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IIASA Research Memorandum
June 1975

RM-75-25

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J. Casti*

1. Introduction

One of the great satisfactions associated with any scientific investigation comes at the moment when one realizes that the subject under investigation has deep and unexpected connections with basic problems in an entirely different corner of the scientific forest. Our objective in this report is to present one such unexpected pairing: the theories of optimal linear filtering and neutron transport in a rod.

Of course, it has been known for many years that linear least-squares filtering and optimal control theory have strong interconnections through the Duality Principle. However, the results presented here seem to have a much more recent origin, essentially dating from the papers of Casti-Tse [1] and Sidhu-Casti [2], which treat a restricted version of the present topic. In retrospect, it seems surprising that development of these ideas did not follow immediately upon the heels of the Duality Principle since, as will be seen, all of our results *could* have been derived directly from this source. The fact that the classical problems of transport theory are stated as Fredholm integral equations, rather than as variational problems is the most likely source for the delay in uncovering the "other life" of filtering theory. In any case, the discovery that these two theories are identical leads to a wide variety

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of new interpretations of the classical Kalman filtering scheme, as well as to the introduction of many new functions which promise to provide new insights into the standard approaches, together with significant computational advantages.

Since the majority of readers of this paper will not be well-versed in both filtering and transport theory, we provide a brief introduction to each topic in Sections 2 and 3, respectively, before proceeding to our main results in Sections 4-9. The final section presents a discussion of many side topics and areas which seem promising for future research.

2. Linear Least-Squares Filtering Theory

The standard Kalman filtering set-up is the following: we observe a noisy signal z,

$$z(t) = H(t)x(t) + v(t) \quad , \qquad 0 \leq t \leq T \quad , \tag{1}$$

which is assumed to be the output of the linear system

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \quad , \qquad x(0) = x_0 \quad , \tag{2}$$

driven by a noise process u(t). Here x, u, v are n, m, and p-dimensional vectors, respectively with F, G, H being continuous matrix functions of t on $0 \leq t \leq T$. The observation noise v and the system noise u are assumed to satisfy the following statistical assumptions:

$$
\begin{aligned}
E[v] &= E[u] = 0 \quad , \\
E[v(t)v'(s)] &= R(t)\delta(t - s) \quad , \\
E[u(t)u'(s)] &= Q(t)\delta(t - s) \quad , \\
E[v(t)u'(s)] &= 0 \quad , \qquad 0 \leq t < s \leq T \quad .
\end{aligned} \tag{3}
$$

Furthermore, the covariance matrices Q and R satisfy the definiteness assumptions

$$
Q(t) \geq 0 \quad , \qquad R(t) > 0 \quad , \qquad 0 \leq t \leq T \quad .
$$

In general, the initial state x_0 will also be a stochastic quantity with the statistics

$$
E[x_0] = 0 \quad , \qquad E[x_0 x_0'] = \Gamma \quad .
$$

Thus, the various noises in the system are all independent, zero-mean Gaussian processes.

Within the context of the foregoing situation, we desire to determine an estimate, $\hat{x}(t)$, such that for every constant vector λ we have

$$
E[\lambda'(x(t) - \hat{x}(t))]^2 \leq E[\lambda'(x(t) - (f(Z))]^2
$$

for every measurable functional $f(z)$ such that

$$
E[\hat{x}(t)] = E[x(t)] = E[f(z)] = 0 \quad .
$$

In other words, we wish to choose $\hat{x}(t)$ to minimize the covariance of the error between the true state x and the estimated state $\hat{x}$.

The above problem is completely solved by the Kalman filter [18]. The optimal estimate $\hat{x}(t)$ is generated by the differential equation

$$\frac{dx}{dt} = F(t)\hat{x} + P(t)H'(t)R^{-1}(t)\left[z(t) - H(t)\hat{x}\right] \quad ,$$

$$x(0) = 0 \quad , \tag{4}$$

where P(t) is the solution of the matrix Riccati equation

$$\begin{aligned}\frac{dP(t)}{dt} &= G(t)Q(t)G'(t) + F(t)P + PF'(t) \\ &\quad - PH'(t)R^{-1}(t)H(t)P \quad , \\ P(0) &= \Gamma \quad .\end{aligned} \tag{5}$$

For future reference, we observe that in the stationary case when F, G, H, Q, R are constant matrices and (F,G) is controllable, while (F,H) is observable, the steady-state gain $K_\infty = P(\infty)H'R^{-1}$ is determined by the solution to the algebraic Riccati equation

$$GQG' + FP_\infty + P_\infty F' - P_\infty H'R^{-1}HP_\infty = 0 \quad . \tag{6}$$

Another quantity of interest is the error covariance between the estimate of the current state and the unknown initial state. Defining the optimal least-squares error as

$$e^*(t) = x(t) - \hat{x}(t) \quad ,$$

we have

$$E[e^*(t)e^{*\prime}(0)] = S(t) \quad ,$$

while

$$E[e^*(t)e^{*\prime}(t)] = P(t) \quad .$$

The matrix S(t) satisfies the linear matrix equation

$$\dot{S}(t) = [F(t) - P(t)H'(t)R^{-1}(t)H(t)]S \quad ,$$
$$S(0) = \Gamma \quad . \tag{6a}$$

The steady-state version is

$$(F - P_\infty H'R^{-1}H)S = 0 \tag{6b}$$

for constant F and H.

It should be noted that in some cases the assumption of a system model, required by the Kalman approach, may be untenable. Often all that is available is covariance information on the observation process z(t). In this case, Kailath and Geesey [3] have developed an approach to the filtering problem which also leads to the solution of a linear vector differential equation containing the solution to a matrix Riccati equation as a coefficient. Since our subsequent development will be centered upon the Riccati equation, we shall omit the Kailath-Geesey result for the sake of brevity; however, all the remarks we later make in the context of the Kalman filter may also be transferred to the Kailath-Geesey case upon substitution of the appropriate Riccati equation.

3. One-Dimensional Neutron Transport

We consider an idealized transport process in which n different types of particles move in either direction along a rod of length $t < \infty$. The particles may interact with the medium of which the rod is composed in the manner specified below, but they do not interact with each other. The effect of such an interaction is to change the particles from one type to another traveling in the same or opposite direction, or the interaction may result in a particle being absorbed by the medium. Schematically, we have the situation depicted in Figure 1.

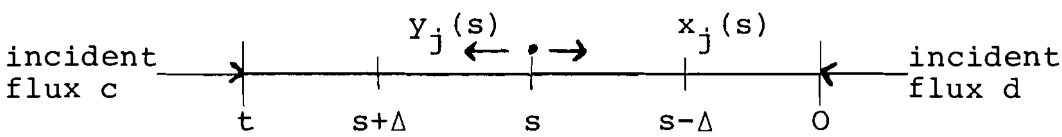

Figure 1. One-dimensional neutron transport.

In a segment of length Δ, we assume that at most one interaction may occur according to the following probabilities:

$a_{ij}(s)\Delta + o(\Delta)$ = the probability that a particle in state j will be transformed into a particle in state i traveling in the same direction, $j \neq i$, upon transversing the interval $[s + \Delta, s]$ going to the right;

$1 - a_{ii}(s)\Delta + o(\Delta)$ = the probability that a particle in state i will remain in state i in the same direction while traversing the interval $[s + \Delta, s]$ going to the right;

$b_{ij}(s)\Delta + o(\Delta)$ = the probability that a particle going to the right in state j will be transformed into a particle in state i traveling in the opposite direction upon traversing the interval $[s + \Delta, s]$.

Similar definitions are made for functions $c_{ij}(s)$ and $d_{ij}(s)$ associated with forward and back scattering for a particle moving to the left through $[s + \Delta, s]$. Also, we define

$\overline{e}_{ii}(s)\Delta$ = the probability that a particle of type i is absorbed in traversing the interval $[s + \Delta, s]$ moving to the right,

while $\overline{f}_{ii}(s)$ represents the analagous function for left-moving particles. In general, we assume that all functions are piecewise continuous. We suppose, also, that there are particles of all n types incident at both ends of the rod, represented by the vectors c and d as depicted in Figure 1.

Introducing the equilibrium functions

$x_i(s)$ = the expected intensity of flux of particles of type i moving to the right at point s, $0 \leq s \leq t$;

$y_i(s)$ = the expected intensity of flux of particles of type i moving to the left,

a simple "input/output" analysis [4] shows that

$$-\dot{x}_i(s) = -a_{ii}x_i(s) + \sum_{j \neq i} a_{ij}x_j(s) + \sum_{j=1}^{n} d_{ij}y_j(s) - \sum_{j=1}^{n} \bar{e}_{jj}x_j(s)$$

$$\dot{y}_i(s) = -c_{ii}y_i(s) + \sum_{j \neq i} c_{ij}y_j(s) + \sum_{j=1}^{n} b_{ij}x_j(s) - \sum_{j=1}^{n} \bar{f}_{jj}y_j(s) \quad (7)$$

$$x_i(t) = c_i \quad ,$$

$$y_i(0) = d_i \quad , \qquad i = 1,2,\ldots,n \quad . \quad (8)$$

Thus, the fluxes x and y satisfy the linear two-point boundary value problem (7) with boundary conditions given by (8).

Remark: To facilitate our discussion in the next section, we have kept the notation t to indicate the spatial length of the rod. Thus, when we speak of x(s), y(s) as equilibrium functions, we are acknowledging the fact that, generally speaking, the above situation will be a time-dependent transport process and the fluxes x and y will be functions of time, as well as position. Our considerations here are only with respect to the "steady-state" distributions.

Adopting the notation

$$A = \begin{bmatrix} -a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & -a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & -a_{nn} \end{bmatrix} \qquad C = \begin{bmatrix} -c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & -c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & -c_{nn} \end{bmatrix}$$

$$B = [b_{ij}] \ , \quad D = [d_{ij}] \ , \quad \bar{E} = [\bar{e}_{ij}] \ , \quad \bar{F} = [\bar{f}_{ij}] \ ,$$

the two-point boundary value problem (7)-(8) now assumes the form

$$\begin{aligned} \dot{y} &= Bx + (C - \bar{F})y \ , \quad & y(\Theta) &= d \ , \\ -\dot{x} &= (A - \bar{E})x + Dy \ , \quad & x(t) &= c \ . \end{aligned} \tag{9}$$

In addition to the internal fluxes, which may be physically unmeasurable, we are also interested in the more meaningful quantities, the reflected, transmitted, and absorbed intensities. Let

$r_{ij}(t)$ = the intensity of flux in state i which emerges from the left end of a rod length t due to an incident flux of unit intensity in state j at t,

$t_{ij}(t)$ = the analagous transmitted intensity function,

$\bar{l}_{ij}(t)$ = the intensity of absorbed flux in state i resulting from an incident flux of unit intensity at in state j, i, j = 1,2,...,n.

Application of standard "particle-counting" methods in invariant imbedding [4] yields the matrix equations for $R = [r_{ij}]$, $T = [t_{ij}]$, $\bar{L} = [\bar{l}_{ij}]$:

$$\dot{R}(t) = B(t) + RA(t) + C(t)R + RD(t)R \quad , \tag{10}$$

$$\dot{T}(t) = T[A(t) + D(t)R] \quad , \tag{11}$$

$$\dot{\bar{L}}(t) = \bar{E}(t) + \bar{F}(t)R + \bar{L}[A(t) + D(t)R] \quad , \qquad t > 0 \quad , \tag{12}$$

with the initial conditions

$$R(0) = 0 \quad ,$$

$$T(0) = I \quad ,$$

$$\bar{L}(0) = 0 \quad . \tag{13}$$

The special case when the matrices A, B, C, D are symmetric with A = C, B = D corresponds to the <u>isotropic</u> scattering problem, while the case A, B, C, D constant is the condition for a <u>homogeneous</u> rod.

It is not difficult to show that the matrices R, T, L are non-negative and satisfy the conservation law

$$\mathscr{M}(R + T + L) = \mathscr{M} \quad , \tag{14}$$

where $\mathscr{M}$ is the constant matrix

$$\mathscr{M} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} .$$

This fact is an immediate consequence of the relations

$$\mathscr{M}(A + B + \bar{E}) = 0 \quad ,$$

$$\mathscr{M}(C + D + \bar{F}) + 0 \quad .$$

(The interested reader is urged to consult [4] for all relevant details and further references).

The observant reader will note a certain asymmetry in the above situation since we have assumed a unit incident flux at the left end of the rod t and no incident flux at the end 0 (i.e. $d_i = 0$). Although it is not too important for physical considerations, for our later mathematical development it is convenient to also consider the opposite situation, (i.e. $c = 0$, $d \neq 0$). It is not too difficult to see that the Riccati equation (10) transforms to

$$\dot{U}(t) = D(t) + UC(t) + A(t)U + UB(t)U \quad , \tag{15}$$

corresponding to the reflected intensity at $t = 0$, while the new transmission matrix S(t) satisfies the equations

$$\dot{S}(t) = [C(t) + R(t)D(t)]S \quad , \qquad S(0) = I \quad , \tag{16}$$

$$\dot{S}(t) = S[C(t) + B(t)U(t)] \quad , \qquad S(0) = I \quad , \tag{17}$$

depending upon which reflection matrix is employed. For completeness, we can also express our previous transmission matrix T in terms of U as

$$\dot{T} = [A(t) + U(t)B(t)]T \quad , \qquad T(0) = I \quad . \tag{18}$$

Physically, the forms (15) and (16) (or (17)) arise if we add an infinitesimal slab of thickness Δ to the rod at the right end, while the forms (10) and (18) (or (11)) appear if the addition takes place at the left end of the rod.

The physical interpretation of the functions U, R, S, T immediately allow us to state the formulas

$$\dot{U} = TD(t)S \quad , \tag{19}$$

$$\dot{R} = SB(t)T \quad , \tag{20}$$

which we call the Stokes' relations due to early discoveries by Stokes of similar relations for reflection and transmission coefficients for light rays impinging on slabs. The relations (19) and (20) are derived by considering the possible ways in which a neutron can be reflected from either end of a rod of length t. In a later section, we shall have occasion to utilze the following algebraic relations which, using (19)-(20), are rewrites of Eqs. (10) and (15):

$$SB(t)T = B(t) + RA(t) + C(t)R + RD(t)R \quad , \tag{21}$$

$$TD(t)S = D(t) + UC(t) + A(t)U + UB(t)U \quad . \tag{22}$$

From Eqs. (11), (16)-(18), we also note the useful relations

$$T(A + DR) = (A + UB)T \quad , \tag{23}$$

$$S(C + BU) = (C + RD)S \quad . \tag{24}$$

The utility of the physical model is readily apparent at this point as a direct derivation of (19)-(20) is not particularly straightforward, although the physical situation is quite clear.

For a homogeneous medium, the distributions of reflected, transmitted, and absorbed intensities for a semi-infinite rod are obtained from Eqs. (10)-(12) by solving the algebraic matrix equations

$$B + RA + CR + RDR = 0 \quad , \tag{25}$$

$$T(A + DR) = 0 \quad , \tag{26}$$

$$\overline{E} + \overline{F}R + \overline{L}(A + DR) = 0 \quad , \tag{27}$$

for R, T, and L. By fairly straightforward arguments, it can be shown [5] the matrices $A, B, \ldots, \overline{F}$ ensure that unique physically meaningful solutions to (25)-(27) exist. The conservation law (14) is particularly useful in establishing the existence of such a solution.

4. System Isomorphisms and Physical Parallels

Our review of the physical models completed, we are now in position to point out some of the equivalences that exist between the linear filtering problem and neutron transport in a rod.

First of all, consider the case of isotropic, conservative scattering in the transport problem ($E = F = 0$). In this case, the entire scattering process is defined by the matrix Riccati equation (10) for the reflection matrix. Also, it is clear that knowledge of the matrix Riccati equation (5) for the error covariance of the current state completely defines the Kalman filtering problem. Thus, we have the associations

$$B \leftrightarrow GQG', \quad A \leftrightarrow F', \quad C \leftrightarrow F, \quad D \leftrightarrow -(H'R^{-1}H) \quad .$$
$$R \leftrightarrow P, \quad T \leftrightarrow S',$$

where the first object in each pairing is the transport quantity, the second its filtering theory counterpart.

Upon introducing the Hamiltonian matrix

$$H(\overline{z}, w, t) = -\tfrac{1}{2}(G'(t)\overline{z}, Q(t)G'(t)\overline{z}) - (w, F'(t)\overline{z}) + \tfrac{1}{2}(H(t)w, R^{-1}(t)H(t)w),$$

one immediately recognizes the two-point boundary value problem (9) for the internal fluxes x and y as none other than the classical canonical differential equations

$$\frac{d\bar{z}}{dt} = \frac{\partial H}{\partial w} = -F'(t)\bar{z} + H'(t)R^{-1}(t)H(t)w \quad ,$$

$$\frac{dw}{dt} = \frac{-\partial H}{\partial z} = G(t)Q(t)G'(t)\bar{z} + F(t)w \quad ,$$

where we have made the identifications

$$x \leftrightarrow \bar{z} \quad , \qquad y \leftrightarrow w \quad .$$

At this juncture, parallel relationships have been found for most of the basic functions of linear filtering and neutron transport; however, some important quantitites remain unidentified. These are Γ, the covariance of the initial-state error in the filtering problem, and $\bar{E}$, $\bar{F}$, $\bar{L}$, the quantitites associated with absorption of particles in the rod. It is fairly easy to make the transport identification of Γ, since it is just the initial condition for the Riccati matrix. Transferring to the transport setting, we see that a non-zero initial condition on R must, for physical reasons, represent a reflecting surface at the right end of the rod (t = 0). If we let

z_{ij} = the intensity of flux in state i which is back-scattered from the reflecting surface at t = 0, due to an incident flux of unit intensity in state j on a rod of zero length, we have

$$R(0) = Z = [z_{ij}] \quad ,$$

leading to the pairing

$$Z \leftrightarrow \Gamma \quad .$$

A suitable filtering theory interpretation of the absorption quantities $\bar{L}$, $\bar{E}$, $\bar{F}$ is a bit more obscure, but not much. In the standard filtering problem we have defined the optimal error as

$$e^*(t) = x(t) - \hat{x}(t) \quad .$$

It is relatively straightforward to see that e^* satisfies the equation

$$\frac{de^*}{dt} = F(t)e^* + G(t)u - K(t)[x(t) + H(t)e^*] \quad ,$$

$$e^*(0) = x_0 \quad ,$$

where $K(t) = P(t)H'(t)R^{-1}(t)$. If we let

$$W(t) = E[e^*(t)e^{*'}(0)] \quad ,$$

the covariance between the error at time t and the initial error, and further postulate that the noise processes $u(t)$, $v(t)$ are correlated with the initial error $e(0)$ as

$$E[G(t)u(t)e^{*\prime}(0)] = M(t) \quad ,$$

$$E[H'(z)R^{-1}(t)v(t)e^{*\prime}(0)] = N(t) \quad ,$$

then we obtain the matrix equation

$$\frac{dW(t)}{dt} = [F(t) - K(t)H(t)]W + M(t) - P(t)N(t) \quad ,$$

$$W(0) = \Gamma \quad .$$

Aside from the sign on $N(t)$ and the possibly non-zero initial condition, this is equation (12) for the absorption function $\bar{L}$ under the identifications

$$\bar{L} \leftrightarrow W \quad , \qquad \bar{E} \leftrightarrow M \quad , \qquad \bar{F} \leftrightarrow -N \quad .$$

Notice that in the filtering version, there is no natural need to assume M, N diagonal. Thus, the filtering equation for W represents a generalization of the absorption function $\bar{L}$ of transport theory.

To summarize our results so far, we have

Table 1. Neutron transport filtering theory pairs.

Transport Quantity	Filtering Quantity
A	F'
B	GQG'
C	F
D	$-(H'R^{-1}H)$
R	P
T	S'
x	$\overline{z}$
y	w
Z	Γ
$\overline{L}$	W
$\overline{E}$	M
$\overline{F}$	- N

Remarks:

i) from Table 1 we note the somewhat surprising fact that there appears to be no direct transport quantity that one can associate with the state x of the filtering model. Given the absolutely basic role that the state plays in the Kalman filtering problem, this anomaly merits further study. Of course, the superficial reason for this situation is that the filtering "input" matrix G has no direct counterpart in transport, composing only a part of the scattering quantity B. It's clear that the special filtering problem with Q = I will yield a parallel to the state x in transport by factoring the matrix B. What is not clear is what the physical interpretation of the resulting quantity should be, if any.

ii) It is interesting to compare the transport and filtering conditions which lead to existence (and uniqueness) of physically meaningful solutions to the two basic equations (5) and (10) for the Riccati matrices P and R. In the filtering situation, the standard conditions are $Q \geq 0$, $R > 0$, (F, G) controllable, (F, H) observable, while the transport conditions are B, D having non-negative off-diagonal elements, plus the probability conditions $\mathcal{M}(A + B) = \mathcal{M}(C + D) = 0$ (conservative case). Mathematically, these are quite different conditions but yet, in a certain sense, they say the same thing: a positivity condition plus an additional constraint on the internal system structure yields a global solution to the appropriate Riccati equation. It is tantalizing to speculate as to whether this observation can be exploited to yield new asymptotic results in both filtering and transport theory under "unnatural" operating circumstances, i.e. when Q, R may not be covariances and/or when A, B, C, D are not probability matrices. We shall say more on this later.

5. Dimensionality Reduction and Generalized X-Y Functions

The entire line of investigation pursued in this paper was originally motivated by the observation [6] and exploitation [7-9] of the fact that certain special functions introduced into transport theory by Chandrasekhar could be used to significantly reduce the computational burden associated with certain optimal filtering and control processes. As motivation for continued study of the transport/filtering isomorphism, we shall review some of these results in this section. For the remainder of this section, we shall restrict attention to problems with constant coefficient matrices or, what is the same thing, to neutron transport through a homogeneous rod.

We begin with the following general lemma from which all else follows directly.

Riccati lemma [7-8]: Let the n x m matrix R(x) satisfy the matrix Riccati equation

$$\frac{dR}{dx} = A + BR + RC + RDR \quad , \qquad R(0) = F \quad ,$$

where A, B, C, D, F are constant matrices of the appropriate sizes. Furthermore, let

$$\text{rank } Z \ (= A + BF + FC + FDF) = p \quad ,$$

$$\text{rank } D = r \quad ,$$

and assume Z, D are factored as

$$Z = Z_1 Z_2 \quad ,$$

$$D = GH \quad ,$$

where Z_1, Z_2, G, H are of sizes n x p, p x m, m x r, r x n, respectively. Then R satisfies the algebraic relation

$$BR(x) + R(x)C = L_1(x)L_2(x) - K_1(x)K_2(x) - A \quad ,$$

where L_1, L_2, K_1, K_2 are matrices of sizes n x p, p x m, m x r, r x n satisfying the equations

$$\frac{dL_1}{dx} = [B + K_1(t)H]L_1 \quad , \qquad L_1(0) = Z_1 \quad ,$$

$$\frac{dL_2}{dx} = L_2[C + GK_2(t)] \quad , \qquad L_2(0) = Z_2 \quad ,$$

$$\frac{dK_1}{dx} = L_1L_2G \quad , \qquad K_1(0) = FG \quad ,$$

$$\frac{dK_2}{dx} = HL_1L_2 \quad , \qquad K_2(0) = HF \quad .$$

The importance of the Riccati lemma is that the functions K_1, K_2 have the definitions

$$K_1 = RG \quad , \qquad K_2 = HR \quad .$$

In most applications where the Riccati equation occurs, it turns out that these functions represent important physical variables, often more basic to the problem than the Riccati function itself [7]. Thus, if p and r are small as compared with n, m, the L-K system of the lemma represents far fewer equations with which to calculate the functions K_1, K_2 than the original Riccati equation for R. Also, the number of equations can be further reduced if $F = F'$, $A = A'$, $D = D'$, and $B = C'$, the usual type of symmetry condition that occurs in applied problems. In this case, $L_1 = L_2'$, $K_1 = K_2'$, halving the size of the L-K system. Since the functions L_i, K_i are generalizations of the basic X-Y functions introduced into radiative transfer decades ago by Chandrasekhar and Ambartsumian, we call L and K generalized X and Y functions.

Another important relationship coming out of the proof of the Riccati lemma is that

$$\dot{R}(t) = L_1(t)L_2(t) \quad ,$$

L_1, L_2 being <u>defined</u> as

$$L_1 = \mathscr{U}Z \quad , \qquad L_2 = Z_2\mathscr{V} \quad ,$$

with $\mathscr{U}(t)$ and $\mathscr{V}(t)$ satisfying the equations

$$\dot{\mathscr{U}} = (B + RD)\mathscr{U} \quad , \qquad \mathscr{U}(0) = I \quad , \tag{28}$$

$$\dot{\mathscr{V}} = \mathscr{V}(C + DR) \quad , \qquad \mathscr{V}(0) = I \quad . \tag{29}$$

Comparing Eqs. (28)-(29) with Eqs. (11) and (16) and making the identification of matrices

Transport Problem	Riccati lemma
B	A
C	B
A	C
D	D ,

we see that

$$\mathscr{U}(x) \equiv S(x) \quad , \tag{30}$$

$$\mathscr{V}(x) \equiv T(x) \quad . \tag{31}$$

These equations, in conjunction with the algebraic Stokes' relations (19)-(20) and the Riccati lemma, suggest that a substantial reduction in the computing effort necessary to obtain R and T may be achieved if the backscattering matrices B and D have low rank. We shall indicate in a moment what can be done, but first it is illuminating to re-examine the physical situation.

Recalling the definitions of the entries in the reflection and transmission matrices R and T (or U and S), the (i-j)-element tells us the intensity of emerging particles in state i, due to a unit incident particle in state j. With this interpretation, it is natural to ask about the total intensity of reflected or transmitted particles of type i, given a mixture of incident intensities ω_j, that is, instead of a single unit intensity of a given type j, we have an incident flux composed of several different types of particles with respective intensities ω_j, $j = 1,\ldots,N$. To describe this situation (in the case $d = 0$), we let

$X_i(t)$ = the total intensity of particles in state i, reflected from a rod of length t upon which ω_j particles of type j are incident at the left end, $j = 1,\ldots,N$;

$Y_i(t)$ = the total intensity of particles in state i transmitted through a rod of length t upon which ω_j particles of type j are incident at the left end, $j = 1,\ldots,N$.

Clearly, by linearity

$$X_i(t) = \sum_{j=1}^{N} R_{ij}(t)\omega_j \quad ,$$

$$Y_i(t) = \sum_{j=1}^{N} T_{ij}(t)\omega_j \quad , \qquad i = 1,\ldots,N \quad ,$$

or, in vector-matrix form

$$X(t) = R(t)m \quad ,$$

$$Y(t) = T(t)m \quad ,$$

where m is the $N \times 1$ vector whose jth entry is ω_j.

The relevance of the Riccati lemma in the foregoing context is clear. For example, to obtain an equation for $X(t)$, we use the relation

$$\dot{R} = L_1 L_2$$

to see that

$$\dot{X}(t) = L_1(t) L_2(t) m \quad , \tag{32}$$

with $X(0) = 0$ (if there is no reflector at $t = 0$). In fact, the situation is even simpler if the backscattering matrix D factors as $D = GH$, with G happening to equal m. In this case $X(t) \equiv K_1(t)$ and the auxiliary equation (32) is redundant. A similar argument can be used to derive an equation for $Y(t)$ only this time it is necessary to apply the Riccati lemma to the reflection equation (15) for $U(t)$ and utilize Eqs. (18) and (31). The result is

$$\dot{Y} = (A + UB)Y \quad , \qquad Y(0) = m \quad , \tag{33}$$

which simplifies upon application of the lemma to decompose U.

Another useful piece of new information which the Riccati lemma supplies to the transport situation is a set of algebraic formulas relating the basic observable quantities R, U, S, T to the potentially much lower dimensional functions L_1, L_2, K_1, and K_2. First of all, translating the conclusion of the lemma into the neutron transport notation, the formula for the reflection function R is obtained as

$$CR(t) + R(t)A = L_1(t)L_2(t) - K_1(t)K_2(t) - B \quad . \tag{34}$$

Applying the Riccati lemma to the reflection function U and utilizing the earlier relations (23)-(24), we are led to similar algebraic relations for the transmission functions S and T. It is interesting to observe that, although it is possible to obtain algebraic representation formulas for either R or U alone in terms of the lower dimensional functions of whichever version of the lemma is appropriate, it appears that, in general, corresponding formulas for S and T require both reflection functions and their associated reduced dimension auxiliary functions to be used. The only apparent exception to this situation is for an isotropically-scattering, homogeneous rod which we will now pursue both from a transport and filtering theory point of view.

6. Further Simplifications

Imposition of the basic structural property of constancy (or homogeneity of the medium) was shown in the preceding section to lead to significant simplifications of basic transport theory phenomena. A natural supposition is that further simplification will be obtained by superimposing additional structure upon the basic physical processes. In light of the natural symmetry situations that occur in the filtering problem, as well as their interpretation as the simplest possible scattering law in the transport setting, we now explore the consequences of assuming that $A = A'$, $D = D'$, $B = C'$ and $F = F'$ in the Riccati lemma. Examination of the various ramifications of this additional mathematical structure will be our objective in this section.

To begin with, as previously observed the assumed symmetry conditions immediately lead to the properties

$$L_1 = L_2' = L \quad , \tag{35}$$

$$K_1 = K_2' = K \quad . \tag{36}$$

Thus, there is only a single L and a single K function. In addition, it is easy, to see that

$$\mathscr{U} = \mathscr{V}' \quad , \tag{37}$$

an important fact for later results. For purposes of exposition, it is convenient to temporarily separate the implications of constancy, symmetry, and the lemma for the transport and filtering problems. The two streams of results will later converge to yield a unified picture. (Henceforth, we shall use the notations introduced for the physical processes in Sections 2-3, automatically translating the lemma without explicit mention.)

Neutron Transport

As discussed earlier, the assumptions of constancy and symmetry imply a homogeneous medium with an "almost" isotropic scattering law. The precise scattering law $A' = C$ saying that the probability of a right-moving particle being forward scattered from state j to state i equals the probability of a left-moving particle being forward scattered from state i to state j. This is not the type of scattering that we usually associate with isotropy unless $A = C$. Thus, we shall impose this additional condition which, of course, together with $A' = C$ implies A and C are symmetric. In addition, to satisfy the isotropy requirement we must also impose the condition $B = D$. Hence, the homogeneous, isotropically scattering rod is mathematically characterized by the two conditions:

i) A, B, C, D constant, symmetric matrices,

ii) $A = C \quad , \qquad B = D \quad .$

Returning to the equations for the reflection and transmission matrices R, U, S, T, it is easily verified that under isotropic scattering (even in the inhomogeneous case),

$$R = R' = U \quad , \qquad T = T' = S \quad . \tag{38}$$

This situation results in the particularly simple forms for the relations (19)-(24):

$$\dot{R} = TBT \quad , \tag{39}$$

$$TBT - RBR = B + RA + AR \quad , \tag{40}$$

$$T(A + BR) = (A + RB)T \quad . \tag{41}$$

Superimposing homogeneity upon the medium, and assuming that the backscattering law B factors as

$$B = \mathscr{B}\mathscr{B}' \quad ,$$

the functions L and K of the lemma satisfy

$$\dot{L}(t) = [A + K(t)\mathscr{B}']L \quad , \qquad L(0) = \mathscr{B} \quad , \tag{42}$$

$$\dot{K}(t) = LL'\mathscr{B} \quad , \qquad K(0) = 0 \quad . \tag{43}$$

The appropriate equations for the total intensity vectors X and Y are obtained from Eqs. (32)-(33) as

$$\dot{X}(t) = L(t)L'(t)m \quad , \qquad X(0) = 0 \quad , \tag{44}$$

$$\dot{Y}(t) = [A + K(t)\mathscr{B}']Y \quad , \qquad Y(0) = m \quad . \tag{45}$$

We note in passing that the definition of K shows that

$$\mathscr{B}'X(t) = K'(t)m \quad , \tag{46}$$

which, in principle, would allow computation of X directly from the functions L and K by means of the pseudoinverse of $\mathscr{B}'$.

The filtering theory reader may wonder to what extent the simplifications (38)-(46) hold if we consider the scattering law which corresponds to the usual symmetry conditions of filtering, i.e. $A = C'$, $B = B'$, $D = D'$. It is easy to see that in this case

$$R = R' \quad , \qquad U = U' \quad , \qquad S = T' \quad , \tag{47}$$

but, in general, there is no relationship between R and U. However, when considering <u>either</u> R or U, we still obtain a single L and a single K function as before. Thus, even in this "bizarre" physical situation most of the basic simplification achieved by isotropic scattering is retained. It would be interesting to know if there are reasonable physical transport processes which obey this "filtering"-scattering law, rather that the conventional isotropic law. To date the author knows of no such situation. We shall explore more of the mathematical consequences of the filtering-scattering law in the next subsection.

<u>Linear Filtering Theory</u>

Armed with the numerous results of neutron transport, we now focus on the question of what the Riccati lemma has to say about filtering theory.

Recalling the basic Riccati equation (5), the Riccati lemma first gives the equations for L and K in the form

$$\dot{L} = [F - K(t)R^{-\frac{1}{2}}H]\,L \quad , \qquad L(0) = \mathcal{G}^{\frac{1}{2}} \quad , \tag{48}$$

$$\dot{K} = -L(t)L'(t)H'R^{-\frac{1}{2}} \quad , \qquad K(0) = -\Gamma H'R^{-\frac{1}{2}} \quad , \tag{49}$$

where $\mathcal{G}$ is the symmetric, positive semidefinite matrix

$$\mathcal{G} = GQG' + F\Gamma + \Gamma F' - \Gamma H'R^{-1}H\Gamma \quad .$$

It is of special importance to note that the definition of K is

$$K(t) = -P(t)H'R^{-\frac{1}{2}} \quad . \tag{50}$$

Thus, from Eq. (4) we see that the equation of the optimal estimate of the state is

$$\frac{d\hat{x}}{dt} = F\hat{x} - K(t)R^{-\frac{1}{2}}[z(t) - H\hat{x}] \quad , \qquad \hat{x}(0) = 0 \quad . \tag{51}$$

Since K is an n x p matrix, where p is the dimension of the observation process, while L is an n x r matrix where $r = \text{rank}\ \mathcal{G}$, Eqs. (48)-(49) constitute a substantial computational improvement over the Riccati equation (5) if p and r are small relative to n. (<u>Special case</u>: if there is complete knowledge of the initial state x_0, then $\Gamma = 0$ and $\text{rank}\ \mathcal{G} = m$, the dimension of the system noise.)

From Eq. (28) and Eq. (6a) for the error covariance S of the initial and current state, we see that

$$S(t) = \mathscr{U}(t)\Gamma \quad . \tag{52}$$

This relation suggests that if Γ is of low rank, say s, and factors as $\Gamma = \Gamma_1\Gamma'_1$, where Γ_1 is an n x s matrix, it will be possible to compute S by lower-dimensional functions. Define

$$\mathscr{S}(t) = \mathscr{U}(t)\Gamma_1 \quad .$$

Then

$$\dot{\mathscr{S}}(t) = \left[F + K(t)R^{-\frac{1}{2}}H\right]\mathscr{S}(t) \quad , \qquad \mathscr{S}(0) = \Gamma_1 \quad , \tag{53}$$

and

$$S(t) = \mathscr{S}(t)\Gamma_1' \quad . \tag{54}$$

Of course, the lemma also gives the algebraic relation

$$FP(t) + P(t)F' = L(t)L'(t) - K(t)K'(t) - GQG' \quad . \tag{55}$$

There appear to be no additional results that can be squeezed out of the lemma. However, we can now begin to exploit the many relationships earlier derived in the neutron transport situation in order to suggest new relationships in filtering.

In order to explore the potential transfer of the transport relations to filtering, we can theoretically follow two paths:

(1) impose the "isotropic scattering" condition on the filtering matrices and directly transcribe the new results, or (2) use only the natural symmetry conditions of the filtering problem and use the transport relationships to suggest the form that analagous filtering formulas should take. Actually, we have little choice but to follow route (2) since the hypotheses of the first approach lead to physically meaningless filtering problems (negative definite covariance matrices).

At the end of the preceding sub-section, we noted that the symmetry conditions of filtering, i.e. $A = C'$, $B = B'$, $D = D'$, $Z = Z'$, in the context of the transport Riccati equation for R, lead to the conditions $R = R'$, $U = U'$, $S = T'$. Referring to Table 1, we see that these simple facts suggest the desirability of introducing some new quantities into the filtering theory picture, namely, quantities to correspond to the transport functions U and S. Recalling Eqs. (15)-(17) and using Table 1, it is easy to see that these new functions, denoted by $\overline{U}$ and $\overline{S}$, must satisfy the equations

$$\dot{\overline{U}}(t) = -(H'R^{-1}H) + \overline{U}F + F'\overline{U} + \overline{U}GQG'\overline{U} \quad , \tag{56}$$

$$\dot{\overline{S}}(t) = \left[F - P(t)H'R^{-1}H\right]\overline{S} \quad , \tag{57}$$

$$\dot{\overline{S}}(t) = \overline{S}\left[F + GQG'\overline{U}\right] \quad , \tag{58}$$

with

$$\overline{U}(0) = 0 \quad , \tag{59}$$

$$\overline{S}(0) = I \quad . \tag{60}$$

But Eq. (57) is, modulo the initial condition, identical to the earlier derived filtering quantity S, hence $S = \overline{S}\Gamma$. However, Eq. (58) is new. Also, the reader should note that Eq. (56) is not the equation one would obtain by dualizing the filtering problem.

Utilizing the symmetry of P and $\overline{U}$, from Eqs. (57)-(58), we have the interesting algebraic relation

$$\overline{S}(F + GQG'\overline{U}) = (F - PH'R^{-1}H)\overline{S} \quad , \tag{61}$$

the filtering counterpart of (24). Letting $\Gamma^{\#}$ denote the pseudoinverse of Γ (since, in general, Γ is only positive semi-definite), we can rewrite (61) as

$$S(t)\Gamma^{\#}(F + GQG'\overline{U}(t)) = (F - P(t)H'R^{-1}H)S(t)\Gamma^{\#} \quad , \tag{62}$$

valid even for time-varying F, G, H, Q, and R.

With the Riccati lemma proof as background, it is also not hard to see that the Stokes' relations for filtering theory (for constant coefficients) are

$$\dot{\overline{U}}(t) = - \overline{S}(t)H'R^{-1}H\overline{S}(t) \quad , \tag{63}$$

$$\dot{P}(t) = \overline{S}(t)GQG'\overline{S}'(t) \quad . \tag{64}$$

7. Conservation Laws

In neutron transport theory, the conservation law (14), together with the non-negativity of the elements of the basic matrices R, T, $\overline{L}$, enables one to assert that the basic equations (10)-(12) have unique solutions for all $t > 0$ and that the limiting solutions satisfy the algebraic equations (25)-(27). This is an entirely different approach than the one generally followed in filtering theory to prove existence of a steady-state solution to the error covariance equation (5). In addition, the conservation law itself is of some independent interest as a mathematical statement of a basic property of the physical process. Our main objective in this section will be to derive an analagous conservation law for the linear filtering problem.

We first note that the basic symmetry and definiteness assumptions on the matrices F, G, H, Q, R, and Γ are sufficient to insure that the Riccati equation for P has a unique solution for any finite interval. Thus, in contrast to the transport situation, there is no question here of utilizing a conservation law to prove global existence. It is guaranteed for any physically meaningful filtering problem. Our interest, then, will be in deriving a pointwise conservation law, followed by an appeal to the standard results to insure its global validity.

The main new result in this direction is our

<u>Conservation Theorem for Linear Filtering:</u> <u>Let N, M be n x n matrices such that the matrices</u>

$$F' - 3H'R^{-1}H\Gamma - N$$

and

$$GQG' + 3F\Gamma + M$$

<u>have null spaces with a nontrivial intersection spanned by the vectors $\eta_1, \eta_2, \ldots \eta_k$. Form the n x k matrix $\mathcal{N}$ as</u>

$$\mathcal{N} = [\eta_1 \vdots \eta_2 \vdots \ldots \vdots \eta_k] \quad .$$

<u>Then the basic filtering matrices P, S, and W satisfy the conservation law</u>

$$(P(t) + S(t) + W(t))\mathcal{N} = 3\Gamma\mathcal{N} \quad . \tag{65}$$

<u>Proof</u>. From the differential equations (5), (6a), and the equation for W, we have

$$\frac{d}{dt}[(P + S + W - 3\Gamma)\mathcal{N}] = \frac{d}{dt}[(P + S + W)\mathcal{N}]$$

$$= (GQG' + FP + PF' - PH'R^{-1}HP)\mathcal{N}$$

$$+ \ (F - PH'R^{-1}H)S\mathcal{N}$$

$$+ \ [(F - PH'R^{-1}H)W - PN + M]\mathcal{N}$$

$$= F(P + S + W - 3\Gamma)\mathcal{N} + 3F\Gamma\mathcal{N}$$

$$- PH'R^{-1}H(P + S + W - 3\Gamma)\mathcal{N}$$

$$- 3PH'R^{-1}H\Gamma\mathcal{N} + P(F' - N)\mathcal{N}$$

$$+ \ (GQG' + M)\mathcal{N}$$

$$= F(P + S + W - 3\Gamma)\mathcal{N}$$

$$- PH'R^{-1}H(P + S + W - 3\Gamma)\mathcal{N}$$

$$+ P(F' - N - 3H'R^{-1}H\Gamma)\mathcal{N}$$

$$+ \ (GQG' + M + 3F\Gamma)\mathcal{N}$$

$$= (F - PH'R^{-1}H)(P + S + W - 3\Gamma)\mathcal{N} \ .$$

This is an equation of the form

$$\dot{Z} = (F - PH'R^{-1}H)Z \quad , \qquad Z(0) = 0 \quad ,$$

which clearly has the unique solution $Z \equiv 0$ within the domain of existence of $P(t)$. Our earlier remarks show that (65) then holds for all t.

Remarks:

1) If the hypotheses of the theorem can be satisfied with $M = N = 0$, then $W \equiv S$ and the conservation law simplifies to

$$(P + 2S)\mathcal{N} = 3\Gamma\mathcal{N} \ .$$

2) In general, it will be necessary to choose M and N to satisfy the Theorem. In an earlier section, it was shown that $M, N, \neq 0$ corresponds to an assumption that the initial error $e^*(0)$ is correlated with functionals of the two noise processes u and v. This is an unusual assumption in the filtering problem. An alternative point of view, is to regard the equation for W as a purely mathematical artifice introduced to make the Conservation Theorem "work out." Clearly, from this viewpoint any choice of M and N is admissible.

3) Aside from its possible theoretical interest, the Conservation Law (65) may be useful in practical situations in which high accuracy numerical solutions of the Riccati equation (5) are difficult to achieve. Since the equations for S and W are linear, very efficient schemes exist for accurately computing their solutions. The Conservation Law (65) may then be employed to control the accuracy in calculation of P, perhaps in a type of predictor-corrector mode.

8. Steady-State Solutions

An important consideration in both filtering and transport is the analysis and determination of the basic physical quantities when the observation time or rod length approaches $+\infty$. Thus far, we have gnawed around the edges of this question, but in this section we will make it the main course.

Standard results in both filtering and transport show that the "steady-state" equations for the basic quantities are the algebraic relations (25)-(27) and (6), (6b), with the understanding that only the physically meaningful solutions are considered, e.g. the positive-semi-definite solution in (6), the non-negative solution in (25). The basic question we want to consider in this section is whether or not an infinite interval version of the Riccati lemma equations L and K exists. Consideration of the differential equations for L_1, L_2, K_1, K_2 of the lemma shows that the naive approach of setting the derivatives equal to zero yields no useful algebraic relations other that $L_1(\infty)L_2(\infty) = 0$. This is in sharp contrast to the usual situation for the Riccati equation. Our answer to this dilemma is the

Steady-State Theorem: Let R_∞ be any steady-state solution of the algebraic Riccati equation

$$A + BR_\infty + R_\infty C + R_\infty D R_\infty = 0 \quad . \tag{66}$$

Assume that D factors as in the Riccati lemma, i.e. $D = GH$ and let $K_1 = R_\infty G$, $K_2 = HR_\infty$. Then K_1 and K_2 satisfy the algebraic equations

$$\sigma(K_1) = -(G'\otimes I)(I\otimes B + C'\otimes I)^{-1}\,\sigma(A + K_1K_2) \quad , \tag{67}$$

$$\sigma(K_2) = -(I\otimes H)(I\otimes B + C'\otimes I)^{-1}\,\sigma(A + K_1K_2) \quad , \tag{68}$$

where $\otimes$ denotes the usual Kronecker product and σ is the column "stacking" operation, i.e. if

$$A = [a_1 \vdots a_2 \vdots \cdots \vdots a_n] \quad ,$$

then

$$\sigma(A) = (a_{11}\ a_{21}\ \cdots\ a_{n1}\ a_{12}\ a_{22}\ \cdots\ a_{n2}\ \cdots\ a_{nn})' \quad .$$

Proof. Apply σ to Eq. (66), and utilize the property $\sigma(PAQ) = (Q'\otimes P)\sigma(A)$, valid for any P, A, Q for which the product is defined.

Remarks:

If $\{\lambda_i\}$ and $\{\mu_j\}$ are the characteristic roots of B and C, respectively, then we must have $\lambda_i + \mu_j \neq 0$ for all $i, j = 1,\ldots,N$. If not, then $\det\,[I\otimes B + C'\otimes I] = 0$ and Eqs. (67)-(68) are no longer valid.

Applying the Steady-State Theorem to the filtering equation

$$GQG' + F\underset{\infty}{P} + \underset{\infty}{P}F' - \underset{\infty}{P}H'R^{-1}H\underset{\infty}{P} = 0 \quad ,$$

and using the fact that the symmetry conditions yield $K_1 = K_2'$, we can see that the steady-state function $\hat{K}$, defined as $\hat{K} = \underset{\infty}{P}H'R^{-\frac{1}{2}}$, satisfies the equation

$$\sigma(\hat{K}) = (R^{-\frac{1}{2}}H \otimes I)(I \otimes F + F \otimes I)^{-1}\sigma(\hat{K}\hat{K}' - GQG') \quad . \tag{69}$$

The steady-state gain, K_∞, is then given as

$$K_\infty = \hat{K}R^{-\frac{1}{2}} \quad . \tag{70}$$

A more complete discussion of the Steady-State Theorem, together with its implications for the inverse problem of optimal control theory, is given in [10].

9. Integral Equations and Source Functions

In classical radiative transfer in the atmosphere, as well as in filtering theory a la Wiener-Kolmogorov, the hub around which all else revolves is a Fredholm integral equation. In radiative transfer, the basic integral equation for the "source function" describes the rate of production of particles at a particular point in the atmosphere and the auxiliary functions R, T, X, Y and L are expressed as linear functionals of its solution. In filtering theory, the basic Fredholm integral equation is for the so-called impulse-response function, h, in terms of which the optimal estimate is determined as a linear functional of the observed signal, the integrand of the functional being h. Since the source function equations for the rod model of Section 3 have not previously appeared in the transport literature, one of our goals in this section is to give a derivation of the relevant quantities and to show their relationship to the generalized X-Y functions of Section 5. Following this transport theory development, we specialize to the "filtering"-scattering law and show the connection between the integral relations for the rod source function and the basic filtering quantities P and S.

We begin by considering the general matrix Riccati equation

$$\dot{R} = A + BR + RC + RDR \quad , \qquad R(0) = F \quad ,$$

under the same assumptions as the Riccati lemma, i.e. $Z \equiv A + BF + FC + FDF = Z_1 Z_2$, $D = GH$. Our goal is to introduce integral relations in terms of which the generalized X-Y functions, L_1, L_2, K_1, K_2 can be expressed as linear functionals and to relate these integral relations to R. The basic new result in this direction is the

<u>Integral Representation Theorem.</u> <u>Let k(t,s) be the matrix kernel</u>

$$k(t,s) = e^{Bt}Fe^{Cs} + \int_0^{\min(t,s)} e^{B(t-\xi)}Ae^{C(s-\xi)}d\xi \quad , \qquad (71)$$

$$0 \le t,s < \infty \quad ,$$

<u>and let $J^1(t,x)$, $J^2(t,x)$, $U^1(t,x)$, $U^2(t,x)$ satisfy the coupled integral equations</u>

$$J^1(t,x) = e^{B(x-t)}Z_1 + \int_0^x U^2(s,x)He^{B(s-t)}ds\, Z_1 \quad , \qquad (72)$$

$$J^2(t,x) = Z_2e^{C(x-t)} + Z_2\int_0^x e^{C(s-t)}GU^1(s,x)ds \quad , \qquad (73)$$

$$U^1(t,x) = Hk(t,x) + H\int_0^x k(t,s)GU^1(s,x)ds \quad , \qquad (74)$$

$$U^2(t,x) = k(x,t)G + \int_0^x U^2(s,x)Hk(s,t)ds\, G \quad , \qquad (75)$$

$$0 \le t \le x < \infty \quad .$$

<u>Then the functions $R(x)$, $L_1(x)$, $L_2(x)$, $K_1(x)$, $K_2(x)$ of the Riccati lemma satisfy the relations</u>

$$L_1(x) = J^1(0,x) \quad , \tag{76}$$

$$L_2(x) = J^2(0,x) \quad , \tag{77}$$

$$K_1(x) = U^2(x,x) \quad , \tag{78}$$

$$K_2(x) = U^1(x,x) \quad , \tag{79}$$

$$\begin{aligned} R(x) &= k(x,x) + \int_0^x k(x,s)GU^1(s,x)ds \\ &= k(x,x) + \int_0^x U^2(s,x)Hk(s,x)ds \quad , \qquad x \geq 0 \quad . \end{aligned} \tag{80}$$

<u>Proof</u>:

Let

$$M(x) = k(x,x) + \int_0^x k(x,s)GU^1(s,x)ds \quad .$$

Differentiating, we obtain

$$\begin{aligned} \frac{dM}{dx} &= A + B\,k(x,x) + k(x,x)C + k(x,x)D\,M(x) \\ &+ B\int_0^x k(x,s)GU^1(s,x)ds \\ &+ \int_0^x k(x,s)GU^1(s,x)\left[C + DM(x)\right]ds \quad , \end{aligned}$$

where we have used the result

$$U_x^1(t,x) = U^1(t,x)\left[C + GU^1(x,x)\right] \quad , \qquad t \leq x \quad ,$$

which is easily obtained from Eq. (74). Hence, we see that M satisfies the Riccati equation

$$\frac{dM}{dx} = A + BM + MC + MDM \quad , \qquad x > 0$$

with

$$M(0) = F \quad .$$

Thus, by uniqueness $M(x) \equiv R(x)$ and the first equation in (80) is established. The second equation follows in a similar manner using $U^2(t,x)$ instead of U^1. We now obtain relations (78)-(79) through use of (80), direct substitution into the defining equations (74)-(75), and the definitions of K_1 and K_2 from the Riccati lemma. It remains to establish (76)-(77).

Differentiate Eq.(72) with respect to x. This yields

$$\begin{aligned} J^1_x(t,x) &= Be^{B(x-t)}Z_1 + U^2(x,x)He^{B(x-t)}Z_1 \\ &\quad + \int_0^x U^2_x(s,x)He^{B(s-t)}ds\ Z_1 \quad , \\ &= Be^{B(x-t)}Z_1 + U^2(x,x)He^{B(x-t)}Z_1 \\ &\quad + \int_0^x [B + U^2(x,x)H]U^2(s,x)He^{B(s-t)}ds\ Z_1 \\ &= [B + K_1(x)H]J^1(t,x) \quad , \qquad x > t \quad , \end{aligned}$$

where we have used Eq.(78) and the differential equation for U^2. At $x = t = 0$,

$$J^1(0,0) = Z_1 \quad .$$

Thus, $J^1(0,x)$ and $L_1(x)$ satisfy the same initial value problem which, by uniqueness, establishes (76). Relation (77) is obtained in a similar manner.

Remarks

1) The differential equations for J^1, J^2, U^1, U^2 represent a substantial generalization of the L-K functions obtained through the Riccati lemma. In fact, the proof of the Riccati lemma may be carried out beginning with the representation (80) for R and the differential equations for the U and J functions. The details are straightforward, tedious, and are left to the interested reader. The key to this approach is to recognize that $J^1(0,x)J^2(0,x) = \frac{dR}{dx}$.

2) As before, under the usual symmetry conditions for A, B, C, D, F, we have $J^1 = (J^2)'$ and $U^1 = (U^2)'$ thereby reducing the number of basic functions by a factor of two.

3) The functions U^1 and U^2 have been used in [11] to establish equivalences between matrix Riccati equations, Fredholm resolvents, and the Bellman-Krein formula for integral operators. However, the functions J^1 and J^2, which enable us to establish the L-K equations of the Riccati lemma, do not seem to have been in the literature before. It is interesting to note that various "invariant imbedding" techniques for solving Fredholm integral equations [11, 12] rely only upon the U^1 and U^2 functions, with J^1 and J^2 entering only as catalysts to provide an alternative to the Riccati equation (or to the Fredholm resolvent) in those cases where a low rank structure exists.

We now turn our attention to the use of the Integral Representation Theorem in transport and filtering. Since the transport implications are fairly clear, we shall only remark on the results there, concentrating our attention upon the filtering situation.

Transport Relations. The relevance of the Integral Representation Theorem to transport problems is made most apparent by formula (80) for the reflection function R. Upon substitution of the defining equation for k (Eq. (71)), we see how R is generated by the various scattering processes taking place at each interior point of the rod. In essence, Eq.(80) says that the reflected intensity is composed of the reflections from an infinite number of infinitesinally small slabs each reflecting according to the scattering rule defined by A, B, C, D and F. Further integral relationships involving the transmission functions S and T, as well as the "other" reflection function U may also be obtained by arguments similar to the above. Since these will be reported in a subsequent paper, we leave the derivations to the interested reader.

Filtering Theory. Assume the usual filtering theory symmetary conditions. Then it is easy to see that $U^1 = (U^2)'$, and $J^1 = (J^2)'$ and our basic integral quantities are (in filtering notation)

$$J(\sigma,t) = e^{F(t-\sigma)}Z_1 - \int_0^t U(s,t)R^{-\frac{1}{2}}He^{F(s-\sigma)}ds\ Z_1 \quad , \qquad (81)$$

$$U(\sigma,t) = k(t,\sigma)H'R^{-\frac{1}{2}} - \int_0^t U(s,t)R^{-\frac{1}{2}}Hk(s,\sigma)ds\ H'R^{-\frac{1}{2}} \qquad (82)$$

where

$$Z_1 Z_1' = GQG' + F\Gamma + \Gamma F' - \Gamma H' R^{-1} H \Gamma \quad . \tag{83}$$

Defining the new kernel

$$K(t,\sigma) = R^{-\frac{1}{2}} H k(t,\sigma) H' R^{-\frac{1}{2}} \quad ,$$

and recalling that the basic impulse response function h of linear filtering satisfies the Fredholm integral equation

$$h(\sigma,t) = K(t,\sigma) - \int_0^t h(s,t) K(s,\sigma) ds, \qquad \sigma \leq t \quad ,$$

it is an easy consequence of linearity to see that

$$h(\sigma,t) = R^{-\frac{1}{2}} H U(\sigma,t) \quad .$$

Thus, the differential equations for U enable us to calculate the important filtering quantity h. Details of this calculation procedure may be found in [12] along with many numerical examples.

Another useful result that falls out of the Integral Representation Theorem is the expression

$$P(t) = k(t,t) - \int_0^t U(s,t) R^{-\frac{1}{2}} H k(s,t) ds \quad ,$$

obtained from (80). In principle, this representation could be used in lieu of the algebraic formula of the Riccati lemma in order to produce P(t) for some fixed value of t, given the function U. The advantage here is that U is an n x p matrix function, while P is n x n. Alternatively, we could use the result

$$\frac{dP}{dt} = J(0,t)J'(0,t)$$

and a quadrature to obtain P. In either case, far fewer than $O(n^2)$ equations need be integrated if $p << n$.

10. Extensions and Generalizations

In this paper, we have shown connections between the simplest problem of optimal linear filtering and the simplest problem of one-dimensional neutron transport and have used this relationship to derive some new results in filtering theory. Since realistic problems in both filtering and transport often involve features other than the simple situations described above, it is clear that much more work remains in order to ferret out all relevant connections. Let us mention a few transport situations for which results are available that should be transferable to an analagous filtering problem with modest additional effort. Then we shall conclude with a discussion of some open points for future research.

Fission and Fusion - On the basis of the foregoing pages, it may be argued, and rightly so, that a complete equivalence (or isomorphism) between the filtering and transport problems has not been established. Instead, what has been shown is that the two subjects have many common features that *suggest* a possible equivalence if certain obstacles could be cleared away, the most serious one being the actual physical interpretation of the scattering matrices A, B, C, D as probabilities versus the interpretation of the associated filtering quantities F', GQG', F, $-(H'R^{-1}H)$. The argument would be that the usual demands that A, B, C, D be probability matrices is unrealistic for the filtering problem and, hence, there are many filtering problems which do not correspond to any transport problem and conversely. Within the purview of the transport model given in Section 3, this objection is perfectly valid. However, the basic difficulty is not with the equivalence of the two subjects, but rather it is with the physical model. We have chosen the simplest possible transport problem for expository reasons; but, two vitally important features have been omitted: fission and fusion, i.e. the creation and annihilation of particles during the process of interaction with the medium.

The inclusion of fission and fusion into our model requires that we reinterpret the basic scattering matrices A, B, C, D. Instead of their previous probabalistic meanings, we must now assume that their (i,j) entries represent the expected number of particles of type i that appear moving in the appropriate direction as a result of interaction with the medium of a single particle of type j moving in the relevant direction. Clearly, positive entries greater than 1 correspond to fission, negative entries to fusion. With this interpretation of the scattering matrices, it is clear that any filtering problem can be put into 1-1 (but not onto) correspondence with a scattering process.

Variational Formulations - It is, of course, explicit in the formulation of the filtering problem that the optimal estimator $\hat{x}$ is to be chosen in such a way that the mean-square error is minimized, i.e. we must solve a quadratic variational problem, subject to a linear differential side constraint. However, the usual formulation of the transport problem does not proceed from a variational starting point, but rather from the two-point boundary value problem for the internal fluxes. Of course, the two points of view are equivalent but it may prove illuminating and useful to workers in transport theory to consider the subject within this framework.

As an example, the simple one-dimensional rod problem of Section 3 corresponds to the variational problem of minimizing (over u)

$$\int_0^t \left[(z(t), Bz(t)) - (u, D^{\frac{1}{2}}u) \right] dt \quad ,$$

$$\dot{z}(t) = Cz + D^{\frac{1}{2}}u \quad ,$$

if $A = C'$, $D = D'$, $B = B'$, i.e. the symmetry conditions are satisfied. It is interesting to note that the negative sign in the integrand takes this problem outside the usual confines of linear control theory and adds considerable mathematical spice to the situation when fission and fusion are allowed to occur. Also, it is an interesting question as to what variational problem might be associated with the non-symmetric case.

Time Dependent Scattering - Our transport model of Section 3 has dealt only with the spatial aspects of the neutron transport situation, the assumption being that all fluxes, intensities, etc. had reached their "steady-state" values. In contrast, the filtering problem was manifestly dynamic but, on the other hand, no spatial dependencies were considered. These two views are made apparent by the physical meaning attached to the independent variable t. In transport it represented the length of the rod, a spatial variable, while in filtering it was the observation interval, a time variable. For the mathematics, though, all that was important was that the independent variable be one-dimensional.

In an attempt to exploit the above considerations, we should note that the steady-state assumption is not too realistic for many actual transport processes. As a result, a great deal of work has been carried out for time-dependent processes [13]. As one might expect, the basic equations are now partial differential equations of various types, involving both spatial and time variables, in addition to other mathematically complexifying features. For example, if we let

$U(x, t)$ = the expected number of neutrons reflected from a homogeneous rod of length x in the time interval $[0, t]$ due to a single trigger neutron incident at time zero,

it can be shown that U satisfies the equation

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 1 + \int_0^t \frac{\partial U}{\partial t}(x, s)U(x, t - s)ds \quad ,$$

with the initial conditions

$$U(0, t) = 0 \quad , \qquad U(x, 0) = 0 \quad .$$

This is a nonlinear partial differential-integral equation; nevertheless, judicious use of the Laplace transform enables one to reduce its solution to manageable terms. Not surprisingly, a matrix Riccati equation plays a pivotal role in the analysis.

From what has gone before, it is reasonable to conjecture that if the injection of time-dependence into the transport situation leads to tractable equations, the inclusion of spatial dependencies in the filtering set-up should also be approachable by the same techniques.

New Geometries - No real scattering process takes place in a one-dimensional rod. Convenient as it is for analysis, the pressure of reality forces us to consider processes in which the scattering medium has a richer geometrical structure than a line. Many results have been obtained for problems in which the medium is a two-dimensional slab [14], a cylinder [15], a sphere [16], and other more exotic shapes. In the slab case, an analysis of the filtering/transport duality similar to the one of this paper, but more limited in scope, has been carried out with good success [2]. Some of the standard filtering concepts such as inpulse-response functions make their appearance in [2], whose analysis is based upon the fundamental Fredholm integral equations of transport rather than the Riccati equation. In addition, certain infinite-dimensional concepts arise in a natural manner when more general geometries are considered. We conjecture that these transport studies will prove useful in providing new directions in the analysis of infinite-dimensional filtering processes.

Nonlinear Problems - For many applications, the assumption of a linear model in either filtering or transport is not tenable. In these cases, new techniques are necessary. Generally speaking, these new techniques involve some sort of successive approximation/linearization idea. However, in some instances a direct approach is possible.

For example, when we allow particle-particle interaction in the rod problem for a particular type of scattering law, the relevant two-point boundary value problem (in the scalar case) is

$$\dot{u} = \sigma v - buv \quad , \qquad u(0) = 0 \quad ,$$

$$\dot{v} = - \sigma u + buv \quad , \qquad v(t) = d \quad ,$$

where u, v are the internal fluxes, and σ, b are constants. As expected, the reflection function associated with this situation satisfies a quasilinear first-order partial differential equation, the nonlinear analague of the Riccati equation. A detailed analysis of the foregoing problem is given in [17]. Generalizations to any nonlinear two-point boundary value problem may be found in [12].

The point of the above example is that for some types of nonlinear problems it is possible to develop feasible computational and analytic approaches without a direct and immediate appeal to linearization. Instead, we make an appeal to the physics of the situation to guide our attack.

Presumably, many of the same techniques will be applicable to filtering problems.

Discrete-Time Case - The original development of the filtering problem of Section 4 was carried out in discrete-time [18] and, in many situations, this is still the most fruitful way to discuss the problem. Our development has been concentrated on continuous time in order to make contact with the transport literature. However, there has been some transport work on discrete spaces [19] and, if one shifts the physical emphasis from neutrons and atmospheres to transmission lines, the discrete, rather than continuous, is the order of the day [20]. In either case, the parallels we have drawn between filtering and transport remain unchanged in principle, although a few of the formulas take on a different form. As further evidence of this fact, we refer to the interesting paper [21], in which the discrete-time version of the Riccati lemma is given (for the symmetric case).

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