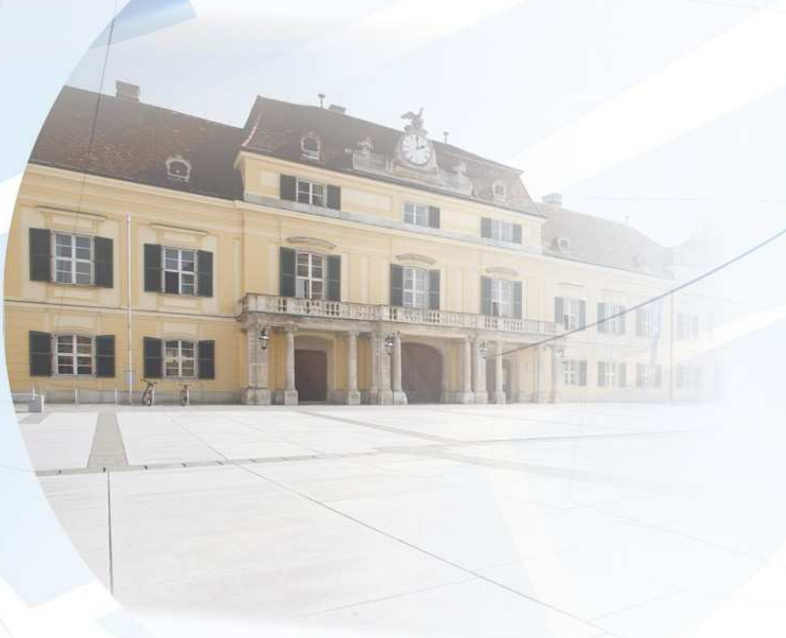
International Institute for
Applied Systems Analysis
I I A S A www.iiasa.ac.at

A Finitely Convergent Duality Theory for Zero-One Integer Programming

Bell, D.E. and Shapiro, J.F.

IIASA Research Memorandum
July 1975



RM-75-33

A FINITELY CONVERGENT DUALITY THEORY FOR ZERO-ONE INTEGER PROGRAMMING

David E. Bell

Jeremy F. Shapiro

July 1975

A Finitely Convergent Duality Theory for Zero-One Integer Programming

David E. Bell* and Jeremy F. Shapiro**

Abstract

Given an integer programming problem, a constructive procedure is presented for generating a finite sequence of increasingly stronger dual problems to the given problem. The last dual problem in the sequence yields an optimal solution to the given integer programming problem. It is shown that this dual problem approximates the convex hull of the feasible integer solutions in a neighborhood of the optimal solution it finds. The theory is applicable to any bounded integer programming problem with rational data.

This paper presents a complete duality for the zero-one integer programming problem

$$\begin{aligned} v &= \min cx \\ \text{s.t.} \quad Ax &= b \qquad\qquad \text{(IP)} \\ x_j &= 0 \text{ or } 1 \quad , \end{aligned}$$

where A is an $m \times n$ matrix, and the coefficients of A and b are integers. We let a_j denote a column A, c_j denote the cost coefficient of this column, and b_i denote the ith component of b.

Fisher and Shapiro [5] give a mathematical programming problem which is a dual to (IP) and which may solve it. In all cases, their IP dual problem provides lower bounds to v, the minimal objective function value in (IP). The greatest lower bound is shown to be at least as great as the lower bound attainable by solving the linear programming relaxation of (IP) ($0 \leq x_j \leq 1$), adding all Gomory cuts, and then solving the restricted linear programming relaxation.

The statement of the integer programming problem in [5] is more general than (IP) because the variables are allowed to take

---

*International Institute for Applied Systems Analysis, Laxenburg, Austria.

**Sloan School of Management, Massachusetts Institute of Technology.

on any non-negative integer values. We have chosen to express the constraints here in equality form for expositional convenience; the theory developed here generalizes without difficulty to any bounded integer programming problem with inequality as well as equality constraints. Since the essence of integer programming is the selection of small integer values for the decision variables, the assumption of boundedness of the variables is not a serious one.

Bell [1,2] gives a constructive method for resolving duality gaps if they appear in the IP dual problem of Fisher and Shapiro. Bell's method involves the construction of a sequence of groups of increasing size, called supergroups, and corresponding IP dual problems. In this paper, we give a different but related method for constructing a finite sequence of supergroups and IP dual problems terminating with one which yields an optimal solution to (IP). The more direct construction is due in part to the absence of technical difficulties presented by infinite sets and sums, and also due to the fact that we do not rely on basic representations of linear programming relations of (IP). Although IP dual problems are actually large scale linear programming problems, they can be solved, at least approximately, without great reliance on the simplex method.

For future reference, we define the set

$$F = \{x \mid Ax = b,\ x_j = 0 \text{ or } 1\} \quad . \tag{1}$$

We allow the possibility that F is empty. It is well known that (IP) can be solved in theory by solving the linear programming problem

$$\begin{aligned} &\min\ cx \\ &\text{s.t.}\quad x \in [F] \end{aligned} \tag{2}$$

where [ ] denotes convex hull. That is because the simplex method applied to (2) will yield an optimal extreme point x* of [F] which implies $x^* \varepsilon F$, and therefore, that it is an optimal solution to (IP). The difficulty with (2) is that [F] is generally difficult if not impossible to characterize. One of the interpretations of the duality theory presented here is that it approximates [F] in the neighborhood of an optimal solution to (IP).

The following section contains the iterative supergroup and IP dual procedure and the proof of convergence. Section 2 contains a short discussion about the relation of the IP dual problems to the set [F]. A numerical example is presented in Section 3. There are two appendices containing the relevant group theoretic constructs.

1. Supergroup Construction and Convergence Proof

The overall strategy presented here for solving (IP) is to construct a sequence of IP dual problems, each one strictly stronger than the previous one, until an optimal solution to (IP) is found, or it is proven infeasible. We show how this is done by: 1) constructing an IP dual problem from a given abelian group; 2) stating the algorithmic principles for solving it; 3) discussing the conditions under which it finds an optimal solution to (IP); and if an optimal solution to (IP) is not found, 4) constructing a supergroup and a stronger IP dual problem. Our development is self-contained and treats only briefly a number of considerations relating the IP dual methods to other IP methods and its practical use in solving IP problems. These considerations are discussed in a series of remarks.

Consider the abelian group

$$G = Z_{q_1} \oplus Z_{q_2} \oplus \cdots \oplus Z_{q_r}$$

where the q_i are positive integers and Z_{q_i} is the cyclic group of order q_i. Except for an initial group $G^0 = Z_1$, the q_i will be greater than one. Clearly, the order of G, denoted by $|G|$, is $\prod_{i=1}^{r} q_i$. Let $\varepsilon_1,\ldots,\varepsilon_m$ be any elements of G, and for any integer m-vector s, let $\phi(s)$ be the element of G given by

$$\phi(s) = \sum_{i=1}^{m} s_i \varepsilon_i$$

The mapping ϕ is used to construct the mathematical programming problem

$$\begin{aligned} v' &= \min cx \\ \text{s.t.}\quad Ax &= b \\ x &\in X \end{aligned}$$

where

$$X = \{x \mid \sum_{j=1}^{n} \alpha_j x_j \equiv \beta, \quad x_j = 0 \text{ or } 1\}$$

and $\alpha_j = \phi(a_j)$, $\beta = \phi(b)$. We use "$\equiv$" to denote group equality ; for the groups in this paper, the equations are systems of congruences.

Lemma 1: Problem (3) and (IP) are equivalent in the sense that

$$F = \{x \mid Ax = b\} \cap X;$$

that is, they have the same solution sets and therefore the same minimal objective function values.

Proof: The proof that $F \supseteq \{x \mid Ax = b\} \cap X$ is trivial given the zero-one constraints in X. Thus, it suffices to show $F \subseteq \{x \mid Ax = b\} \cap X$. To this end, let x be any solution in F. We have $x_j = 0$ or 1 for all j and

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad , \qquad i = 1,\ldots,m \quad .$$

Multiplying each row i by ε_i and summing gives us

$$\sum_{i=1}^{m} \varepsilon_i \sum_{j=1}^{n} a_{ij}x_j \equiv \sum_{i=1}^{m} \varepsilon_i b_i \quad .$$

But the order of summation can be interchanged yielding

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\varepsilon_i\right)x_j \equiv \sum_{i=1}^{m} \varepsilon_i b_i$$

which implies $x \in X$ by the definition of ϕ. ||

Although (3) and (IP) are equivalent problems, the reformulation (3) permits greater resolution when the following IP dual is constructed. For any $u \in R^m$, define the Lagrangean function

$$L(u) = ub + \min_{x \varepsilon X}(c - uA)x \quad . \tag{4}$$

If X is empty, then (IP) is infeasible and we can take $L(u) = +\infty$. Otherwise, it is well known and easily shown that $L(u) \leq v$ and L is a concave continuous function (Rockafellar [22]). The IP dual problem is constructed by finding the best lower bound

$$\begin{aligned} w &= \max L(u) \\ \text{s.t.} \quad & u \in R^m \end{aligned} \tag{D}$$

Although L(u) may be finite for any u, it may be that the maximum in (D) is $+\infty$ in which case (IP) is once again infeasible.

The relationship between the IP dual problem (D) and (IP) that we seek, but may not find, is summarized by the following.

Optimality Conditions: The pair (x^*, u^*) with $x^* \in X$ is said to satisfy the optimality conditions if

1) $L(u^*) = u^*b + (c - u^*A)x^*$
2) $Ax^* = b$.

It is easily shown that if x^* and u^* satisfy the optimality conditions, then they are optimal in (IP) and (D), respectively, and $w = v$. Thus, our strategy in trying to solve (IP) is to compute an optimal solution u^* to the concave unconstrained maximization problem (D) in the hope that we will find an $x^* \varepsilon X$ satisfying 1) and 2) to complement it.

To see how (D) can be optimized we rely on the fact that it is equivalent to the large scale linear programming problem

$$
\begin{aligned}
w = \max\ & \nu \\
\text{s.t.}\ & \nu \le ub + (c - uA)x^t \quad , \qquad t = 1,\ldots,T \qquad (5) \\
& \nu \ \varepsilon\ R^1 \\
& \mu \ \varepsilon\ R^m .
\end{aligned}
$$

where $X = \{x^1,\ldots,x^T\}$ since there can be at most 2^n feasible solutions to (IP). For any $u \ \varepsilon\ R^m$, we can see that the objective function in (5) is

$$\nu(u) = ub + \min_{t=1,\ldots,T} (c - uA)x^t$$

which is simply $L(u)$. Thus (D) and (5) are equivalent problems. The linear programming dual of (5) is

$$
\begin{aligned}
w = \min\ & \sum_{t=1}^{T} (cx^t)\lambda_t \\
\text{s.t.}\ & \sum_{t=1}^{T} (Ax^t)\lambda_t = b \qquad (6) \\
& \sum_{t=1}^{T} \lambda_t = 1 \\
& \lambda_t \ge 0 \quad , \quad t = 1,\ldots,T
\end{aligned}
$$

The primal dual simplex algorithm can be used to solve the pair of linear programming problems (5) and (6) and implicitly gives us the necessary and sufficient conditions for optimality of (D) which we now state.

Consider any $u \varepsilon R^m$, and let

$$T(u) = \{t | L(u) = ub + (c - uA)x^t\} \tag{7}$$

Then the complementary slackness conditions of LP tell us that u* is optimal in (D), equivalently (5), if and only if there is a solution to the system

$$\begin{aligned} &\sum_{t \varepsilon T(u^*)} (Ax^t)\lambda_t = b \\ &\sum_{t \varepsilon T(u^*)} \lambda_t = 1 \\ &\lambda_t \geq 0 \quad , \quad t \varepsilon T(u^*) \quad . \end{aligned} \tag{8}$$

Algorithmically, the primal-dual ascent algorithm given in Fisher and Shapiro [5] proceeds by testing a point $u \varepsilon R^m$ for optimality by trying to establish the conditions (8). If these conditions fail to hold, then a direction of increase of L is found and the algorithm proceeds to a new point u' such that $L(u') > L(u)$. The set T(u) may be large, however, and the algorithm in [5] begins with a subset T'(u) consisting of one or two elements and builds up T(u) until u is proven optimal in (D) or u' is found such that $L(u') > L(u)$.

Suppose the point u* is optimal in (D) and thus the following phase one problem has minimal objective function value equal to zero

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \left(s_i^+ + s_i^-\right) \\ \text{s.t.} \quad & \sum_{t \varepsilon T(u^*)} (Ax^t)\lambda_t + Is^+ - Is^- = b \\ & \sum_{t \varepsilon T(u^*)} \lambda_t = 1 \\ & \lambda_t \geq 0 \quad , \quad t \varepsilon T(u^*) \\ & s^+ \geq 0 \quad , \quad s^- \geq 0 \quad . \end{aligned} \tag{9}$$

Let λ_t^*, $t \in T(u^*)$, denote an optimal solution to (9). The solution $\tilde{x} = \sum_{t \in T(u^*)} \lambda_t^* x^t$ satisfies conditions 1) and 2) of the optimality conditions, and it is an optimal solution to (IP) if it is integer. Our concern is what to do if $\tilde{x}$ is not integer because more than one λ_t^* in (9) is positive.

Remarks: The computational effort required to do the Lagrangean minimization (4) is dependent on $|G|$. The time required is no more than a few seconds for $|G|$ up to 3,000. Computational experience with the primal-dual ascent algorithm on a single IP dual problem is given in Gorry [13]. This algorithm can spend too much time picking a direction of ascent which is an interpretation of the function of problem (9). An alternative approach which can be integrated with the primal-dual is subgradient optimization which generates a sequence of dual solutions $\{u^k\}_{k=1}^{\infty}$ to (D) by using $\lambda^k = b - Ax^t$ as a direction of ascent for some $t \in T(u^k)$. The new dual solution in this direction is $u^{k+1} = u^k + \theta_k \lambda^k$ and if $\theta_k \to 0^+$ and $\sum \theta_k \to +\infty$, then it can be shown that $L(u^k) \to w$ (Poljak [19]). Finite convergence to any value $\bar{w} < w$ can be achieved if

$$\theta_k = \rho_k \frac{(\bar{w} - L(u^k))}{||\lambda^k||}$$

where $||\lambda^k||$ denotes the Euclidean norm and $\varepsilon < \rho_k \leq 2$ for $\varepsilon > 0$ (Poljak [21]). Subgradient optimization has worked very well on large scale linear programming problems similar to (5) for approximating discrete optimization problems (Held and Karp [15], Held, Wolfe and Crowder [16], Marsten, Northup and Shapiro [17]). A hybrid computational approach is indicated using subgradient optimization as an opening strategy followed by the primal-dual or some other exact algorithm. The exactness is required for construction of effective supergroups. ||

The crucial observation for modifying (D) when it fails to solve (IP) is that (IP) is equivalent to another integer programming problem

$$\min \sum_{t=1}^{T} (cx^t)\lambda_t$$

$$\text{s.t.} \quad \sum_{t=1}^{T} (Ax^t)\lambda_t = b \qquad (10)$$

$$\sum_{t=1}^{T} \lambda_t = 1$$

$$\lambda_t \geq 0 \quad \text{and integer.}$$

The linear programming problem (6) is the linear programming relaxation of (10), $\tilde{x}$ is an optimal solution to (6), and u^* is optimal in (5), the linear programming dual of (6). The supergroup is constructed from the optimal $(m + 1) \times (m + 1)$ basis B for (9) which is optimal in (6) (artificials are at zero). By rearranging rows and columns of B, we can write it in the form

$$\begin{array}{ll} \text{variables} & \begin{array}{cc} s_1^{\pm} \cdots\cdots s_{m-K}^{\pm} & \lambda_1 \cdots\cdots\cdots \lambda_{K+1} \end{array} \\ B = & \left(\begin{array}{ccc|ccc} \pm 1 & & & \sum_{j=1}^{n} a_{1j}x_j^1 & \cdots & \sum_{j=1}^{n} a_{1j}x_j^{K+1} \\ & \ddots & & \vdots & & \\ & & \pm 1 & \sum_{j=1}^{n} a_{m-K,j}x_j^1 & \cdots & \sum_{j=1}^{n} a_{m-K,j}x_j^{K+1} \\ \hline & & & \sum_{j=1}^{n} a_{m-K+1,j}x_j^1 & \cdots & \sum_{j=1}^{n} a_{m-K+1,j}x_j^{K+1} \\ & & & & \vdots & \\ & & & \sum_{j=1}^{n} a_{m,j}x_j^1 & \cdots & \sum_{j=1}^{n} a_{m,j}x_j^{K+1} \\ & & & 1 & \cdots & 1 \end{array}\right) \end{array} \qquad (11)$$

Lemma 2: If only one λ_k is positive in the optimal basic solution corresponding to B defined in (11), then the corresponding solution x^k is optimal in (IP). On the other hand, if more than one λ_k is positive, then all the x^k defining B are infeasible in (IP).

Proof: If only one λ_k is positive, then $\lambda_k = 1$ and we have $Ax^k = b$. Since B is an optimal basis for the IP dual problem in the form (6), we have $x^k \in \bar{\underline{X}}$ and $L(u^*) = u^*b + (c - u^*A)x^k$. Thus, (x^k, u^*) satisfy the optimality conditions and x^k is optimal in (IP).

In the second case, suppose to the contrary that x^1 satisfies $Ax^1 = b$. This implies

$$\begin{pmatrix} b \\ 1 \end{pmatrix} \lambda_1 + \sum_{k=2}^{K+1} \begin{pmatrix} Ax^k \\ 1 \end{pmatrix} \lambda_k = \begin{pmatrix} b \\ 1 \end{pmatrix} ,$$

or,

$$\sum_{k=2}^{K+1} \begin{pmatrix} Ax^k \\ 1 \end{pmatrix} \lambda_k = \begin{pmatrix} b \\ 1 \end{pmatrix} (1 - \lambda_1) .$$

This in turn implies (since $\lambda_1 < 1$)

$$\sum_{k=2}^{K+1} \begin{pmatrix} Ax^k \\ 1 \end{pmatrix} \frac{\lambda_k}{1-\lambda_1} = \begin{pmatrix} b \\ 1 \end{pmatrix} = \begin{pmatrix} Ax^1 \\ 1 \end{pmatrix} ;$$

that is $\begin{pmatrix} Ax^1 \\ 1 \end{pmatrix}$ can be written as a linear combination of the other columns in B which is impossible since B is a basis.||

The basis B induces a group H by application of the Smith Reduction procedure as described in Appendix 1, and the supergroup we seek is $G \oplus H$. Indirectly, H can be thought to be derived from the lack of integrality in the optimal basic solution to the linear programming relaxation to (10). However, the construction is direct and its validity does not depend on the original motivation.

Remarks: The new group constraints derived from H may well imply many of those originally derived from G, causing some redundancy. In this case only a subgroup G' of G need be combined with H to form the supergroup. By a slight reformulation of (10) it may be ensured that the new constraints always imply the old. The details of this are given in Appendix 2. ||

The group induced from B is

$$H = Z_{h_1} \oplus Z_{h_2} \oplus \cdots \oplus Z_{h_s} \tag{12}$$

the mapping of integer vectors onto H is ψ, and $\omega_{m-K+k} = \psi(e_{m-K+k})$, $k = 1,\ldots,K+1$, are images of the unit vectors in H. By our discussion in Appendix 1 we know $\omega_i = 0$, $i = 1,\ldots,m-K$, because the vectors $\pm e_i$, $i = 1,\ldots,m-K$, are basic.

The group H is used to define the set

$$X' = \{x \mid \sum_{j=1}^{n} \alpha_j' x_j \equiv \beta' \quad , \quad x_j = 0 \text{ or } 1\} \tag{13}$$

where $\alpha_j' = \psi(a_j)$, $\beta' = \psi(b)$. As with X, we have $F \subseteq X'$.

The critical property of this construction is given by the following lemma.

Lemma 3: If more than one λ_k is positive in the basis solution corresponding to B defined in (11), then the infeasible solutions $x^1,\ldots,x^{K+1} \in X$ are not contained in X'

Proof: Since the columns $\begin{pmatrix} Ax^1 \\ 1 \end{pmatrix},\ldots,\begin{pmatrix} Ax^{K+1} \\ 1 \end{pmatrix}$ are part of a basis B, by the construction in the appendix, we have

$$\sum_{i=m-K+1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^k\right)\omega_i + \omega_{m+1} \equiv 0$$

or

$$\sum_{j=1}^{n} \left(\sum_{i=m-K+1}^{m} a_{ij}\omega_i \right) x_j^k = -\omega_{m+1} \quad .$$

Since α_j' equals the term in parentheses, we have

$$\sum_{j=1}^{n} \alpha_j' x_j^k = -\omega_{m+1} \quad . \tag{14}$$

On the other hand, the basic solution corresponding to B is not integer (more than one λ_k positive) implying

$$\sum_{i=m-K+1}^{m} b_i\omega_i + \omega_{m+1} \not\equiv 0 \quad ,$$

or

$$\beta' \equiv \sum_{i=m-K+1}^{m} b_i\omega_i \not\equiv -\omega_{m+1} \quad . \tag{15}$$

The lemma is established by comparing (14) and (15). ||

Remarks: The construction of the supergroup can clearly be done with respect to any basis of the form (11) with more than one $\lambda_k > 0$. This is important since convergence to the exact optimal IP dual solution may be slow (see Fisher, Northup, Shapiro [6]). Moreover, only the submatrix in the lower right corner needs to be considered (see Appendix 1). ||

The implication of lemma 3 is that we want to redefine the IP dual problem to require $x \in X \cap X'$ which is equivalent to deriving the IP dual problem from $G \oplus H$.

Iterative Dual Method

Step 0 (Initialization): Start with $G^0 = Z_1$ and construct the IP dual problem to (IP)

$$w^0 = \max_{u \in R^m} L^0(u) \tag{D^0}$$

where

$$L^0(u) = ub + \min_{x \varepsilon X^0} (c - uA)x$$

and

$$X^0 = \{x | x_j = 0 \text{ or } 1\} \quad .$$

Go to Step 1 with $\ell = 0$.

Step 1: Solve the IP dual problem (D^ℓ); if $w^\ell = +\infty$, (IP) is infeasible and the method is terminated. Otherwise, let u^ℓ denote an optimal solution to (D^ℓ) and let x^ℓ be the convex combination of points in $\underline{\bar{X}}^\ell$ satisfying the IP dual optimality conditions (8). If x^ℓ is not integer, go to Step 2.

Step 2: Apply the Smith reduction procedure to construct the supergroup

$$G^{\ell+1} = (G^\ell)' \oplus H \tag{16}$$

where H is the group induced by the basis defined in (11) and $(G^\ell)'$ is the subgroup of G^ℓ producing non-redundant congruence constraints. Use $G^{\ell+1}$ to define $X^{\ell+1}$ given in (3) and construct the dual problem

$$\begin{aligned} w^{\ell+1} &= \max L^{\ell+1}(u) \\ &\text{s.t.} \quad u \; \varepsilon \; R^m \quad . \end{aligned} \tag{17}$$

Return to Step 1.

Remarks: The initial IP dual problem (D^0) is simply the linear programming relaxation of (IP) when $x_j = 0$ or 1 is replaced by $0 \leq x_j \leq 1$ (see Nemhauser and Ullman [19]).

Theorem 1: The iterative dual procedure converges finitely to an optimal solution to (IP), or proves (IP) is infeasible.

Proof: The solution of each IP dual problem (D^ℓ) defined over $X^\ell \subseteq X^0$ is finite because X^ℓ is finite implying (D^ℓ) is a linear programming problem. If the solution $x^\ell = \sum_{t \varepsilon T(u^\ell)} \lambda_t x^t$ from (9)' is not integer, then the Smith reduction procedure, which is finite, produces a new set $X^{\ell+1}$ satisfying $|X^{\ell+1}| \leq |X| - 2$. This is because nonintegrality of x^{ℓ} implies at least two λ_t are positive in (11) which in turn implies by lemma 3 that at least two solutions from X are eliminated in the construction of $X^{\ell+1}$. Clearly, since X^0 is finite, this reduction process must terminate finitely with an optimal solution to (IP) in the case $F \neq \phi$, or, with $w^{\ell} = +\infty$ for some ℓ in the case $F = \phi$. ||

<u>Remarks</u>: Total enumeration of the set $X^0 = \{x | x_j = 0 \text{ or } 1\}$ would be another finitely convergent procedure for (IP). The strength of the IP dual approach is that the sets X^{ℓ} are considerably smaller than X^0, and the computation of $L^{\ell}(u)$ for various values of u does not explicitly use more than a small fraction of X^{ℓ}. The practical imperfection in the IP dual theory is that $|G^{\ell}|$ may grow too large. Some measures are possible to effectively reduce $|G^{\ell}|$; suggestions are given in [12] and [13]. Another approach is to approximate any large group by a group of fixed order, say 2000 (Burdet and Johnson [4]). Another positive feature of the IP dual approach are the monotonically increasing lower bounds $L^{\ell}(u)$ for use in branch and bound (see Fisher and Shapiro [5]).

<u>Remarks</u>: The IP duality theory can be combined with Benders' method to achieve a duality theory for mixed integer programming. Details are given in [24].

2. <u>Relation to the Convex Hull of Feasible Integer Solutions</u>

The equivalence between dualization and convexification of mathematical programming problems (Magnanti, Shapiro and Wagner [17]) permits us to give a convex analysis interpretation of the results of the previous section.

<u>Lemma 4</u>:

$$\{x | x = \sum_{t=1}^{T} x^t \lambda_t, \ \ \lambda_t \text{ feasible in the IP dual problem}\}$$

$$= \{x | Ax = b, \ \ 0 \leq x_j \leq 1\} \cap [X] \quad .$$

Proof: The proof is straightforward and therefore omitted. ||

Theorem 2: The IP dual problem is equivalent to

$$w = \min cx$$
$$\text{s.t.} \quad x \in \{x \mid Ax = b, \ 0 \leq x_j \leq 1\} \cap [X] \quad . \tag{18}$$

Proof: The IP dual problem can be written as the linear programming problem (6). In view of lemma 4, (18) follows immediately. ||

Thus, a given IP dual problem approximates (IP) by minimizing the objective function over the intersection of the linear programming feasible region with the convex polyhedron [X]. When the IP dual problem solves (IP) in the sense that the optimality conditions are found to hold, then [X] has cut off enough of the LP feasible region for an optimal solution to (IP) to be discovered. Thus, in this case the IP dual problem has found a local approximation to [F], the convex hull of feasible integer solutions, in the neighborhood of an optimal solution. The exact nature of this approximation is an area of future research; polyhedra similar to [X] have been studied by Gomory [11]. Bell and Fisher [3] have considered the approximation of [F] by intersecting the corner polyhedra corresponding to different LP bases.

3. Numerical Example

Consider the zero-one IP problem

$$v = \min - 3x_1 - 7x_2 + 2x_3 + 4x_4 - x_5 + 8x_6 + 12x_7 + 2x_8 + 12x_9 + 20x_{10}$$
$$\text{s.t.} \quad 2x_1 - x_2 + x_3 + 3x_4 - 2x_5 + 2x_6 + 5x_7 - 2x_8 + x_9 + 0x_{10} = 1$$
$$- 4x_1 - 3x_2 + 2x_3 + 1x_4 + 1x_5 + 1x_6 + 2x_7 - 2x_8 + 0x_9 + x_{10} = -4$$
$$x_j = 0 \text{ or } 1, \quad j = 1,2,\ldots,10 \quad . \tag{19}$$

For the purpose of the example we will initialize the algorithm by solving the linear programming relaxation of (19) and use its optimal basis to form the first set of group equations. The optimal solution is $x_1 = x_2 = x_3 = 1$, $x_4 = \frac{1}{5}$, $x_5 = \frac{4}{5}$ all other variables being at zero. The optimal basis is

$$\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \tag{20}$$

which induces the group $G^1 = Z_5$ and the congruence

$$4x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 4x_6 + 4x_7 + 4x_8 + 1x_9 + 2x_{10} \equiv 3 \pmod 5 \tag{21}$$

is added to (19) preparatory to creating the IP dual problem (D^1).

The dual has optimal solution $u^1 = \left(\frac{5}{3}, -1\right)$ with dual objective value $w^1 = -\frac{8}{3}$. The optimal x solution is $x_2 = 1$, $x_1 = x_4 = x_9 = \frac{1}{3}$ which has two representations in X^1 and for the sake of illustration we will proceed with the algorithm in each case.

One representation is as a linear combination of $x_2 = 1$ and $x_1 = x_2 = x_4 = x_9 = 1$ with optimal basis to (8) of

$$\begin{pmatrix} -1 & 5 & 0 \\ -3 & -6 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{22}$$

which induces the group $G^2 = Z_6$ and the congruence

$$4x_1 + x_2 + 2x_3 + 3x_4 + 5x_5 + x_6 + 4x_7 + 2x_8 + 2x_9 + 3x_{10} = 2 \pmod 6$$

This is added to (21) to form the next dual problem (D^2). This dual has optimal solution $u^2 = (0,0)$ yielding $x_1 = x_3 = x_8 = 1$, the optimal solution to (19) by the optimality conditions.

The other representation of the x solution to (D^1) is as a linear combination of $x_2 = 1$, $x_1 = x_2 = x_9 = 1$ and $x_2 = x_4 = 1$ with optimal basis to (8) of

$$\begin{pmatrix} -1 & 2 & 2 \\ -3 & -2 & -7 \\ 1 & 1 & 1 \end{pmatrix}$$

which induces $G^2 = Z_{15}$ and the congruence

$$11x_1 + 2x_2 + 10x_3 + 0x_4 + 10x_5 + 11x_6 + 11x_7 + x_8 + 4x_9 + 3x_{10} = 7 \pmod{15} \tag{23}$$

Notice that congruence (23) implies that of (21) which is thus redundant. (D^2) may be formed using only (23) as the group constraints. It is equivalent to adding the further group constraint

$$x_1 + x_2 + 2x_3 + 0x_4 + 2x_5 + x_6 + x_7 + 2x_8 + 2x_9 + 0x_{10} = 2 \pmod{3} \tag{24}$$

to (D^1).

Once again the optimal solution is $u^2 = (0,0)$ with optimal x solution of $x_1 = x_3 = x_8 = 1$.

APPENDIX 1

Smith Reduction Procedure

Smith's reduction procedure for integer matrices is at the heart of the supergroup and iterative IP dual methods. Although this procedure has appeared in the literature (e.g. Wolsey [25] or Garfinkel and Nemhauser [8]), we review it briefly here emphasizing a few subtle points.

Consider the linear system

$$Py = p$$

where P is an $m \times n$ $(n \geq m)$ matrix of rank m and P and p are integer. We wish to characterize the integer solutions to this linear system. Let B be any basis of P (B is $m \times m$ of rank m) and partition P into (B,N) and y into (y_B, y_N). Solving for the dependent variables y_B in terms of the independent variables y_N gives us

$$y_B = B^{-1}p - B^{-1}Ny_N \quad .$$

In general, B is not unimodular implying the integer vectors y_N which make y_B integer are a subset of the set of all $n - m$ integer vectors.

The requirement that (y_B, y_N) be integer is written as

$$B^{-1}Ny_N \equiv B^{-1}p \qquad (\text{mod } 1) \tag{25}$$

$$y_N \text{ integer.}$$

The Smith procedure reduces this linear system of m congruences by working with a diagonal matrix $\Delta = RBC$, where R and C are $m \times m$ unimodular matrices. The exact form of this matrix is

$$\Delta = \begin{pmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & q_1 & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & \cdot \\ & & & & & & & & & & q_r \end{pmatrix}$$

where q_i is integer, $q_i \geq 2$, $q_i | q_{i+1}$, $i = 1,\ldots,r$, and $|\det B| = \prod_{i=1}^{r} q_i$. For future reference, we let Z_{q_i} denote the cyclic group of order q_i. Substituting $C\Delta^{-1}R$ for B^{-1} in (25), we obtain from (25)

$$C\Delta^{-1}RNy_N \equiv C\Delta^{-1}Rp \pmod 1$$

$$y_N \text{ integer.}$$

Since C is unimodular, this last system of congruences is equivalent to

$$\Delta^{-1}RNy_N \equiv \Delta^{-1}Rp \pmod 1 \qquad (26)$$
$$y_N \text{ integer .}$$

Let $\tilde{a}_{ij}$ be the integer coefficients of the $m \times (n - m)$ matrix RN, and let $\tilde{p}_i$ be the integer coefficients of the $m \times 1$ vector Rp. For the first $m - r$ congruences in (26), the diagonal elements of Δ^{-1} are 1 and therefore these congruences are

$$\sum_{j=m+1}^{n} \tilde{a}_{ij}y_j \equiv \tilde{p}_i \pmod 1 \qquad i = 1,\ldots,m-r$$

which are not restrictive because y_j are required to be integer. On the other hand, the last r congruences in (26) are of the form

$$\frac{1}{q_i} \sum_{j=m+1}^{n} \tilde{a}_{m-r+i,\,j}y_j \equiv \frac{1}{q_i}\tilde{p}_{m-r+i} \pmod 1, \quad i = 1,\ldots,r$$

or,

$$\sum_{j=m+1}^{n} \tilde{a}_{m-r+i,\,j}y_j \equiv \tilde{p}_{m-r+i} \pmod{q_i}, \qquad i = 1,\ldots,r$$

which restricts the permissable integer y_j values because $q_i \geq 2$. For convenience, and without loss of generality, we can replace the $\tilde{\alpha}_{m-r+i,j}$ by the unique $\alpha_{ij} \in Z_{q_i}$ satisfying $\alpha_{ij} \equiv a_{m-r+i,j} \pmod{q_i}$ and $\tilde{p}_{m-r+i}$ by $\beta_i \in Z_{q_i}$ satisfying $\beta_i \equiv \tilde{p}_{m-r+i} \pmod{q_i}$. In conclusion, the condition we seek is

$$\begin{array}{c} (y_B, y_N) = (B^{-1}p - B^{-1}Ny_N, y_N) \\ \text{is integer} \\ \Longleftrightarrow \\ \sum_{j=m+1}^{n} \alpha_{ij}y_j \equiv \beta_i \pmod{q_i}, \qquad i = 1,\ldots,r \\ y_j \text{ integer}, \qquad j = m+1,\ldots,n \end{array} \tag{27}$$

For our construction in Section 2, we dispense with the basis representation $y_B = B^{-1}p - B^{-1}Ny_N$ and work with the abelian group $G = Z_{q_1} \oplus \cdots \oplus Z_{q_r}$ induced by B. Let ϕ be the mapping taking arbitrary m-vectors s into elements $\phi(s) \in G$. This mapping is given by the last r rows of $R = (\rho_{ij})$. To see this, note that in the construction above, the integer vector $s = (s_1,\ldots,s_m)$ is mapped into $\sigma = (\sigma_1,\ldots,\sigma_r) = \phi(s) \in G$ where

$$\sigma_k \equiv \sum_{i=1}^{m} \rho_{m-r+k,i} s_i \pmod{q_k}, \qquad k = 1,\ldots,r \quad .$$

The vector s can also be represented as $s = \sum_{i=1}^{m} s_i e_i$, where e_i is the ith unit vector in R^m. This implies

$$\sigma = \phi(s) = \sum_{i=1}^{m} s_i \phi(e_i) = \sum_{i=1}^{m} s_i \varepsilon_i \quad ,$$

where

$$\phi(e_i) = \varepsilon_i = (\varepsilon_{1i},\ldots,\varepsilon_{ri}) \text{ and}$$

$$\varepsilon_{ki} \equiv \rho_{m-r+1,i} \pmod{q_k} \; k = 1,\ldots,r; \; i = 1,\ldots,m \quad .$$

There are three important observations about this construction used in Section 2. First, the element $\beta \in G$ is not zero, the the identity element, if $B^{-1}p$ is not integer. If it were, then $y_j = 0$ for $j = m + 1,\ldots,n$, would be a solution in (27) which is not possible.

Second, the vectors a_k of the basis B are mapped into $\phi(a_k) = 0$. To see this, since $C\bar{\Delta}^1 R = B^{-1}$, we have

$$\Delta^{-1} Ra_k = C^{-1} B^{-1} a_k - C^{-1} e_k \quad .$$

Thus, since C is unimodular the last r components of $\Delta^{-1} Ra_k$ are integer, or the last r components of Ra_k are integer multiples of $q_1,\ldots,q_r$ implying $\alpha_k = \phi(a_k) \equiv 0$.

Finally, suppose there are columns of B corresponding to unit vectors e_i or $-e_i$,

$$B = \begin{pmatrix} \pm 1 & & & & & \\ & \pm 1 & & & & \\ & & \cdot & & & Q \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \pm 1 \\ 0 & & & & & \hat{B} \end{pmatrix}$$

where the unit vectors are on rows $1,\ldots,t$. Then, by elementary column operations

$$B' = BC = \begin{pmatrix} 1 & & & & & & & & 0 \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ 0 & & & & & & & & \hat{B} \end{pmatrix}$$

implying the reduction can be performed on $\hat{B}$. Moreover, $\varepsilon_i = \phi(e_i) = 0$, $i = 1,\ldots,t$, since $\pm e_i$ is in the basis B.

APPENDIX 2

A Modification of the Procedure

The modification that follows has the advantage that at each iteration all the non-redundant group constraints are generated directly from the current optimal basis, but has the disadvantages that work may be repeated and the number of constraints increases by one at each iteration.

Step 0 (Initialization). Start with $G^0 = Z_1$ and construct the IP dual problem to (IP)

$$w^0 = \max_{u \in R^m} L^0(u) \qquad (D^0)$$

where

$$L^0(u) = ub + \min_{x \in X^0}(c-uA)x$$

and

$$X^0 = \{x | x_j = 0 \text{ or } 1\} \quad .$$

Go to Step 1 with $\ell = 0$, B_0 as the identity matrix.

Step 1: Solve the IP dual problem (D^ℓ); if $w^\ell = +\infty$ (IP) is infeasible and the method is terminated. Otherwise let u^ℓ denote an optimal solution to (D^ℓ) and let x^ℓ be the convex combination of points in X^ℓ satisfying the IP dual optimality conditions and let $M_{\ell+1}$ be the optimal LP basis to

$$\min \sum_{i=1}^{m+\ell} (s_i^+ + s_i^-)$$

$$\text{s.t.} \sum_{t \in T(u^\ell)} B^{-1} \begin{pmatrix} Ax^t - b \\ \theta_\ell \end{pmatrix} \lambda_t + Is^+ - Is^- = 0 \qquad (9')$$

$$\sum_{t \varepsilon T(u^{\ell})} \lambda_t = 1$$

$$\lambda_t \geq 0 \quad \text{for} \quad t \; \varepsilon \; T(u^{\ell})$$

$$s^+ \geq 0 \quad , \quad s^- \geq 0 \quad .$$

where θ_{ℓ} is a column of ℓ zeros.

We may take $x^{\ell} = \sum_{t \varepsilon T(u^{\ell})} \lambda_t^* \, x^t$ where λ^* is optimal in (9'). If x^{ℓ} is integral, then it is optimal and the method terminates. If x^{ℓ} is not integral, go to Step 2.

Step 2: Apply the Smith reduction procedure to construct the supergroup $G^{\ell+1}$ induced by the basis

$$B_{\ell+1} = \begin{pmatrix} B_{\ell} & 0 \\ 0 & 1 \end{pmatrix} M_{\ell+1} \quad .$$

Use $G^{\ell+1}$ to define $X^{\ell+1}$ given in (13) and construct the dual problem

$$\begin{aligned} w^{\ell+1} &= \max \; L^{\ell+1}(u) \\ \text{s.t.} \quad & u \; \varepsilon \; R^m \end{aligned} \qquad (D^{\ell+1})$$

Return to Step 1.

Theorem 3: At each step of the procedure $X^{\ell+1} \subseteq X^{\ell}$.

Proof: Note that $M_{\ell+1}$ is integral since

$$B_{\ell}^{-1} \begin{pmatrix} Ax^t - b \\ \theta_{\ell} \end{pmatrix}$$

is integral for all $x^t \; \varepsilon \; X^{\ell}$ by definition of X^{ℓ}.

$X^{\ell+1}$ is defined by the equations

$$M_{\ell+1}^{-1}\left(B_{\ell}^{-1}\begin{pmatrix}Ax^{t}-b\\ \theta_{\ell}\end{pmatrix}\\ 1\right) \equiv M_{\ell+1}^{-1}\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{(mod 1)}$$

and since $M_{\ell+1}$ is integral, this implies that

$$\left(B_{\ell}^{-1}\begin{pmatrix}Ax^{t}-b\\ \theta_{\ell}\end{pmatrix}\\ 1\right) \equiv \begin{pmatrix}0\\1\end{pmatrix} \qquad \text{(mod 1)}$$

and hence $X^{\ell+1} \subseteq X^{\ell}$. ||

Note thus that a homomorphism from $G^{\ell+1}$ to G^{ℓ} induced by $M_{\ell+1}$ exists.

References

[1] Bell, D.E. "The Resolution of Duality Gaps in Discrete Optimization." Tech Report No. 81, Operations Research Center, Massachusetts Institute of Technology, 1973.

[2] Bell, D.E. "Improved Bounds for Integer Programs: A Supergroup Approach." IIASA RM-73-5. International Institute for Applied Systems Analysis, Laxenburg, Austria, 1973. To appear in SIAM Journal on Applied Mathematics.

[3] Bell, D.E. and Fisher, M.L. "Improved Integer Programming Bounds Using Intersections of Corner Polyhedra." To appear in Mathematical Programming (1975).

[4] Burdet, C. and Johnson, E.L. "A Subadditive Approach to the Group Problem of Integer Programming." Mathematical Programming Study 2 (1974), 51-71.

[5] Fisher, M.L. and Shapiro, J.F. "Constructive Duality in Integer Programming." SIAM Journal on Applied Mathematics, 27 (1974), 31-52.

[6] Fisher, M.L., Northup, W.D. and Shapiro, J.F. "Using Duality to Solve Discrete Optimization Problems: Theory and Computational Experience." Operations Research Center Working Paper No. OR 030-74, Massachusetts Institute of Technology, 1974. To appear in Mathematical Programming.

[7] Fuchs, L. Abelian Groups. Pergamon Press, 1960.

[8] Garfinkel, R.S. and Nemhauser, G.L. Integer Programming, Wiley-Interscience, 1972.

[9] Geoffrion, A.M. "Lagrangean Relaxation for Integer Programming." Mathematical Programming Study 2 (1974), 82-114.

[10] Glover, F. "Integer Programming over a Finite Additive Group." SIAM Journal on Control, 7 (1969), 213-231.

[11] Gorry, G.A."Some Polyhedra Related to Combinatorial Problems." Linear Algebra and Applications, 2 (1969), 451-558.

[12] Gorry, G.A. and Shapiro, J.F. "An Adaptive Group Theoretic Algorithmic for Integer Programming." Management Science, 17 (1971), 285-306.

[13] Gorry, G.A., Northup, W.D. and Shapiro, J.F. "Computational Experience with a Group Theoretic Integer Programming Algorithm." Mathematical Programming, 4 (1973), 171-192.

[14] Halfin, S. "Arbitrary Complex Corner Polyhedra are Dense in R^n." SIAM Journal on Applied Mathematics, 23 (1972), 157-163.

[15] Held, M. and Karp, R.M. "The Travelling-Salesman Problem and Minimum Spanning Trees: Part II." Mathematical Programming, 1 (1971).

[16] Held, M., Wolfe, P. and Crowder, H. "Validation of Subgradient Optimization." Mathematical Programming, 6 (1974), 102-88.

[17] Magnanti, T.L., Shapiro, J.F. and Wagner, M.H. "Generalized Linear Programming Solves the Dual." Operations Research Center Working Paper No. OR 019-73, Massachusetts Institute of Technology, 1973.

[18] Marsten, R.E., Northup, W.D. and Shapiro, J.F. "Subgradient Optimization for Set Partitioning and Covering Problems." In preparation.

[19] Nemhauser, G and Ullman, Z. "A Note on Generalized Lagrange Multiplier Solution to an Integer Programming Problem." Operations Research, 16 (1968), 450-452.

[20] Poljak, B.T. "A General Method for Solving Extremum Problems." Soviet Mathematics Doklady, 8 (1967), 563-597.

[21] Poljak, B.T. "Minimizational of Unsmooth Functionals." U.S.S.R. Computational Mathematics and Mathematical Physics, 9 (1969), 4-29.

[22] Rockafellar, T.R. Convex Analysis. Princeton University Press, 1970.

[23] Shapiro, J.F. "Generalized Lagrange Multipliers, in Integer Programming." Operations Research, 19 (1971), 68-76.

[24] Shapiro, J.F. "A New Duality Theory for Mixed Integer Programming." Operations Research Center Working Paper No. OR 033-74, Massachusetts Institute of Technology, 1974. To appear in the Proceedings of the Congress on Mathematical Programming and Its Applications, Rome, April 1974.

[25] Wolsey, L.A. "Mixed Integer Programming: Discretization and the Group Theoretic Approach." Tech Report No. 42, Operations Research Center, Massachusetts Institute of Technology, 1969.