On Some Optimal Control Problems Arising from Project Management

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1. Introduction

The complexity of modern industrial and governmental enterprises with the consequent increase in the quantity and sophistication of managerial decisions, on the one hand, and the fact that the payoffs from good decisions are greater than ever before, on the other hand, offer a challenge to build up conforming scientific methods for decision making.

The systems programmed approach or the programmed control method is a practical method to manage large and complex systems. In general it is a feed-back decision-making process which implies many time planning processes and consists of such elements as forecasting, formulation of goals and objectives, collection of available alternative strategies to achieve the goals, selection of the best alternative, realization of the strategy, comparison of the results with predicted outcome, new forecasting, reformulation of goals and so on [1, 2, 4, 5, 6].

Thus the process includes both formal (strict) and informal (heuristic) procedures.

To find the application and to emphasize the necessity for using rigorous mathematical methods in the decision-making process in the economic systems, we shall briefly describe some decision-making elements and the corresponding mathematical models.

2. Planning Procedures for Decision Making in Economic Systems.

Let us consider some given economic system. Here we shall not concentrate any system properties and only assume that the system satisfies the complex system definition given in [7].

In general the system has a multilevel hierarchical structure. The directive body or top management identifies and formulates goals of the system. In the end goal determination is a heuristic process. The formulation of alternative goal patterns depends essentially on alternative future environments , internal state of the system and preferences of the directive body.

The identification and formulation of goals may be facilitated if one uses such techniques as scenario writing and analysis of future environments. They help to select appropriate goals from the standpoint of top management for the most probable future events. Here one may use estimates of resources necessary for achieving the goals. The estimates should be obtained on the basis of aggregated growth models of system development. As a result, the top management formulates the goals of the system.

The next problem one faces is elaboration of programs. Here we define a purpose program as a long-range plan under which action may be taken toward the goals and the action itself. The program's elaboration consists of two stages. The first stage is a hierarchical representation of the goals. For this purpose social and economic goals, needs, objectives, missions and tasks are listed in decreasing order of generality. For every task all kinds of technological means have to be found, including for instance, all kinds of R and D projects, to perform the tasks. Thus a tree of all kinds of means in fact known or conceivable must be developed to attain the goals.

Analizing the tree one determines a set of jobs (operations) to be performed to achieve every given goal. Some of these jobs can exclude each other. Thus we get a number of alternative programs which realize various strategies of goal attainment.

Every job of a program is to be assigned the resources to be utilized and the time or intensity of the job's performance. In general some jobs may give birth to new kinds of resources. Such jobs simulate building of new industries.

The second stage of the program's elaboration is the calculation of the system development programs.

It will be shown that this problem is reduced to a special scheduling problem. Its peculiarity is that resource supply is not given in advance and may be found only in the course of scheduling calculation. As a result one obtains some variants of development programs.

Every variant is a schedule of the job's performance (including R and D programs), the resources supply schedule and resource allocation over time.

The top management selects the most appropriate version of the program sets obtained. Note that none of the program versions may suit the directive body. Then, the goal formulation procedure will have to begin again.

The program accepted is the base for planning at the next (lower) hierarchy levels.

In so far as the program includes preliminary data on R and D dynamics and the development of new technologies, one has grounds for predicting changes in technologies and corresponding expenditures.

Apparently one can do a quite reliable forecasting for the sum time interval of the period required to accomplish R and D projects and the period of experimental production. This sum time interval determines the planning horizon.

The program sets the tasks for industrial capacities growth in a national economy and the tasks for ultimate product outputs over a period of planning.

The national economy planning may be realized on the basis of dynamic economic models (for example see [3]). In the next section we shall consider the simplest model of this type. As a result we obtain quantative indices of growth and load plans for production capacities. These plans determine the management program for industries.

Let us consider a model of program performance which is necessary for preliminary program selection and evaluation by the top management.

3. The Program Performance Model and the Industrial Model.

According to the above definition the program P is a set of operations (jobs) $\{a_1,...,a_N\}$ which should be carried out to achieve the system goals. In a social system the programs are usually formulated by groups of various experts. So the operations are defined by experts, who use the language of the corresponding science. The concept of an operation includes some object, some transformation of this object and some conditions to be provided to realize the transformation. For example, one may consider as the operation the designing of some device or its testing.

The model of the program may be written in various ways. Unlike [2], where the model is written in the language of multidimensional logic, here we use the finite-difference equations to describe this model.

Every operation a_i is characterized by the number (mark) z_f^i which depends on the concrete content of the operation. For example, it may be the time necessary to complete the job or the amount of the job measured in some terms, etc.

The program performance is considered in discrete time $t = 0, 1, \cdots$ and consists of the performnance of all the operations $a_1, \ldots, a_n$ included in the program.

The state of the job i at a given instant is characterized by the number $z^i(t)$. So, if $z^i = 0$, the job a_i has not yet begun. If $0 < z^i < z_f^i$, the job is partially performed. If $z^i \geq z_f^i$ the job is completed.

The state of the program is described by the vector $z = (z^1, \ldots, z^N)$. Its dimension N is equal to the number of jobs in the program.

Let us consider the program to be completed, if

$z(t) \geqq z_f$ where $z_f = (z_f^1, \ldots, z_f^N)$.

Call $u^i(t)$ the performance intensity (speed) of the job a_i, i.e. the rate at which the operation number z^i increases within the period $[t, t+1]$. The performance intensity of the program at the given period t is the vector.

$$u(t) = (u^1(t), \ldots, u^N(t)).$$

Certain components of this vector may be zero. This implies that the jobs with the relevant numbers are not carried out within the given period.

The intensity vector may be regarded as a control of the program performance.

The performance of the jobs is usually subjected to certain constraints. We divide them into two constraint groups.

The first group is (α) constraints. These constraints describe interrelations between the performance of the operations and they include constraints of a logical kind (constraints on the sequence of the operation's performance). In general they may be written in the following form

$$u(t) \in U \ (z(t), t)$$

where

$U \equiv$ a set of admissable intensities at the moment t, when the state of the program is $z(t)$.

The second group is (β) constraints. This is the group of constraints on the amounts of resources.

Let the system produce and consume M different resources. While being performed the job i requires certain resources. This demand depends on the intensity with which the job is carried out.

Besides, new resources may be generated as a given operation is completed because the operation itself may be a production process. Let R^i be the M-space resource vector required to perform the job i at the period t.

We assume that the job i consumes k-th resource if $R_k^i > 0$, and the job produces the resource if $R_k^i < 0$. So the total amount of resources which the program consumes at the period t is equal to $\sum_{i=1}^{N} R^i$. This amount depends on the state of the program z(t) and the intensity u(t). Therefore the (β) constraints may be written as

$$\sum_{i=1}^{N} R^i(z(t), u(t), t) \leqq Q(t) \quad .$$

where Q(t) ≡ inflow resources to carry out the program. Thus to calculate the program performance means to make up a list which designates at each period t the vector of intensities with which jobs of the program should be performed.

The performance of the program can be controlled because usually there are a number of various ways to carry out the jobs. Indeed there are a number of schedules which satisfy the restrictions of (α) and (β) groups. Therefore we may formulate the problem as follows: select a schedule which best suits the purposes of the top management. In some cases the management requires that the program be completed in the minimum time. In other cases the time of fulfilling the program is given in advance and the states of this program are ranked according to the preferences of management. Then the calculation of this program is reduced to the following problem: find a schedule which transfers the program to the most preferable state for a given time.

Thus, in general, the best schedule u(t) is regarded as that in which a given objective function I is minimized.

The peculiarity of the problem is that inflow of resources (vector Q(t)) is unknown in advance and can be determined during the process of the schedule calculation. The resources inflow is provided by the manufacturing processes and the amounts depend on the capacities of the various industries.

The operation of those industries may be described in a similar (to the program performance model) manner.

The resources supply plan for the industries is regarded as a support program. In this case the jobs to be carried out are unknown in advance and are to be determined during the process of schedule calculation on the basis of the most appropriate ("best") resources provision of the purpose program.

To illustrate the main idea we consdier here the simpliest linear multibranch industrial model:

Let the system economy be subdivided into L producing sectors.

A balance equation is written as

$$q(t+1) = q(t) + x(t) - A(t)x(t) - y(t) - w(t) - Q(t)$$

where

$x(t) = (x^1(t),\dots,x^L(t));$

$x^i(t)$ ≡ a(gross) output of the i-th industry within the period $[t, t+1]$ (the duration of the production cycle is considered to be equal to 1).

$q(t)$ ≡ the vector of resource reserves within the same period;

$A(t)$ ≡ matrix L × L with elements $a_{ij}(t)$;

$a_{ij}(t)$ ≡ input coefficient of product of sector i into sector j (the quantity of the output of sector i absorbed by sector j per unit of its total output);

$w(t)$ ≡ given vector of consumer goods;

$y(t)$ ≡ investment vector;

$Q(t)$ ≡ portion of the final product which is sent into the purpose program.

The amount of the gross product (output) is restricted by the capacities of the industries

$$s(t+1) = s(t) + r(t),$$

$$r(t) = F(y(t), y(t-1),\dots,y(t-\ell)) \quad .$$

where

$s(t) = (s^1(t),\ldots,s^L(t))$;

$s^i(t) \equiv$ capacity of the i-th industry (sector) within t-th period;

$r(t) = (r^1(t),\ldots,r^L(t))$;

$r^k(t) \equiv$ increase of the i-th industrial capacity during the period $[t, t+1]$;

$F \equiv$ given linear operator which takes into account lags between investments and their realization.

Here we assume ℓ to be the uniform value for lag in all industries. In addition initial values for s and q are given. In this model we consider x, y and Q to be controls.

We state the problem of calculation of the development program as:

find u, x, y, Q, T, such that

$$I(z, u) \rightarrow \text{Min}$$

under the constraints

$$z(t+1) = z(t) + u(t) \quad , \quad z(0) = 0 \qquad (1)$$

$$z(T) \geqq z_f \quad , \qquad (2)$$

$$(\alpha) \quad u(t) \in U(z(t), t) \quad , \quad u \geqq 0 \qquad (3)$$

$$(\beta) \quad \sum_{j=1}^{N} R_j(z(t), u(t), t) \leqq Q(t) \quad , \qquad (4)$$

$$q(t+1) = Q(t) + x(t) - A(t)x(t) - y(t) - Q(t) \quad , \qquad (5)$$

$$x(t) \leqq s(t) \quad , \qquad (6)$$

$$s(t+1) = s(t) + r(t) \quad , \qquad (7)$$

$$r(t) = F(y(t), y(t-1),\ldots,y(t-\ell)) \quad , \qquad (8)$$

$$s(0) = s_0, \ q(0) = q_0 \quad , \qquad (9)$$

$$y(-1) = y^0_{-1}, \ y(-2) = y^0_{-2},\ldots,y(-\ell) = y^0_{-\ell} \qquad (10)$$

$$q(t) \geqq 0, \ x(t) \geqq 0, \ r(t) \geqq 0, \ w(t) \geqq 0 \qquad (11)$$

Further we refer to this problem as the main one.

Note that the connection between industrial and program models is realized only over parameters Q(t). Below we consider an approach to carry out joint calculation of the models (to solve the main problem). This approach takes into account the "weak connection" between these two models.

The main problem is to solve for every alternative program. As a result we obtain information about optimal job performance and the resource inflow schedule (Q(t)) in terms of limited industrial capacities.

4. The Joint Calculation of Industrial and Program Models

Let us consider a simple computational approach to solve the main problem which is based on the utilization of standard linear programming methods. We assume the planning interval to be given. Its length is equal to T and the objective function I is defined as

$$I(z(T)) = \sum_{j=1}^{N} \lambda_j (z_f^j - z^j(T))^2 \quad .$$

We interpret I as the distance between two points z(T) and z_f in N-dimensional space. $\lambda_j > 0$ is the relative "weight" of the j-th operation in the program. Thus our aim is to minimize the distance.

The iterative process of solving the problem is as follows:

1. To calculate the plan of the program performance or jobs schedule $u^{(0)}$ when all input resources are available

$(Q(t) = +\infty$ for all $t)$. That is, we solve the problem

$$I \rightarrow \text{Min}$$

s.t. (1) - (3).

As a result we obtain the plan when all jobs begin at their early start times. During the process we calculate the resources requirement of the program - $Q^{(0)}(t)$, $t \in [0,T]$ $R^{(0)}$ determines the ("ideal") structure of resources demand in the program.

We denote $I^{(0)}$ as the value of the objective function corresponding to $u^{(0)}$. This number is the low limit for the optimal value of objective function I^{*} under (α) and (β) constraints.

2. Solve the Problem

$$\sum_{t=0}^{T-1} \rho(t) \rightarrow \text{Max}$$

s.t. (6) - (11),

$$\rho(t+1) \doteq \rho(t) + x(t) - A(t)x(t) - y(t) - w(t) - \rho(t)\, R^{(0)}(t),$$

$$0 \leq \rho(t) \leq 1 \qquad (12)$$

where $\rho(t)$ is a scalar.

Here we are trying to adjust the industry output to satisfy the input resources demand of the purpose program. Let $\rho^{(0)}_{(t)}$, $x^{(0)}(t)$, $y^{(0)}$, $\rho^{(0)}$ be the solution of the problem.

3. If $\rho^{(0)}(t) = 1$ for all t, the main problem is solved. The optimal solution is $\xi^{(0)} = (u^{(0)}, Q^{(0)}, x^{(0)}, y^{(0)}, \rho^{(0)})$ where $Q^{(0)}(t) = R^{(0)}(t)$, $t \in [0,T]$.

Otherwise, if for at least one t $\rho^{(0)}(t) < 1$, we set

$$Q^{(1)}(t) = \rho^{(0)}(t)\ R^{(0)}(t)$$

and go on to 4.

4. The Problem

$$I \rightarrow \text{Min}$$

s.t. (1) - (4)

is to be solved where $Q(t) = Q^{(1)}(t)$.

The solution is denoted as $u^{(1)}$ and the corresponding value of the objective function $I^{(1)}$. At each step we calculate the resource deficit as the difference between the resources demand to carry out all (admissable by (α) constraints) jobs and the given resources inflow $Q^{(1)}(t)$ at the same moment. We denote this difference as $\Delta Q(t)$. It may happen that at some moments $\Delta Q_i(t) < 0$. This means we have the surplus of the i-th resource at these moments.

$I^{(1)}$ is the upper limit for the optimal value of the objective function.

The computations are considered to be finished when the difference between $I^{(1)}$ and $I^{(0)}$ is less than some given small positive number ε_1.

In this case we assume that the problem is solved and $\xi^{(1)} = (u^{(1)}, Q^{(1)}, x^{(0)}, y^{(0)}, \rho^{(0)})$ is the optimal solution.

Otherwise, if $I^{(1)} - I^{(0)} > \varepsilon_1$, go on to 5.

5. Let the (k-1)-th iteration be completed. We describe k-th iteration.

The following problem is to be solved

$$\sum_{t=0}^{T-1} \rho(t) \to \text{Max}$$

s.t.

$$\rho(t+1) = \rho(t) + x(t) - A(t)x(t) - y(t) - w(t)$$

$$- (Q^{(k-1)}(t) + \rho(t)\ \Delta Q^{(k-1)}(t) \quad , \qquad (6) - (11)$$

$$Q^{(k-1)}(t) + \rho(t)\ \Delta Q^{(k-1)} \geqq 0 \quad ,$$

$$\rho(t) \geqq 0 \quad .$$

We denote the solution as $\rho^{(k)}(t)$.

If it holds that

$$\max_{0 \le t \le T-1} \rho^{(k)}(t) < \varepsilon_2$$

then $\xi^{(k-1)} = (u^{(k-1)}, Q^{(k)}, x^{(k)}, y^{(k)}, \rho^{(k)})$ is the optimal solution of the main problem where

$$Q^{(k)}(t) = Q^{(k-1)} + \rho^{(k)}(t)\ \Delta Q^{(k-1)}(t), \quad t \in |0,T| ;$$

$\varepsilon_2 \equiv$ given small positive number.

Otherwise, we let

$$Q^{(k)}(t) = Q^{(k-1)}(t) + \rho^{(k)}(t)\ \Delta Q^{(k-1)}(t)$$

and go on to 6.

6. Solve the problem

$$I \to \min$$

s.t. (1) - (4),

where $Q(t) = Q^{(k)}(t)$. Go back to 5.

The solution of the problem and the value of the objective function are $\xi(k)$, $I^{(k)}$.

We have arrived at the next set of inequalities

$$I^{(1)} \geq I^{(2)} \geq \cdots \geq I^{(k)} \geq I^{(0)}$$

because at each iteration we either increase or at least do not decrease the set of admissable controls in the problem (1) - (4).

It should be noted that instead of (13) other criteria may be used to abbreviate computations, e.g.

$$I^{(k-1)} - I^{(k)} \leq \varepsilon_1 \quad .$$

The problems to be solved in 1, 4, and 6 are dynamic scheduling problems and the problems to be solved in 2, and 5 are linear dynamic programming problems.

In conclusion we emphasize that the efficiency of the decision making procedures described is essentially dependent upon providing efficient calculations for the models presented. Usually there are several alternative programs which should be compared. Moreover each program consists of a large number of operations (even though aggregated) and the program requires many kinds of resources. Thus such calculations are impossible and inconceivable without the development of appropriate numerical methods, their computer realization and testing.

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