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Polynomials of First and Second Degree

The general noncommutativity of quadriforms forces one to change his interpretation of roots of a polynomial. Both a linear and a quadratic equation make this clear. For example, one must distinguish between

$$ax = b$$

and

$$xa = b \quad .$$

If $|a|^2 > 0$, then the solutions to these are

$$x = a^{-1}b$$

and

$$x = ba^{-1}$$

respectively.

Turning to a quadratic, the situation quickly becomes considerably more complicated. The familiar formula for the solution to

$$ax^2 + bx + c = 0$$

depends heavily on commutativity, in particular, that $bx = xb$. Hence it does not carry over to general quadriforms. First, let us get rid of the leading coefficient and confine our attention to the simplified form

$$x^2 + bx + c = 0 \quad .$$

We expect, in general, two roots, say u and v. Then the equation may be written

$$(x-u)(x-v) = 0 \quad .$$

Expanding this, we get

$$x^2 - (ux + xv) + uv = 0 \quad .$$

Thus it appears that one is a "left root" and the other a "right root" which doesn't make much sense since how would either one satisfy the equation alone.

Suppose we interpret the linear term in simplified form as

$$\frac{1}{2}(bx + xb) \quad .$$

If we do this, then the usual quadratic formula is valid since one can "complete the square". Note that

$$(x + \frac{1}{2})^2 = x^2 + \frac{1}{2}(bx + xb) + \frac{1}{4}b^2 \quad .$$

Using this,

$$x^2 + \frac{1}{2}(bx + xb) + c = 0$$

becomes

$$(x + \frac{1}{2}b) = \frac{1}{4}b^2 - c$$

or

$$x = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c}) \quad .$$

Substituting this back in the quadratic will prove its validity. Using the x derived:

$$x^2 = \frac{1}{4}(b^2 + b^2 - 4c \mp (b\sqrt{b^2 - 4c} + \sqrt{b^2 - 4c}\,b))$$

$$\frac{1}{2}bx = \frac{1}{4}(-b^2 \pm b\sqrt{b^2 - 4c})$$

$$\frac{1}{2}xb = \frac{1}{4}(-b^2 \pm \sqrt{b^2 - 4c}\,b) \quad .$$

Adding

$$x^2 + \frac{1}{2}(bx + xb) = \frac{1}{4}(-4c) = -c$$

which verifies the formula for either choice of sign. Neither need be considered right or left. However, c is not their product but $\frac{1}{2}$ the sum of their products both ways.

An example may clarify the situation. Let

$$b = (3,1,0,1) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$c = (1,1,1,0) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Then $|b|^2 = 7$, $|c|^2 = 1$ and

$$b^2 = \begin{bmatrix} 17 & 6 \\ 6 & 5 \end{bmatrix}, \qquad b^2-4c = \begin{bmatrix} 9 & 10 \\ 2 & 5 \end{bmatrix} = (7,2,-4,6) \quad .$$

Let $w = \sqrt{b^2-4c}$. Then $w_0^2 = \frac{1}{2}(7+5) = 6$, which is allowable. (Note that $|b^2-4c|^2 = 25$). Then

$$w = \frac{1}{\sqrt{6}}(6,1,-2,3) \quad , \quad |w|^2 = 5 \quad .$$

In matrix form,

$$w = \frac{1}{\sqrt{6}} \begin{bmatrix} 7 & 5 \\ 1 & 5 \end{bmatrix}$$

$$x = \frac{1}{2}\left(\begin{bmatrix} -4 & -1 \\ -1 & -2 \end{bmatrix} \pm \frac{1}{\sqrt{6}} \begin{bmatrix} 7 & 5 \\ 1 & 5 \end{bmatrix} \right)$$

Let x_1 be the value with + and x_2 the value with -. Then

$$x_1 x_2 = \begin{bmatrix} 2-\frac{1}{\sqrt{6}} & -1+\frac{2}{\sqrt{6}} \\ 1 & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$x_2x_1 = \begin{bmatrix} 2+\frac{1}{\sqrt{6}} & -1-\frac{2}{\sqrt{6}} \\ 1 & +\frac{1}{\sqrt{6}} \end{bmatrix}$$

and

$$\tfrac{1}{2}(x_1x_2 + x_2x_1) = c$$

For either value

$$x^2 = \tfrac{1}{4}(b^2 + w^2 \mp (bw + wb))$$

$$= \begin{bmatrix} \frac{13}{2} & 4 \\ 2 & \frac{5}{2} \end{bmatrix} \mp \tfrac{1}{4}(bw + wb)$$

$$\tfrac{1}{2}bx = \tfrac{1}{4}(-b^2 \pm bw)$$

$$= \begin{bmatrix} -\frac{17}{4} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{5}{4} \end{bmatrix} \pm \tfrac{1}{4}bw$$

$$\tfrac{1}{2}xb = \tfrac{1}{4}(-b^2 \pm wb)$$

$$= \begin{bmatrix} -\frac{17}{4} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{5}{4} \end{bmatrix} \pm \tfrac{1}{4}wb$$

Hence

$$x^2 + \tfrac{1}{2}(bx + xb) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} = -c$$

Therefore, at least for this example,

$$x^2 + \tfrac{1}{2}(bx + xb) = -\tfrac{1}{2}(x_1x_2 + x_2x_1) = -c$$

We developed the outer members algebraically as well as arithmetically. (In fact, we had already done so previously). How does it happen that c is also a half-sum of transposed products? One can see this from the quadratic formula. We have

$$x_1 = \frac{1}{2}(-b + w) \quad , \qquad x_2 = \frac{1}{2}(-b - w) \quad .$$

so,

$$\begin{aligned} x_1x_2 &= \frac{1}{4}(b^2 + bw - wb - w^2) \\ x_2x_1 &= \frac{1}{4}(b^2 - bw + wb - w^2) \\ \frac{1}{2}(x_1x_2 + x_2x_1) &= \frac{1}{4}(b^2 - w^2) \\ &= \frac{1}{4}(b^2 - b^2 + 4c) = c \end{aligned}$$

Now suppose we have two roots x_1, x_2 and wish to construct the standard quadratic form. We already know that

$$c = \frac{1}{2}(x_1x_2 + x_2x_1) \quad .$$

How shall we compute b? Again referring to the quadratic formula for x,

$$\begin{aligned} x_1 &= \frac{1}{2}(-b + w) \\ x_2 &= \frac{1}{2}(-b - w) \end{aligned}$$

so

$$b = -(x_1 + x_2)$$

To verify,

$$x^2 + \frac{1}{2}(bx + xb) + c = x^2 - \frac{1}{2}((x_1 + x_2)x + x(x_1 + x_2)) + \frac{1}{2}(x_1x_2 + x_2x_1)$$

Letting $x = x_1$ or $x = x_2$ clearly reduces this to zero. Therefore, b is computed just as with straight real or complex numbers, since it is a sum, but is interpreted differently; c is computed differently but reduces to a straight product if x_1 and x_2 commute. Since, of course, it is still true that

$$(x - x_1)(x - x_2) = x^2 - (x_1x + xx_2) + x_1x_2 = 0 \quad ,$$

this form is valid but there are two linear coefficients. The following is also true:

$$(x - x_2)(x - x_1) = x^2 - (x_2x + xx_1) + x_2x_1 = 0 \quad .$$

The standard form derived before is simply half the sum of these two. Hence

$$x^2 + \frac{1}{2}(bx + xb) + c = \frac{1}{2}((x - x_1)(x - x_2) + (x - x_2)(x - x_1)) \quad .$$

It is clear that these half sums of products in opposite order need a name; hence the following definition.

The Arithmetic Mean Product, or AMP, of two quadriform numbers x and y is denoted by x § y and has the value

$$x \S y = \frac{1}{2}(xy + yx) \quad .$$

One may still ask whether the equation

$$x^2 + bx + c = 0$$

has roots. The only answer one can give is that it certainly may but it is a very tedious job to find them. To clearly distinguish cases, we will consider the standard form of quadratic to be

$$x^2 + b \S x + c = 0 \quad .$$

However, given the more general form with a coefficient of x^2, further analysis is required. To begin with, if x^2 has a coefficient, say a, there is no more reason for a and x^2 to commute than for b and x. Hence, logically, the equation should have the form

$$a \S x^2 + b \S x + c = 0 \quad .$$

However, with this definition, there is no way to keep terms in x segregated into right and left products, that is, x becomes trapped. This is avoided with the definition

$$a \S x^2 + a \S (b \S x) + a \S c = 0 \qquad (1)$$

or, setting

$$f(x) = x^2 + b \S x + c \quad ,$$

then the above becomes

$$a \S f(x) = 0 \quad .$$

Unfortunately, this seems unnatural and is not the way coefficients usually arise. Still, it is compatible with pure complex or real numbers, for which the above form reduces to

$$ax^2 + abx + ac = 0$$

or, setting $\bar{b} = ab$, $\bar{c} = ac$,

$$ax^2 + \bar{b}x + \bar{c} = 0$$

from which a can be factored out. Before declaring (1) the standard, general form of a quadratic equation, we need to check the expansion in terms of roots x_1 and x_2. We can begin by ignoring a and using AMP for the factors instead of straight multiplication. The following expansions then occur:

$$(x - x_1)\S(x - x_2) = \frac{1}{2}((x - x_1)(x - x_2) + (x - x_2)(x - x_1))$$

$$= \frac{1}{2}[(x^2 - xx_2 - x_1x + x_1x_2) + (x^2 - xx_1 - x_2x + x_2x_1)]$$

$$= x^2 - \frac{1}{2}((x_1 + x_2)x + x(x_1 + x_2)) + \frac{1}{2}(x_1x_2 + x_2x_1)$$

$$= x^2 - (x_1 + x_2)\S x + x_1 \S x_2 = 0$$

Now setting $b = -(x_1 + x_2)$, $c = x_1 \S x_2$, the next to last line above becomes

$$x^2 + \frac{1}{2}(bx + xb) + c = 0 \quad .$$

Taking the AMP of this with a,

$$a \S x^2 + \frac{1}{2}[\frac{1}{2}(a(bx + xb) + (bx + xb)a)] + a \S c = 0 \quad .$$

It is clear that, if all numbers commute, this reduces to

$$ax^2 + abx + ac = 0 \quad .$$

Therefore, we take (1) to be the standard general form and the following to be the factor form:

$$a \S [(x - x_1) \S (x - x_2)] = 0 \quad . \tag{2}$$

Notice that (2) is entirely analogous to familiar factor forms with AMP replacing multiplication.

We may now also define an AMP form of linear equation, as follows:

$$a \S x = b$$

or

$$\frac{1}{2}(ax + xa) = b$$

Incredible as it seems, this is not solvable with any simple operations. In fact, one must solve a 4×4 system of linear equations. Since this operation is generally needed, we proceed to define and describe it.

In [1], the products u v and v u were compared. Setting u = a and v = x and taking half the sum, one gets; for components of b:

$$\begin{aligned}
a_0x_0 + a_1x_1 - a_2x_2 + a_3x_3 &= b_0\\
a_1x_0 + a_0x_1 \qquad\qquad\qquad &= b_1\\
a_2x_0 \qquad\quad + a_0x_2 \qquad\quad &= b_2\\
a_3x_0 \qquad\qquad\qquad + a_0x_3 &= b_3
\end{aligned}$$

Putting this in matrix-vector form (in real numbers):

$$\begin{array}{c}
\{x_0 \quad x_1 \quad x_2 \quad x_3\}\\
\begin{bmatrix} a_0 & a_1 & -a_2 & a_3 \\ a_1 & a_0 & & \\ a_2 & & a_0 & \\ a_3 & & & a_0 \end{bmatrix}
\end{array}
= \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The determinant is $a_0^2 \cdot |a|^2$. Hence, for a solution, a must be allowable, nonsingular, and here a nonzero leading component. If these conditions are met, we can multiply the last three rows by a_1/a_0, $-a_2/a_0$, a_3/a_0, respectively, and subtract them from the top row. This gives:

$$\begin{array}{c}
\{x_0 \quad x_1 \quad x_2 \quad x_3\}\\
\begin{bmatrix} \frac{|a|^2}{a_0} & 0 & 0 & 0 \\ a_1 & a_0 & & \\ a_2 & & a_0 & \\ a_3 & & & a_0 \end{bmatrix}
\end{array}
= \begin{bmatrix} \bar{b}_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where

$$\bar{b}_0 = \frac{1}{a_0}(a_0 b_0 - a_1 b_1 + a_2 b_2 - a_3 b_3)$$

We can now read off the values of x:

$$x_0 = \frac{a_0 \bar{b}_0}{|a|^2} = \frac{a*b}{|a|^2} \qquad \text{(see below)}$$

$$x_1 = \frac{b_1 - a_1 x_0}{a_0}$$

$$x_2 = \frac{b_2 - a_2 x_0}{a_0}$$

$$x_3 = \frac{b_3 - a_3 x_0}{a_0}$$

We call this operation the <u>extraction of b by a</u> and denote it by

$$x = b/a$$

It is the analogue of division corresponding to AMP. We never use this notation for multiplication by a reciprocal which is either $a^{-1}b$ or ba^{-1}.

It should be noted that, if a § b = c, then c/a = b may be possible while c/b is not defined.

It is also useful to have a name for the quantity $a_0\bar{b}_0$ above. We call this the <u>cross value</u> and denote it by

$$a * b = a_0 b_0 - a_1 b_1 + a_2 b_2 - a_3 b_3 \quad .$$

Hence,

$$a * a = |a|^2 \; .$$

In the complex subset, this is equivalent to multiplication of a complex number by its conjugate. We may also write the linear AMP equation in terms of its root, say x_1. Then

$$a \; § \; (x - x_1) = 0$$

or

$$a \; § \; x = a \; § \; x_1 = b \quad .$$

To illustrate the solution x = b/a to the equation a § x = b, let

$$a = (3,1,0,1) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$b = (4,4,3,1) = \begin{bmatrix} 8 & -2 \\ 4 & 0 \end{bmatrix}$$

Then

$$|a|^2 = 7 \quad , \qquad a * b = 7 \quad .$$

Hence

$$x_0 = \frac{a*b}{|a|^2} = 1$$

$$x_1 = \frac{4-1}{3} = 1$$

$$x_2 = \frac{3-0}{3} = 1$$

$$x_3 = \frac{1-1}{3} = 0$$

or

$$x = (1,1,1,0) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Then, to verify,

$$ax = \begin{bmatrix} 9 & -4 \\ 4 & -1 \end{bmatrix}$$

$$xa = \begin{bmatrix} 7 & 0 \\ 4 & 1 \end{bmatrix}$$

$$\frac{1}{2}(ax + xa) = \begin{bmatrix} 8 & -2 \\ 4 & 0 \end{bmatrix} = b \quad .$$

Cubic Equations

Suppose we have a cubic equation in factor form with leading coefficient of unity. Taking a clue from quadratics, we tentatively write this as

$$(x - x_1) \ \S\ (x - x_2) \ \S\ (x - x_3) = 0 \quad .$$

We first need to ascertain the meaning of two AMPs which are not nested. Let a,b,c be any three quadriforms. Then, one interpretation is:

$$\begin{aligned} a \ \S\ b \ \S\ c &= \tfrac{1}{2}(ab + ba) \ \S\ c \\ &= \tfrac{1}{4}(abc + bac + cab + cba) \quad . \end{aligned}$$

If we multiply the last pair first,

$$\begin{aligned} a \ \S\ b \ \S\ c &= a \ \S\ \tfrac{1}{2}(bc + cb) \\ &= \tfrac{1}{4}(abc + acb + bca + cba) \quad . \end{aligned}$$

These are not the same since, of the six permutations of a,b,c only abc and cba appear in both. In other words AMPs are not associative. On the other hand if we attempt to nest the AMPs, then we must give some preference to the numbers. None of these situations is satisfactory.

If we write the general equation in fully nested form, we get

$$x^3 + b \ \S\ (x^2 + c \ \S\ (x + d)) = 0$$

or

$$x^3 + b \ \S\ (x^2 + c \ \S\ x + c \ \S\ d) = 0$$

which looks like x^3 followed by a full quadratic, that is, factoring the quadratic for two roots, x_1 and x_2,

$$x^3 + b \ \S\ [(x - x_1) \ \S\ (x - x_2)] = 0$$

or

$$x^3 + b \ \S\ [x^2 - (x_1 + x_2) \ \S\ x + x_1 \ \S\ x_2] = 0 \quad .$$

Then

$$c = -(x_1 + x_2) \quad , \qquad c \ \S\ d = x_1 \ \S\ x_2$$

or

$$d = (x_1 \S x_2)/c \quad .$$

However, there are two troubles with this. First, we have selected two roots for special treatment; second, the x^3 term seems unconnected and, in any event, we would have to arbitrarily assign an order of multiplication of the roots.

If we use straight multiplication of the factors, we get a form which satisfies each root. Thus, after expanding and collecting terms,

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

becomes

$$x^3 - (x_1x^2 + xx_2x + x^2x_3) + (x_1x_2x + x_1xx_3 + xx_2x_3) - x_1x_2x_3 = 0 \quad .$$

It is readily seen that x_1, x_2 and x_3 all satisfy this equation. But it is also obvious that the roots have received arbitrary treatment. Hence we conclude that there is no practical and viable form for general polynomials of degree higher than the second.

Cubic and Higher, and Fractional, Roots

Even though there seems no way to handle cubic and higher order equations, it is desirable to have some method for finding cube roots and roots of higher order, and also fractional roots. In real arithmetic, the use of logorithms is the most practical way. In [1], it was shown that logorithms (i.e., the inverse of a generalization of exponentiation) are not additive in general. The difficulty essentially reduces to the following observation. Any generalization of ln z will involve, in some manner, $\sinh^{-1} x$ which is not periodic as is $\sin^{-1} x$. In fact, powers of sinh x involve expressions in binomial coefficients as shown below. (Fractions are not reduced so the binomial coefficients stand out).

$(\sinh x)^n =:$

$$
\begin{aligned}
n = 2{:}\quad & \tfrac{1}{2}\cosh 2x - \tfrac{2}{4} \\
3{:}\quad & \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x \\
4{:}\quad & \tfrac{1}{8}\cosh 4x - \tfrac{4}{8}\cosh 2x + \tfrac{6}{16} \\
5{:}\quad & \tfrac{1}{16}\cosh 5x - \tfrac{5}{16}\sinh 3x + \tfrac{10}{16}\sinh x \\
6{:}\quad & \tfrac{1}{32}\cosh 6x - \tfrac{6}{32}\cosh 4x + \tfrac{15}{32}\cosh 2x - \tfrac{20}{64} \\
7{:}\quad & \tfrac{1}{64}\sinh 7x - \tfrac{7}{64}\sinh 5x + \tfrac{21}{64}\sinh 3x - \tfrac{35}{64}\sinh x \\
8{:}\quad & \tfrac{1}{128}\cosh 8x - \tfrac{8}{128}\cosh 6x + \tfrac{28}{128}\cosh 4x - \tfrac{56}{128} \\
& \cosh 2x + \tfrac{70}{256}
\end{aligned}
$$

Clearly, this is intractable for a basic computational tool.

We first note that there are some special numbers which have very simple powers. For example, if

$$v_0 = 1 \quad , \qquad v_2^2 - v_1^2 - v_3^2 = 0 \quad ,$$

then

$$v^2 = (1,\ 2v_1,\ 2v_2,\ 2v_3)$$
$$v^3 = (1,\ 3v_1,\ 3v_2,\ 3v_3)$$

Examples are:

$$v = (1,3,5,4) \qquad v^2 = (1,6,10,8) \quad , \quad v^3 = (1,9,15,12),\ldots$$
$$v = (1,5,13,12) \quad , \quad v^2 = (1,10,26,24) \quad , \; v^3 = (1,15,39,36),\ldots$$

In fact, more generally, if $v_2^2 - v_1^2 - v_3^2 = 0$ and $v_0 \neq 0$, then

$$v^n = v_0^{n-1}(v_0,\ nv_1,\ nv_2,\ nv_3) \quad .$$

(Note that signs alternate by n if $v_0 < 0$). Curiously, this includes the case of pure reals but not pure complex. Suppose $w = v^n$ and $w_2^2 - w_1^2 - w_3^2 = 0$, with $w_0 \neq 0$ and positive if n is even. Then,

$$w_2^2 - w_1^2 - w_3^2 = n^2(v_2^2 - v_1^2 - v_3^2) = 0 \quad .$$

Hence

$$v_0 = \sqrt[n]{w_0}$$

and

$$v_p = \frac{w_p}{v_0^{n-1} n} \quad , \qquad p = 1,2,3.$$

Notice that only real n-th roots of w_0 may be used for the general case. Complex roots can be given a valid representation if w is pure real, but otherwise non-commutative factors occur. For example, the three cube roots of -1 are:

$$(-1,\ 0,\ 0,\ 0) \quad , \quad (\tfrac{1}{2},\ 0,\ \tfrac{1}{2}\sqrt{3},\ 0) \quad , \quad (\tfrac{1}{2},\ 0,\ -\tfrac{1}{2}\sqrt{3},\ 0) \quad .$$

The three cube roots of unity are formed by changing the sign of the first component of the above. These are all allowable numbers; unallowable roots are in addition to these.

If w is pure real or pure complex, then real or complex logs and exponentials can be used and the results written in quadriform format. Hence let us assume that not both w_1 and w_3 are zero and that $w_2^2 - w_1^2 - w_3^2 \neq 0$ since the opposite situation was covered above. Now if w has a square root and is not singular, i.e.,

$$w_0 + |w| > 0$$

then one of its square roots has a square root but not necessarily both. If $v = \sqrt{w}$, then

$$v_0 = \frac{1}{\sqrt{2}} \sqrt{w_0 + |w|}$$

$$v_p = \frac{w_p}{2v_0} \quad , \qquad p = 1,2,3 \quad .$$

Hence,

$$|v|^2 = \frac{1}{2}(w_0 + |w|) + \frac{1}{4v_0^2} (-w_1^2 + w_2^2 - w_3^2)$$

and

$$4v_0^2|v|^2 = 2(|w|^2 + w_0|w|) = 2|w|(w_0 + |w|) > 0 \quad .$$

Thus for the positive choice for v_0, $v_0 + |v| > 0$, but not necessarily for the negative choice. Hence w has at least two fourth roots. Then it also has at least two eighth roots, and so on.

We will now restrict our attention to cube roots. Suppose w as above and $v^3 = w$. If one cubes a general v, the following formulas are obtained:

$$v_0(v_0^2 + 3v_1^3 - 3v_2^2 + 3v_3^2) = w_0$$

$$v_1(3v_0^2 + v_1^2 - v_2^2 + v_3^2) = w_1$$

$$v_2(3v_0^2 + v_1^2 - v_2^2 + v_3^2) = w_2$$

$$v_3(3v_0^2 + v_1^2 - v_2^2 + v_3^2) = w_3 \quad .$$

Let $D = 3v_0^2 + v_1^2 - v_2^2 + v_3^2$. The following two lemmas are useful.

Lemma 1: For at least one root v,

$$v_1 = v_2 = v_3 = 0 \Leftrightarrow w_1 = w_2 = w_3 = 0 \quad .$$

Proof: If $w_1 = w_2 = w_3 = 0$, w is real and, for at least one root,

$$v = (\sqrt[3]{w_0}, 0, 0, 0) \quad .$$

If $v_1 = v_2 = v_3 = 0$, clearly $w_1 = w_2 = w_3 = 0$ from the above formulas.

Lemma 2: $D = 0$ for any root $\Rightarrow w$ is real.

Proof: Suppose $D = 0$ for any root. Then $w_1 = w_2 = w_3 = 0$.

Therefore, if w is not real, $D \neq 0$ and

$$v_p = \frac{w_p}{D} \quad , \qquad p = 1,2,3 \quad .$$

In other words, the last three components of the root are directly proportional to the corresponding components of w. We also have

$$v_0(3D - 8v_0^2) = w_0$$

so $w_0 \neq 0 \Rightarrow v_0 \neq 0$, $w_0 = 0 \Rightarrow$ either $v_0 = 0$ or

$$3D = 8v_0^2 \quad .$$

Let us take these by cases. We assume $|w|^2 > 0$.

$w_0 = 0,\ v_0 = 0$ Then $D = -|v|^2$ or $-D = |w|^{2/3}$.

We can thus compute v_p from w_p.

The above case gives only one root for $w_0 = 0$. The other two corresponding roots are given by the next case.

$w_0 = 0,\ v_0 \neq 0$ Then $v_0(4v_0^2 - 3|v|^2) = w_0$. But $|v|^2 = |w|^{2/3}$.

Hence, since $w_0 = 0$ and $v_0 \neq 0$,

$$v_0^2 = \frac{3}{4}|w|^{2/3} > 0$$

Then

$$D = 4v_0^2 - |v|^2 = 2|w|^{2/3} > 0$$

and we can compute v_p from w_p for $p \neq 0$. Note that there are two values for v_0 but only one each for v_p, $p \neq 0$. This gives the other two roots corresponding to the first case.

$w_0 \neq 0$ Then $v_0 \neq 0$ but

$$4v_0^3 - 3|w|^{2/3}\ v_0 - w_0 = 0 \quad .$$

Let $a = -\frac{3}{4}|w|^{2/3}$, $b = -\frac{1}{4}w_0$. Then we have the reduced cubic equation

$$v_0^3 + av_0 + b = 0 \quad .$$

The discriminant for this is

$$\frac{b^2}{4} + \frac{a^3}{27} = \frac{1}{64}\ (w_0^2 - |w|^2) \quad .$$

Thus we have the following three subcases for the third case:

(a) $w_0^2 > |w|^2$. There is only one real root for v_0 and hence only one root for w. It never occurs for a pure complex number.

(b) $w_0^2 = |w|^2$. There are three real roots but at least two are equal. Note that this is the case $w_2^2 - w_1^2 - w_3^2 = 0$. It never occurs for a pure complex (nonreal) number.

(c) $w_0^2 < |w|^2$. There are three unequal real roots.

The following example illustrates subcase (a).

$$w = (5,3,7,8) \quad , \qquad |w|^2 = |w| = |w|^{2/3} = 1 \quad .$$

$$w_0^2 = 25 > |w|^2 \quad .$$

Let

$$a = -\frac{3}{4} \quad , \qquad b = -\frac{1}{4}w_0 = -\frac{5}{4}$$

and we must solve

$$v_0^3 - \frac{3}{4}v_0 - \frac{5}{4} = 0 \quad .$$

The discriminant is

$$\frac{25}{64} - \frac{1}{64} = \frac{3}{8}$$

The real root is A+B where

$$A = \sqrt[3]{\frac{5}{8} + \sqrt{\frac{3}{8}}} \quad , \qquad B = \sqrt[3]{\frac{5}{8} - \sqrt{\frac{3}{8}}} \quad .$$

$$\sqrt{\frac{3}{8}} = .61238$$

$$A^3 = 1.23738 \quad , \qquad B^3 = .01262$$

$$A = 1.0736 \quad , \qquad B = .23282$$

$$v_0 = 1.3064 \quad .$$

Now

$$D = 4v_0^2 - |w|^{2/3} = 6.827181988-1 = 5.827181988$$

and

$$v_1 = \frac{3}{5.8267} = .514828521$$

$$v_2 = \frac{7}{5.8267} = 1.201266549$$

$$v_3 = \frac{8}{5.8267} = 1.372876056$$

Subcase (b) is best handled by the method discussed previously, based on $w_2^2 - w_1^2 - w_3^2 = 0$. For example:

$$w = (1,4,5,3) = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}$$

$$w_0^2 = |w|^2 = 1$$

$$v = (1, \frac{4}{3}, \frac{5}{3}, 1) = \frac{1}{3}\begin{bmatrix} 7 & -2 \\ 8 & -1 \end{bmatrix}$$

$$v^3 = \frac{1}{27}\begin{bmatrix} 135 & -54 \\ 216 & -81 \end{bmatrix} = w$$

Thus v is the value of one of the roots. It would seem there should be two other equal roots, and, if they have to be found, the method of subcase (a) should be applied. Note that these two equal roots would correspond to complex values for which the imaginary part vanishes since A = B. To try to find the other root(s) here we can proceed as follows. Let

$$a = -\frac{3}{4} \quad , \qquad b = -\frac{1}{4}$$

$$A = B = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$$

Then A+B corresponds to the root already found. The other has

$$v_0 = -\frac{A+B}{2} = -\frac{1}{2} \quad .$$

Then

$$D = 4(v_0)^2 - 1 = 0 \quad .$$

Since this violates Lemma 2, it cannot correspond to roots of w. The difficulty arises from our evaluation of $|w|^{2/3}$ which has complex roots. However, they cannot be used in a meaningful way and hence there is only one valid root of w if the discriminant is nonnegative.

For subcase (c), the usual trigonometric solution for v_0 can be applied. Consider, for instance, the following.

$$w = (3,4,5,1) \quad , \qquad |w|^2 = 17 \quad .$$

Let

$$a = -\frac{3}{4}\sqrt[3]{17} \quad , \qquad b = -\frac{3}{4} \quad .$$

Then

$$\frac{b^2}{4} + \frac{a^3}{27} = \frac{9}{64} - \frac{17}{64} = -\frac{1}{8} \quad .$$

We must find ϕ so that

$$\cos\phi = \sqrt{\frac{b^2}{4} \Big/ \left(-\frac{a^3}{27}\right)} = \sqrt{\frac{9}{64} \Big/ \frac{17}{64}} = \frac{3}{\sqrt{17}} = .727606875$$

Hence

$$\phi = 43.31385668^{0} \quad .$$

then

$$\phi_1 = \frac{1}{3}\phi \quad , \qquad \cos\phi_1 = \;\; .968418219$$

$$\phi_2 = \phi_1 + 120^{0} \quad , \qquad \cos\phi_2 = -.700136446$$

$$\phi_3 = \phi_1 + 240^{0} \quad , \qquad \cos\phi_3 = -.268281771$$

also,

$$2\sqrt{-\frac{a}{3}} = 6/\sqrt{17} = 1.603521621$$

Then the three values of v_O are

$$v_O^{(1)} = r\cos\phi_1 = \;\; 1.552879552$$

$$v_O^{(2)} = r\cos\phi_2 = -1.122683929$$

$$v_O^{(3)} = r\cos\phi_3 = -.430195621$$

The corresponding values of $D = 4v_O^2 - |w|^{2/3}$ are:

$$D^{(1)} = 7.074458022$$

$$D^{(2)} = 2.470395226$$

$$D^{(3)} = 1.831008502$$

Hence we have the three roots:

$$v^{(1)} = (1.552879552,\ .565414338,\ .706767923,\ .141353585)$$

$$v^{(2)} = (-1.122683929,\ 1.619174113,\ 2.023967642,\ .404793528)$$

$$v^{(3)} = (-.430195621,\ -2.184588436,\ -2.730735543,\ -.546147109)$$

Computer calculated cubes, using single precision floating point, returned w to five decimal places exactly for all three roots.

Hence, if one uses enough precision, almost any fractional power can be computed using products of repeated square and cube roots. Fortunately, commutativity holds among roots. Although algebraic analysis is intractable, this has been investigated empirically with computer routines. For example, if $u^2 = w$ and $v^3 = w$, then $uv = w^{5/6}$. A number of examples are in the appendix.

Straight Power Series

One can, of course, take the formal definitions of straight power series without parameters and apply them to quadriform variables. For example, sin x, cos x, e^x and other such series can be evaluated. The algebra becomes exceedingly hard to follow. The first five powers of a quadriform x can be derived from x^2 as follows.

$$\begin{aligned}
x &= x \\
x^2 &= 2x_0x - |x|^2\ t_0 \\
x^3 &= 2x_0x^2 - x|x|^2\ t_0 = (4x_0^2 - |x|^2)x - 2x_0|x|^2\ t_0 \\
x^4 &= (x^2)^2 = 4x_0^2x^2 - 4x_0|x|^2x + |x|^4\ t_0 \\
&= 8x_0^3x - 4x_0^2|x|^2\ t_0 - 4x_0|x|^2x + |x|^4\ t_0 \\
&= (8x_0^3 - 4x_0|x|^2)x - (4x_0^2|x|^2 - |x|^4)\ t_0 \\
x^5 &= x^4x = 16x_0^4x - 8x_0^3|x|^2\ t_0 - 8x_0^2|x|^2x + 4x_0|x|^4\ t_0 \\
&\quad - 4x_0^2|x|^2x + |x|^4x \\
&= (16x_0^4 - 12x_0^2|x| + |x|^4)x - (8x_0^3|x|^2 - 4x_0|x|^4)\ t_0
\end{aligned}$$

The rule of formation for the coefficient of t_0 is evident: for x^n it is the coefficient of x in x^{n-1} multiplied by $-|x|^2$. The coefficient of x is more complicated. The first term is $2^{n-1}x_0^{n-1}$. The others are related to the coefficient of t_0 in x^{n-1}, where its first term is multiplied by n-2 and there is a sign alteration.

The functions sin x, cos x, and e^x have been programmed taking 6 terms for sin and cos and 10 for e^x. Some typical cases are shown in the Appendix. It is interesting to compare e^x by power series with the functions E(v) and H(v) in [1]. It is evident that if $|x| < 1$, the series converge fairly quickly but are affected by the value of x_0. The rules of formation indicated above give a vivid picture of the infrastructure of quadriform, and hence complex, functions with a power series expansion. The continued weaving of coefficients back and forth forms an extremely "hard cloth".

Binomial expansions and most parameterized series are not tractable due to noncommutativity. This does not affect straight power series since all multiplications are of a number either by itself or by a scalar.

The function ln x has also been programmed for quadriforms. Results are shown in the Appendix. Three reductions of the argument are used which amount to use of real parameters.

Simultaneous Linear Equations

It seems possible to have two styles of simultaneous linear equations: one using straight multiplication, the other with AMPs. Let us consider two equations in two unknowns in each style, first with straight multiplication.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Then, performing obvious quadriform arithmetic:

$$x_1 + a^{-1}_{11}a_{12}x_2 = a_{11}^{-1}b_1 \qquad (|a_{11}|^2 > 0)$$

$$(a_{22} - a_{21}a_{11}^{-1}a_{12})x_2 = b_2 - a_{21}a_{11}^{-1}b_1$$

Now if the latter coefficient of x_2 is nonsingular, we can solve for x_2, and then for x_1 in the upper equation. It will be helpful to calculate the determinant of the quadriform matrix. Since the coefficients are themselves matrices, we can write the whole thing as a 4×4 matrix.

$$\begin{bmatrix} a_{011}+a_{111} & -a_{211}+a_{311} & a_{012}+a_{112} & -a_{212}+a_{312} \\ a_{211}+a_{311} & a_{011}-a_{111} & a_{212}+a_{312} & a_{012}-a_{112} \\ a_{021}+a_{121} & -a_{221}+a_{321} & a_{022}+a_{122} & -a_{222}+a_{322} \\ a_{221}+a_{321} & a_{021}-a_{121} & a_{222}+a_{322} & a_{022}-a_{122} \end{bmatrix}$$

To simplify notation, we rewrite this temporarily as:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

Now the determinant of each 2×2 is the square of the absolute value of a coefficient; for example,

$$c_{11}c_{22} - c_{12}c_{21} = |a_{11}|^2 \quad .$$

Unfortunately, this has nothing to do with the determinant of the entire matrix, as is evident from expansion by minors. On the other hand, treating the quadriform system as a matrix product is valid. Let A be the first matrix above and define X and B as follows:

$$\underset{\sim}{X} = \begin{bmatrix} x_{01}+x_{11} & -x_{21}+x_{31} \\ x_{21}+x_{31} & x_{01}-x_{11} \\ x_{02}+x_{12} & -x_{22}+x_{32} \\ x_{22}+x_{32} & x_{02}-x_{12} \end{bmatrix} , \qquad B = \begin{bmatrix} b_{01}+b_{11} & -b_{21}+b_{31} \\ b_{21}+b_{31} & b_{01}-b_{11} \\ b_{02}+b_{12} & -b_{22}+b_{31} \\ b_{22}+b_{32} & b_{02}-b_{12} \end{bmatrix}$$

Then the real matrix product

$$AX = B$$

is the same as the quadriform matrix product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

since the latter can be regarded as a partitioning of the former. Hence it must also be true that

$$X = A^{-1}B$$

provided A^{-1} exists. Therefore a solution depends on the real determinant, not the quadriform determinant. Hence by writing A in the (c_{ij}) form, and similar forms for X and B, any $n \times n$ system of quadriforms can be treated as a $(2n) \times (2n)$ system of reals. Only one precaution is needed in applying matrix algebra: a quadriform row is not the matrix transpose of a quadriform column. That is,

$$(X_1, X_2) \neq X' \quad .$$

Instead, the individual 2×2 blocks must be used intact, for example:

$$(X_1, x_2) = \begin{bmatrix} x_{01} + x_{11} & -x_{21} + x_{31} & x_{02} + x_{12} & -x_{22} + x_{32} \\ x_{21} + x_{31} & x_{01} - x_{11} & x_{22} + x_{32} & x_{02} - x_{12} \end{bmatrix} \quad .$$

The same principle applies, of course, to A.

Note that noncommutativity is not a consideration except in the case of transposition discussed above. The reason is, obviously, that matrix multiplication is noncommutative anyway, so it is just more of the same.

Now consider a system with AMPs.

$$a_{11} \S x_1 + a_{12} \S x_2 = b_1$$

$$a_{21} \S x_1 + a_{22} \S x_2 = b_2 \quad .$$

Expanding,

$$a_{11}x_1 + x_1a_{11} + a_{12}x_2 + x_2a_{12} = 2b_1$$

$$a_{21}x_1 + x_1a_{21} + a_{22}x_2 + x_2a_{22} = 2b_2$$

Clearly, this does not lead to a 4×4 system in reals. Instead, one must proceed with extraction operations, from the quadriform system.

$$x_1 + (a_{12} \S x_2)/a_{11} = b_1/a_{11}$$

$$x_1 + (a_{22} \S x_2)/a_{21} = b_2/a_{21}$$

In order to subtract the first from the second, we would now have to develop a complete set of algebraic operations with AMP and extraction. Obviously, this would get complicated and we will not pursue it. It seems clear that quadriforms lend themselves much more to simultaneous linear systems than to polynomials. The AMP and extraction operations arose from a need to overcome noncommutativity. This is not necessary with a linear system and straight multiplication.

Further investigations along the lines of this paper appear to have diminishing value for the effort. The next paper in this series will turn attention to functions of a quadriform variable to see if some analytic theory can be developed.

APPENDIX

Numerical Examples

```
examples of u*v and u (amp) v

input values in locations 1 to 7
        1      1.00000      2.00000      4.00000      3.00000
        2      2.00000      1.00000      3.00000      3.00000
        3      1.00000      1.00000      1.00000      1.00000
        4      4.00000      3.00000      2.00000      1.00000
        5      1.00000      2.00000      3.00000      4.00000
        6      4.00000     -3.00000      2.00000     -1.00000
        7      1.00000      0.00000      1.00000      0.00000

   n        sq(abs.val)   abs.value     modulus     hypermodulus   magnitude
   1          4.00000      2.00000      4.12311      3.60555       5.47723
   2          3.00000      1.73205      3.60555      3.16228       4.79583
   3          0.00000      0.00000      1.41421      1.41421       2.00000
   4         10.00000      3.16228      4.47214      3.16228       5.47723
   5        -10.00000                   3.16228      4.47214       5.47723
   6         10.00000      3.16228      4.47214      3.16228       5.47723
   7          2.00000      1.41421      1.41421      0.00000       1.41421

 star product of         1 and       2 =
           3.00000
 star product of         3 and       3 =
           0.00000
 star product of         4 and       5 =
           0.00000
 star product of         4 and       6 =
          30.00000
 star product of         7 and       7 =
           2.00000

amp = (uv+vu)/2 of  1and  2stored in 11 =
    1.00000     5.00000    11.00000     9.00000
amp = (uv+vu)/2 of  3and  3stored in 12 =
    2.00000     2.00000     2.00000     2.00000
amp = (uv+vu)/2 of  4and  5stored in 13 =
    8.00000    11.00000    14.00000    17.00000
amp = (uv+vu)/2 of  4and  6stored in 14 =
    2.00000     0.00000    16.00000     0.00000
amp = (uv+vu)/2 of  7and  7stored in 15 =
    0.00000     0.00000     2.00000     0.00000

   n        sq(abs.val)   abs.value     modulus     hypermodulus   magnitude
  11         16.00000      4.00000     11.04536     10.29563      15.09967
  12          0.00000      0.00000      2.82843      2.82843       4.00000
  13       -150.00000                  16.12452     20.24846      25.88436
  14        260.00000     16.12452     16.12452      0.00000      16.12452
  15          4.00000      2.00000      2.00000      0.00000       2.00000
 command :
```

run 1

above products with ordinary multiplication in both orders

product of	1 and	2 stored in		21 =
1.00000	2.00000	8.00000	7.00000	
product of	2 and	1 stored in		22 =
1.00000	8.00000	14.00000	11.00000	
product of	3 and	3 stored in		23 =
2.00000	2.00000	2.00000	2.00000	
product of	4 and	5 stored in		24 =
8.00000	6.00000	4.00000	12.00000	
product of	5 and	4 stored in		25 =
8.00000	16.00000	24.00000	22.00000	
product of	4 and	6 stored in		26 =
2.00000	4.00000	16.00000	-12.00000	
product of	6 and	4 stored in		27 =
2.00000	-4.00000	16.00000	12.00000	

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
21	12.00000	3.46410	8.06226	7.28011	10.86278
22	12.00000	3.46410	14.03567	13.60147	19.54482
23	0.00000	0.00000	2.82843	2.82843	4.00000
24	-100.00000		8.94427	13.41641	16.12452
25	-100.00000		25.29822	27.20294	37.14835
26	100.00000	10.00000	16.12452	12.64911	20.49390
27	100.00000	10.00000	16.12452	12.64911	20.49390

example of non-commutative extraction

input in locations 1 and 2
amp = (uv+vu)/2 of 1and 2stored in 3 =

1.05000	0.50000	-1.00000	0.00000	
1	1.00000	0.10000	1.00000	0.10000
2	0.00000	0.50000	-1.00000	0.00000
3	1.05000	0.50000	-1.00000	0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	1.98000	1.40712	1.41421	0.14142	1.42127
2	0.75000	0.86603	1.00000	0.50000	1.11803
3	1.85250	1.36107	1.45000	0.50000	1.53379

division of	3 by	1 stored in		4 =
-0.00000	0.50000	-1.00000	0.00000	

vector 2 is unallowable for division
command :

set 1
<
quad

```
examples of general quadratics

p(x)=a$(x**2 + b$x + c) = 0
  a =         1.00000      0.00000      0.00000      0.00000
  b =         2.00000      0.00000      0.00000      0.00000
  c =        -1.00000      0.00000      0.00000      0.00000
 x1=
     0.41421     0.00000     0.00000     0.00000
 x2=
    -2.41421     0.00000     0.00000     0.00000
  p(x1) =        -0.00000      0.00000      0.00000      0.00000
  p(x2) =         0.00000      0.00000      0.00000      0.00000

p(x)=a$(x**2 + b$x + c) = 0
  a =         1.00000      0.00000      0.00000      0.00000
  b =         2.00000      0.00000      0.00000      0.00000
  c =         1.00000      0.00000      0.00000      0.00000
 x1=
    -1.00000     0.00000     0.00000     0.00000
 x2=
    -1.00000     0.00000     0.00000     0.00000
  p(x1) =         0.00000      0.00000      0.00000      0.00000
  p(x2) =         0.00000      0.00000      0.00000      0.00000

p(x)=a$(x**2 + b$x + c) = 0
  a =         1.00000      0.00000      0.00000      0.00000
  b =         2.00000      0.00000      0.00000      0.00000
  c =         1.00000      0.00000      1.00000      0.00000
 x1=
    -0.29289     0.00000    -0.70711     0.00000
 x2=
    -1.70711     0.00000     0.70711     0.00000
  p(x1) =         0.00000      0.00000      0.00000      0.00000
  p(x2) =         0.00000      0.00000      0.00000      0.00000

p(x)=a$(x**2 + b$x + c) = 0
  a =         1.00000      0.10000      1.00000      0.10000
  b =         2.00000      0.00000      0.00000      0.00000
  c =         1.00000      0.00000      1.00000      0.00000
 x1=
    -0.26494     0.15461    -0.53626     0.15461
 x2=
    -0.74516    -0.05360     1.54636    -0.05360
  p(x1) =         0.00000      0.00000      0.00000      0.00000
  p(x2) =         0.00000      0.00000      0.00000      0.00000

there is no solution of quadratic equation
p(x)=a$(x**2 + b$x + c) = 0
  a =     1.0     0.1     1.0     0.1
  b =     0.0     0.5    -1.0     0.0
  c =     0.0     0.0     1.0    -0.5
  determinant is negative

p(x)=a$(x**2 + b$x + c) = 0
  a =         1.00000      0.10000      1.00000      0.10000
  b =         0.00000      0.50000     -1.00000      0.00000
  c =         0.00000      0.00000      2.00000     -0.50000
 x1=
     0.55934    -0.34977    -1.19297     0.97534
 x2=
    -0.02904    -0.20326     1.66267    -1.02837
  p(x1) =         0.00000      0.00000      0.00000      0.00000
  p(x2) =         0.00000      0.00000     -0.00000      0.00000
 command :
int I 3
integer         I=          3
 command :
set I
<
roots
```

examples of cube roots
in each case the argument is in location 1.
the 3 cube roots are in locs. 2, 3, 4.
(if 3, 4 are zero, only one allowable root exists)
the results of cubing the roots are in locs 5, 6, 7.

cube roots of minus unity

1	-1.00000	0.00000	0.00000	0.00000
2	-1.00000	0.00000	0.00000	0.00000
3	0.50000	0.00000	0.86603	0.00000
4	0.50000	0.00000	-0.86603	0.00000
5	-1.00000	0.00000	0.00000	0.00000
6	-1.00000	0.00000	-0.00000	0.00000
7	-1.00000	0.00000	0.00000	0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	1.00000	1.00000	1.00000	0.00000	1.00000
2	1.00000	1.00000	1.00000	0.00000	1.00000
3	1.00000	1.00000	1.00000	0.00000	1.00000
4	1.00000	1.00000	1.00000	0.00000	1.00000

cube roots of complex unit i

1	0.00000	0.00000	1.00000	0.00000
2	0.00000	0.00000	-1.00000	0.00000
3	0.86603	0.00000	0.50000	0.00000
4	-0.86603	0.00000	0.50000	0.00000
5	0.00000	0.00000	1.00000	0.00000
6	0.00000	0.00000	1.00000	0.00000
7	0.00000	0.00000	1.00000	0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	1.00000	1.00000	1.00000	0.00000	1.00000
2	1.00000	1.00000	1.00000	0.00000	1.00000
3	1.00000	1.00000	1.00000	0.00000	1.00000
4	1.00000	1.00000	1.00000	0.00000	1.00000

cube roots of (1 + i)

1	1.00000	0.00000	1.00000	0.00000
2	1.08422	0.00000	0.29051	0.00000
3	-0.79370	0.00000	0.79370	0.00000
4	-0.29051	0.00000	-1.08422	0.00000
5	1.00000	0.00000	1.00000	0.00000
6	1.00000	0.00000	1.00000	0.00000
7	1.00000	0.00000	1.00000	0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	2.00000	1.41421	1.41421	0.00000	1.41421
2	1.25992	1.12246	1.12246	0.00000	1.12246
3	1.25992	1.12246	1.12246	0.00000	1.12246
4	1.25992	1.12246	1.12246	0.00000	1.12246

command :

general quadriforms with only one allowable root

1	1.00000	0.90000	1.00000	-0.90000
2	0.90521	0.35248	0.39164	-0.35248
3	0.00000	0.00000	0.00000	0.00000
4	-0.00000	0.00000	0.00000	0.00000
5	1.00000	0.90000	1.00000	-0.90000
6	0.00000	0.00000	0.00000	0.00000
7	0.00000	0.00000	0.00000	0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.38000	0.61644	1.41421	1.27279	1.90263
2	0.72432	0.85107	0.98631	0.49848	1.10512
3	0.00000	0.00000	0.00000	0.00000	0.00000
4	0.00000	0.00000	0.00000	0.00000	0.00000

1	27.00000	8.00000	8.00000	-8.00000
2	2.96973	0.30133	0.30133	-0.30133
3	0.00000	0.00000	0.00000	0.00000
4	0.00000	0.00000	0.00000	0.00000
5	27.00000	8.00000	8.00000	-8.00000
6	0.00000	0.00000	0.00000	0.00000
7	0.00000	0.00000	0.00000	0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	665.00000	25.78759	28.16026	11.31371	30.34798
2	8.72852	2.95441	2.98498	0.42615	3.01525
3	0.00000	0.00000	0.00000	0.00000	0.00000
4	0.00000	0.00000	0.00000	0.00000	0.00000

general quadriform cube roots

1	27.00000	8.00000	9.00000	-3.00000
2	3.00364	0.29570	0.33266	-0.11089
3	-1.59236	7.20955	8.11074	-2.70358
4	-1.41128	-7.50525	-8.44341	2.81447
5	27.00000	8.00000	9.00000	-3.00000
6	27.00004	7.99995	8.99994	-2.99998
7	27.00004	8.00005	9.00007	-3.00004

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	737.00000	27.14775	28.46050	8.54400	29.71532
2	9.03280	3.00546	3.02201	0.31580	3.03846
3	9.03281	3.00546	8.26558	7.69980	11.29631
4	9.03282	3.00546	8.56054	8.01561	11.72744

1	0.00000	0.90000	2.00000	-0.90000
2	0.00000	-0.67409	-1.49797	0.67409
3	1.00068	0.33704	0.74899	-0.33704
4	-1.00068	0.33704	0.74899	-0.33704
5	0.00000	0.90000	2.00000	-0.90000
6	-0.00000	0.90000	2.00000	-0.90000
7	0.00000	0.90000	2.00000	-0.90000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	2.38000	1.54272	2.00000	1.27279	2.37065
2	1.33514	1.15548	1.49797	0.95331	1.77559
3	1.33514	1.15548	1.24993	0.47665	1.33773
4	1.33514	1.15548	1.24993	0.47665	1.33773

command :

examples of commutativity of roots

u and 3 cube roots of u

1	1.00000	0.90000	2.00000	-0.90000	
2	1.15819	0.23287	0.51748	-0.23287	
3	-0.92478	0.46871	1.04159	-0.46871	
4	-0.23341	-0.70158	-1.55907	0.70158	
n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	3.38000	1.83848	2.23607	1.27279	2.57294
2	1.50074	1.22505	1.26854	0.32932	1.31059
3	1.50074	1.22505	1.39288	0.66286	1.54257
4	1.50074	1.22505	1.57644	0.99218	1.86269

positive square root and fourth root of u

5	1.19132	0.37773	0.83941	-0.37773	
6	1.12854	0.16735	0.37190	-0.16735	
n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
5	1.83848	1.35590	1.45734	0.53419	1.55216
6	1.35590	1.16443	1.18824	0.23667	1.21158

first cube root times fourth root and vice versa, = 7/12 root

7	1.19256	0.45663	1.01473	-0.45663	
8	1.19256	0.45663	1.01473	-0.45663	
n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
7	2.03486	1.42649	1.56585	0.64577	1.69378
8	2.03486	1.42649	1.56585	0.64577	1.69378

reciprocals of cube and fourth roots, products both ways, = -7/12 root

9	0.77175	-0.15517	-0.34482	0.15517	
10	0.83232	-0.12343	-0.27428	0.12343	
11	0.58607	-0.22440	-0.49867	0.22440	
12	0.58607	-0.22440	-0.49867	0.22440	
n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
9	0.66634	0.81630	0.84528	0.21944	0.87330
10	0.73751	0.85879	0.87635	0.17455	0.89356
11	0.49143	0.70102	0.76951	0.31735	0.83238
12	0.49143	0.70102	0.76951	0.31735	0.83238

7/12 root times -7/12 root

13	1.00000	0.00000	0.00000	0.00000

command :

examples of sin v and cos v

v, sin v, cos v, (sin v)**2, (cos v)**2, sum of squares

1	1.00000	0.50000	0.40000	0.30000
2	0.76687	0.26212	0.20970	0.15727
3	0.49240	-0.40823	-0.32658	-0.24494
4	0.63756	0.40202	0.32162	0.24121
5	0.36245	-0.40202	-0.32162	-0.24121
6	1.00000	0.00000	-0.00000	-0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.82000	0.90554	1.07703	0.58310	1.22474
2	0.53862	0.73391	0.79502	0.30568	0.85176
3	0.12247	0.34996	0.59086	0.47607	0.75879
4	0.29011	0.53862	0.71408	0.46883	0.85424
5	0.01500	0.12247	0.48457	0.46883	0.67425
6	1.00000	1.00000	1.00000	0.00000	1.00000

same as above with singular v

1	0.70710	0.70710	0.70710	0.70710
2	0.49388	0.49388	0.49388	0.49388
3	0.57798	-0.42202	-0.42202	-0.42202
4	0.48784	0.48784	0.48784	0.48784
5	0.51216	-0.48784	-0.48784	-0.48784
6	1.00000	-0.00000	-0.00000	-0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.00000	0.00000	0.99999	0.99999	1.41420
2	0.00000	0.00000	0.69845	0.69845	0.98776
3	0.15596	0.39491	0.71565	0.59683	0.93186
4	0.00000	0.00000	0.68991	0.68991	0.97568
5	0.02432	0.15596	0.70732	0.68991	0.98806
6	1.00000	1.00000	1.00000	0.00000	1.00000

complex projection of v is on lower diagonal

1	0.50000	0.30000	-0.50000	0.30000
2	0.49630	0.26636	-0.44393	0.26636
3	0.90848	-0.14551	0.24252	-0.14551
4	0.19114	0.26439	-0.44065	0.26439
5	0.80886	-0.26439	0.44065	-0.26439
6	1.00000	-0.00000	0.00000	-0.00000

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.32000	0.56569	0.70711	0.42426	0.82462
2	0.30150	0.54909	0.66588	0.37669	0.76504
3	0.84180	0.91750	0.94029	0.20578	0.96255
4	0.09090	0.30150	0.48032	0.37390	0.60869
5	0.70863	0.84180	0.92110	0.37390	0.99410
6	1.00000	1.00000	1.00000	0.00000	1.00000

command :

examples of e**v and ln v

u, v, e**u, e**v

1	0.50000	0.40000	0.30000	0.20000
2	0.40000	0.10000	0.60000	0.30000
3	1.74024	0.67165	0.50373	0.33582
4	1.30205	0.14280	0.85681	0.42840

u+v, e**(u+v), e**u * e**v, e**v * e**u

5	0.90000	0.50000	0.90000	0.50000
6	2.08811	1.16724	2.10103	1.16724
7	2.07406	1.19496	1.90716	0.67925
8	2.07406	1.05109	2.38672	1.68632

(e**u * e**v)**1/2, (e**v * e**u)**1/2

9	1.50560	0.39684	0.63335	0.22557
10	1.50560	0.34906	0.79261	0.56002

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.14000	0.37417	0.58310	0.44721	0.73485
2	0.42000	0.64807	0.72111	0.31623	0.78740
3	2.71828	1.64872	1.81168	0.75092	1.96114
4	2.22554	1.49182	1.55867	0.45158	1.62277
5	1.12000	1.05830	1.27279	0.70711	1.45602
6	6.04965	2.45960	2.96219	1.65073	3.39108
7	6.04965	2.45960	2.81762	1.37452	3.13501
8	6.04965	2.45960	3.16198	1.98708	3.73451
9	2.45960	1.56831	1.63339	0.45647	1.69598
10	2.45960	1.56831	1.70149	0.65990	1.82497

(e**u * e**v)**1/2 * (e**v * e**u)**1/2

11	2.02967	0.94713	2.00344	1.08932

log e**u = u, log e**v = v, log e**(u+v) = u+v

12	0.50000	0.40000	0.30000	0.20000
13	0.40000	0.10000	0.60000	0.30000
14	0.90000	0.50000	0.90000	0.50000

log (e**u * e**v), log (e**v * e**u)

15	0.90000	0.51293	0.81863	0.29156
16	0.90000	0.45117	1.02448	0.72384

log ((e**u * e**v)**1/2), log ((e**v * e**u)*1/2)

17	0.45000	0.25646	0.40932	0.14578
18	0.45000	0.22559	0.51224	0.36192

log((e**u * e**v)**1/2 * (e**v * e**u)**1/2)

19	0.90000	0.40921	0.86559	0.47064

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
11	6.04965	2.45960	2.85190	1.44350	3.19641
12	0.14000	0.37417	0.58310	0.44721	0.73485
13	0.42000	0.64807	0.72111	0.31623	0.78740
14	1.12000	1.05830	1.27279	0.70711	1.45602
15	1.13206	1.06398	1.21662	0.59000	1.35213
16	1.13206	1.06398	1.36366	0.85294	1.60843
17	0.28301	0.53199	0.60831	0.29500	0.67607
18	0.28301	0.53199	0.68183	0.42647	0.80422
19	1.17028	1.08180	1.24869	0.62366	1.39578

command :