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Experience Accumulation for Decision Making in Multivariate Time Series

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Preface

In particular, the situation is often met when a stochastic process has to be controlled, the model of which is a priori not well known. In this case the decision maker (or decision making device - controller) has to determine sequentially the input on the basis of his limited knowledge of the process and to learn the process at the same time. The present paper deals with the process identification under these conditions and under the assumption that the mathematical model of the process is a priori known up to a finite number of unknown parameters. The uncertainty of the parameters is characterized by subjective probability distributions. Recursive relations are derived for the evolution of this distribution when the amount of observed data is growing. The general theory is elaborated, including practical algorithms for the case when the input-output relation is describable by a multivariate (auto-) regression model no parameter of which (except the order) is a priori known.

Experience Accumulation for Decision Making in Multivariate Time Series

Abstract

A dynamic stochastic system with multivariate input and multivariate output, possibly controlled in a closed loop, is considered. It is assumed that the input-output relation is describable by a model of a given structure but with a finite set of unknown parameters. The uncertainty of the parameters is characterized by the subjective probability density function. Functional recursion relations are derived describing the evolution of this subjective p.d.f. when it is successively conditioned by the observed data. A self-reproducing form of the conditional p.d.f. is found for the case, when the process is describable by a multivariate regression model and no parameter - except the order - is a priori known. This makes it possible to reduce the functional recursion into an algebraic recursion which is easy to perform.

1. Introduction

A multivariate stochastic process with ν-dimensional output $\{y(t);t=1,2,...\}$ and μ-dimensional input $\{u_{(t)};t=1,2,...\}$ is considered. The inputs $u_{(t)}$, $t \geq 1$, are accessible to the decision maker and may be used to influence the output of the process $y_{(t)}$, $t \geq 1$, in order to achieve some desired goal which may be, for instance, the minimization of some criterion characterizing the quality of the process. The time indexing of random variables is chosen so that the time sequence of inputs and outputs is

$$u_{(1)}, y_{(1)}, u_{(2)}, y_{(2)}, \ldots, u_{(t)}, y_{(t)}, \ldots \quad .$$

Thus, the output $y_{(t)}$ is not known when $u_{(t)}$ is applied and therefore cannot be taken into account when the decision concerning $u_{(t)}$ is taken. The decision maker can only make use of past outputs and inputs known to him when the decision is made.

If the following notation for the sets of random variables is introduced

$$y^{(\tau)} \triangleq \{y_{(\tau)}, y_{(\tau-1)}, y_{(\tau-2)}, \ldots, y_{(2)}, y_{(1)}\}$$

$$u^{(\tau)} \triangleq \{u_{(\tau)}, u_{(\tau-1)}, u_{(\tau-2)}, \ldots, u_{(2)}, u_{(1)}\} \quad ,$$

the general form of the control law - which is to be determined by the decision maker - can be written as the conditional probability density function (c.p.d.f.)

$$p(u_{(t)}|y^{(t-1)}, u^{(t-1)}) \quad . \tag{1.1}$$

In the special case, when the applied control law is deterministic

$$u_{(t)} = f_{(t)}(y^{(t-1)}, u^{(t-1)}) \quad ,$$

the c.p.d.f. (1.1) degrades into the Dirac δ-function

$$p(u_{(t)},|y^{(t-1)}, u^{(t-1)}) = \delta(u_{(t)} - f_{(t)}(y^{(t-1)}, u^{(t-1)}))$$

where $f_{(t)}$ is the function which is to be determined by the decision maker.

To be able to perform this task in some optimal way the decision maker needs to know the c.p.d.f.

$$p(y_{(t)}|y^{(t-1)}, u^{(t)}) \quad . \tag{1.2}$$

In this paper it will be assumed that the process is describable by a model which defines the c.p.d.f. (1.2) up to a finite set of unknown parameters K, i.e., that only the c.p.d.f. (for any K)

$$p(y_{(t)}|K, y^{(t-1)}, u^{(t)}) \tag{1.3}$$

is apriori known. The purpose of the present paper is to answer the following question: What is the optimal way for the decision maker to collect information about the unknown parameters which are contained in observable input-output data? How has he to accumulate his experience in order to improve his performance?

If the true values of the parameters K are not known the c.p.d.f. (1.3) cannot be directly used by the decision maker. The unknown parameters have to be eliminated first. This can be done in the following way.

$$p(y_{(t)}|y^{(t-1)},u^{(t)}) = \int p(y_{(t)}|K,y^{(t-1)},u^{(t)})p(K|y^{(t-1)},u^{(t)})dK \qquad (1.4)$$

where

$$p(K|y^{(t-1)},u^{(t)}) \qquad (1.5)$$

is the subjective c.p.d.f., as defined e.g., in [2], which reflects the uncertainty of the parameters. The integration in (1.4) is taken over all possible values of the parameter set K. However, to be able to perform this integration the c.p.d.f. (1.5) has to be known. Thus, the problem of first importance is to find the way this c.p.d.f. can be calculated for each t.

The paper is organized in the following manner.

In the next Section the general recursion relations for the evolution of the c.p.d.f. (1.5) will be derived (Theorem 1) for the "natural conditions of control" which are defined and discussed.

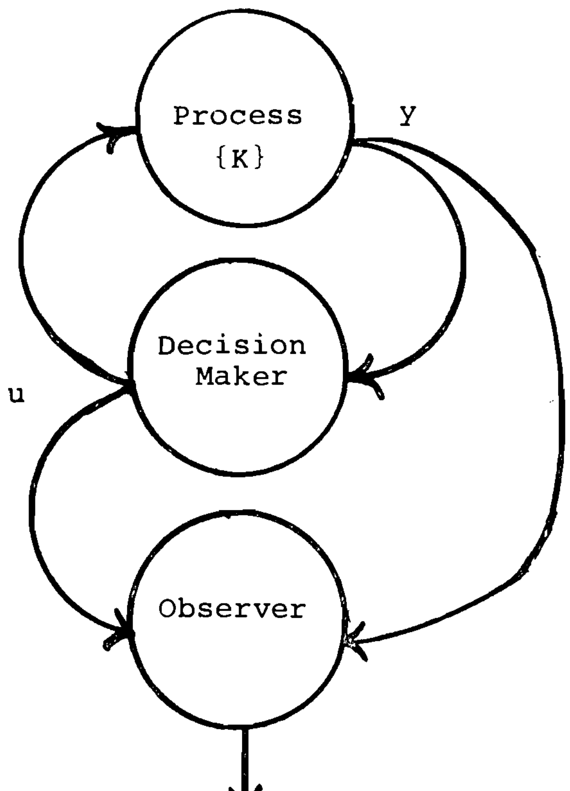
In Section 3 the general recursion relations are applied to the particular case of a multivariate regression model of given order but with unknown parameters. The self-reproducing form of the subjective c.p.d.f. for the set of unknown parameters is found and analyzed. The practical advantage of the self-reproducing form is that it makes it possible to reduce the functional recursion into an algebraic recursion which is easy to perform. These results are summarized in Theorem 2. The self-reproducing form of the c.p.d.f. (1.2) is given by Theorem 3.

The topic of Section 4 is the question of how prior information about the unknown parameters can be respected in the starting subjective p.d.f. Section 5 is then the concluding section.

2. Evolution of Conditional Probability Density Functions

According to the relation (1.4) all information relevant for the forecast and/or control of the output of a stochastic process with an unknown parameter set K can be expressed through the c.p.d.f. (1.5). In this Section general recursion relations will be derived which make it possible to update this c.p.d.f. under certain, very general conditions which will be called "natural conditions of control." Before the precise mathematical definition of these conditions is given, some basic concepts have to be clarified.

Consider the information flow diagram in Figure 1. In this diagram the decision maker and the observer are distinguished. The decision maker determines for each t the input of the process



$p(K|y^{(t-1)},u^{(t)})$

Figure 1.

$u_{(t)}$. The input $u_{(t)}$ is followed by the output of the process $y_{(t)}$. The observer has the possibility to observe both the inputs and the outputs of the process starting with t = 1.

To make our consideration as general as possible, we shall not consider a particular type of the process model at this stage. We shall only assume that the c.p.d.f.

$$p(y_{(t)}|K,y^{(t-1)},u^{(t)})$$

is known to the observer up to a finite set of unknown parameters K. The task of the observer is to determine

$$p(K|y^{(t-1)},u^{(t)})$$

for each $t > 1$. The initial (unconditioned) subjective p.d.f. $p(K)$ reflects the observer's initial belief in the likelihood of possible values of the unknown parameters and he has to update his opinion according to new data which he is obtaining sequentially in real time.

In a general case the information available for the observer may be different from that available to the decision maker. If, for instance, either the parameters K were known to the decision maker or the decision maker had more experimental data at his disposal and the observer knew his strategy, the observer could also gain some information about the parameters K from the single action of the decision maker. However, when the decision maker and the observer are the same person - mathematically speaking, when they operate on the same σ-algebra - nothing can be gained from the single $u_{(t)}$. One cannot learn solely from one's own action without getting the response.

Throughout the rest of the paper it will be assumed that the decision maker and the observer are identical, i.e., that the decision maker has both to learn the process and to control it. These conditions may be mathematically defined as follows.

Definition 1. The conditions under which

$$p(K|y^{(t-1)},u^{(t)}) = p(K|y^{(t-1)},u^{(t-1)}) \tag{2.1}$$

holds will be called *the natural conditions of control.*

From (2.1) it immediately follows that

$$p(u_{(t)}|K,y^{(t-1)},u^{(t-1)}) = p(u_{(t)}|y^{(t-1)},u^{(t-1)}) \qquad (2.2)$$

as it can be seen from the second equality in the following relation.

$$p(K,u_{(t)}|y^{(t-1)},u^{(t-1)}) = p(u_{(t)}|y^{(t-1)},u^{(t-1)})p(K|y^{(t-1)},u^{(t)})$$

$$= p(K|y^{(t-1)},u^{(t-1)})p(u_{(t)}|K,y^{(t-1)},u^{(t-1)}) \quad .$$

The equality (2.2) may well be used as a definition of natural conditions of control instead of (2.1). In this equality the fact is reflected that the action of the decision maker may not depend on the unknown parameter set K when all information contained in the past history of the process and in the initial subjective p.d.f. p(K) is considered. In other words, the process input $u_{(t)}$ may depend on the unknown parameters K only through the past history of the process $\{y^{(t-1)},u^{(t-1)}\}$ which is known to the decision maker at the time instant when the decision concerning $u_{(t)}$ is taken. When this information is considered in the condition part of the c.p.d.f. (2.2) the conditioning on K is redundant. Notice that the equality (2.2) does not hold if any $y_{(\tau)}$ or $u_{(\tau)}$ of the set $\{y_{(\tau)},u_{(\tau)};t>\tau\geq 1\}$ is omitted in the conditional part of the c.p.d.f.'s in (2.2).

The equalities (2.1) and (2.2) make it possible to derive the desired recursion in a straightforward way. Consider the joint p.d.f. of $y_{(t)}$, $u_{(t)}$ and K conditioned by the past history of the process $\{y^{(t-1)},u^{(t-1)}\}$ and rewrite it in the following two ways.

$$p(y_{(t)},u_{(t)},K|y^{(t-1)},u^{(t-1)})$$

$$= p(K|y^{(t-1)},u^{(t-1)})p(u_{(t)}|K,y^{(t-1)},u^{(t-1)})p(y_{(t)}|K,y^{(t-1)},u^{(t)})$$

$$= p(u_{(t)}|y^{(t-1)},u^{(t-1)})p(y_{(t)}|y^{(t-1)},u^{(t)})p(K|y^{(t)},u^{(t)}) \qquad (2.3)$$

When the equalities (2.1) and (2.2) are considered, the following result is obtained from the second equality in (2.3) and from (1.4).

Theorem 1. Under natural conditions of control the evolution of the c.p.d.f. (2.1) is determined by the recursive relation

$$p(K|y^{(t)},u^{(t)}) = \frac{p(y_{(t)}|K,y^{(t-1)},u^{(t)})}{p(y_{(t)}|y^{(t-1)},u^{(t)})} p(K|y^{(t-1)},u^{(t-1)}) \tag{2.4}$$

where

$$p(y_{(t)}|y^{(t-1)},u^{(t)}) = \int p(y_{(t)}|K,y^{(t-1)},u^{(t)})p(K|y^{(t-1)},u^{(t-1)})dK \ . \tag{2.5}$$

In the next Section this general result is applied to the particular case of the multivariate regression model.

3. Self-reproducing Forms of Conditional Probability Distributions for Multivariate Regression Models

The functional recursion relations derived in the previous Section may be applied to any process model defining the c.p.d.f.

$$p(y_{(t)}|K,y^{(t-1)},u^{(t-1)}) \tag{3.1}$$

where K is the set of unknown model parameters. In general it may be difficult to perform this calculation as the whole function

$$p(K|y^{(t-1)},u^{(t-1)}) \tag{3.2}$$

has to be recalculated for each t according to the recursion (2.4) and (2.5). However, if such a form of the c.p.d.f. (3.2) can be found that remains unchanged, up to a finite set of its parameters, when t is growing, the functional recursion (2.4) and (2.5) can be reduced to an algebraic recursion which considerably simplifies the calculation. The c.p.d.f.'s having this property are called self-reproducing, or one says that they form a conjugate family of distributions [2,5].

In this Section it will be assumed that the relation between the μ-dimensional input u and the ν-dimensional output y of the process is describable by the multivariate (auto-) regression model

$$y_{(t)} = \sum_{i=1}^{n} A_i y_{(t-1)} + \sum_{i=0}^{n} B_i u_{(t-i)} + c + e_{(t)} \tag{3.3}$$

where $\{e_{(t)};t=1,2,\ldots\}$ is a sequence of mutually independent gaussian random vectors with zero mean

$$E[e_{(t)}] = 0$$

and covariance matrix

$$\mathrm{Cov}[e_{(t)}] = E[e_{(t)}e_{(t)}^T] = R \quad . \tag{3.4}$$

A_i, B_i are matrix-valued regression coefficients of appropriate dimensions and c is a ν-vector. However, it is not assumed that the parameters A_i, B_i, c and R are apriori known.

It is convenient to write the multivariate regression model (3.3) in the following compact form

$$y_{(t)} = P^T z_{(t)} + e_{(t)} \tag{3.5}$$

where $z_{(t)}$ is a column vector of dimension

$$\rho = n(\nu+\mu) + \mu + 1 \tag{3.6}$$

and P is a $(\rho\times\nu)$-matrix of regression coefficients. The following arrangement of $z_{(t)}$ and P may have some advantages

$$z_{(t)}^T = [u_{(t)}^T, y_{(t-1)}^T, u_{(t-1)}^T, \ldots, y_{(t-n)}^T, u_{(t-n)}^T, 1] \tag{3.7}$$

$$P^T = [B_0, A_1, B_1, \ldots, A_n, B_n, c] \quad . \tag{3.8}$$

It is also convenient to consider the precision matrix [2,5]

$$\Omega = R^{-1} \tag{3.9}$$

to be the unknown parameter instead of R itself. Thus, the set of unknown parameters in our particular case of multivariate regression models will be

$$K = \{P,\Omega\} \tag{3.10}$$

and the c.p.d.f. (3.1), defined by the regression model for $t > n$, may be written as follows.

$$p(y_{(t)}|P,\Omega,y^{(t-1)},u^{(t)}) = (2\pi)^{-\frac{\nu}{2}}|\Omega|^{\frac{1}{2}}\exp\{-\tfrac{1}{2}(y_{(t)}-P^Tz_{(t)})^T\Omega(y_{(t)}-P^Tz_{(t)})\} \quad . \tag{3.11}$$

The goals of this Section are:

(i) to find and analyze the self-reproducing form of

$$p(P,\Omega|y^{(t)},u^{(t)}) \tag{3.12}$$

and related c.p.d.f.'s

$$p(P|y^{(t)},u^{(t)}) \quad , \quad p(\Omega|y^{(t)},u^{(t)}) \quad , \quad p(P|\Omega,y^{(t)},u^{(t)}) \quad ,$$

(ii) to derive the algebraic recursions for updating of the parameters of these c.p.d.f.'s,

(iii) to find the c.p.d.f. $p(y_{(t)}|y^{(t-1)},u^{(t)})$ which does not contain the unknown model parameters and may be used to forecast and/or control the output.

The results concerning the items (i) and (ii) are summarized in Theorem 2, item (iii) is addressed in Theorem 3. As the proofs of these Theorems are rather involved they are left to Appendices A and B.

Theorem 2. If the process, describable by the multivariate regression model (3.5) with gaussian random component $e_{(t)}$, is controlled under natural conditions (see Definition 1) and if $\mathcal{M}_\nu$ denotes the set of all positive definite matrices of dimension ν, then

(a) the self-reproducing c.p.d.f. (3.12) is

$$p(P,\Omega|y^{(t)},u^{(t)})$$

$$= \alpha_{(t)} |\Omega|^{\frac{\theta_{(t)}}{2}} \exp\left\{-\frac{1}{2}\,\mathrm{tr}\left(\Omega\begin{bmatrix}-I_\nu\\ P\end{bmatrix}^T V_{(t)}\begin{bmatrix}-I_\nu\\ P\end{bmatrix}\right)\right\} \text{ for } \Omega \in \mathcal{M}_\nu \qquad (3.13)$$

$$p(P,\Omega|y^{(t)},u^{(t)}) = 0 \quad \text{for all } \Omega \notin \mathcal{M}_\nu \quad ,$$

where $\theta_{(t)}$ *is a scalar parameter determined by the recursion relation*

$$\theta_{(t)} = \theta_{(t-1)} + 1 \quad , \qquad (3.14)$$

$V_{(t)}$ *is a* $(\nu+\rho)\times(\nu+\rho)$*-matrix for which the following recursion holds*

$$V_{(t)} = V_{(t-1)} + \begin{bmatrix}y_{(t)}\\ z_{(t)}\end{bmatrix}\begin{bmatrix}y_{(t)}\\ z_{(t)}\end{bmatrix}^T \quad , \qquad (3.15)$$

I_ν *means the identity matrix of dimension* $(\nu\times\nu)$ *and* $\alpha_{(t)}$ *is the normalizing factor independent of matrix-valued variables* P *and* Ω *(see (g) for formula);*

(b) the maximum of the c.p.d.f. (3.13) lies in the point $P = \hat{P}_{(t)}$, $\Omega = \hat{\Omega}_{(t)}$

$$\hat{P}_{(t)} = V_{z(t)}^{-1} V_{zy(t)} \qquad (3.16)$$

$$\hat{\Omega}_{(t)} = \theta_{(t)} \Lambda_{(t)}^{-1} \qquad (3.17)$$

where $V_{z(t)}$ *and* $V_{zy(t)}$ *are the submatrices of* $V_{(t)}$ *partitioned in the following way*

$$V_{(t)} = \underbrace{\begin{bmatrix}V_{y(t)} & V_{zy(t)}^T\\ V_{zy(t)} & V_{z(t)}\end{bmatrix}}_{\nu \quad\quad \rho} \begin{matrix}\}\nu\\ \}\rho\end{matrix} \qquad (3.18)$$

and

$$\Lambda_{(t)} = V_{y(t)} - V_{zy(t)}^{T} V_{z(t)}^{-1} V_{zy(t)} \quad , \tag{3.19}$$

(c) *the marginal distribution of the uncertain matrix* Ω *is the Wishart distribution*

$$p(\Omega|y^{(t)},u^{(t)}) = \gamma_{(t)} |\Omega|^{\frac{\theta_{(t)}-\rho}{2}} \exp\{-\frac{1}{2}\,\mathrm{tr}(\Omega\Lambda_{(t)})\} \text{ for } \Omega \in \mathcal{M}_\nu \tag{3.20}$$

$$p(\Omega|y^{(t)},u^{(t)}) = 0 \text{ for } \Omega \notin \mathcal{M}_\nu$$

where $\gamma_{(t)}$ *is the normalizing factor given in (g);*

(d) *the conditional distribution of the uncertain regression-coefficients matrix P given the precision matrix* Ω *is*

$$p(P|\Omega,y^{(t)},u^{(t)}) = (2\pi)^{-\frac{\rho}{2}} |V_{z(t)}|^{\frac{\rho}{2}} |\Omega|^{\frac{\rho}{2}}$$

$$\cdot \exp\{-\frac{1}{2}\,\mathrm{tr}(\Omega[P-\hat{P}_{(t)}]^{T} V_{z(t)} [P-\hat{P}_{(t)}])\} \quad ; \tag{3.21}$$

(e) *the marginal probability distribution of the uncertain matrix P is*

$$p(P|y^{(t)},u^{(t)}) = \beta_{(t)} |I_\nu + \Lambda_{(t)}^{-1} [P-\hat{P}_{(t)}]^{T} V_{z(t)} [P-\bar{P}_{(t)}]|^{-\frac{\theta_{(t)}+\nu+1}{2}} \tag{3.22}$$

where $\beta_{(t)}$ *is the normalizing factor given in (g);*

(f) *instead of using the formulae (3.16) and (3.19) for each t the characteristics* $\hat{P}_{(t)}$, $\Lambda_{(t)}$ *and*

$$C_{(t)} = V_{z(t)}^{-1} \tag{3.23}$$

may be updated directly by the following algebraic recursion relations

$$\hat{e}_{(t)} = y_{(t)} - \hat{P}^T_{(t-1)} z_{(t)} \tag{3.24}$$

$$g_{(t)} = C_{(t-1)} z_{(t)} \tag{3.25}$$

$$\xi_{(t)} = z^T_{(t)} C_{(t-1)} z_{(t)} = z^T_{(t)} g_{(t)} \tag{3.26}$$

$$\hat{P}_{(t)} = \hat{P}_{(t-1)} + \frac{1}{1+\xi_{(t)}} g_{(t)} \hat{e}^T_{(t)} \tag{3.27}$$

$$\Lambda_{(t)} = \Lambda_{(t-1)} + \frac{1}{1+\xi_{(t)}} \hat{e}_{(t)} \hat{e}^T_{(t)} \tag{3.28}$$

$$C_{(t)} = C_{(t-1)} - \frac{1}{1+\xi_{(t)}} g_{(t)} g^T_{(t)} \quad ; \tag{3.29}$$

(g) the factors $\alpha_{(t)}$, $\beta_{(t)}$ *and* $\gamma_{(t)}$ *normalize the c.p.d.f.'s (3.13), (3.22) and (3.20) in such a way that*

$$\int p(X|\cdot)\,dX = 1$$

where X stands for $\{P,\Omega\}$ *or P or* Ω, *respectively, are given by the formulae*

$$\alpha_{(t)} = \frac{|\Lambda_{(t)}|^{\frac{\theta_{(t)}-\rho+\nu+1}{2}} |V_{z(t)}|^{\frac{\nu}{2}}}{2^{\frac{(\theta_{(t)}+\nu+1)\nu}{2}} \pi^{\frac{(2\rho+\nu-1)\nu}{4}} \prod_{j=1}^{\nu} \Gamma\left(\frac{\theta_{(t)}-\rho+2+\nu-j}{2}\right)}$$

$$= (2\pi)^{-\frac{\rho\nu}{2}} |V_{z(t)}|^{\frac{\nu}{2}} \gamma_{(t)} \tag{3.30}$$

$$\beta_{(t)} = \frac{\prod_{j=1}^{\nu} \Gamma\left(\frac{\theta_{(t)}+2+\nu-j}{2}\right)}{\pi^{\frac{\rho}{2}} \prod_{j=1}^{\nu} \Gamma\left(\frac{\theta_{(t)}-\rho+2+\nu-j}{2}\right)} \frac{|V_{z(t)}|^{\frac{\nu}{2}}}{|\Lambda_{(t)}|^{\frac{\rho}{2}}} \tag{3.31}$$

$$\gamma_{(t)} = \frac{|\Lambda_{(t)}|^{\frac{\theta_{(t)}-\rho+\nu+1}{2}}}{2^{\frac{(\theta_{(t)}-\rho-\nu+1)\nu}{2}} \pi^{\frac{\nu(\nu-1)}{4}} \prod_{j=1}^{\nu} \Gamma\left(\frac{\theta_{(t)}-\rho+2+\nu-j}{2}\right)} \; . \tag{3.32}$$

Remark 3.1. In most practical applications the normalizing factors given by complicated formulae in (g) are not required to be known.

Remark 3.2. Notice that the c.p.d.f. (3.21) is actually a gaussian distribution for a vector, obtained by stacking the matrix P column by column, with covariance matrix $\Omega^{-1} \otimes V_{z(t)}^{-1}$ where $\otimes$ denotes the Kronecker product.

Remark 3.3. For a univariate case, when $\nu = 1$ and Ω is a scalar, the Wishart distribution (3.20) turns into Γ-distribution

$$p(\Omega|y^{(t)},u^{(t)}) = \gamma_{(t)}\Omega^{\frac{\theta_{(t)}-\rho}{2}} \exp\left\{-\frac{\Lambda_{(t)}}{2}\Omega\right\} \tag{3.33}$$

where $\Lambda_{(t)}$ defined by (3.19) is a scalar and

$$\gamma_{(t)} = \frac{\left(\frac{\Lambda_{(t)}}{2}\right)^{\left(\frac{\theta_{(t)}-\rho}{2}+1\right)}}{\Gamma\left(\frac{\theta_{(t)}-\rho}{2}+1\right)} \quad . \tag{3.34}$$

If the distribution for uncertain variance $\sigma^2 = E(e_{(t)}^2)$ is of interest, it can be obtained from (3.33) through simple transformation $\sigma^2 = \Omega^{-1}$ which gives

$$p(\sigma^2|y^{(t)},u^{(t)}) = \gamma_{(t)}(\sigma^2)^{-\left(\frac{\theta_{(t)}-\rho}{2}+2\right)} \exp\left\{-\frac{\Lambda_{(t)}}{2\sigma^2}\right\} \quad . \tag{3.35}$$

Remark 3.4. Notice that in the univariate case, when $\nu = 1$ and P is a ρ-vector, (3.22) turns into a ρ-dimensional Student distribution with $(\theta_{(t)}-\rho+2)$ degrees of freedom the mean of which is

$$E[P|y^{(t)},u^{(t)}] = \hat{P}_{(t)} \tag{3.36}$$

and the covariance matrix is

$$\mathrm{Cov}[P|y^{(t)},u^{(t)}] = \frac{\Lambda_{(t)}}{\theta_{(t)} - \rho} C_{(t)} \quad . \tag{3.37}$$

Remark 3.5. It is numerically advantageous to propagate the triangular Cholesky square root $C^{\frac{1}{2}}_{(t)}$,

$$C_{(t)} = C^{\frac{1}{2}}_{(t)}[C^{\frac{1}{2}}_{(t)}]^T \quad ,$$

instead of $C_{(t)}$ itself. See [3,4] for the algorithm and more detailed discussion of numerical aspects. It also may be of great numerical advantage to update directly the Cholesky square root of $V^{-1}_{(t)}$ which can be done by the same algorithm [4].

Theorem 3. If the assumptions of Theorem 1 are fulfilled the probability of the output $y_{(t)}$ given the past history of the process $\{y^{(t-1)}, u^{(t)}\}$ but not the parameters P and Ω is distributed according to the ν-dimensional Student distribution with $(\theta_{(t-1)}-\rho+2)$ degrees of freedom

$$p(y_{(t)}|y^{(t-1)},u^{(t)}) \tag{3.38}$$

$$= \kappa_{(t)}\left[1 + (y_{(t)}-\hat{P}^T_{(t-1)}z_{(t)})^T \frac{\Lambda^{-1}_{(t-1)}}{1+\xi_{(t)}} (y_{(t)} - \hat{P}^T_{(t-1)}z_{(t)})\right]^{-\frac{\theta_{(t-1)}-\rho+2+\nu}{2}}$$

where $P_{(t-1)}$, $\Lambda_{(t-1)}$ and $\xi_{(t)}$ are defined by (3.16), (3.19) and (3.26) and may be calculated recursively according to formulae (3.24) to (3.29). The normalizing factor of the Student c.p.d.f. (3.38) is

$$\kappa_{(t)} = \frac{\Gamma\left(\frac{\theta_{(t-1)}-\rho+2+\nu}{2}\right)}{\pi^{\frac{\nu}{2}}\,\Gamma\left(\frac{\theta_{(t-1)}-\rho+2}{2}\right)} \left|\frac{\Lambda^{-1}_{(t-1)}}{1+\xi_{(t)}}\right|^{\frac{1}{2}} \tag{3.38}$$

The proof of Theorem 3 is given in Appendix B.

4. Prior Information

In this Section the question will be discussed how the prior information about the possible values of the parameters of the multivariate regression model can be incorporated.

The regression model (3.3) defines the c.p.d.f. (3.12) only for $t > n$ and therefore the recursion according to Theorem 1 for $K = \{P,\Omega\}$ may start with $p(K|y^{(n)},u^{(n)})$. The self-reproducing form (3.13) of this c.p.d.f. has two parameters, matrix $V_{(n)}$ and scalar $\theta_{(n)}$, into which the prior information about the unknown parameters has to be inserted. The matrix $V_{(n)}$ itself determines three characteristics $\hat{P}_{(n)}$, $C_{(n)}$, $\Lambda_{(n)}$ and conversely for given $\hat{P}_{(n)}$, $C_{(n)}$ and $\Lambda_{(n)}$ the matrix $V_{(n)}$ can be composed according to (3.18) where

$$V_{z(n)} = C^{-1}_{(n)} \tag{4.1}$$

$$V_{zy(n)} = V_{z(n)}\hat{P}_{(n)} \tag{4.2}$$

$$V_{y(n)} = \Lambda_{(n)} + V^T_{zy(n)}C_{(n)}V_{zy(n)} = \Lambda_{(n)} + \hat{P}^T_{(n)}V_{z(n)}\hat{P}_{(n)} \; . \tag{4.3}$$

According to the part (b) of Theorem 2 $\hat{P}_{(n)}$ can be chosen directly as the most likely value of P. However, the remaining characteristics cannot be determined so easily.

First, we shall consider the case with single output, $\nu = 1$, when $\Lambda_{(n)}$ and $\Omega = 1/\sigma^2$ are scalars. We shall assume that the prior information is expressed through

$$\hat{P}_{(n)} = E[P] \quad , \qquad E[\sigma^2|P = \hat{P}_{(n)}]$$

and

$$\mathrm{Cov}[P] \quad , \qquad \mathrm{Var}[\sigma^2|P=\hat{P}_{(n)}] \; .$$

It is a lengthy, but easy exercise to prove that the following relations hold.

$$E[\sigma^2|P = \hat{P}_{(n)}] = \frac{1}{\hat{\Omega}_{(n)}} = \frac{\Lambda_{(n)}}{\theta_{(n)}} \tag{4.4}$$

$$\mathrm{Var}[\sigma^2|P = \hat{P}_{(n)}] = \frac{2\Lambda^2_{(n)}}{\theta^2_{(n)}(\theta_{(n)}-2)} = \frac{2}{(\theta_{(n)}-2)\hat{\Omega}^2_{(n)}} \tag{4.5}$$

$$\mathrm{Cov}[P] = \frac{\Lambda_{(n)}}{\theta_{(n)} - \rho} C_{(n)} \quad . \tag{4.6}$$

From these relations we get

$$\theta_{(n)} = 2\left[\frac{(E[\sigma^2|P=\hat{P}_{(n)}])^2}{\mathrm{Var}[\sigma^2|P=\hat{P}_{(n)}]} + 1\right] \tag{4.7}$$

$$\Lambda_{(n)} = \theta_{(n)} E[\sigma^2|P=\hat{P}_{(n)}] \tag{4.8}$$

$$C_{(n)} = \frac{\theta_{(n)}}{\Lambda_{(n)}} \mathrm{Cov}[P] \quad . \tag{4.9}$$

Notice that in the case when the "prior estimate" of σ^2, i.e., $E[\sigma^2|P=P_{(n)}]$, must be considered as very uncertain, i.e., when $\mathrm{Var}\ [\sigma^2|P=P_{(n)}] \to \infty$, the formula (4.7) gives a simple result $\theta_{(n)} = 2$.

Notice also that high uncertainty of the "prior estimate" $\hat{P}_{(n)}$ can be modelled by a diagonal matrix $C_{(n)}$ with large numbers on its diagonal.

The multivariate case, $\nu > 1$, can be handled as ν univariate regression models if the "prior estimate" of the covariance matrix R is assumed to be diagonal, however, the convariances of the "prior estimates" of R and P cannot be chosen independently in order to maintain the advantageous self-reproducing forms of c.p.d.f.'s given in Theorem 2.

5. Conclusion

The paper solves the problem of real-time identification of a multivariate stochastic process which is controlled in a closed feedback loop. It is assumed that the model of the process is known up to a finite set of parameters which are time-invariant but unknown. The introduction of the fairly general "natural conditions of control" (Definition 1) made it possible to derive the functional recursion for the evolution of the conditional subjective-probability density characterizing the uncertainty of the unknown parameters.

The practical use of the derived functional recursion may be considerably simplified when the self-reproducing forms of the conditional probability densities can be found for the particular type of the model. This has been done for the case of a multi-variate regression model (with an auto-regressive component). In this way the functional recursion was reduced to an algebraic recursion which is similar to recursive least squares.

APPENDIX A - PROOF OF THEOREM 2

In this appendix the assertions of Theorem 2 will be proved sequentially as they appear in the theorem.

Proof of assertion (a)

The denominator on the right-hand side of (2.4) does not depend on the variables $K = \{P,\Omega\}$. Hence, the following proportionality holds

$$p(P,\Omega|y^{(t)},u^{(t)}) \propto p(P,\Omega|y^{(t-1)},u^{(t-1)})p(y_{(t)}|P,\Omega,y^{(t-1)},u^{(t)}) \quad , \quad \text{(A.1)}$$

where $\propto$ means equality up to a factor not depending on the variables P and Ω. The second c.p.d.f. on the right-hand side of (A.1) is given by (3.11) and may be written in the following form

$$p(y_{(t)}|P,\Omega,y^{(t-1)},u^{(t)})$$

$$= (2\pi)^{-\frac{\nu}{2}}|\Omega|^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\,\mathrm{tr}\left(\Omega\begin{bmatrix}-I_\nu\\ P\end{bmatrix}^T\begin{bmatrix}y_{(t)}\\ z_{(t)}\end{bmatrix}\begin{bmatrix}y_{(t)}\\ z_{(t)}\end{bmatrix}^T\begin{bmatrix}-I_\nu\\ P\end{bmatrix}\right)\right\} \quad . \quad \text{(A.2)}$$

The substitution (A.2) and (3.13) into (A.1) proves that the c.p.d.f. (3.13) is self-reproducing and in the same time proves the recursions (3.14) and (3.15).

Proof of assertion (b)

It is easy to verify that the following sequence of equalities holds

$$\begin{bmatrix}-I_\nu\\ P\end{bmatrix}^T V_{(t)}\begin{bmatrix}-I_\nu\\ P\end{bmatrix} = \begin{bmatrix}-I_\nu\\ P\end{bmatrix}^T\begin{bmatrix}V_{y(t)} & V_{zy(t)}^T\\ V_{zy(t)} & V_{z(t)}\end{bmatrix}\begin{bmatrix}-I_\nu\\ P\end{bmatrix}$$

$$= V_{y(t)} - P^TV_{zy(t)} - V_{zy(t)}P + P^TV_{z(t)}P$$

$$= (P-\hat{P}_{(t)})^TV_{z(t)}(P-\hat{P}_{(t)}) + \Lambda_{(t)} \quad ,$$

where $\hat{P}_{(t)}$ and $\Lambda_{(t)}$ are matrices defined by (3.16) and (3.19). Using this rearrangement the c.p.d.f. (3.13) may be rewritten in the form

$$p(P,\Omega|y^{(t)},u^{(t)}) = \alpha_{(t)}|\Omega|^{\frac{\theta(t)}{2}} \exp\left\{-\frac{1}{2}\operatorname{tr}(\Omega\Lambda_{(t)})\right\} \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\left(\Omega[P-\hat{P}_{(t)}]^T V_{z(t)} [P-\hat{P}_{(t)}]\right)\right\} . \quad (A.3)$$

Only the last factor of this expression depends on P and its maximum -- equal to 1 as both Ω and $V_{z(t)}$ are positive semidefinite -- is reached by $P = \hat{P}_{(t)}$ for any $\Omega \in \mathcal{M}_\nu$. Thus, it remains to prove that $\Omega = \hat{\Omega}_{(t)}$, where $\hat{\Omega}_{(t)}$ is defined by (3.17), maximizes the remaining part of (A.3)

$$|\Omega|^{\frac{\theta_{(t)}}{2}} \exp\left\{-\frac{1}{2}\operatorname{tr}(\Omega\Lambda_{(t)})\right\} .$$

This proof is given in [1;§3.2].

Proof of assertion (c)

The marginal distribution of Ω given $\{y^{(t)},u^{(t)}\}$ can be obtained by integration

$$p(\Omega|y^{(t)},u^{(t)}) = \int p(P,\Omega|y^{(t)},u^{(t)})dP . \quad (A.4)$$

After the substitution of (A.3) into (A.4) we obtain

$$p(\Omega|y^{(t)},u^{(t)}) \propto |\Omega|^{\frac{\theta_{(t)}}{2}} \exp\left\{-\frac{1}{2}\operatorname{tr}(\Omega\Lambda_{(t)})\right\} \int \exp\left\{-\frac{1}{2}\operatorname{tr}\left(\Omega(P-\hat{P}_{(t)})^T V_{z(t)} (P-\hat{P}_{(t)})\right)\right\} dP . \quad (A.5)$$

As both Ω and $V_{z(t)}$ must be positive semidefinite they may be expressed as products

$$\Omega = (\Omega^{\frac{1}{2}})^T \Omega^{\frac{1}{2}} , \qquad V_{z(t)} = (V_{z(t)}^{\frac{1}{2}})^T V_{z(t)}^{\frac{1}{2}} ,$$

where $\Omega^{\frac{1}{2}}$ and $V_{z(t)}^{\frac{1}{2}}$ are real (not unique) matrices. The exponent in the integrand (A.5) can be written in the following form

$$\mathrm{tr}\left(\Omega(P - \hat{P}_{(t)})^T V_{z(t)} (P - \hat{P}_{(t)})\right) = \|V_{z(t)}^{\frac{1}{2}} (P - \hat{P}_{(t)})\Omega^{\frac{1}{2}}\|^2$$

where $\|\cdot\|^2$ denotes the square of the euclidean norm (sum of squares of all entries). Thus, using the substitution

$$X = V_{z(t)}^{\frac{1}{2}} (P - \hat{P}_{(t)})\Omega^{\frac{1}{2}} \tag{A.6}$$

the integral in (A.5) can be calculated as the product of integrals

$$\prod_{ij} \int \exp\left\{\frac{1}{2} X_{ij}^2\right\} dX_{ij} = (2\pi)^{\frac{\rho\nu}{2}} \quad . \tag{A.7}$$

However, the Jacobian of the transformation (A.6) has to be determined. In order to find this Jacobian let us decompose the transformation (A.6) into two steps.

$$Y = V_{z(t)}^{\frac{1}{2}} (P - \hat{P}_{(t)}) \tag{A.8}$$

$$X = Y\Omega^{\frac{1}{2}} \quad . \tag{A.9}$$

If $Y^{[j]}$ denotes the j^{th} column in Y, then

$$Y^{[j]} = V_{z(t)}^{\frac{1}{2}} (P^{[j]} - \hat{P}_{(t)}^{[j]})$$

and

$$dY^{[j]} = dY_{1j}\, dY_{2j} \ldots dY_{\rho j} = |V_{z(t)}^{\frac{1}{2}}|\, dP^{[j]} \quad .$$

But in the same way all ν columns of the matrix P are transformed and thus we have

$$dY = dY^{[1]} dY^{[2]} \ldots dY^{[\nu]} = |V_{z(t)}^{\frac{1}{2}}|^{\nu} dP = |V_{z(t)}|^{\frac{\nu}{2}} dP \quad .$$

If the second transformation (A.9) is written in the transposed form

$$X^T = (\Omega^{\frac{1}{2}})^T Y$$

we get into a similar situation,

$$dX = dX^T = |\Omega|^{\frac{\rho}{2}} dY = |\Omega|^{\frac{\rho}{2}} |V_{z(t)}|^{\frac{\nu}{2}} dP \quad . \qquad (A.10)$$

Using (A.10) and (A.7) the integral in (A.5) may be expressed as follows:

$$\int \exp\left\{-\frac{1}{2}\,\mathrm{tr}\left(\Omega[P-\hat{P}_{(t)}]^T V_{z(t)} [P-\hat{P}_{(t)}]^T\right)\right\} dP = (2\pi)^{\frac{\rho\nu}{2}} |\Omega|^{-\frac{\rho}{2}} |V_{z(t)}|^{-\frac{\nu}{2}} \quad . \qquad (A.11)$$

The substitution of this integral into (A.5) completes the proof. The formula (3.32) for the normalizing factor $\gamma_{(t)}$ of the Wishart distribution may be found e.g. in [1] or [6].

Proof of assertion (d)

Obviously, the following relation holds

$$p(P|\Omega, y^{(t)}, u^{(t)}) = \frac{p(P,\Omega|y^{(t)}, u^{(t)})}{p(\Omega|y^{(t)}, u^{(t)})} \quad . \qquad (A.12)$$

Substitution of (A.3) and (3.20) into (A.12) gives

$$p(P|\Omega, y^{(t)}, u^{(t)})$$

$$= \frac{\alpha_{(t)}}{\gamma_{(t)}} |\Omega|^{\frac{\rho}{2}} \exp\{-\frac{1}{2}\,\mathrm{tr}\,\Omega[P-\hat{P}_{(t)}]^T V_{z(t)} [P-\hat{P}_{(t)}]\} \qquad (A.13)$$

Considering the integral (A.11) we can see that

$$\frac{\alpha_{(t)}}{\gamma_{(t)}} (2\pi)^{\frac{\rho\nu}{2}} |V_{z(t)}|^{-\frac{\nu}{2}} = 1 \quad .$$

Thus, we have proved (3.21) and the formula (3.30) for $\alpha_{(t)}$ at the same time.

Proof of assertion (e)

To prove assertion (e) of Theorem 2 we have to perform the integration

$$p(P|y^{(t)},u^{(t)})$$

$$= \int p(P,\Omega|y^{(t)})d\Omega \tag{A.14}$$

$$= \alpha_{(t)} \int |\Omega|^{\frac{\theta_{(t)}}{2}} \exp\left\{-\frac{1}{2}\operatorname{tr}(\Omega[P - \hat{P}_{(t)}]^T V_{z(t)} [P - \hat{P}_{(t)}] + \Lambda_{(t)})\right\} d\Omega \quad ,$$

where the integral is taken over all set $\mathcal{M}_\nu$ of positive definite matrices of dimension $\nu\times\nu$. For convenience, we can make use of the fact that the normalizing factor $\gamma_{(t)}$ of the Wishart distribution (3.20) is known [1], [6]. Hence we have

$$\int |\Omega|^{\frac{\theta_{(t)}-\rho}{2}} \exp\left\{-\frac{1}{2}\operatorname{tr}(\Omega\Lambda_{(t)})\right\} d\Omega = \gamma_{(t)}^{-1} \quad .$$

But this integral is of the same type as (A.14). Thus, the integral in (A.14) can be evaluated simply by the replacement of $\theta_{(t)}$ instead of $\theta_{(t)} - \rho$ and $([P - \hat{P}_{(t)}]^T V_{z(t)}] + \Lambda_{(t)})$ instead of $\Lambda_{(t)}$ in the formula (3.32) for $\rho_{(t)}$. In this way the c.p.d.f. (3.22) is derived and at the same time the formula (3.31) for the normalizing factor $\beta_{(t)}$ is proved.

Proof of assertion (f)

The derivation of the recursion relations for the characteristics $\hat{P}_{(t)}$, $C_{(t)}$ and $\Lambda_{(t)}$ may be found in [3].

The formulae for the normalizing factors given in assertion (e) have been already proved.

APPENDIX B - PROOF OF THEOREM 3

The c.p.d.f. (3.3) might be derived in a straightforward way by integration (2.5). However, it is more convenient to exploit the relation

$$p(y_{(t)}|y^{(t-1)},u^{(t)}) = \frac{p(y_{(t)}|P,\Omega,y^{(t-1)},u^{(t)})\,p(P,\Omega|y^{(t-1)},u^{(t-1)})}{p(P,\Omega|y^{(t)},u^{(t)})} , \quad \text{(B.1)}$$

which follows from (2.4). By substitution of (3.13) and (A.2) into (B.1) and considering the relations (3.14) and (3.15) we obtain

$$p(y_{(t)}|y^{(t-1)},u^{(t)}) = (2\pi)^{-\frac{\nu}{2}} \frac{\alpha_{(t-1)}}{\alpha_{(t)}} .$$

Making use of formulae (3.30) and (3.14) we have

$$p(y_{(t)}|y^{(t-1)},u^{(t)})$$
$$= \pi^{-\frac{\nu}{2}} \frac{\Gamma\left(\frac{\theta_{(t-1)} - \rho + 2 + \nu}{2}\right)}{\Gamma\left(\frac{\theta_{(t-1)} - \rho + 2}{2}\right)} \frac{\left|V_{z(t-1)}\right|^{\frac{\nu}{2}}}{\left|V_{z(t)}\right|^{\frac{\nu}{2}}} \frac{\left|\Lambda_{(t-1)}\right|^{\frac{\theta_{(t-1)}-\rho+1+\nu}{2}}}{\left|\Lambda_{(t)}\right|^{\frac{\theta_{(t-1)}-\rho+2+\nu}{2}}} . \quad \text{(B.2)}$$

The determinant $\left|V_{z(t)}\right|$ in (B.2) may be expressed in the following way

$$\left|V_{z(t)}\right| = \left|V_{z(t-1)} + z_{(t)}z^T_{(t)}\right| = \left|V_{z(t-1)}\right|\left(1 + z^T_{(t)}V^{-1}_{z(t-1)}z_{(t)}\right) .$$

Making use of (3.23) and (3.26) we have

$$\left|V_{z(t)}\right| = \left|V_{z(t-1)}\right|(1+\xi_{(t)}) . \quad \text{(B.3)}$$

Similarily for the determinant $\left|\Lambda_{(t)}\right|$ we obtain from (3.28)

$$\left|\Lambda_{(t)}\right| = \left|\Lambda_{(t-1)}\right| \left(1 + \hat{e}^{T}_{(t)} \frac{\Lambda^{-1}_{(t-1)}}{1 + \xi_{(t)}} \hat{e}_{(t)}\right) . \tag{B.4}$$

Substitution of (B.3) and (B.4) into (B.2) completes the proof.

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