION TO THE ELECTORAL COLLEGE

The Electoral College and its intriguing game-theoretic aspects have been the subject of investigations for a number of years. Mann and Shapley [11] computed the relative strengths of the states on the basis of the Shapley-Shubik index [17], and showed that in fact the states' relative strength is very nearly proportional to their actual electoral votes. While one might question the appropriateness of the Shapley-Shubik index in the present context (since it is based on the notion of an n-person game played *between the states* rather than a 2-person game played *for the electoral votes of the states*, nevertheless, it is an interesting fact that the defensive equilibrium obtained from the present model is proportional to the electoral voting weights of the various states. Yet another approach to estimating the relative power of the various states is due to Banzhaf [2]. Colantoni, Levesque, and Ordeshook [5] analyze some of the statistical evidence in the light of several different possible assumptions about the candidate's objective functions and voter responses and conclude that a "modified" type of proportional allocation seems as plausible as various other hypotheses. Brams and Davis [3,4] develop an approach called the "$3/2$'s rule" which leads to the conclusion that large states are favored out of proportion to their size. A host of other models, analyses and arguments have been presented over the years to show various purported advantages or disadvantages of the Electoral College; further references to this literature may be found in [4].

The Electoral College may be described as a weighted voting game in which the 50 states and the District of Columbia play the role of the voters. The total weight is 538, and a majority of 270 is required to win. In terms of the lobbying model developed in the preceding section, an appealing feature of the competition for electoral votes is that campaign fund allocations for this purpose are *legal* (within certain ground rules), and therefore data are more readily available. A second feature is that minimum expectations do not seem to play an important role, in part because the process *is* legal: any campaign expenditure on a state, however small, is certainly "acceptable"; the only question might be whether expenditures below a certain threshold "do any good". This latter point may in fact impose, *implicitly*, certain lower bounds $\underset{\sim}{p}^{0}$, but for present purposes we will assume that $\underset{\sim}{p}^{0} = 0$.

For this problem it is clear that the two-lobbyist model is more relevant than that of one lobbyist acting without opposition. We shall be interested in the situation in which one of the parties has "substantially" more funds than the other. In this case "substantially" means *roughly twice as much* (or, more precisely, more than 538/270 times as much; see the following section).

The advantage of this case is that it has an equilibrium solution in pure strategies, whereas in the case of equal resources, for example, the strategies are probability distributions that are difficult (though, in principle, possible) to compute. Moreover, precisely the unequal resources case has arisen frequently in U.S. Presidential elections.* In Table 2 are shown aggregate expenditures by the Republican and the Democratic National Committees for every Presidential election from 1948 to 1968. Also shown are expenditures by Labor National Committees, which, we may assume, contributed most of their resources in support of Democratic candidates. It will be seen that the Republicans outspent the Democrats by a ratio of approximately 2 to 1 in 1952, 1956, and 1968 (and the Republicans won in each of these years). The data are probably not wholly reliable, so that these must be regarded as estimates, especially in view of Labor's uncertain contribution to specific candidates. Thus, in analyzing the 1968 data (the only one of the above three for which detailed state-by-state information is available) we shall assume that the Republicans were *approximately* in the position

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*This situation may change, however, due to recent electoral law reforms in the United States.

Table 2. National Committee Expenditures 1948-1968 ($US).

	1948	1952	1956
Republicans	3,686,779	12,229,239	13,220,144
Democrats	2,266,261	5,121,698	6,492,634
Labor	1,291,733	2,070,350	1,805,482
	1960	**1964**	**1968**
Republicans	12,950,232	19,314,796	29,563,337
Democrats	11,800,979	13,348,791	13,577,715
Labor	2,450,944	3,816,242	7,631,868

Source: [18].

of having enough resources to employ a purely "defensive" strategy, even though the above numbers do not prove this. Under the above assumptions, an efficient strategy for the Republicans in these years would have been to allocate funds according to the defensive equilibrium calculated from (23), where a is the amount of funds available to the Democrats (assumed known).

In fact, it may be shown that in 1968,

(27) *the defensive equilibrium allocation for the Electoral College is precisely proportional to the voting strengths of the various states, i.e. to their respective numbers of Electors.*

This result is established computationally and depends on the particular distribution of voting strengths in 1968. A similar result holds for the 1972 voting strengths. See also the following section, where an explicit relationship between the defensive equilibrium and voting weight is established for decisive simple majority games.

By way of comparison with other indices, it is interesting to note that the Shapley-Shubik (SS) values for the Electoral College [10,11] are also very nearly proportional to the states' actual voting strengths. The largest states have an SS value of only about 5% more than their voting strength, whereas the smallest states have a value of only about 3% less than their voting strength. The SS values are therefore very close to those given by the present theory, and are probably statistically indistinguishable from them.

The Brams-Davis $^3/_2$'s rule, on the other hand, predicts a very substantial bias in favor of large states. The Brams-Davis model postulates that, in any state i, an (undecided) voter's "probability of voting" Republican (say) is equal to $r_i/(r_i + d_i)$, where r_i is the amount spent by the Republicans, d_i the amount spent by the Democrats, in state i. Each of the two candidates is supposed to have the objective of *trying to maximize his total electoral vote*. It may be seen however (as Brams and Davis themselves remark) that this is not the same as trying to maximize one's chance of winning, which is presumably the candidates' true objective, and the one assumed in this model. Consider an example having three states and "electoral" votes 10,10,1 respectively, with a majority of 11 required to win. Then the objective of trying to maximize total electoral vote will lead, incorrectly, to an overconcentration on the first two states. This results from the fact that the third player is in reality just as "strong" as the two others, even though his weight is much different.

An equilibrium solution in pure strategies will, generally, exist only in the rather extreme case that one candidate has more than n times the resources of the other, n being the number of states. The "$^3/_2$'s rule", however, is derived by assuming that both candidates have *equal resources*, and moreover that they *exactly match each other's expenditure in every state*. Thus it does not really apply to the parties' positions in the 1968 election. (However, since Brams and Davis use the *total number of appearances of a candidate and his running mate* as a measure of resource allocation, the candidates did have equal resources in this sense.)

Expenditure data by party and state for the 1968 Presidential election do not seem to be available. As a rather crude substitute, we will use expenditure for local (nonnetwork) *political broadcasts* by party and state, as reported by the Federal Communications Commission [6]. While these data aggregate Presidential campaign spending with Congressional and some local races, the Presidential spending represents the principal component. (Moreover, expenditures by candidates from the same party tend to reinforce each other, due to the "coat-tail" effect.) Expenditures on nationwide (i.e. network) campaign broadcasts are not included; if they were, they would presumably be allocated to the various states in proportion to the audiences receiving them.

Consistent with the parties' respective overall budgets (Table 2), total spending on Presidential campaign broadcasts by the Republicans was about twice that of the Democrats ([6,p.1]).

According to (27), a theoretically optimum strategy for the Republicans would have been to spend in proportion to a state's number of electoral votes. The expenditure data for 1968, aggregated by large, medium, and small states following Brams' and Davis' classification (see Table 3), indicate that

Table 3. Campaign Allocation Bias by State Size - 1968.

	Equilibrium Allocation	Actual Allocation: Republicans**	$^3/_2$'s Allocation
Large states*	39%	48%	57%
Medium states*	32%	30%	27%
Small states*	29%	22%	15%

*Large states are those having more than 20 electoral votes (7 in 1968), medium states are those having between 10 and 20 electoral votes (14 in 1968), and small states are those with less than 10 electoral votes (30 in 1968) [4].

**Nonnetwork broadcast spending, including Congressional and some local races.

the Republicans chose a strategy favoring the large states at the expense mainly of the small states. The solution deviates from proportionality, but by less than the $3/2$'s rule predicts. Given the dominant financial position of the Republicans in 1968, a possible conclusion is that they may have overallocated funds to the large states in their spending strategy.

7. RESULTS FOR WEIGHTED VOTING GAMES

The Electoral College is a special case of a weighted voting game and the equilibria both with and without opposition enjoy special regularity properties for this type of game. Specifically, suppose G is a weighted voting game with representation $(q;\ w_1, w_2, \ldots, w_n)$ and that $w_1 \geqq w_2 \geqq \cdots \geqq w_n$.

(28) *Theorem. There is a defensive equilibrium* $\bar{\underset{\sim}{p}}$ *for* G *against* a *such that* $\bar{p}_1 \geqq \bar{p}_2 \geqq \cdots \geqq \bar{p}_n$.

The proof is obtained by selecting an optimum $\bar{\underset{\sim}{p}}$ for the linear program (23) having as few "bad pairs" as possible, where (i,j) is a *bad pair* if $i < j$ and $\bar{p}_i < \bar{p}_j$. In fact we may choose $\bar{\underset{\sim}{p}}$ and (i,j) such that $\bar{p}_h \geqq \bar{p}_j$ for all $h < i$ and $\bar{p}_k \leqq \bar{p}_i$ for all $k > j$. Letting $\varepsilon = (\bar{p}_j - \bar{p}_i)/2$ it may then be shown that $\bar{\bar{\underset{\sim}{p}}} = (\bar{p}_1, \ldots, \bar{p}_i + \varepsilon, \ldots \bar{p}_j - \varepsilon, \ldots \bar{p}_n)$ is an optimum having fewer bad pairs than $\bar{\underset{\sim}{p}}$. Indeed, we have $\bar{\bar{\underset{\sim}{p}}} \geqq \underset{\sim}{0}$. Moreover, for all $S \in \mathcal{S}$, $\bar{\bar{\underset{\sim}{p}}}(S) \geqq \bar{\underset{\sim}{p}}(S) \geqq a$ *unless* perhaps $i \notin S$, $j \in S$. But then, since $w_i \geqq w_j$, $S' = S + i - j \in \mathcal{S}$, so $\bar{\underset{\sim}{p}}(S) = \bar{\underset{\sim}{p}}(S') - \bar{p}_i + \bar{p}_j \geqq a$, whence $\bar{p}(S) \geqq a + 2\varepsilon$ so $\bar{\bar{\underset{\sim}{p}}}(S) \geqq a$ and $\bar{\bar{\underset{\sim}{p}}}$ is feasible. Therefore $\bar{\bar{\underset{\sim}{p}}}$ is optimal, since $\sum_i \bar{\bar{p}}_i = \sum_i \bar{p}_i$. □

We say that a set of minimum expectations $\underset{\sim}{p}^0$ is *regular* for the weighted voting game G (as above) if $p_1^0 \geqq p_2^0 \geqq \cdots \geqq p_n^0$. The following is a strengthening of a theorem in [19]. The proof is similar to that of (28).

(29) *Theorem. If* G *has no veto players and minimum expectations are regular, then there is an equilibrium without opposition* $\bar{\underset{\sim}{p}}$ *satisfying* $\bar{p}_1 \geqq \bar{p}_2 \geqq \cdots \geqq \bar{p}_n$.

In the preceding section we noted the fact that the 1968 voting weights for the Electoral College were actually *proportional* to the defensive equilibrium, and similarly for the 1972 weights. This is a rather special situation, since, in general, a weighted voting game may have several *different* representations as a weighted voting game, but normally only one defensive equilibrium. However for an important class of games, namely the decisive simple majority games, more can be said.

(30) *Theorem. If* G *is a decisive simple majority game and* $\bar{p}$ *is a defensive equilibrium against resource level* $a > 0$, *then* $(a; \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ *is a representation for* G.

Proof: Let G have representation $(q; w_1, w_2, \ldots, w_n)$. We can choose $q = \min_{S \in \mathcal{S}} w(S)$. Since $q > 0$ by definition, define $\hat{w} = (a/q)w$. Then $\hat{w}$ satisfies

$$\hat{w} \geqq 0 \quad ,$$

(31) $$\hat{w}(S) \geqq a \quad \text{if and only if} \quad S \in \mathcal{S} \quad .$$

In particular, $\hat{w}$ is *feasible* for (23); moreover, by choice of q, (31) holds as an *equality* for *some* $S' \in \mathcal{S}$, so

$$\sum_i \hat{w}_i = \sum_{i \in S'} \hat{w}_i + \sum_{i \in N-S'} \hat{w}_i < 2a \quad .$$

Let $\bar{p}$ be an *optimum* to (23). To show that $\bar{p}$ is a set of weights for G we must show that

$$\bar{p}(S) \geqq a \quad \text{if and only if} \quad S \in \mathcal{S} \quad .$$

Now $\bar{p}(S) \geqq a$ for all $S \in \mathcal{S}$, by feasibility. If $\bar{p}(S) \geqq a$ for some $S \notin \mathcal{S}$, then since G is a decisive simple majority game, $N - S \in \mathcal{S}$ and $\bar{p}(N-S) \geqq a$. But then $\sum_i \bar{p}_i \geqq 2a$ whereas $\sum_i \hat{w}_i < 2a$, contradicting optimality.□

8. CONCLUSION

The intent of this paper has been two-fold. First, given lobbying and campaigning as a fact of life in the political sphere, we ask how should a calculating lobbyist allocate his funds most effectively to achieve his goals? Two models were presented: the case of one lobbyist acting unopposed, and the case of two opposing lobbyists; and each was shown to lead to a certain concept of equilibrium payments to voters. These solutions may find a practical application by practitioners of lobbying and campaigning. But the models also have a theoretical interest: they provide a new approach to the problem of finding a normative measure of power in a voting system. In fact, two new measures of power were defined. While they are related in certain ways, their differences also point to the importance of considering the context of the problem in which power is to be measured.

Of course, the models we have described are idealized, and--some may say--too cynical: surely not everybody's vote can be bought. Suppose then that some voters are immune to influence, that their minds are made up (pro or con). Let $U \subseteq N$ be the set of voters *a priori* for the measure, V the set of voters irrevocably against. (In the case of lobbying without opposition, we might say that the minimum expectation of every voter in U is zero, whereas the minimum expectation of every voter in V is plus infinity.) If U wins, or $N - V$ loses, a lobbyist can do nothing. Otherwise, a lobbyist (or opposing lobbyists) can try to influence the "swing" voters by proceeding as if the game were

$$G' = (N - U \cup V, \mathcal{S}') \quad ,$$

where

$$\mathcal{S}' = \{S \subseteq N - U \cup V: S \cup U \in \mathcal{S}\} \quad ,$$

and prices are determined accordingly.

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