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Abstract

Generalizations of the Chandrasekhar-Ambartsumian X-Y functions of radiative transfer are used to give a new representation of the Bellman-Krein formula for the Fredholm resolvent, as well as to represent the Volterra factors of the Gohberg-Krein factorization theory for Fredholm integral operators. It is shown that the new formulas have direct connections to matrix Riccati equations and may be used to advantage in various imbedding procedures used to calculate the Fredholm resolvent.

1. Introduction

In an important paper, Schumitzky [11] demonstrated the equivalence between matrix Riccati equations and Fredholm resolvents, in the sense that the existence of one on a certain interval [a,x] implied the existence of the others, and conversely. Later, Kailath [8] established some similar results in the context of linear filtering theory. In both cases, basic results related to factorization theory a la Gohberg-Krein [7], the Bellman-Krein formula [1,10], and Wiener-Hopf equations were noted, along with implications for invariant imbedding procedures used to numerically solve Fredholm integral equations [2,5,6].

Our objective in this note is to utilize recent developments in the factorization of matrix Riccati equations for the derivation of new representation formulas for the Bellman-Krein equation and the Volterra factors. In contrast to the results in Schumitzky [11], we will obtain representations involving functions of a single independent variable, thereby eliminating the so-called "expanding grid" phenomenon encountered in numerical calculation of the Fredholm resolvent by means of the Bellman-Krein equation.

2. Problem Statement

We consider the family of integral operators $\{T_x\}$ on the space $C_p[a,x]$ of p-dimensional vectors whose components are complex-valued continuous functions on $[a,x]$. The operators T_x are defined by

$$(T_x f)(t) = H\int_a^x k(t,s)Gf(s)ds \quad , \qquad a \le t \le x \quad ,$$

where H and G are constant $p \times n$, $n \times p$ matrices, respectively and $k(t,s)$ is a continuous $n \times n$ matrix kernel defined on $t,s \ge a$ by the equation

$$k(t,s) = e^{(t-a)B}Fe^{(a-s)C} + \int_a^{\min(t,s)} e^{(t-\xi)B}Ae^{(\xi-s)C}d\xi \quad ,$$
$$t,s \ge a \quad .$$

Here A, B, C, F are $n \times n$ constant matrix functions. For future reference, we define the $n \times n$ matrix $D = GH$.

It has been shown in Schumitzky [11] that if the matrix functions $R(x)$, $U(t,x)$, $V(t,x)$ satisfy the equations

$$\frac{dR}{dx} = A + BR + RC + RDR \quad , \qquad R(a) = F \quad ,$$

$$\frac{\partial}{\partial x}U(t,x) = U(t,x)[C + DR] \quad , \qquad U(t,t) = HR(t) \quad ,$$

$$\frac{\partial}{\partial x}V(s,x) = [B + RD]V(s,x) \quad , \qquad V(t,t) = R(t)G \quad , \qquad s,t \le x \quad ,$$

then the Fredholm resolvent $\mathcal{K}(t,s,x)$ of T_x satisfies the equation

$$\frac{\partial}{\partial x}\mathcal{K}(t,s,x) = U(t,x)DV(s,x) \quad , \qquad a \le t,s \le x \quad , \qquad (*)$$

$$= \mathcal{K}(t,x,x)\mathcal{K}(x,s,x)$$

$$\mathcal{K}(t,x,x) = U(t,x)G \quad ,$$

$$\mathcal{K}(x,s,x) = HV(s,x) \quad .$$

(Remark: The second form of (*) is the well-known Bellman-Krein formula.) In addition, the functions U and R may be used to numerically solve the integral equation

$$f(t,x) = g(t) + (T_x f)(t)$$

by means of the initial-value system

$$\frac{\partial}{\partial x} f(t,x) = U(t,x) G f(x,x) \quad , \qquad f(t,t) = f(t) + HW(t) \quad ,$$

$$\frac{dW}{dt} = [B + RD]W + R(t)Gg(t) \quad , \qquad W(a) = 0 \quad ,$$

$$a \leq t \leq x \quad .$$

The results of Schumitzky [11] also made contact with the Gohberg-Krein theory of factorization for integral operators. If we factor the integral operator T_b as

$$(I - T_b)^{-1} = (I + \bar{\mathscr{T}}^{+})(I + \bar{\mathscr{T}}^{-}) \quad ,$$

where $\bar{\mathscr{T}}^{\pm}$ are continuous Volterra kernels on $[a,b] \times [a,b]$, then we have the representations

$$\bar{\mathscr{T}}^{+}(t,s) = \begin{cases} U(t,s)G \quad , & t \leq s \leq b \\ 0 \quad , & t > s \end{cases}$$

$$\bar{\mathscr{T}}^{-}(t,s) = \begin{cases} HV(t,s) \quad , & t > s \\ 0 \quad , & a \leq t \leq s \quad . \end{cases}$$

Similar results are also reported in Casti [6] and Kailath [8].

3. Basic Lemma

Given the central importance of the Riccati function R, and the associated linear functions U and V, our goal will be to derive alternate representations for the basic quantities $U(t,x)G$ and $HV(t,x)$ which involve only functions of a single variable. This will simultaneously simplify the computational procedure and provide a streamlined representation for the Fredholm resolvent (via the Bellman-Krein equation) and the Volterra factors.

A pivotal role in our development is played by the following lemma:

Riccati Lemma [4,9]. *Let R satisfy the matrix Riccati equation*

$$\frac{dR}{dx} = A + BR + RC + RDR \quad , \qquad R(a) = F \quad ,$$

with A, B, C, D, F constant matrices. Furthermore, assume the matrices

$$Z = A + BF + FC + FDF \quad ,$$

$$D = GH \quad ,$$

have ranks p, r, respectively, with Z factored as

$$Z = Z_1 Z_2 \quad .$$

Then R admits the representation

$$BR(x) + R(x)C = L_1(x)L_2(x) - K_1(x)K_2(x) - A \quad ,$$

where L_1, L_2, K_1, K_2 are $n \times p$, $p \times n$, $n \times r$, $r \times n$ matrix functions, respectively, satisfying the equations

$$\frac{dL_1}{dx} = [B + K_1(x)H]L \quad , \qquad L_1(a) = Z_1 \quad ,$$

$$\frac{dL_2}{dx} = L_2[C + GK_2(x)] \quad , \qquad L_2(a) = Z_2 \quad ,$$

$$\frac{dK_1}{dx} = L_1L_2G \quad , \qquad K_1(a) = FG \quad ,$$

$$\frac{dK_2}{dx} = HL_1L_2 \quad , \qquad K_2(a) = HF \quad .$$

Remarks: i) The primary importance of the Riccati Lemma is in the definitions of the functions L_1, L_2, K_1, K_2:

$$\frac{dR}{dx} = L_1(x)L_2(x) \quad , \qquad K_1(x) = R(x)G \quad , \qquad K_2(x) = HR(x) \quad ,$$

which comes out of the proofs of the Lemma in [4,9].

ii) Another important fact emerging from the proof of the Riccati Lemma is the representation

$$L_1(x) = \mathscr{V}(x)Z \quad ,$$

$$L_2(x) = Z_2\mathscr{U}(x) \quad ,$$

where $\mathscr{V}$, $\mathscr{U}$ satisfy the linear matrix equations

$$\frac{d\mathscr{V}}{dx} = [B + R(x)D]\mathscr{V} \quad , \qquad \mathscr{V}(a) = I \quad ,$$

$$\frac{d\mathscr{U}}{dx} = \mathscr{U}[C + DR(x)] \quad , \qquad \mathscr{U}(a) = I \quad .$$

We shall make use of these identities in stating our main results.

4. Representation Formulas

We now apply the Riccati Lemma to prove our main Resolvent Representation Theorem. Assume the kernel of

<u>the integral operator T_x is such that the matrix Z has full rank, i.e.</u>

$$\det(A + BF + FC + FDF) \neq 0 \quad .$$

<u>Then the basic quantities $\mathcal{K}(t,x,x)$, $\mathcal{K}(x,t,x)$ defining the Bellman-Krein formula satisfy the relations</u>

$$\mathcal{K}(t,x,x) = K_2(t)Z_2^{-1}L_2(x)G$$

$$\mathcal{K}(x,t,x) = HL_1(x)Z_1^{-1}K_1(t) \quad ,$$

<u>where L_1, L_2, K_1, K_2 are as in the Riccati Lemma</u>.

<u>Proof</u>. By linearity, we have

$$U(t,x) = HR(t)\mathcal{U}(x) = K_2(t)\mathcal{U}(x)$$

$$V(t,x) = R(t)G\mathcal{V}(x) = K_1(t)\mathcal{V}(x) \quad .$$

Thus, since

$$\mathcal{K}(t,x,x) = U(t,x)G \quad ,$$

we see that

$$\mathcal{K}(t,x,x) = HR(t)\mathcal{U}(x)G$$

$$= K_2(t)\mathcal{U}(x)G$$

$$= K_2(t)Z_2^{-1}L_2(x)G \quad .$$

A similar proof establishes the result for $\mathcal{K}(x,t,x)$.

The following corollary establishes a similar representation for the Volterra factors.

<u>Corollary. The Volterra factors associated with the integral operator T_b satisfy the relations</u>

$$\mathcal{V}^{+}(t,s) = \begin{cases} K_2(t)Z_2^{-1}L_2(s)G , & t \leq s \leq b , \\ 0 , & t > s \end{cases}$$

$$\mathcal{V}^{-}(t,s) = \begin{cases} HL_1(s)Z_1^{-1}K_1(t) , & t > s , \\ 0 , & a \leq t \leq s . \end{cases}$$

5. Discussion

We have seen that under the generic condition

$$\det(A + BF + FC + FDF) \neq 0 ,$$

the Fredholm resolvent and the Volterra factors associated with the family of Fredholm integral operators $\{T_x\}$ may be expressed in terms of easily computable functions of a single independent variable. In some problems of practical interest, however, the generic condition is not satisfied. In these situations, a judicious use of one of the many pseudoinverses in the literature (see Ben-Israel [3]) may be used in place of the inverses Z_1^{-1}, Z_2^{-1}, to provide analogous results.

It should also be explicitly pointed out that the foregoing results made no use of any symmetry that may be present in the kernel of the integral operator. An inspection of the relevant equations immediately shows that in this case the relation $L_1 = L_2'$, $K_1 = K_2'$ hold, thereby reducing the basic system of equations by a factor of two.

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