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City Sizes, Morphology and Interaction

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CITY SIZES, MORPHOLOGY, AND INTERACTION

W. R. Tobler

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On City Sizes and Growth

It has frequently been observed that the built up area of a city can be approximated very well ($R^2 \approx .90$) by a formula of the form

$$A = \alpha P^{\beta}$$

where A is the built up area, α and β are numerical coefficients, and P is the number of people in the town. It follows that the population density of a town is

$$D = \frac{1}{\alpha} P^{1-\beta}$$

and that the rate of expansion of the town is

$$\frac{dA}{dP} = \alpha\beta \; P^{\beta-1}$$

Clearly this relation can be inverted to estimate the population from the growth in area. If a town is circular then it follows that its radius is

$$r = \sqrt{\frac{\alpha}{\pi}} \; P^{\frac{\beta}{2}}$$

and its rate of radial growth is

$$\frac{dr}{dp} = \frac{\beta}{2}\sqrt{\frac{\alpha}{\pi}} \; P^{\frac{\beta}{2} - 1}$$

Previous estimates of the coefficients have shown variations in different cultures, Egyptian and Japanese towns being approximately sixteen times more compact than similar sized cities in West European cultures, (Tobler, 1969). For the latter Boyce (1963) and Nordbeck (1965) obtain approximately

$$A = 0.003848 \; P^{0.88} \qquad (A \text{ in km}^2)$$

whereas Maher (1973) gives

$$A = 0.001539 \; P^{0.87}$$

for Canadian cities. In all of these cases the estimates of the coefficients were obtained from cross section data. It was therefore considered worth examining at least one city through time. The attached graph shows the logarithm of the area of Ann Arbor plotted against the logarithm of the population. The area in this instance is the legal area (in acres); this of course does not coincide with the functional built up area of the city. The growth has been continuous so that connecting the dots in population size order also gives the temporal path, though not in equal increments. For the last two decades the data have been available annually.

The graph clearly splits into two periods,

Pre WWII:

$$A = 1.0117\ P^{0.2625}, \qquad (A = km^2)$$

and

Post WWII:

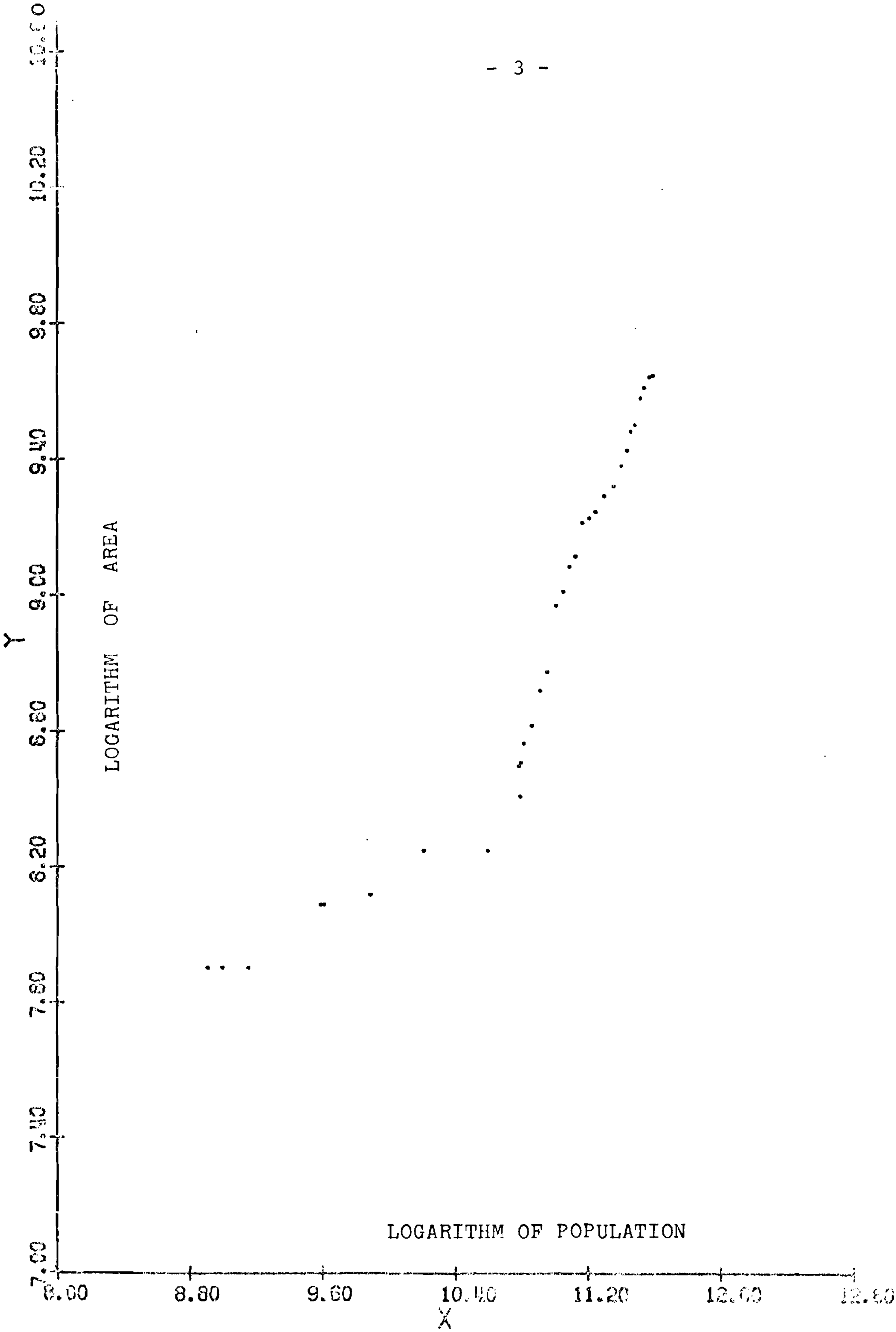
$$A = 0.0000029\ P^{1.4625}$$

where I have estimated the coefficients graphically. Both periods seem to display log-linear behavior, but of dramatically different slopes. If this phenomenon is generally true perhaps this allometric relation can be used to monitor structural changes in the urban system.

W. Tobler,

Laxenburg,

4 February, 1975.


Y
LOGARITHM OF AREA
7.00
7.40
7.80
8.20
8.60
9.00
9.40
9.80
10.20
X
LOGARITHM OF POPULATION
8.00
8.80
9.60
10.40
11.20

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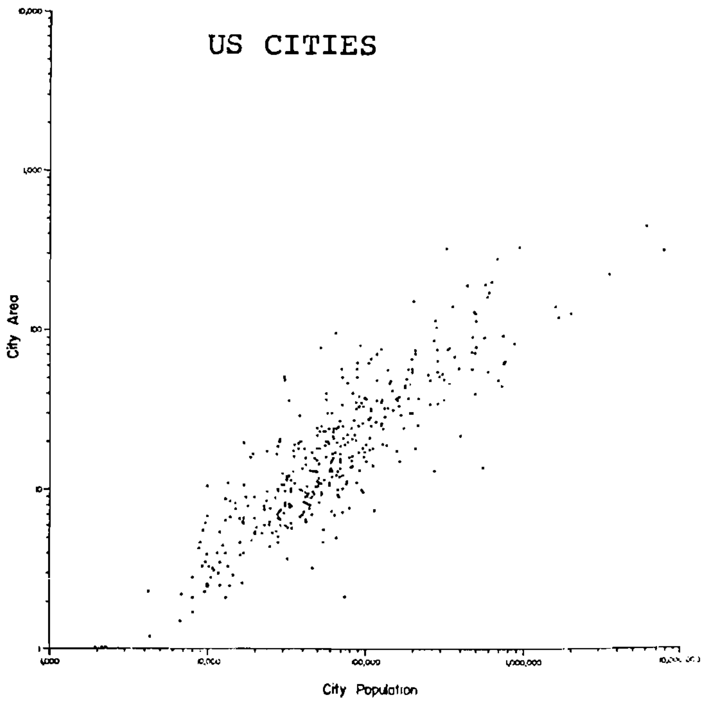
R. Boyce, "Changing Patterns of Land Use Consumption" The Professional Geographer, 15, No. 2, (1963) pp. 19-24.

C.A. Maher, "Urban Form and City Size: An Ontario Example", pp. 37-46, of L. Bourne, R. MacKinnon, and J. Simmons, eds., The Form of Cities in Central Canada, Research Pub. #11, Department of Geography, University of Toronto 1973.

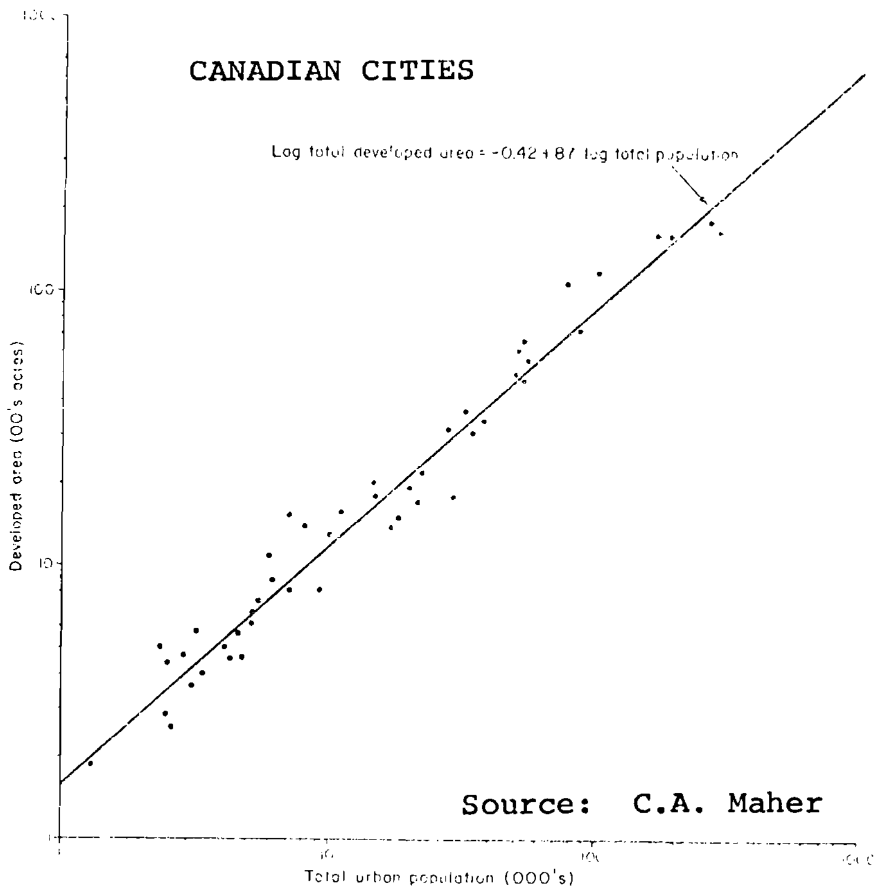
S. Nordbeck, The Law of Allometric Growth, Discussion Paper #7, MICMOG, Ann Arbor, 1965.

W. Tobler "Satellite Confirmation of Settlement Size Coefficients", Area Vol. 1, #3 (1969), pp. 30-34.

The literature on this topic is summarized in Tobler (1969), op cit; for persons not familiar with this work the two graphs below may be helpful. Also observe that one isolated person requires a radius of 35 meters, by the formula.



Source: B. Berry, F. Horton, Geographic Perspectives on Urban Systems, Prentice Hall, 1970, p.65.



Source: C.A. Maher

A Difficulty in Urban Morphology Description

(I) The density of population (persons kilometers^{-2}) within a city is describable as a decreasing function of distance from the center of the city; usually

$$D(r) = Ae^{-br} ,$$

although Gaussian curves, Gamma functions, Bessel functions, modified Pareto functions, cylindrical, conical, parabolic, and cosine functions, and several others, have also been proposed. There are some versions which treat radially asymmetric cases, and some consider changes over time. A and b are numerical coefficients.

(II) The radius of a circular city is describable by

$$R = \alpha N^{\beta}$$

where

N is the total population;

R is the distance to the edge of the city from its center;

alpha and beta are numerical coefficients.

(III) The equation in (I) does not define an edge for the city. But (II) clearly specifies the distance at which the edge occurs. Empirical evidence seems compelling for both equations.

(IV) If we know the rural density, call it F, then this can be used to locate the edge of the city, using I:

$$F = Ae^{-bR}$$

and thus

$$R = -\frac{1}{b} \ln \left(\frac{F}{A}\right)$$

and, alternatively, if we know R then the rural density could be calculated.

For Paris, Bussière gives

N = 7,6000,000 people

R = 27.5 kilometers

A = 54,892 persons km^{-2}

b = 0.211

Thus

F = 165.77 persons km^{-2}

For cities Nordbeck gives

$\alpha = 0.035$

$\beta = 0.44$

Thus, from II,

R = 37.29 km

and one would then have, from I,

F = 21.01 persons km^{-2}.

The two results do not seem very compatible with each other. If Bussière's data are used then Nordbeck's coefficient α can be re-calibrated to obtain:

	Population	Radius	α
Paris	7,600,000	27.5	0,0258
Lyons	850,000	14.3	0.0352
Marseilles	800,000	13.0	0.0329
Toronto	1,700,000	20.0	0.0363
Toulouse	340,000	5.0	0.0184
Auxerre	28,000	1.8	0.0199

The agreement with Nordbeck's result is good in three of the six cases. These also agree with the value of 3/8 found by Stewart and Warntz.

Bussière also shows that

$$\lim_{r \to \infty} \int_0^{2\pi} \int_0^r r\, D(r)\, dr\, d\theta = \frac{2\pi A}{b^2}$$

which for his data on Paris yields a population of 7,746,829 people, 2% greater than the actual value. This is the population outside his "edge" for the region of Paris. In his other cases he over or under estimates the total populations by comparable amounts, except for Toronto, where the population beyond the edge amounts to ten percent of the total. He does not make clear how he chose the edges for the urban regions which he studied. This is a pity because his is otherwise one of the outstanding treatments on the topic. Presumably Bussière takes the "edge" of the city to be the point at which the cumulative population curve begins to fluctuate erratically. More attention needs to be paid to the boundary.

Stewart and Warntz also assert that the central density of a city, A, is equal to

$$A = k \cdot N^{\frac{1}{2}}.$$

Using Bussière's data the constant k becomes for

Paris	k = 19.9
Lyons	29.9
Marseilles	47.3
Toronto	8.0
Toulouse	53.9
Auxerre	54.8

A rather variable constant. Winsborough claims that A is a function of the age and type of city.

It has also been asserted (e.g. Weiss) that the exponent b is a function of the size of the city ($C = 1/2$):

$$b = aN^{-C} .$$

A logarithmic plot of Bussières' data shows this relationship. If A, b, and R were all such simple functions of N, then this one number would completely specify the morphology of a city. But they appear to vary according to the geographical context, being relatively stationary within large culture regions, but still depending on the geographical location relative to all other place in the same region. But Bussière has recently shown that A and b Plot as a straight line over time for several cities. Since they are related in the following way:

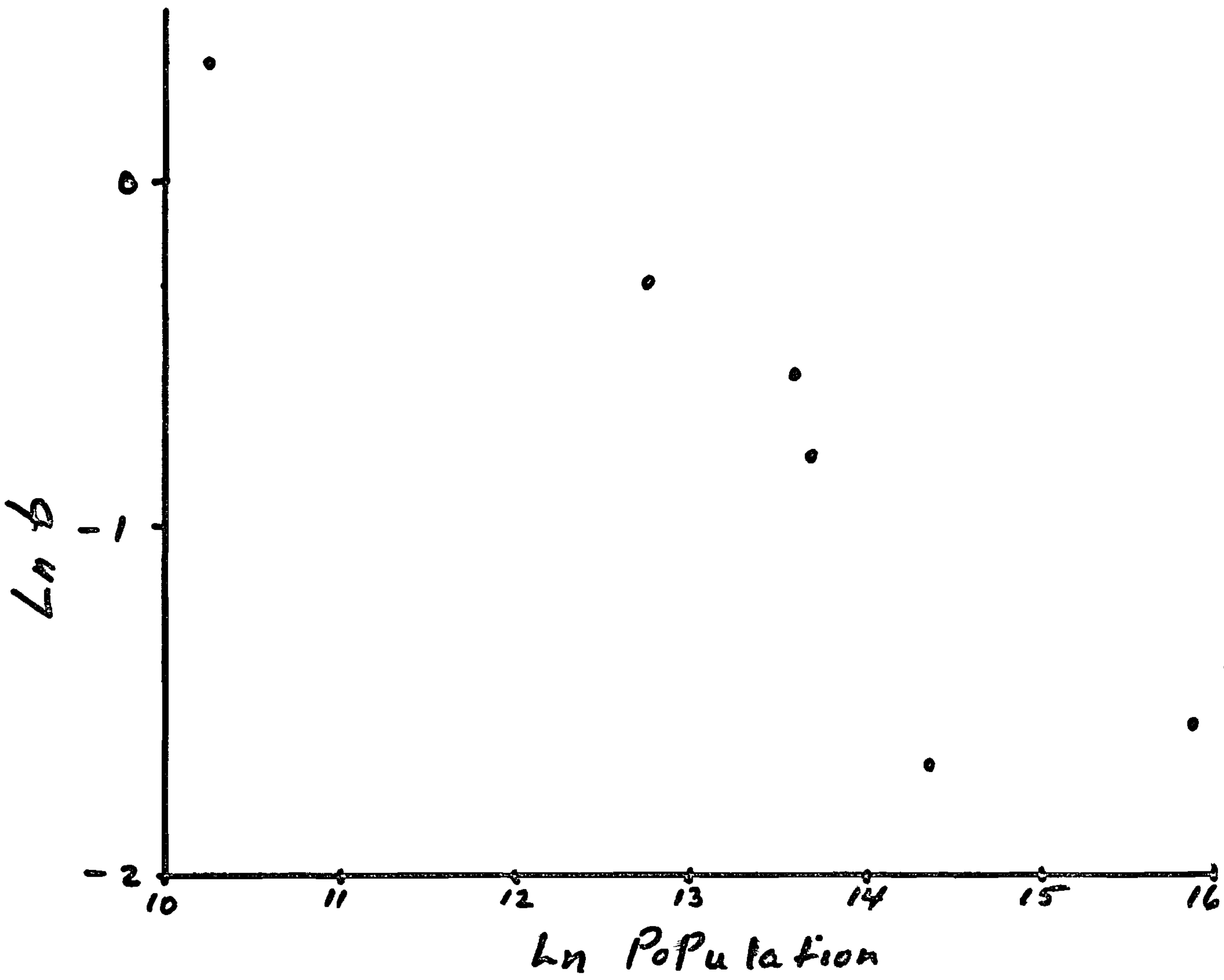
$$N = \frac{2\pi A}{b^2}$$

these graphs are a type of phase diagram for urban growth.

Thus some improvement is being made in our empirical under standing of urban morphology.

W. Tobler,
Laxenburg,
20th February, 1975.

b as a function of N



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A Re-examination of Some Old Data on Linguistic Borders and Spatial Interaction

The classic study of the effects of a linguistic border on the amount of interaction between geographic areas was published some years ago by MacKay (1). More recent work on this topic has been presented *inter alia*, by Gould and White (2). The present note is based on data assembled, and also analyzed, by Lill (3). The results are not as definative as I had hoped, but the data were too enticing, and I could not resist a quick look. Tables I and II give the number of passengers travelling on a portion of the Austrian Nordwestbahn during the year 1889. The selected section of the line runs NNW from Vienna past Prague to the then border of the Austro-Hungarian empire near Tetschen; see Figure I. The underlined places were asserted by Lill to be Czech speaking. The locations not underlined were German speaking. Vienna was, of course, cosmopolitan and a goodly proportion of its inhabitants spoke Czech. Lill's analysis of these data, he concludes, demonstrates a propensity to interact which is greater between linguistically similar places. But he does not do this in quite the same way as do MacKay or Gould and White. The thrust of Lill's paper, of course, is a thorough derivation, verification and application of what we now call the gravity model of interaction. Only a small section of the work is devoted to the study of linguistic effects. What I have done here is to have replotted some of his data in the currently fashionable manner. All of the data in the accompanying tables are extracted from Lill's paper. The greatest shortcoming of the data, I believe, is that they represent only the passengers of the one railroad company. For example, there existed another railroad connecting Prague and Vienna, to the west of the line shown in Figure I. Lill's original map shows several of the other possible routes of travel. Unfortunately the passenger counts for these other lines are not available. From Vienna to Kolin the line shown seems to have been the most direct connection. An asterisk in table I indicates places which are junction points, as far as I can tell from Lill's map.

The present border between Austria and Czechoslovakia bisects the section Retz-Znaim; my touristic map shows no current railroad connecting the two. Since these borders were supposed to have been based on local preferences, it is to be asked whether Lill's linguistic knowledge was correct. The borders were also supposed to end wars; clearly they were a failure! I have not examined any of the data collected by the Wilson peace commissions, though it should be available in some archive. They may even have used interaction data as a part of their deliberations.

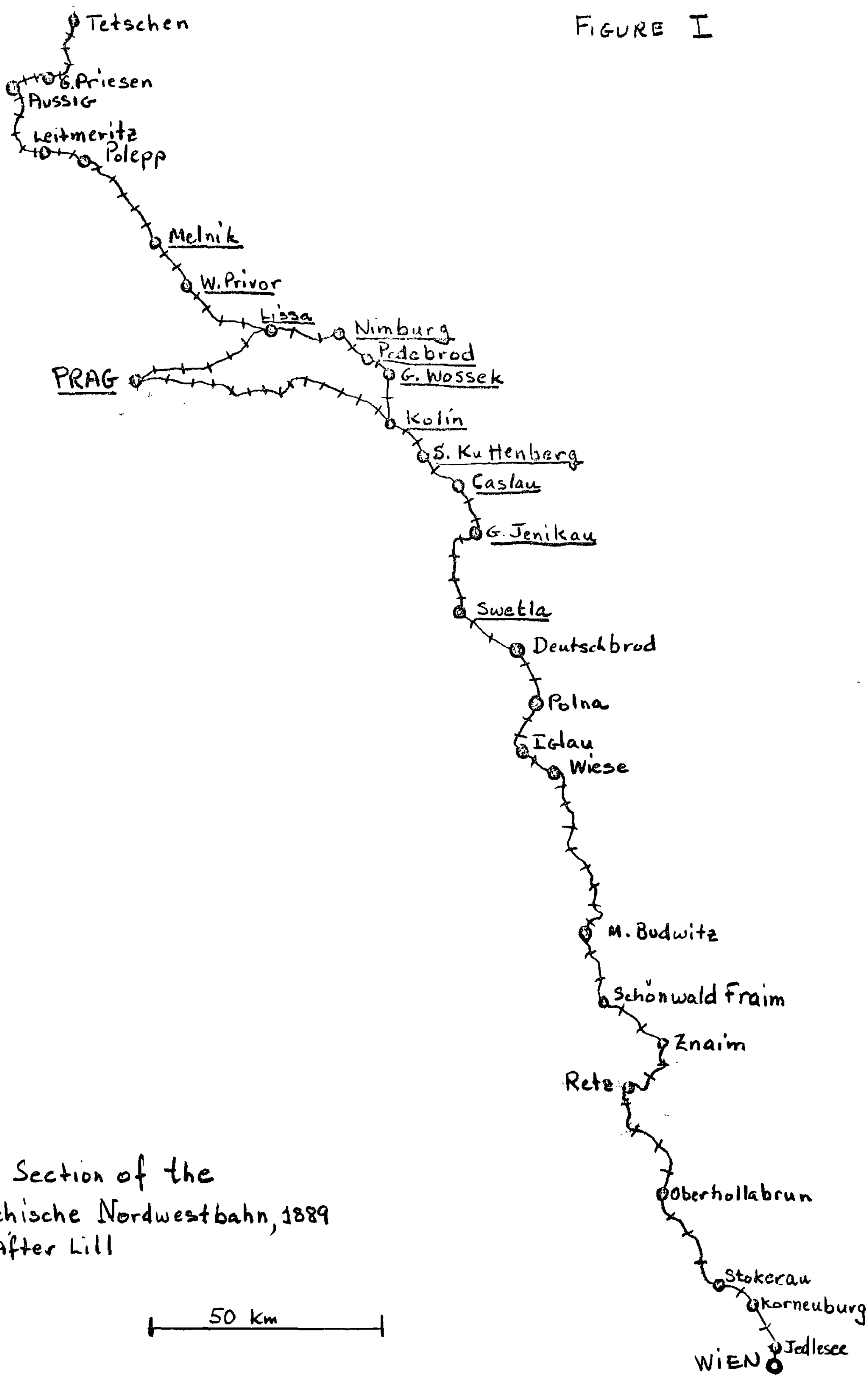
A plot (Figure II) of a portion of the data from Table I suggests that Gross Priesen and Tetschen really were German speaking. The anomolous values for Gross Wossek, Nimburg, and perhaps some others, may be explainable by the status of these as transfer stations, connecting to trains going to other parts of the empire. I have not sketched regression lines, so as to avoid biasing the reader's eye. Interestingly the slopes of the two lines which I would draw are different, and are not just a spatial translation. Znaim would fall at the intersection of my lines, suggesting that it is indeed a border town. The rate of decay of passenger traffic appears steeper beyond this point, especially if Gross Priesen and Tetschen are dis-regarded. Analysis of covariance, discriminant analysis, and so on, are statistical techniques of greater elaboration which may be used on these data. I find it difficult to detect any effect other than distance decay in the figure giving interaction with Prague.

W. Tobler,
Laxenburg,
30 Jan. 1975.

TABLE I	1890 Population	From Wien RR km	RR Passengers 1889	
			From Wien	To Wien
Wien *	705,400	0	--	--
Jedlesee *	2,312	6	37,798	--
Korneuburg	14,245	15	103,758	105,257
Stockerau	14,240	23	66,399	66,725
Oberhollabrunn	14,081	52	21,941	22,550
Retz	11,204	85	8,943	--
Znaim (Znojmo) *	31,803	103	19,303	18,240
Schönwald-Fraim (Šumná)	20,010	121	2,055	--
M. Budwitz (Moravske Budejovice)	28,750	139	4,665	5,530
Wiese (Luka n Jihlavou)	10,424	186	623	--
Iglau (Jihlava) *	51,049	198	7,726	8,997
Polna	18,716	210	838	--
Deutschbrod (Havlíčkův Brod) *	29,666	225	2,585	2,633
Swetla (Svetlá n Sázavou)	21,489	241	427	--
Goltsch Jenikau (Golčův Jenikov)	16,035	267	350	--
Caslau (Čáslav)	33,247	279	681	928
Sedletz Kuttenberg (Kutna Hora)	43,944	289	555	655
Kolin *	35,187	299	530	618
Gross Wossek *	1,897	309	215	346
Podebrod (Poděbrady)	12,701	315	128	--
Nimburg (Nymburk) *	11,937	323	528	502
Lissa (Lysa n Labem) *	9,801	338	149	--
Wschetat Privor (Všetaty) *	6,084	363	13	--
Melnik	22,851	375	99	--
Polepp	8,771	400	31	--
Leitmeritz Stadt (Litoměriçe)	25,438	409	535	696
Aussig (Usti n Labem) *	10,583	434	1,040	1,220
Gross Priesen	6,000	444	20	--
Tetschen (Děčin)	27,825	458	8,846	6,618

TABLE II	RR km From Prag	RR Passengers 1889		Population 1890
		From Prag	To Prag	
Prag	0	--	--	205,100
Kolin	74	822	1,301	35,187
S. Kuttenberg	85	5,856	6,773	43,944
Caslau	95	6,234	6,273	33,247
G. Jenikau	106	1,852	--	16,035
Swetla	132	1,087	--	21,489
Deutschbrod	149	1,579	1,762	29,666
Polna	163	342	--	18,716
Iglau	175	1,506	1,585	51,049
Wiese	187	23	--	10,424
M. Budwitz	234	149	139	28,750
Schönwald-F	252	11	--	20,010
Znaim	274	265	309	31,803
Retz	288	14	--	11,204
Oberhollabrunn	321	14	15	14,081
Stockerau	350	21	22	14,240
Korneuburg	358	27	46	14,245
Jedlesee	367	4	--	2,312
Wien	373	1,596	1,647	705,400

FIGURE I



A Section of the
Oesterreichische Nordwestbahn, 1889
After Lill

Figure II
Interaction with Vienna

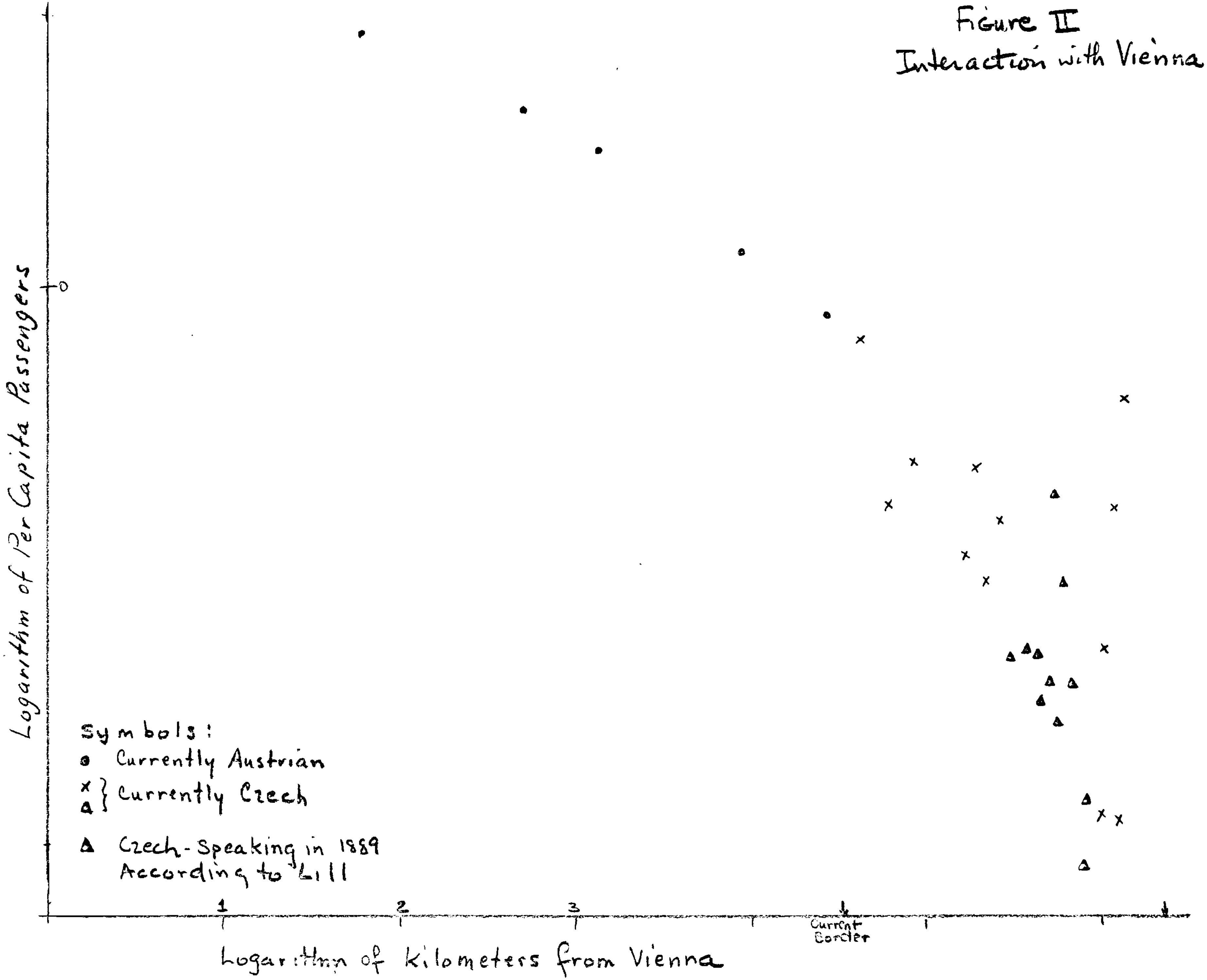
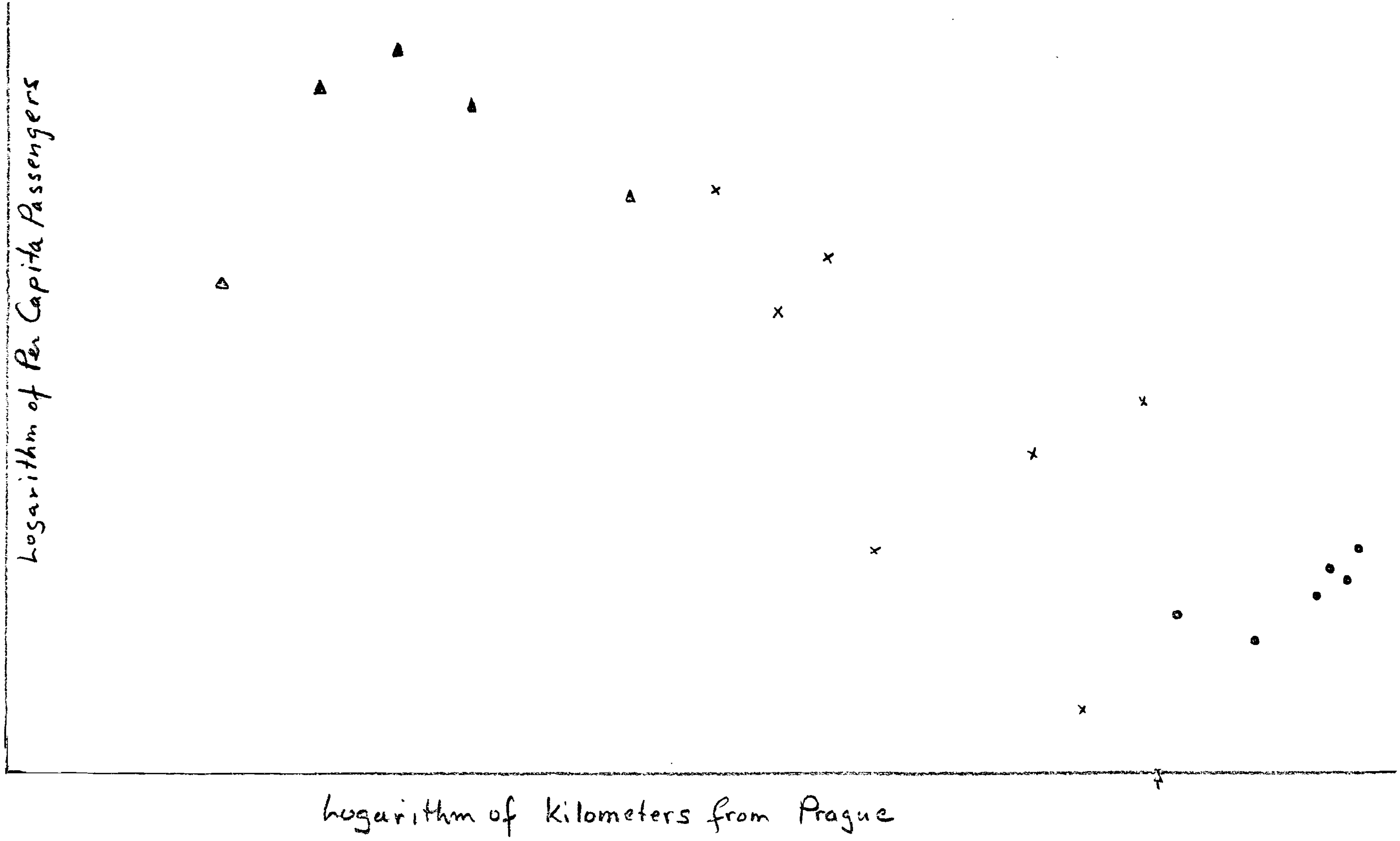


Figure III
Interaction with Prague



Acknowledgements:

My thanks to E. Löser for procuring the Lill paper and to J-M Gambrelle for converting and plotting the data.

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