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Abstract

Recent analysis and computational results for the solution of linear dynamics-quadratic cost control processes are presented. It is shown that, if the number of system inputs and outputs is less than the number of state variables, a substantial reduction in computing effort may be achieved by utilizing the new equations, termed "generalized X-Y" functions over the standard matrix Riccati equation solution.

In addition to the basic X-Y equations, the paper also discusses the reduced algebraic equation for infin-infinite-interval problems, infinite-dimensional problems, the discrete-time case, and Kalman filtering problems. Numerical experiments are also reported.

I. Introduction

In the symbiosis that has existed between the calculus of variations and optimal control theory for the past few decades, a central role has been reserved for those variational problems whose Euler-Lagrange equations are linear. A great wealth of information concerning such problems has been obtained by the work of many mathematicians and control engineers. As a result of this effort, we now possess very explicit characterizations of the properties of the solutions of these problems, detailed existence and iniqueness theorems, sophisticated computational procedures for determining optimal trajectories and controls, and much more. In view of these successes, one might reasonably conjecture that the quadratic criteria-linear dynamics control/

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variational problem has been successfully laid to rest and that no problems of any real mathematical substance remain, particularly for the subclass of constant coefficient systems. Our purpose in this report is to show that such a conjecture, reasonable as it appears at first glance, must be rejected on the basis of developments over the past few years and that many problems of surprising mathematical depth and practical importance remain in this most classical of variational problems.

We shall consider the problem of minimizing the integral

$$J = \int_t^T [(x,Qx) + (x,Su) + (u,Ru)]ds + (x(T),Mx(T))$$

over all piecewise continuous vector functions $u(t)$, $t \leq s \leq T$. The functions x and u are n and m-dimensional vector functions, respectively, and are assumed to be related by the linear differential system

$$\frac{dx}{ds} = Fx + Gu \quad , \qquad x(t) = c \quad .$$

Here Q, S, R, F, G, M are constant matrices of appropriate sizes. Usually, we shall require the definiteness assumptions $M, Q \geq 0$, $R > 0$, although any equivalent conditions that ensure a unique minimizing u for all $t \leq T$ would serve our purposes equally well.

Utilizing either the Maximum Principle [1], dynamic programming [2], or other means, it is well known that the minimum value of J is characterized as

$$J_{min}(t) = (c,P(t)c) \quad ,$$

where $P(t)$ is an $n \times n$ matrix satisfying the matrix Riccati differential equation

$$\frac{-dP}{dt} = Q + PF + F'P - (PG + S)R^{-1}(PG + S)' \quad , \qquad P(T) = M \quad .$$

The minimizing control function $u^*(t)$ is given in feedback form as

$$u^*(t) = R^{-1}(G'P(t) + S')x(t) \quad .$$

Until recently, the basic mathematical features of our basic problem remained very much as outlined above with the principal advances being computational procedures designed to exploit sparseness in the system matrices, weak coupling in the dynamics, and so forth, most developments being of a somewhat ad hoc character. The classical paper by Kalman [3] gives a good account of the mathematical picture, while numerous articles in the IEEE Transactions on Automatic Control, Automation and Remote Control, SIAM Control Journal, and other periodicals may be consulted for subsequent advances.

From a system-theoretic viewpoint, the state of affairs sketched above is somewhat puzzling since no explicit use is made of the concept of system inputs and outputs. Of course, this is not unexpected from a classical mathematical standpoint since work in the calculus of variations has never explicitly recognized the engineering notions of state, input, and output in its formulations and approaches to variational problems; however, control engineers have utilized the important conceptual advantages to be gained by explicitly distinguishing between inputs, outputs, and states for many years in their studies of feedback control systems.

To formulate the above variational problem in explicit input/output form, assume the systems output is a p-dimensional linear function of the state, i.e.

$$y(t) = Hx(t) \quad ,$$

where H is a p x n constant matrix. Further, suppose we desire to minimize the weighted sum of the p outputs and m inputs (controls)

$$\bar{J} = \int_t^T [(y,\bar{Q}y) + (y,\bar{S}u) + (u,\bar{R}u)]ds + (y(T),\bar{M}y(T)) \quad .$$

Then it is easy to see that the problem with criteria J is equivalent to the problem with criteria $\bar{J}$ under the identifications

$$Q = H'\bar{Q}H \quad , \qquad S = H'\bar{S} \quad , \qquad R = \bar{R} \quad , \qquad M = H'\bar{M}H \quad .$$

Our principal aim in this paper is to exploit the situation in which the number of inputs and outputs is less than the dimension of the state space. We will show that in this situation, the optimal feedback control law may be characterized by a set of no more than $n(p + m)$ ordinary differential equations of non-Riccati type, and that the new equations possess inherent analytic and computational features not present in the usual Riccati set-up. The original results of this type were obtained in [4] in the context of linear filtering, while the control theoretic versions were announced in [5]. In addition to the basic finite-interval results, we shall also present a new "steady-state" theorem characterizing the optimal infinite-interval feedback law by means of a system of nm nonlinear algebraic equations, as contra ted with the $n(n + 1)/2$ equations of the algebraic Riccati equation. Our presentation concludes with results of numerical experiments using the new equations, and a discussion of related topics such as optimal linear filtering, infinite dimensional problems, time-dependent systems, and discrete-time problems.

II. The Riccati Lemma

From the preceding discussion, it is clear that most of the useful information concerning the linear quadratic variational problem is present in the matrix Riccati equation for P. Our development will be based upon this fact and the following crucial lemma:

Riccati Lemma [6,7]: Let $\Pi(t)$ be the solution of the matrix Riccati differential equation

$$\frac{d\Pi}{dt} = A + B\Pi + \Pi C + \Pi D\Pi \quad , \qquad \Pi(a) = F \quad ,$$

where A, B, C, D, F are constant n x n matrices. Further, assume

i) $\operatorname{rank} Z (= A + BF + FC + FDF) = p$,

ii) $\operatorname{rank} D = r$,

and that Z and D are factored as

$$Z = Z_1 Z_2 \quad , \qquad D = GH \quad ,$$

with Z_1, Z_2, G, H being constant matrices of sizes n x p, p x n, n x r, r x n, respectively. Then $\Pi(t)$ admits the representation

$$B\Pi(t) + \Pi(t)C = L_1(t)L_2(t) - K_1(t)K_2(t) - A \quad ,$$

where L_1, L_2, K_1, K_2 satisfy the initial-value system

$$\frac{dL_1}{dt} = [B + K_1(t)H]L_1(t) \quad , \qquad L_1(a) = Z_1 \quad ,$$

$$\frac{dL_2}{dt} = L_2(t)[C + GK_2(t)] \quad , \qquad L_2(a) = Z_2 \quad ,$$

$$\frac{dK_1}{dt} = L_1(t)L_2(t)G \quad , \qquad K_1(a) = FG \quad ,$$

$$\frac{dK_2}{dt} = HL_1(t)L_2(t) \quad , \qquad K_2(a) = HF \quad .$$

The proof of this lemma follows by differentiating $\Pi(t)$, and using the properties of linear matrix equations. Full details may be found in [6,7].

The importance of the Riccati lemma for practical problems resides in the following two facts:

1) The system of equations for L_1, L_2, K_1, K_2 represents $2n(p + r)$ equations with known initial conditions. Thus, if $p + r < n/2$, there are fewer equations than in the matrix Riccati equation for $\Pi(t)$;

2) in the proof of the lemma, the definitions

$$\frac{d\Pi}{dt} = L_1(t)L_2(t) \quad ,$$

$$K_1(t) = \Pi(t)G \quad ,$$

$$K_2(t) = H\Pi(t)$$

are used. This is of paramount importance since in almost all practical problems what is desired is not the solution of the Riccati equation itself, but rather a linear functional of the solutions (compare the expression for the optimal feedback gain above). In fact, what is usually required is precisely the function $K_1(t)$ (or $K_2(t)$). Thus, the L - K system enables us to calculate directly the relevant physical quantity, totally bypassing the usual Riccati equations.

Before passing on to the specific case of the linear-quadratic control problem, a few supplementary remarks concerning the Riccati lemma are in order:

i) if the symmetry conditions

$$A = A' \quad , \quad B = C' \quad , \quad D = D' \quad , \quad F = F'$$

are satisfied, one can easily show that

$$L_1 = L_2' \quad , \quad K_1 = K_2' \quad .$$

Thus, the L - K system is reduced to $n(p + r)$ equations in this case;

ii) in the event $\Pi(t)$ exists over the semi-infinite interval $(0, \infty)$, the standard approach to determine $\Pi(\infty)$ is to set $\frac{d\Pi}{dt} = 0$ and solve the resulting algebraic Riccati equation. However, the same technique fails for the L - K system since the equations for K_1 and K_2 do not contain L_1 or L_2;

iii) the functions L_1, L_2, K_1, K_2 are substantial generalizations of the X and Y functions introduced into radiative transfer by Ambartsumian and Chandrasekhar in the 1940's [8,9]. For this reason, they have been termed "generalized X-Y functions" [6] or "Chandrasekhar-type algorithms" [7] in the recent literature on this problem;

iv) unless F is of special structure (as it is in radiative transfer, for example, where F is diagonal) the representation formula of the lemma, while of some theoretical interest, is only of practical value for computing $\Pi(t)$ itself if the solution is desired at only a small number of values of t. However, as just remarked, in most practical cases what is desired is not Π

but a functional of Π, with $\Pi(t)$ itself being needed at only one, or a small number of t values.

III. Generalized X-Y Functions in Control Theory

We now return to the control problem posed in section I and apply the Riccati lemma to obtain an appropriate L - K system. The relevant Riccati equation here is

$$\frac{-dP}{dt} = H'\bar{Q}H + PF + F'P - (PG + H'\bar{S})R^{-1}(PG + H'\bar{S})' \quad ,$$

$$P(t) = H'\bar{M}H \quad .$$

We may apply the Riccati lemma to the above equation by making the identification of quantities

Riccati Lemma	Control Problem
A	$H'(\bar{S}R^{-1}\bar{S}' - Q)H$
B	$H'\bar{S}R^{-1}G' - F'$
C	$GR^{-1}\bar{S}'H - F$
D	$GR^{-1}G'$
F	$H'\bar{M}H$
Z	$H'[\bar{S}R^{-1}\bar{S}' - Q + \bar{M}HGR^{-1}\bar{S}' + \bar{S}R^{-1}G'H'\bar{M} + \bar{M}HGR^{-1}G'H'\bar{M}]H - H'\bar{M}HF - F'H'\bar{M}H$

Since the symmetry conditions of remark i) are satisfied, we have $L_1 = L_2'$ and $K_1 = K_2'$. Thus, the appropriate L - K system for the control process is

$$(*) \qquad \frac{dL}{dt} = [H'\bar{S}R^{-1}G' - F' + K'(t)R^{-1/2}G']L(t) \quad , \qquad L(T) = Z^{1/2} \quad ,$$

$$\frac{dK}{dt} = R^{-1/2}G'L(t)L'(t) \quad , \qquad K(t) = R^{-1/2}G'H'\bar{M}H$$

Here we have made use of the definition $K(t) = R^{-1/2}G'P(t)$. Thus, the optimal feedback law $u^*(t)$ is given by

$$u^*(t) = -[R^{-1/2}K(t) + R^{-1}\bar{S}'H]x(t) \quad .$$

Again, several comments are called for:

i) elementary properties of rank show that rank $Z \leq 3p$ and, if $\bar{M} = 0$, rank $Z = p$. Also, rank $GR^{-1}G' \leq m$. Thus, the L - K system (*) represents, at most, $n(3p + m)$ equations suitable for computing the basic feedback quantiy $K(t)$. If $3p + m < (n + 1)/2$, this is a fewer number of equations than in the usual Riccati equation. Thus, we see how the number of inputs and outputs to the dynamical system directly, and possibly dramatically, enter into the complexity of the mathematical equations describing its regulation. The L - K system confirms ones intuitive feeling that a system with only a small number of input and output channels should be "easier" to deal with than a system possessing a richer set of possibilities for interaction with the external world;

ii) in the frequently occurring case $R = I$, $\bar{S} = 0$, the function $K(t)$ becomes exactly the optimal feedback gain function and, in any case, we always have the important relation

$$\frac{dP}{dt} = L(t)L'(t) \quad ,$$

which may be used to calculate $P(t)$ by quadrature for a fixed value of t, i.e.

$$P(t) = \int_t^T L(s)L'(s)ds + H'\bar{M}H$$

as an alternative to inverting the representation formula of the Riccati lemma.

To demonstrate the utility of the L - K system in practice, several comparative numerical experiments were performed, calculating the optimal feedback gain by the L - K system and by the standard Riccati equation. In all experiments, the matrices $\overline{S} = 0$, $R = I$, $\overline{M} = 0$. Thus, the critical matrix $Z = -Q$. In the first set of experiments the matrices F and G were chosen in the forms

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

so that F is completely specified by its characteristic values. Furthermore, it was assumed that Q was a rank one matrix $Q = (iq)(iq')$, where q had entries chosen randomly. The experiments consisted in integrating both the Riccati and the L - K systems over the interval [0,1] to a prescribed degree of accuracy using both variable-and fixed-step size integration procedures. Various systems of state dimensions n = 4, 8, and 16 were investigated. The variable step method was the GBS extrapolation procedure [10], while a standard Runge-Kutta routine was used with a fixed step h = 0.02 for the fixed-step integration. A local discretization error of 0.5×10^{-4} was employed in the variable step made. The results for n = 4 and 16 are displayed in Tables 1-2.

Table 1. Computing Times* (in Secs.) for n = 4.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
1,1.4,1.5,9.5	0.137	0.440	3.21	0.063	0.189	3.00
-5.5±3.5i,8.4,-12.4	0.155	0.475	3.06	0.048	0.198	4.13
3.3,-7.5.-0.2±9.2i	0.161	0.972	6.04	0.059	0.200	3.39
-0.1±i,-0.2±9.2i	0.170	0.891	5.24	0.063	0.199	3.16
1.75,-8,-8,-8	0.191	0.737	3.86	0.067	0.204	3.04
-8,-8,-8,-8	0.180	0.908	5.04	0.072	0.198	2.75

Table 2. Computing Times* (in Secs.) for n = 16.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
-0.5,-0.8,1,1±i, 2,2±i,-3,3±i, -4,4±2i,5,-6	1.664	16.723	10.05	0.561	4.150	7.40
-0.5,-0.8,1,-1.4, 1±i,1.5,2±i,5,-6, -8,-8,-8,-8,9.5	1.565	13.491	8.62	0.589	4.156	7.06
-0.5,-0.8,1,1±i,2, 2±i,-3,-4,5,-6, -7,8,0.1±3.3i	1.052	12.753	12.12	0.580	4.151	7.16

*All computing times are for computations carried out by Dr. O. Kirschner on a CDC Cyber 74 computer.

The most significant point about Tables 1-2 is not the fact that the LK-system produced the optimal feedback law faster than the P-system, but the magnitude of the improvement. On a purely equation-counting basis, one would have expected an improvement of approximately 1.25 for $n = 4$ and 4.25 for $n = 16$, taking account of the symmetry of P. Instead, we see computational improvements of two to three times greater than the theoretical expectation. Two possibilities immediately suggest themselves to account for this observation: i) the special structure of F and G are somehow particularly favorable for the LK-system; and/or ii) the LK-system possesses much better analytic properties than the P-system, thereby admitting fewer numerical operations and, in the variable-step mode, much larger integration steps.

To test the foregoing hypotheses, two additional sets of experiments were performed. The first involved retaining the structure of F and G, but choosing a Q matrix of full rank. Thus, in this case there will be twice as many equations in the LK-system as in the P-system. This experiment was designed to partially test conjecture (ii). The results for $n = 4$ and 16 are given in Tables 3-4. The second experiment involved returning to a rank one Q matrix but now selecting the components of F, G, and Q to be random numbers of absolute value less than 1. These results for $n = 4$ and 16 are given in Tables 5-6.

Table 3. Computing Times for n = 4, Q = full rank.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
1,-1.4,1.5,9.5	0.857	0.400	0.47	0.322	0.204	0.63
-5.5±3.5i,8.4,12.4	1.000	0.738	0.74	0.331	0.199	0.60
3.3,-7.5,-0.2±9.2i	0.713	0.943	1.32	0.314	0.231	0.74
-0.1±i,-0.2±9.2i	0.957	0.718	0.75	0.325	0.178	0.55
1.75,-8,-8,-8	0.867	0.733	0.85	0.316	0.268	0.85
-8,-8,-8,-8	1.076	1.246	1.17	0.325	0.208	0.64

Table 4. Computing Times for n = 16, Q = full rank.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
-0.5,-0.8,1,1±i, 2,2±i,-3,3±i,-4, 4±2i,5,-6	63.201	26.699	0.42	18.096	5.559	0.31
-0.5,-0.8,1,-1.4, 1±i,1.5,2±i,5,-6, -8,-8,-8,-8,9.5	60.458	25.598	0.42	18.216	5.425	0.30
-0.5,-0.8,1,1±i,2, 2±i,-3,0.1±3.3i,-4, 5,-6,-7,8	60.909	30.367	0.50	17.696	5.848	0.33

Table 5. Computing Times for n = 4, F, G, Q = random.

	Variable Step			Fixed Step		
Case No.	LK	P	factor	LK	P	factor
1	0.321	1.111	3.46	0.088	0.215	2.44
2	0.341	0.647	1.90	0.094	0.213	2.27
3	0.368	0.985	2.68	0.082	0.225	2.74
4	0.346	1.354	3.91	0.093	0.216	2.32
5	0.273	0.762	2.79	0.081	0.215	2.65
6	0.457	0.686	1.50	0.099	0.219	2.21

Table 6. Computing Times for n = 16, F, G, Q = random.

	Variable Step			Fixed Step		
Case No.	LK	P	factor	LK	P	factor
1	1.452	22.024	15.17	0.890	5.522	6.20
2	1.667	27.474	16.48	0.871	5.547	6.37
3	1.602	16.574	10.35	0.901	5.560	6.17
4	1.802	22.752	12.63	0.906	5.568	6.15

The overwhelming conclusions to be drawn from Tables 1-6 are that the LK-system not only yields a smaller system of equations if the number of system inputs and outputs is small, but also possesses a more favorable analytic structure. This last point is well illustrated by the variable-step experiments when, for example, in the case $n = 16$, a theoretical factor of between 4 and 5 is expected while the observed factor is between 10 and 16 (Table 6). Even in the fixed-step mode the LK-system exceeds theoretical expectations due to the requirement of performing fewer numerical operations in a single integration step than that required for the P-systems.

IV. Infinite Interval Case

Many problems of control and estimation require the optimal gain function over the semi-infinite interval $[(0, \infty)$ or $(-\infty, T)]$. When calculating optimal gains using the P-system, it is an easy matter to obtain the relevant algebraic Riccati equations for $P(\infty)$ simply by setting $\dot{P} = 0$. This yields the quadratic matrix equation

$$(\dagger) \quad H'\bar{Q}H + PF + F'P - (PG + H'\bar{S})R^{-1}(PG + H'\bar{S})' = 0 \quad .$$

Under conditions of controllability and observability on the system (F, G, H) a unique, positive semi-definite solution to $(\dagger)$ exists [11].

The situation for the LK-system is not quite so simple. Examination of Eq. (*) shows that the standard approach of setting $\dot{L} = \dot{K} = 0$ in order to obtain the appropriate algebraic equation for $L(\infty)$, $K(\infty)$ yields only the information $L(\infty) = 0$. This is because the function $K(t)$ does not appear in the equation for $\dot{K}$. To overcome this difficulty, we employ a useful lemma from matrix theory.

<u>Lemma. Let P, A, Q be any three matrices for which the product PAQ is defined. Then</u>

$$\sigma(PAQ) = (Q'\otimes P)\sigma(A) \quad ,$$

where $\otimes$ denotes the Kronecker product and σ is the operator which "stacks" the columns of a matrix into a column vector, i.e. if $A = [a_{ij}]$, then

$$\sigma(A) = (a_{11},a_{21},\ldots,a_{n1},a_{12},a_{22},\ldots,a_{n2},\ldots,a_{nn})' \quad .$$

Proof. Direct component-by-component verification of the asserted relation.

Using the lemma, we may manipulate (†) to obtain an equation for $K(\infty)$. The main result is the

Steady-State Theorem [12]: Assume the matrix $F - GR^{-1}\bar{S}'H$ has no purely imaginary characteristic roots and no real characteristic roots symmetric relative to the origin. Then the optimal steady state gain $K(\triangleq K(\infty))$ satisfies the algebraic equation

$$\sigma(K) = (I\otimes R^{-1/2}G')[(F - GR^{-1}\bar{S}'H)'\otimes I + I\otimes(F - GR^{-1}\bar{S}'H)']^{-1} \cdot$$
$$\sigma(K'K - H'(\bar{Q} - \bar{S}R^{-1}\bar{S}')H) \quad .$$

Proof. Collecting terms in (†), we see that

$$H'(\bar{Q} - \bar{S}R^{-1}\bar{S}')H + P(F - GR^{-1}\bar{S}'H) + (F - H'\bar{S}R^{-1}G')P - PGR^{-1}G'P = 0 \quad .$$

Now apply σ to both sides of this equation and use the characteristic value hypothesis to see that

$$\sigma(P) = [(F - GR^{-1}\bar{S}'H)'\otimes I + I\otimes(F - GR^{-1}\bar{S}'H)']^{-1}\sigma(K'K - H'(\bar{Q} - \bar{S}R^{-1}\bar{S}')H) \quad ,$$

where we use the definition $K = R^{-1/2}G'P$. Next, apply σ to the definition of K obtaining

$$\sigma(K) = (I\otimes R^{-1/2}G')\sigma(P) \quad ,$$

completing the proof.

The importance of the Steady-State Theorem is that it gives an algebraic equation in the nm variables of K, rather than the $n(n + 1)/2$ variables of P. Also, for many purposes of analysis the form of the equation for K is more desirable since the effects of the three basic system quantities F, G, and Q are separated in a multiplicative manner rather than in the additive manner of the algebraic Riccati equation.

Since the final utility of the Steady-State Theorem is measured in the improvement in computing time it affords as compared with the algebraic Riccati equation, several comparative numerical experiments were performed. The basic approach was to integrate both the LK- and P-systems to a value of t^* for which the terminating condition was that a change of less than 1.0×10^{-4} be observed in all components of L, K, and P in going from $t^* - \Delta$ to t^*. These values of L, K, and P were then used as initial approximations in a modified Newton iteration procedure [13] to compute the solutions of both the algebraic equation for K and the algebraic Riccati equation. The computing times reported in Tables 7-8 are only for the iterative scheme and do not include the preliminary generation of the initial approximations. However, the computing times for the K-equation do include inversion of the matrix $(F - GR^{-1}\bar{S}'H)'\otimes I + I\otimes(F - GR^{-1}\bar{S}'H)$. The stopping criteria for the iteration scheme was that the residuals have ℓ_2-magnitude less than 0.5×10^{-4}. In all experiments, the matrices were

$$F = \text{earlier companion form} \quad , \qquad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} ,$$

$$R = I \quad , \qquad \bar{S} = 0 \quad , \qquad H = 0 \quad , \qquad \bar{Q} = \text{random rank one matrix} \quad .$$

The results, computed on the CDC Cyber 74 computer were

Table 7. Computing Times for n = 4.

roots of F	K-equation	P-equation	factor (theory 2.5)
1,-1.4,1.5,9.5	0.123	0.431	3.50
-5.5±3.5i,8.4,-12.4	0.123	0.757	6.15
3.3,-7.5,-0.2±9.2i	0.119	0.818	6.87
-0.1±i,-0.2±9.2i	0.087	0.575	6.61
1.75,-8,-8,-8	0.102	0.706	6.92
-8,-8,-8,-8	0.073	0.783	10.73

Table 8. Computing Times for n = 8.

roots of F	K-equation	P-equation	factor (theory 4.5)
1,-1.4,1.5,3.3 -7.5,-0.2±9.2i,9.5	2.680	13.447	5.02
1,2,2.5,2.5,2.5, 2.5,-3,-4	2.894	22.486	7.77
1,-1.4,1.5,-8, -8,-8,-8,9.5	2.982	12.973	4.35
0.5,-3.5,4,-4.5, -7,8,0.1±3.3i	2.826	15.280	5.41
1,-0.1±i,-1.4, 1.5,0.2±9.2i,9.5	2.711	14.792	5.46

Just as in the finite interval case, we see that the observed improvement factor for the K-equation is greater than the theoretically predicted factor based on a count of the number of equations. Again, this is explainable (intuitively) only if the K-equation has a "smoother" analytic structure than the algebraic Riccati equation.

Additional results on the K-equation, including implications for the "inverse problem" of optimal control theory are reported in [12].

V. Infinite-Dimensional Problems

Following one of the major trends in modern control theory, we may extend the foregoing results to the case of distributed parameter problems with almost no additional effort. Formally identifying the matrices F, G, H, $\bar{Q}$, R, $\bar{S}$, $\bar{M}$ with operators on appropriate Hilbert spaces, the finite-interval equations (*) have been rigorously established in [14], together with numerical examples. Although the results are not yet complete, there seems little reason to doubt the validity of an infinite-dimensional version of the Steady-State Theorem.

One of the most interesting aspects of the infinite-dimensional results is that, in contrast to the finite-dimensional situation, the LK-system almost always results in a major computational reduction over the operator Riccati equation. This is due to the fact that in virtually every distributed parameter problem there are only a finite number of places where input signals may be applied and where observations may be made. Thus, although the state space may be infinite-dimensional, the input and output spaces are finite-dimensional. Since the LK-system is constructed to exploit this fact, a substantial computational savings is realized. In fact, from a theoretical viewpoint all that is required to realize a computational savings is for the input and output spaces to be proper subspaces of the state space, although the finite-dimensional case is the one of principal practical importance.

VI. Other Directions and Some Open Problems

For reasons of space, we have confined our attention to only the most basic results in the dimensionality reduction possibilities offerd by the LK-approach to quadratic cost-linear dynamics-time-invariant control processes. Several additional results have been obtained and, as one would hope, a number of interesting mathematical questions have arisen which remain unsettled at the current time (Summer 1975). In this section, we shall briefly sketch some of these developments.

Linear Filtering Theory - the well known connections between quadratic cost-linear dynamics control processes and the optimal filtering of a signal in the presence of additive white noise [15] make it no surprise that the results presented in this paper have natural filtering theory counterparts. In fact, the historical development of the LK-equation was initially carried out in this context [4]. Since the results are virtually identical to the control versions, we shall not elaborate upon them here but refer to the interesting papers of Kailath [7], Sidhu [16], and Lindquist [17] for details.

Time-Dependent Problems - our presentation has been confined to those problems involving constant coefficient matrices in the Riccati equation. In fact, the proof of the Riccati lemma falls through in the time-varying case. The question is to what degree, if any, the dimensionality reduction provided by the LK-system may be extended to the time-dependent case. This question is of particular importance for nonlinear problems in which the linearized version involves time-dependent coefficients. While the final verdict is not yet in on this basic issue, several preliminary results indicate partial success. At the expense of computing generalized inverses, control processes have been treated under the restriction of constant G in [18]. Also, several filtering theory results [19,20] include the possibility of time-varying coefficients in the parameter matrices.

<u>Discrete-Time Problems</u> - a number of interesting control problems, particulary those associated with sampled-data systems, are most conveniently stated in the discrete-time framework. For example, a typical problem of this sort is to minimize

$$J = \sum_{k=0}^{N-1} [(x_K, Qx_K) + (u_K, u_K)] + (x_N, Mx_N)$$

subject to the difference equation

$$x_{K+1} = Fx_K + Gu_K \quad , \qquad x_0 = c \quad .$$

The appropriate discrete-time Riccati difference equation describing the minimum value of J and the optimal feedback law is

$$P_i = (F'P_{i+1}F + Q) - (G'P_{i+1}F)'(I + G'P_{i+1}G)^{-1}(GP_{i+1}F) \quad ,$$

$$P_N = M \quad .$$

In the context of a control problem, the appropriate discrete-time version of the LK-system for the above Riccati equation has not yet been presented; however, the filtering theory results [21] and those cited in the previous subsection contain low-dimensional discrete-time equation suitable for computation of the optimal filter gain function. Undoubtedly, minor modifications of this work will yield the appropriate equations for the control problem.

<u>Connections with Transport Theory</u> - the historical starting point for all of the reduced dimension results presented (and alluded to) above was the analogy between the Riccati equations for the optimal gains and a certain basic Riccati equation appearing in the field of radiative transfer in the atmosphere. Special cases of our LK-system had been obtained by Chandrasekhar [9] in connection with the problem of reducing the calculation

of the radiative transfer Riccati system to that of computing vector functions. However, the underlying reason for the success of this effort remained unclear until the appearance of the Riccati lemma. For infinitely thick atmospheres, an integral equation version of our Steady-State Theorem had been obtained eve earlier by Ambartsumian [8] in his study of the Milne problem.

In light of the origins of the LK-reduction, it is reasonable to ask whether or not other basic results from transport theory may play a role in control and filtering processes and, conversly, whether filtering and control techniques can be of use in the study of transport phenomena. Several preliminary investigations have been made to explore these questions and, not surprisingly, the results are in the affirmative. The work of Sidhu, Tse, and Casti [22,26] studies the parallels between atmospheric transport processes and optimal filtering, while the paper [23] investigates the connections between neutron transport and filtering theory. In both cases, many new insights into filtering processes are obtained along with suggestions for new analytic and computational approaches. In the opposite direction, the papers [24,25] present some new results in transport theory motivated by filtering consideration.

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