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STOCHASTIC CONTROL FOR LINEAR DISCRETE-TIME DISTRIBUTED-LAG MODELS

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PREFACE

Models with distributed-delay variables arise in many subjects of interest to IIASA. They occur for example in economic planning as the distributed-lag policy model, in time-series analysis as the ARIMA process, and in population and agricultural planning as the age-dependent regenerative process. Derivation of optimal estimation and control procedures for such models is the subject of this paper.

Stochastic Control for Linear
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1. INTRODUCTION

An important class of linear-quadratic Gaussian problems has lagged variables in the dynamics or the observations: problems where process behavior depends on the past trajectory for example, where control action is retarded, or where information is delayed. For such problems in continuous time a fairly comprehensive theory is available (see for example Koivo (1974), Kwong and Willsky (1977), Arthur (1977)); for discrete time no satisfactory comprehensive theory as yet exists, but certain procedures are available for numerical solution (Chow (1975), Aoki (1976)).

Both the Chow and the Aoki procedures redefine the state vector to one of higher dimension to transform the original lagged problem into an equivalent, but larger, non-lagged problem. Standard results then apply. While these methods are convenient they suffer drawbacks. Transition matrices for the equivalent problem are large and sparse, with side dimension N determined by the duration of the longest lags. Calculation of the Riccati sequence then requires operations of order N^3 at each step. Also, since results are expressed in terms of the new, non-lagged problem, much of the special structure of the time-lag controller and estimator is obscured.

It would be better from both computational and theoretical points of view to derive results in terms of the original problem and in non-sparse form. For continuous-time problems this is possible, using the so-called Carathéodory and maximum-principle-Fredholm techniques. These, however, are ill-suited to discrete time and to problems with delays in the control: we cannot apply them here. One way to derive non-sparse results for discrete-time delay problems would be to use a direct dynamic programming argument (see Arthur (1977)). A second and yet more straightforward derivation is proposed in this paper. We translate the problem into equivalent non-lagged form and apply standard theory, then use careful matrix partitioning to reexpress the solution in terms of the variables and matrices of the original problem. The results are then in the non-sparse form we want: the qualitative structure of the time-lag controller and estimator stands out clearly; Riccati calculations are reduced to order N^2; and the discrete-time Riccati equations correspond almost term for term to those for the known continuous-time case--the connection between the two becomes clear.

The problem treated is general: distributed lags may occur in dynamics and observations in both state and control variables. Results apply not only to design of discrete-time filters and controllers, but to numerical solution of continuous-time problems which are discretized at the outset.

2. THE DISTRIBUTED-LAG PROBLEM

We study linear processes that evolve according to the distributed-lag dynamics:

$$x_{i+1} = \sum_{\theta=0}^{k} A_i(\theta)x_{i-\theta} + \sum_{\theta=1}^{h} B_i(\theta)u_{i-\theta} + C_iu_i + \omega_i \quad , \tag{1}$$

where a linear measurement of past states and controls is available:

$$z_i = \sum_{\theta=0}^{k} H_i(\theta)x_{i-\theta} + \sum_{\theta=1}^{h} G_i(\theta)u_{i-\theta} + \psi_i \quad . \tag{2}$$

The distributed-lag dynamics of this process include single lags as a special case, and the observations include pure informational delay as a special case. The usual notation applies: x is an n-dimensional vector describing the state, u an m-dimensional vector of policy instruments, z a p-dimensional vector of observations. The parameter matrices are assumed known and nonrandom. All disturbance or error vectors throughout the paper, unless stated otherwise, are distributed normally, are independent of each other, and have zero mean. Expectations E[ ] are taken over appropriate states, observations, and, where necessary, controls. I_n will denote an identity matrix of dimension n. The process disturbance ω_i and measurement error ψ_i have variances Ω_i and Ψ_i (the latter matrix is assumed positive definite). Initial values $x_0,\ldots,x_{-k}$, and $u_{-1},\ldots,u_{-h}$, are assumed to be distributed normally with given means and variances. Subsequent estimation is conditioned on this initial information.

We wish to choose controls u_i at times 0 to T-1 to minimize

$$J = E\left[\sum_{i=0}^{T-1} (x_i' Q_{0_i} x_i + u_i' R_i u_i) + x_T' Q_{0_T} x_T\right] , \tag{3}$$

where the expectation E is taken over all states and observations; Q_0 is assumed positive semidefinite and R positive definite. Z_i will denote $\{z_0,\ldots,z_i\}$, the information available at time i.

In most applications the implementation of controls is imperfect. The *actual value* of the controls u_i will deviate from the *intended value* u_i as in

$$u_i = \mathit{u}_i + \nu_i , \tag{4}$$

where implementation error ν_i has variance T_i. Usually there is no need to consider this type of error separately--it can be subsumed into general process error by substituting the intended for the actual policy value in the dynamics. With lags in the control, however, this procedure would cause sequential correlation of process errors. Instead we substitute the intended control only partially into the dynamics, by writing

$$x_{i+1} = \sum_{\theta=0}^{k} A_i(\theta) x_{i-\theta} + \sum_{\theta=1}^{h} B_i(\theta) u_{i-\theta} + C_i \mathit{u}_i + \lambda_i \tag{5}$$

with $$\lambda_i = \omega_i + C_i \nu_i . \tag{6}$$

Composite error, λ_i, now has mean zero and variance $\Omega_i + C_i T_i C_i'$. We thus retain the advantages of sequentially uncorrelated process noise, λ_i, and perfectly known control, u_i, at the price of including past controls which are not perfectly known. These must be estimated, as must the state, at each step. (Note that the problem is unchanged by substituting u_i for u_i in the performance criterion. $E[\sum u_i' R_i u_i] + \sum \mathrm{Tr}(R_i T_i)$ replaces $E[\sum \mathrm{u}_i' R_i \mathrm{u}_i]$ and since the trace term is constant it does not affect the solution.)

3. THE EQUIVALENT PROBLEM AND SOLUTION

To solve the problem, we first translate it to an equivalent non-lagged form and apply standard results. Define y_i, the *history* of the system at time i, to be

$$[x_i', \ldots, x_{i-k}' \mid u_{i-1}', \ldots, u_{i-h}']' ,$$

the vector obtained by combining the *state history* (state lagged variables) with the *control history* (control lagged variables). We take the history as the new "state" of the equivalent system.

The history evolves according to

$$\begin{bmatrix} x_{i+1} \\ \vdots \\ x_{i-k+1} \\ \hline u_i \\ \vdots \\ u_{i-h+1} \end{bmatrix} = \left[\begin{array}{cccc|cccc} A_i(0) & \cdots & & A_i(k) & B_i(1) & \cdots & & B_i(h) \\ I_n & & & & & & & \\ & \ddots & & & & & 0 & \\ & & I_n & 0 & & & & \\ \hline & & & 0 & 0 & & & \\ & & & & I_m & & & \\ & 0 & & & & \ddots & & \\ & & & & & & I_m & 0 \end{array}\right] \begin{bmatrix} x_i \\ \vdots \\ x_{i-k} \\ \hline u_{i-1} \\ \vdots \\ u_{i-h} \end{bmatrix} + \begin{bmatrix} C_i \\ 0 \\ \hline I_m \\ 0 \end{bmatrix} u_i + \begin{bmatrix} \lambda_i \\ 0 \\ \hline \nu_i \\ 0 \end{bmatrix} \qquad (7)$$

with observations

$$z_i = [H_i(0),\dots,H_i(k) \mid G_i(1),\dots,G_i(h)] \begin{bmatrix} x_i \\ \cdot \\ \cdot \\ \cdot \\ x_{i-k} \\ \hline u_{i-1} \\ \cdot \\ \cdot \\ \cdot \\ u_{i-h} \end{bmatrix} + \psi_i \quad . \tag{8}$$

Writing the history vector as y_i, the problem is now in the standard non-lagged form

$$y_{i+1} = \tilde{\Phi}_i y_i + \tilde{C}_i u_i + \xi_i \quad , \tag{9}$$

$$z_i = \tilde{H}_i y_i + \psi_i \quad , \tag{10}$$

where ξ_i has variance Ξ_i.

It remains to rewrite the criterion in this form. Define Q_i (positive semidefinite) to be

$$\left[\begin{array}{c|c|c} Q_0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

where the partitions are taken to correspond to

$$[x_i' \mid x_{i-1}',\dots,x_{i-k}' \mid u_{i-1}',\dots,u_{i-h}'] \quad .$$

The problem then becomes: choose $u_i(Z_i)$ to minimize

$$J = E\left[\sum_{i=0}^{T-1} (y_i' Q_i y_i + u_i' R_i u_i) + y_T' Q_T y_T\right] . \qquad (11)$$

Results for this problem are standard. They may be found for example in Meier, Larson and Tether (1971). For our later use we summarize them briefly here:

1. The optimal control policy is linear in the conditional mean of the state, $\hat{y}_{i|i}$ ($\equiv \underset{Z_i}{E}[y_i]$: the " $|_i$ " notation means conditioned on all information available at time i):

$$u_i = -P_i^{-1} D_i \hat{y}_{i|i} . \qquad (12)$$

The control gain matrices are

$$P_i = (\tilde{C}_i' K_{i+1} \tilde{C}_i + R_i) > 0 , \qquad (13)$$

$$D_i = \tilde{C}_i' K_{i+1} \tilde{\Phi}_i , \qquad (14)$$

where K_i is the solution to the Riccati difference system

$$K_i = Q_i + \tilde{\Phi}_i' K_{i+1} \tilde{\Phi}_i - \tilde{\Phi}_i' K_{i+1} \tilde{C}_i (\tilde{C}_i' K_{i+1} \tilde{C}_i + R_i)^{-1} \tilde{C}_i' K_{i+1} \tilde{\Phi}_i ; \quad K_T = Q_T . \qquad (15)$$

2. The conditional mean evolves according to the Kalman filter equation

$$\hat{y}_{i|i} = \hat{y}_{i|i-1} + F_i g_i , \qquad (16)$$

where g_i is the measurement residual

$$g_i = z_i - \tilde{H}_i \hat{y}_{i|i-1} . \qquad (17)$$

The prediction $\hat{y}_{i|i-1}$ is extrapolated from $\hat{y}_{i|i}$ by

$$\hat{y}_{i|i-1} = \tilde{\Phi}_{i-1} \hat{y}_{i-1|i-1} + \tilde{C}_{i-1} u_{i-1} ; \quad \hat{y}_{0|-1} = E[y_0] . \qquad (18)$$

The prediction-error covariance matrix, $S_i \equiv E[(y_i - \hat{y}_{i|i-1})(y_i - \hat{y}_{i|i-1})']$, propagates according to

$$S_{i+1} = \tilde{\Phi}_i S_i \tilde{\Phi}_i' + \Xi_i - \tilde{\Phi}_i S_i \tilde{H}_i' (\tilde{H}_i S_i \tilde{H}_i' + \Psi_i)^{-1} \tilde{H}_i S_i \tilde{\Phi}_i' \quad . \qquad (19)$$

The optimal filter gain, F_i, is given by

$$F_i = S_i \tilde{H}_i' (\tilde{H}_i S_i \tilde{H}_i' + \Psi_i)^{-1} \quad . \qquad (20)$$

We now have a solution in terms of variable y_i and sparse matrices $\tilde{\Phi}$, $\tilde{C}$, etc. In principle the problem is "solved". Note however that computation of K_i and S_i would require sparse-matrix multiplications of the form $\tilde{\Phi}' K \Phi$ at each step (order $(nk + mh + n)^3$ multiplications). In the next two sections we reduce such operations significantly and reexpress the above results in terms of the original problem variables and matrices.

4. OPTIMAL CONTROL POLICY

In terms of the original problem, the conditional mean $\hat{y}_{i|i}$ is reexpressed as $[\hat{x}_{i|i}', \ldots, \hat{x}_{i-k|i}' \mid \hat{u}_{i-1|i}', \ldots, \hat{u}_{i-h|i}']'$ where the notation $\hat{x}_{i-\theta|i}$ is read as the estimate of $x_{i-\theta}$ given all information available at time i.

We now partition K_i and D_i:

$$K_i = \begin{bmatrix} K_{0_i} & K_{1_i} \\ K_{1_i}' & K_{2_i} \end{bmatrix} \quad ; \qquad D_i = [V_i \mid W_i] \quad .$$

(The submatrices K_{0_i} and V_i correspond to the state history, $x_i, \ldots, x_{i-k}$; K_{2_i} and W_i correspond to the control history, $u_{i-1}, \ldots, u_{i-h}$.)

We may now obtain the optimal control law in terms of the matrices of the original problem, by substituting for D_i and $\hat{y}_{i|i}$

in (12). This yields:

$$u_i = -P_i^{-1}\left\{\sum_{\theta=0}^{k} V_i(\theta)\hat{x}_{i-\theta|i} + \sum_{\theta=1}^{h} W_i(\theta)\hat{u}_{i-\theta|i}\right\} . \quad (21)$$

The optimal policy is a feedback law, linear in the current estimates of the state and control histories.

By substituting the original problem matrices for $\tilde{\Phi}$ and $\tilde{C}$ in (13) and (14) and multiplying out, we obtain the gain matrices P_i, V_i, and W_i:

$$P_i = C_i'K_{0_{i+1}}(0,0)C_i + C_i'K_{1_{i+1}}(0,1) + K_{1_{i+1}}'(1,0)C_i + K_{2_{i+1}}(1,1) + R_i$$

$$V_i(\theta) = C_i'K_{0_{i+1}}(0,0)A_i(\theta) + C_i'K_{0_{i+1}}(0,\theta+1) + K_{1_{i+1}}'(1,0)A_i(\theta) + K_{1_{i+1}}'(1,\theta+1) \quad (22)$$

$$W_i(\theta) = C_i'K_{0_{i+1}}(0,0)B_i(\theta) + C_i'K_{1_{i+1}}(0,\theta+1) + K_{1_{i+1}}'(1,0)B_i(\theta) + K_{2_{i+1}}(1,\theta+1) .$$

Finally the Riccati difference system (15) is expanded to yield a recursion for the submatrices K_0, K_1, K_2:

$$K_{0_i}(\theta,\phi) = A_i'(\theta)K_{0_{i+1}}(0,0)A_i(\phi) + A_i'(\theta)K_{0_{i+1}}(0,\phi+1) + K_{0_{i+1}}(\theta+1,0)A_i(\phi) + Q_{0_i}\delta_0(\theta,\phi) + K_{0_{i+1}}(\theta+1,\phi+1) - V_i'(\theta)P_i^{-1}V_i(\phi)$$

$$K_{1_i}(\theta,\phi) = A_i'(\theta)K_{0_{i+1}}(0,0)B_i(\phi) + A_i'(\theta)K_{1_{i+1}}(0,\phi+1) + K_{0_{i+1}}(\theta+1,0)B_i(\phi) + K_{1_{i+1}}(\theta+1,\phi+1) - V_i(\theta)P_i^{-1}W_i(\phi) \quad (23)$$

$$K_{2_i}(\theta,\phi) = B_i'(\theta)K_{0_{i+1}}(0,0)B_i(\phi) + B_i'(\theta)K_{1_{i+1}}(0,\phi+1) + K_{1_{i+1}}'(\theta+1,0)B_i(\phi) + K_{2_{i+1}}(\theta+1,\phi+1) - W_i'(\theta)P_i^{-1}W_i(\phi)$$

with end conditions $K_{0_T}(\theta,\phi) = Q_{0_T}\delta_0(\theta,\phi)$; $K_{1_T} = K_{2_T} = 0$.

(In the above results the indices θ,ϕ are taken over 0 to k or 1 to h as appropriate. The symbol $\delta_0(\theta,\phi) = 1$ if θ and ϕ are zero; $\delta_0(\theta,\phi) = 0$ otherwise. Where undefined matrices occur, e.g., $K_0(k+1,0)$, they are taken as zero.)

The control law parameters may be precomputed. Only the estimates of the lagged variables need then be fed back in real time to determine the optimal control.

In the case of state lags only (where $B(\theta)\equiv 0$), the results simplify: W, K_1, K_2 disappear. Where there are control lags only ($A(\theta)\equiv 0$), V, K_1, and K_0 except for $K_0(0,0)$ disappear.

5. THE OPTIMAL FILTER-SMOOTHER

We now translate the filter results of Section 3 to a form that fits the original lagged problem.

Partition F_i and S_i as

$$F_i = \begin{bmatrix} M_i L_i^{-1} \\ N_i L_i^{-1} \end{bmatrix} \quad ; \quad S_i = \begin{bmatrix} S_{0i} & S_{1i} \\ S_{1i}' & S_{2i} \end{bmatrix} ,$$

where L_i is defined as

$$L_i = [\tilde{H}_i S_i \tilde{H}_i' + \Psi_i] > 0 .$$

(The submatrix dimensions of M_i and S_{0i} correspond to the state history, those of N_i and S_{2i} to the control history.)

Now, substituting for $\hat{y}_{i|i}$, the Kalman filter of (16) becomes at each stage an estimator for the history:

$$\begin{aligned} \hat{x}_{i-\theta|i} &= \hat{x}_{i-\theta|i-1} + M_i(\theta)L_i^{-1} g_i , \quad i = 0,\dots,k \\ \hat{u}_{i-\theta|i} &= \hat{u}_{i-\theta|i-1} + N_i(\theta)L_i^{-1} g_i , \quad i = 1,\dots,h . \end{aligned} \tag{24}$$

The history estimates are updated at each stage by combining the previous-stage estimate with the new information g_i--they are improved sequentially as new information comes in, where g_i (the residual) is obtained from (17) as

$$g_i = z_i - \sum_{\theta=0}^{k} H_i(\theta)\hat{x}_{i-\theta|i-1} + \sum_{\theta=1}^{h} G_i(\theta)\hat{u}_{i-\theta|i-1} \quad . \qquad (25)$$

The prediction equation (18) reduces to

$$\hat{x}_{i+1|i} = \sum_{\theta=0}^{k} A_i(\theta)\hat{x}_{i-\theta|i} + \sum_{\theta=1}^{h} B_i(\theta)\hat{u}_{i-\theta|i} + C_i\hat{u}_{i|i} \; , \qquad (26)$$

$$\hat{u}_{i|i} = u_i \quad ,$$

with initial conditions $\hat{x}_{-\theta|-1} = E[x_{-\theta}]$, $\hat{u}_{-\theta|-1} = E[u_{-\theta}]$.

The above equations (24) to (26) make up a recursion system for the estimates of the state and control histories. The filter for the equivalent non-lagged problem has now become a filter-smoother (an estimator of present and past values) for the original lagged problem.

It remains to specify the filter-smoother gain matrices. Equation (20) and the definitions of F_i and L_i yield

$$\begin{aligned} L_i &= H_i S_{0_i} H_i' + H_i S_{1_i} G_i' + G_i S_{1_i}' H_i' + G_i S_{2_i} G_i' + \Psi_i > 0 \; , \\ M_i &= S_{0_i} H_i' + S_{1_i} G_i' \; , \qquad (27) \\ N_i &= S_{1_i} H_i' + S_{2_i} G_i' \; . \end{aligned}$$

We now expand (19) to arrive at a recursive system for the submatrices of S_i:

$$S_{0_{i+1}}(\theta+1,\phi+1) = S_{0_i}(\theta,\phi) - M_i(\theta)L_i^{-1}M_i'(\phi) \quad ,$$

$$S_{1_{i+1}}(\theta+1,\phi+1) = S_{1_i}(\theta,\phi) - M_i(\theta)L_i^{-1}N_i'(\phi) \quad , \qquad (28)$$

$$S_{2_{i+1}}(\theta+1,\phi+1) = S_{2_i}(\theta,\phi) - N_i(\theta)L_i^{-1}N_i'(\phi)$$

(again with indices θ,ϕ taken over the appropriate range 0 to k, or 1 to h). Note that $S_{0_{i+1}}(\theta+1,\phi+1)$ and $S_{0_i}(\theta,\phi)$ are both the estimate-error covariance matrices for $x_{i-\theta}$, $x_{i-\phi}$. But $S_{0_{i+1}}$ is conditioned on Z_i, while S_{0_i} is conditioned on Z_{i-1}. Equations (28) therefore update the covariance of the history estimates. Since the negative term is positive semidefinite, the covariances cannot increase as additional information is brought in.

The equations (28) are used with the expanded form of (19) to yield the error covariance matrices of the prediction $\hat{x}_{i+1|i}$ and $\hat{u}_{i|i}$ with the other estimates:

$$S_{0_{i+1}}(0,0) = \sum_{\theta=0}^{k}\sum_{\phi=0}^{k} A_i(\theta)S_{0_{i+1}}(\theta+1,\phi+1)A_i'(\phi) + \sum_{\theta=0}^{k}\sum_{\phi=1}^{h} A_i(\theta)S_{1_{i+1}}(\theta+1,\phi+1)B_i'(\phi)$$

$$+ \sum_{\theta=1}^{h}\sum_{\phi=0}^{k} B_i(\theta)S_{1_{i+1}}'(\theta+1,\phi+1)A_i'(\phi) + \sum_{\theta=1}^{h}\sum_{\phi=1}^{h} B_i(\theta)S_{2_{i+1}}(\theta+1,\phi+1)B_i'(\phi) + \Lambda_i \quad ,$$

$$S_{0_{i+1}}(0,\phi) = \sum_{\theta=0}^{k} A_i(\theta)S_{0_{i+1}}(\theta+1,\phi) + \sum_{\theta=1}^{h} B_i(\theta)S_{1_{i+1}}'(\theta+1,\phi) \quad , \quad \phi = 1,\ldots,k$$

$$S_{1_{i+1}}(0,1) = \sum_{\theta=0}^{k} A_i(\theta)S_{1_{i+1}}(\theta+1,1) + \sum_{\theta=1}^{h} B_i(\theta)S_{2_{i+1}}(\theta+1,1) + C_iT_i \quad , \qquad (29)$$

$$S_{1_{i+1}}(0,\phi) = \sum_{\theta=0}^{k} A_i(\theta)S_{1_{i+1}}(\theta+1,\phi) + \sum_{\theta=1}^{h} B_i(\theta)S_{2_{i+1}}(\theta+1,\phi) \quad , \quad \phi = 2,\ldots,h$$

$$S_{2_{i+1}}(1,1) = T_i \; ; \quad \text{otherwise } S_1,S_2 = 0 \; .$$

Recursion of S is initialized by equating $S_0(\theta,\phi)$, $S_1(\theta,\phi)$, $S_2(\theta,\phi)$ at time 0 to $Cov(x_{-\theta},x_{-\phi})$, $Cov(x_{-\theta},u_{-\phi})$, $Cov(u_{-\theta},u_{-\phi})$. Since filter gain and covariance equations do not depend on real-time values, they may be computed in advance. Only the past-history estimates need be computed on line.

The filter-smoother derived above specializes to that of Mishra and Rajamani (1975) for the state-variable distributed-lag case they consider.

6. REMARKS AND EXTENSIONS

We have obtained an optimal controller and estimator expressed in terms of the original problem. The resulting gain matrix expressions in (22), (23) and (27) to (29) seem more lengthy than those for the equivalent problem, but they require multiplications of order $(nk + mh + n)^2$ rather than $(nk + mh + n)^3$ at each step.

The time-lag structure of the controller and estimator is clear from (21) and (24) to (26). In contrast to the no-lag case, the controller does not use a once-only estimate of each variable; instead it exploits the fact that lagged variables remain in the dynamics for some time, and during this time the system can "learn" by mixing in new information. For this reason, if estimation lags are shorter than dynamics lags, estimation must still proceed back to the dynamics lag-limits. The controller acts on changing but constantly improving lagged-variable estimates. Note that in cases of informational delay the estimator is constructed to "predict" those lagged variables that have not yet entered direct observation. These "predictions" improve as time progresses.

The discrete-time matrix Riccati results above correspond almost term by term to those for the continuous-time case. Extra terms are present however due to the discrete time interval. It is therefore not possible to obtain the discrete results by discretization of the continuous results; it is possible, however, to go in the other direction. The discrete results can yield the continuous ones by appropriate passage to the limit (see Arthur (1977)).

Some extensions of the problem are worth noting briefly. For example the results are easily modified to the case of a time-lagged criterion. Also, varying lag-limits may be accommodated by replacing k and h by k(i) and h(i), provided k(i) and h(i) do not lengthen by more than one unit per unit time. Otherwise the maximum lag duration can serve as k or h.

The above results carry over to the infinite-horizon, time-invariant regulator case as long as the properties *strong controllability* and *strong observability* are met. That is, we must be able to simultaneously control and consistently estimate not just the present state x_i but the entire history, $x_i,\ldots,x_{i-k}$, $u_{i-1},\ldots,u_{i-h}$. (Cf. for example Thowsen (1977), or Delfour and Mitter (1972).) These properties then guarantee (a) existence of optimal controls and optimal estimator given an infinite horizon, (b) asymptotic stability of the closed estimator-feedback controller system, (c) convergence of the gain matrices to stationary values.

7. CONCLUSIONS

Discrete-time stochastic control results were presented for LQG problems with distributed lags in dynamics and observations. Optimal controls are linear in the estimates of past states and controls, and an optimal filter-smoother obtains and updates these estimates in linear fashion. Gain-matrix calculations are faster than in the usual high-dimensional methods, and the discrete-time results show close correspondence to those for the continuous-time case.

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