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Casti, J.L. and Kirschner, O.

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I. Introduction

A new approach to the determination of optimal feedback gains for linear dynamics-quadratic cost control process (or, equivalently, for linear, least-squares filtering problems) has been presented in a sequence of earlier papers [1-6].

The foundation of this approach is the exploitation of certain redundancies that occur in the components of the matrix Riccati equation ordinarily used to solve this class of problems. These redundancies, or dependencies, arise due to the fact that the input and output spaces of the problem are usually of much lower dimension than the state space. Thus, the system's internal "action" is projected into lower-dimensional spaces where external interaction takes place and this projection may be utilized to derive equations for the feedback gain matrix which explicitly incorporate the dimensions of the input and output spaces.

The works cited above have all been analytical. The appropriate equations have been developed for both the finite and infinite-interval problems, but no numerical investigations into the efficiency of the new equations vis-a-vis the usual Riccati equation have been reported. While the new equations certainly merit study on purely analytical grounds, it appears that their primary advantage is computational. Thus, a detailed numerical study of the properties of the new equations is required.

In this report, we present the results of a semi-comprehensive set of calculations carried out to examine a number of questions. In particular, we were concerned with the points:

i) what is the precise magnitude of the computational speed-up for representative problems for both finite and semi-infinite time horizons;

*International Institute for Applied Systems Analysis, Laxenburg, Austria.

**Technische Universität, Vienna, Austria.

ii) are the new equations computationally stable;

iii) what type of computational algorithms seem to be most effective in solving the new equations;

iv) for the finite-interval case, what are the relative merits of variable vs. fixed-step size algorithms.

While our results do not give a complete resolution of these broad issues, they do strongly indicate certain trends and, perhaps more importantly, they suggest areas for future experimentation. The general picture that emerges, however, is that the new algorithms offer the possibility for substantial computational savings in many problems. Perhaps the most surprising feature of our results is that a computational savings may be achieved even in cases where the standard matrix Riccati equation has fewer component equations to solve than in the X-Y system, suggesting certain analytical questions for future study. In addition, we have found the new equations to be computationally more stable than the Riccati system in almost every case.

In short, our experiments have uncovered no hidden computational obstacles to the routine use of the reduced dimension algorithms for the fast computation of feedback gains in filtering and control problems in which the input and output space dimensions are small relative to the dimension of the state. The succeeding sections will deal with this claim. In section II we briefly review both the basic problem and the Riccati and X-Y systems for its solution. Section III deals with finite-interval problems, section IV empirically treats the question of numerical stability, and in section V we study the algebraic problem characterizing the infinite-interval case. Finally, the paper concludes in section VI with a discussion of the results and various topics meriting study in the near future.

II. The Basic Problem and Equations

For purposes of numerical experimentation, we consider minimization of

$$J = \int_t^T [(x,Qx) + (u,u)]dt_1 \quad , \tag{1}$$

over all piecewise continuous functions $u(t_1)$, $t \leq t_1 \leq T$. The vector functions x and u are related by the linear differential system

$$\dot{x} = Fx + Gu \quad , \qquad x(t) = c \quad . \tag{2}$$

Here F and Q are n x n constant matrices while G is an n x m constant matrix. To avoid stability difficulties in the case $t = -\infty$, we further assume that Q is positive semidefinite and of full rank, and that (F,G) is controllable, while $(F,\sqrt{Q})$ is observable.

Well known results [7] show that the minimizing control law, $u^*(t)$, is given in feedback form as

$$\begin{aligned} u^*(t) &= -G'P(t)x(t) \quad , \\ &= -K(t)x(t) \quad , \end{aligned} \tag{3}$$

where P(t) is the solution of the matrix Riccati equation

$$\begin{aligned} -\frac{dP}{dt} &= Q + PF + F'P - PGG'P \quad , \qquad t < T \quad , \\ P(T) &= 0 \quad . \end{aligned} \tag{4}$$

Furthermore, the minimum value of J is

$$J_{min}(t) = (c,P(t)c) \quad .$$

In recent works [1], it has been shown that if the number of system inputs, m, and outputs, rank of Q, are small relative to the state dimension, n, then the optimal feedback gain K(t) may be calculated by a non-Riccati system of differential equations involving far fewer than the n(n + 1)/2 equations in the matrix Riccati equation (4). Specifically, the new system of equations is

$$\frac{dL}{dt} = -(F' - K'G')L \quad , \qquad L(T) = \sqrt{Q} \quad ,$$
$$\frac{dK}{dt} = -G'LL' \quad , \qquad K(T) = 0 \quad . \tag{5}$$

The most important points to observe about the system (5) are

i) since Q is assumed to be of full rank, we may identify a system output matrix with $(\sqrt{Q}')$, *i.e. the original problem is equivalent to minimizing*

$$\int_t^T [(y,y) + (u,u)]dt_1 \quad ,$$

where

$$y = (\sqrt{Q}')x \quad ;$$

ii) if rank Q = p, then $\sqrt{Q}$ *is an n x p matrix and the system (5) represents n(p + m) equations for computing K. If p + m < (n + 1)/2, this represents a reduction in number over the n(n + 1)/2 equations required for the Riccati system (4);*

iii) while the system (5) only supplies the feedback gain K directly, and not P itself, proofs establishing (5) show that the auxiliary function L is related to P as

$$\frac{dP}{dt} = -L(t)L'(t) \quad . \tag{6}$$

Thus, to obtain values of P for selected t-values, one could perform the quadrature

$$P(t) = \int_t^T L(s)L'(s)ds \quad ,$$

or invert the algebraic relation

$$P(t)F + F'P(t) = K'(t)K(t) + L(t)L'(t) - Q$$

which follows from the Riccati equation (4), relation (6), and the definition of K. However, in practice what is generally important is determination of K, so we shall not consider numerical experimentation on obtaining P in this report.

III. Finite Interval Results

In this section, we consider integration of the Riccati system (4) and the LK-system (5) over the finite interval [0,1]. The first set of experiments consists of choosing the matrices

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad Q = \text{rank } 1 = (iq)(iq')$$

Thus, (F,G) is in control canonical form, while Q is of low rank with random entries. For this case, state dimensions $n = 4$, 8, and 16 were investigated leading to Riccati systems of sizes 10, 36, and 136 equations, respectively. The corresponding LK-systems have 8, 16, and 32 equations, respectively, so a naive equation-counting approach would suggest a computational improvement factor of approximately 1.25, 2.25, and 4.25, for the three cases. The numerical integrations were performed in fixed step mode using a classical Runge-Kutta routine of order 4 with step size 0.02 and in a variable-step mode using the Gragg-Bulirsch-Stoer (GBS) extrapolation routine [8,13] with a local discretization error of 0.5×10^{-4}. The results, computed on a CDC Cyber 74 computer, are given in Tables 1-3.

Table 1. Computing times (in seconds) for n = 4.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
1,-1.4,1.5,9.5	0.137	0.440	3.21	0.063	0.189	3.00
-5.5±3.5i,8.4,-12.4	0.155	0.475	3.06	0.048	0.198	4.13
3.3,-7.5,-0.2±9.2i	0.161	0.972	6.04	0.059	0.200	3.39
-0.1±i,-0.2±9.2i	0.170	0.891	5.24	0.063	0.199	3.16
1.75,-8,-8,-8	0.191	0.737	3.86	0.067	0.204	3.04
-8,-8,-8,-8	0.180	0.908	5.04	0.072	0.198	2.75

Table 2. Computing times (in seconds) for n = 8.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
1,-1.4,1.5,3.3,-7.5, -0.2±9.2i,9.5	0.448	2.622	5.85	0.168	0.801	4.77
-0.4,-0.5,-0.76,-0.9, 2,2.5,±8	0.305	2.263	7.42	0.166	0.799	4.81
1,2,2.5,2.5,2.5,2.5, -3,-4	0.500	2.849	5.70	0.160	0.808	5.05
1,-1.4,1.5,-8,-8,-8, -8,9.5	0.494	2.242	4.54	0.155	0.817	5.27
0.5,0.1±3.3i,-3.5,4, -4.5,-7,8	0.413	3.741	9.06	0.163	0.927	5.69
1,-0.1±i,-1.4,1.5, -0.2±9.2i,9.5	0.750	3.240	4.32	0.160	0.798	4.99

Table 3. Computing times (in seconds) for n = 16.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
-0.5,-0.8,1,1±i, 2,2±i,-3,3±i, -4,4±2i,5,-6	1.664	16.723	10.05	0.561	4.150	7.40
-0.5,-0.8,1,-1.4, 1±i,1.5,2±i,5,-6, -8,-8,-8,-8,9.5	1.565	13.491	8.62	0.589	4.156	7.06
-0.5,-0.8,1,1±i,2, 2±i,-3, 0.1±3.3i, -4,5,-6,-7,8	1.052	12.753	12.12	0.580	4.151	7.16

The most significant point about Tables 1-3 is not the fact that the LK-system produced the optimal feedback law faster than the P-system, but the magnitude of the improvement. On a purely equation-counting basis, one would have expected an improvement factor or approximately 1.25 for n = 4, 2.25 for n = 8, and 4.25 for n = 16, taking account of the symmetry of P. Instead, we see computational improvements of two to three times greater than the theoretical expectation. Two possiblities immediately suggest themselves to account for this observation: i) the special structure of F and G are somehow particularly favorable for the LK-system; and/or ii) the LK-system possesses much better analytic properties than the P-system, thereby admitting fewer numerical operations and, in the variable-step mode, much larger integration steps.*

To test the foregoing hypotheses, two additional sets of experiments were performed. The first involved retaining the structure of F and G, but choosing a Q matrix of full rank. Thus, in this case there will be twice as many equations in the LK-system as in the P-system. This experiment was designed to partially test conjecture (ii). The results are given in Tables 4-6. The second experiment involved returning to a rank one Q matrix but now selecting the components of F, G, and $\sqrt{Q}$ to be random numbers of absolute value less than 10. These results are given in Tables 7-9.

Table 4. Computing times (in seconds) for $n = 4$, Q = full rank.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
1,-1.4,1.5,9.5	0.857	0.400	0.47	0.322	0.204	0.63
-5.5±3.5i,8.4,-12.4	1.000	0.738	0.74	0.331	0.199	0.60
3.3,-7.5,-0.2±9.2i	0.713	0.943	1.32	0.314	0.231	0.74
-0.1±i,-0.2±9.2i	0.957	0.718	0.75	0.325	0.178	0.55
1.75,-8,-8,-8	0.867	0.733	0.85	0.316	0.268	0.85
-8,-8,-8,-8	1.076	1.246	1.17	0.325	0.208	0.64

*A rough count of the number of numerical operations involved in updating the right-hand sides of the LK and P systems shows $(n^3 + 2n^2)(A + M)$ for the P-system and $3n^2M + n^2A$ for the LK-systems, where A = additions, M = multiplications. Thus, the ratio (setting $\alpha = M/A$) is $\rho = (n + 2)(\alpha + 1)/(3\alpha + 1)$. Thus, for α between 1 and ∞, we have $2 < \rho < 3$ for $n = 4$, while $6 < \rho < 9$ for $n = 16$, giving good aggreement with Tables 1 and 3. The authors are grateful to Prof. Jean Abadie for these estimates.

Upon examination of the actual program used to compute Tables 1-3, the actual count of operations was $(2n^3 + \frac{5n^2}{2})(M + A)$ for the P-equation and $n^2(4M + 3A)$ for the LK-systems. Thus, the appropriate ratios are 2.8 for $n = 4$, 4.6 for $n = 8$, and 8.7 for $n = 16$, which agrees quite well with the fixed-step integrations.

Table 5. Computing times (in seconds) for
n = 8, Q = full rank.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
1,-1.4,1.5,3.3,-7.5, -0.2±9.2i,9.5	5.695	3.715	0.65	2.242	0.947	0.42
1,2,2.5,2.5,2.5, 2.5,-3,-4	5.990	3.641	0.61	2.224	0.965	0.43
1,-1.4,1.5,-8, -8,-8,-8,9.5	8.699	4.153	0.48	1.999	0.948	0.47
0.5,0.1±3.3i,-3.5,4, -4.5,-7,8	6.636	4.945	0.75	2.418	0.962	0.40
1,-0.1±i,-1.4, 1.5,-0.2±9.2i, 9.5	5.952	3.797	0.64	2.187	0.943	0.43

Table 6. Computing times (in seconds) for
n = 16, Q = full rank.

	Variable Step			Fixed Step		
roots of F	LK	P	factor	LK	P	factor
-0.5,-0.8,1,1±i, 2,2±i,-3,3±i,-4, 4±2i,5,-6	63.201	26.699	0.42	18.096	5.559	0.31
-0.5,-0.8,1,-1.4, 1±i,1.5,2±i,5,-6, -8,-8,-8,-8,9.5	60.458	25.598	0.42	18.216	5.425	0.30
-0.5,-0.8,1,1±i,2, 2±i,-3,0.1±3.3i,-4, 5,-6,-7,8	60.909	30.367	0.50	17.696	5.848	0.33

Table 7. Computing times (in seconds) for
n = 4, F,G,Q = random.

	Variable Step			Fixed Step		
Case No.	LK	P	factor	LK	P	factor
1	0.321	1.111	3.46	0.088	0.215	2.44
2	0.341	0.647	1.90	0.094	0.213	2.27
3	0.368	0.985	2.68	0.082	0.225	2.74
4	0.346	1.354	3.91	0.093	0.216	2.32
5	0.273	0.762	2.79	0.081	0.215	2.65
6	0.457	0.686	1.50	0.099	0.219	2.21

Table 8. Computing times (in seconds) for
n = 8, F,G,Q = random.

	Variable Step			Fixed Step		
Case No.	LK	P	factor	LK	P	factor
1	1.136	6.115	5.38	0.243	0.945	3.89
2	1.278	6.796	5.32	0.267	0.944	3.54
3	1.514	6.545	4.32	0.250	0.955	3.82
4	1.303	6.635	5.09	0.271	0.954	3.52

Table 9.* Computing times (in seconds) for $n = 16$, F,G,Q = random.

	Variable Step			Fixed Step		
Case No.	LK	P	factor	LK	P	factor
1	1.452	22.024	15.17	0.890	5.522	6.20
2	1.667	27.474	16.48	0.871	5.547	6.37
3	1.602	16.574	10.35	0.901	5.560	6.17
4	1.802	22.752	12.63	0.906	5.568	6.15

The overwhelming conclusions to be drawn from Tables 1-9 are that the LK-system not only yields a smaller system of equations if the number of system inputs and outputs is small, but also possesses a more favorable analytic structure. This last point is well illustrated by the variable-step experiments when, for example, in the case $n = 16$, a theoretical factor of between 4 and 5 is expected while the observed factor is between 10 and 16 (Table 9). Even in the fixed-step mode the LK-system exceeds theoretical expectations due to the requirement of performing fewer numerical operations in a single integration step than that required for the P-systems. As a point in passing, all results show a significant difference in computing times between the variable and the fixed-step procedures, with the variable-step mode being greater. This is due to the widely differing magnitudes of the roots of F and the substantially greater overhead costs associated with execution of the variable step size computer program.

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*The interval length of calculations for this case was 50/512, with the Runge-Kutta integration carried out with a step size of $h = 1/512$.

IV. Numerical Stability

As prelude to an investigation of the infinite interval problem, experiments were performed to empirically check the numerical stability of the LK-system, as opposed to that of the Riccati system. Using a preditor-corrector method of order 7-8, both systems were integrated to an interval length at which the "steady-state" can be assumed to have been reached. To numerically decide at what interval length t* this "saturation" condition is first satisfied, two different criteria were used:

I: relative change in all components of P less than 0.5×10^{-4}
absolute value of components of L less than 0.5×10^{-4} in going from t* - 1 to t*

II: relative change in components of P and K less than 0.5×10^{-4} in going from t* - 1 to t*.

In all experiments, the matrices F, G, and Q were those used in the calculation of Tables 1-3. The results are shown in Tables 10-14.

Table 10. Saturation interval length using stopping rule I, n = 4.

Case No.*	LK	P
1	20.	11.
2	6.	2.
3	11.	4.
4	103.	44.
5	13.	6.
6	4.	9.

*In Tables 10-14, the Case Numbers correspond to those in Tables 1-5.

Table 11. Saturation interval lenths using stopping rule I, n = 8.

Case No.	LK	P
1	36.	23.
2	44.	7.
3	32.	39.
4	37.	6.
5	176.	33.
6	92.	17.

Table 12. Saturation interval lengths using stopping rule I, n = 16.

Case No.	LK	P
1	52.	stiff
2	56.	stiff
3	89.	25.

Table 13. Saturation interval lengths using stopping rule II, n = 4.

Case No.	LK	P
1	11.	12.
2	3.	3.
3	5.	5.
4	35.	35.
5	7.	7.
6	4.	4.

Table 14. Saturation interval lengths using stopping rule II, n = 8.*

Case No.	LK	P
1	29.	stiff
2	9.	12.
3	26.	stiff
4	3.	stiff
5	46.	51.
6	17.	17.

Tables 10-14 show that when similar stopping criteria are used, the LK-system reaches the "right" steady-state solutions at least as fast as the Riccati system, and often faster. It is important to note that in some of our cases, particularly as n became large, it was not possible to obtain a satisfactory steady-state solution using the Riccati system as the numerical error became too great due to the widely varying characteristic roots of F. However, the LK-system still gave satisfactory results for these cases (Tables 12-14).

The tentative conclusion to be drawn from these limited investigations is that the LK-system possesses numerical stability properties at least as strong as those enjoyed by the Riccati equation, and probably stronger. However, further analysis will be needed before precise estimates can be given.

V. Infinite Interval Case

Many problems of control and estimation require the optimal gain function K over the semi-infinite interval $[-\infty,T]$. When calculating gains using the Riccati system (4), it is an

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*The case n = 16 was not calculated due to the numerical instability of the P-equation; however, the LK-system might still give good results.

easy matter to obtain the relevant algebraic Riccati equation for $P(\infty)$ by simply setting $\dot{P} = 0$. This yields the quadratic matrix equation

$$Q + PF + F'P - PGG'P = 0 \quad , \tag{7}$$

for the matrix P.

The situation for the LK-system is not quite so simple as examination of Equation (5) will show. The approach of setting $\dot{L} = \dot{K} = 0$ in order to obtain the appropriate algebraic equation for $L(\infty)$ and $K(\infty)$ yields only the information $L(\infty) = 0$, but gives no equation for $K(\infty)$. This is because $K(t)$ does not appear in the equation for $\dot{K}$. However, a minor trick from matrix theory salvages the situation and it has been shown [9] that the appropriate equation is

$$\sigma(K) = (I \otimes G')(F' \otimes I + I \otimes F')^{-1} \sigma(K'K - Q) \quad , \tag{8}$$

where $\otimes$ denotes the usual Kronecker product while σ is the operator which "stacks" the columns of a matrix into a single column vector, i.e. if $A = [a_{ij}]$, then

$$\sigma(A) = (a_{11}, a_{22}, \ldots, a_{n1}, a_{12}, a_{22}, \ldots, a_{nn})' \quad .$$

The important point to note about Equations (7)-(8) is that Equation (7) represents $n(n + 1)/2$ algebraic equations in the components of P while Equation (8) consists of nm components of K. Thus, if $m < (n + 1)/2$, there are fewer equations in (8) than (7).

To check on the relative efficiency of using (7) or (8) to calculate the optimal gain, experiments were performed for the cases $n = 4, 8$ with F, G, and Q as in the finite-interval calculations. Thus, F is specified by its characteristic roots (choosen so that $F' \otimes I + I \otimes F'$ is invertible) and G is a single column vector $(m = 1)$.

The basic approach was to integrate both the LK- and P-systems to a value of t* for which the terminating condition was that stopping rule I be observed. These values of K and P were then used as initial approximations in a modified Newton iteration procedure [10] to compute the solutions of both the algebraic equation for K and the algebraic Riccati equation. The computing times reported in Tables 15-16 are only for the iterative scheme and do not include the preliminary generation of the initial approximations. However, the computing times for the K-equation do include inversion of the matrix $F'\otimes I + I\otimes F'$. The stopping criteria for the iteration scheme was that the residuals have ℓ_2-magnitude less than 0.5×10^{-4}. The results are

Table 15. Computing times (in seconds) for n = 4.

roots of F	K-equation	P-equation	factor (theory 2.5)
1,-1.4,1.5,9.5	0.123	0.431	3.50
-5.5±3.5i,8.4,-12.4	0.123	0.757	6.15
3.3,-7.5,-0.2±9.2i	0.119	0.818	6.87
-0.1±i,-0.2±9.2i	0.087	0.575	6.61
1.75,-8,-8,-8	0.102	0.706	6.92
-8,-8,-8,-8	0.073	0.783	10.73

Just as in the finite interval case, we see that the observed improvement factor for the K-equation is greater than the theoretically predicted factor based on a count of the number of equations. Again, this is explainable (intuitively) only if the K-equation has a "smoother" analytic structure than the algebraic Riccati equation.

Additional results on the K-equation, including implications for the "inverse problem" of optimal control theory are reported in [9].

Table 16. Computing times (in seconds) for n = 8.

roots of F	K-equation	P-equation	factor (theory 4.5)
1,-1.4,1.5,3.3, -7.5,-0.2±9.2i,9.5	2.680	13.447	5.02
1,2,2.5,2.5,2.5, 2.5,-3,-4	2.894	22.486	7.77
1,-1.4,1.5,-8, -8,-8,-8,9.5	2.982	12.973	4.35
0.5,0.1±3.3i,-3.5, 4,-4.5,-7,8	2.826	15.280	5.41
1,-0.1±i,-1.4, 1.5,-0.2±9.2i,9.5	2.711	14.792	5.46

VI. Discussion and Future Work

The above results, limited as they have been by constraints of time, money, and other interests, strongly suggest that the LK-approach to linear control processes be further investigated. Certainly, many more numerical results on a variety of problems are necessary before any definitive statements can be made about the relative merits of the LK-system versus the Riccati equation. Among the many topics which present themselves for attention, we feel those on the following list are of particular importance:

i) *analytic studies of the LK-structure* - it has been observed above that the LK-system seems to possess certain analytic features which enable variable-step integration routines to take larger steps with the same accuracy than on the Riccati systems. A primary point to investigate is what these analytic features are, how frequently they can be expected to occur in real problems, what their connections are with the rank conditions defining the sizes of L and K, and so on. Presumably, a satisfactory answer to these questions will also shed light on the observed numerical stability reported in Section IV.

ii) *Sparseness in F, G, Q* - many problems of practical interest involve system matrices F, G, and Q which contain a high proportion of zero entries. It would be a worthwhile exercise to investigate the frequency with which such problems also possess the low rank features giving rise to small LK-systems, and also the numerical properties associated with such problems. We have seen that even if the LK-system contains more equations than the Riccati system, a computing time reduction may still be possible. It is of interest to know if such situations are in any way connected with sparseness properties in F, G, and/or Q.

iii) *Iterative Procedures for Steady-State Equation* - our results on the infinite-interval problem were obtained with the help of a quasi-Newton procedure designed for general systems of nonlinear algebraic equations. However, as is known for the algebraic Riccati equation, it is possible [11] to develop special procedures which exploit the specific structure of the problem to generate fast, accurate solutions. A similar line of investigation should be pursued for the steady-state equation (8). An important first step in this program would be to develop reasonably general criteria for the initial approximation which would insure convergence of a Newton scheme.

iv) *Infinite-Dimensional Problems* - many problems of contemporary interest in control theory center about so-called "distributed" parameter systems, in which the system dynamics are described by a partial differential equation, delay-differential equation, or an equation of even more exotic type. It has been observed [12] that the LK-approach affords a significant computational reduction in these problems, generally even greater than in the finite-dimensional case and a few feasibility calculations have been performed. It would be of great value to carry out an extensive set of numerical experiments on this class of problems in much the same way as we have done above for the finite-dimensional situation. The authors are aware of some steps in this direction currently being taken at IRIA

Laboratories in France and at the Research Center for Applied Mathematics at the University of Montreal. Preliminary results seem encouraging but any definitive statements await completion of the experiments.

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