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III. Optimal Control of Dam Storage

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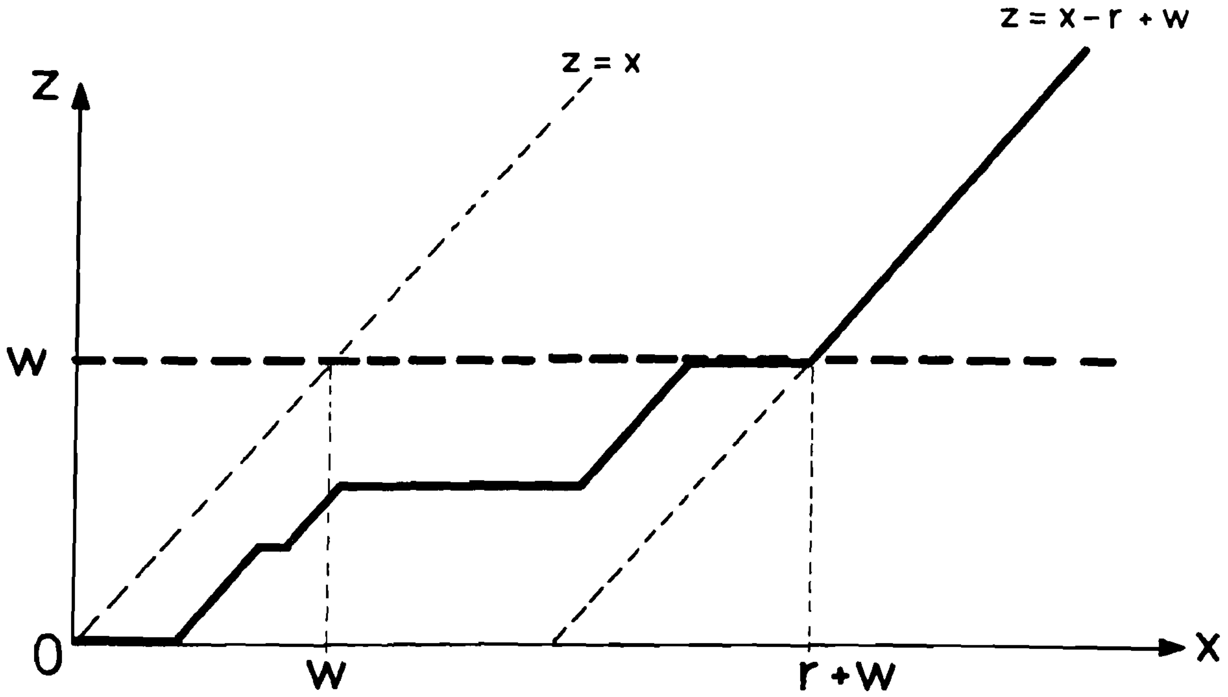
Abstract

Some stochastic aspects of dam storage theory are considered in this paper. In particular optimal control based on some reliable lower estimates of unknown (uncertain) system parameters with the corresponding operational program is developed. Also, statistical equilibrium in dam storage (random) processes are analyzed and general conditions for such a phenomenon are established.

I. A water reservoir operation depends on a proper time period $(t_o, t_o + T)$ and is usually based on a so-called operational graph. This can be represented by a monotone function $z = z(x)$ which shows the amount of water to be released during the considered period $(t_o, t_o + T)$, if the total volume of available water will be x (see Figure 1).

Of course, one does not release the corresponding amount of water z all at once; its distribution over time depends, in particular, on water demands per time unit and channel capacity. If these river basin characteristics are constant during the considered time period $(t_o, t_o \pm T)$, then a local operation policy may be of the following type: The amount of water Δz_t per time unit Δt released with constant discharge at the current time interval $(t, t + \Delta t)$ is

$$\Delta z_t = \min \{c, z(x_t) - z_t\} \quad ,$$



x = AVAILABLE WATER (INITIAL VOLUME + INFLOW);

w = WATER DEMAND;

z = WATER RELEASE;

r = (OPERATIONAL) RESERVOIR CAPACITY.

FIGURE 1.

where

x_t = water available during the period (t_o,t);

z_t = water already released from the reservoir during the period (t_o,t);

c = operational constant limited by channel capacity.

Generally, a reservoir is designed to meet water demands as well as to prevent floods. Thus the operational graph must be chosen according to proper multiobjective decision making. Water demands and flood possibility obviously depend on time, so one has to determine the operational graph $z_k = z_k(x_k)$, for each time period $(t_k,t_k + T)$, $t_k = t_o + kT$; $k = 0,1,\ldots.$

Let us set $T = 1$ and let y_k be the reservoir volume at the beginning of k-period and ξ_k be the total inflow

$$\xi_k = \int_{t_k}^{t_k+T} \dot{\xi}_t \, dt \quad ,$$

where $\dot{\xi}_t$ is the inflow per time usually identified with the so-called hydrograph.

We have

$$x_k = y_k + \xi_k \quad , \quad y_{k+1} = x_k - z_k \quad , \quad k = 0,1,\ldots. \qquad (1)$$

Suppose for each period $(t_k,t_k + T)$ we are given a total water demand w_k, and the loss function $f_k(z_k)$ reflects loss in the case of water deficit $w_k - z_k$ (see Figure 2). The problem is to determine optimal reservoir operation taking into account not only the current water demands, but also possible future water deficits and floods.

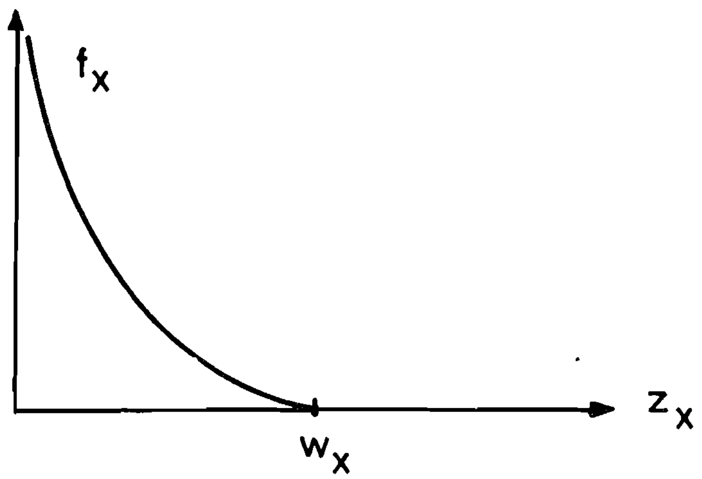


FIGURE 2.

Because of a water channel's limited capacity, a flood is usually connected with a high peak hydrograph on a comparatively small base-time Δ (see Figure 3). Flood damage seems to be a function of the corresponding high peak hydrograph; but by using approximations shown in Figure 3, one may estimate the damage by a proper function $f(\eta,\delta)$ of two parameters η and δ.

Flood damage, as we understand it, is usually incomparably high with respect to the water deficit loss. Thus it seems reasonable to assume that a proper reservoir capacity for a flood catchment can be established disregarding water demands. Let r_k be the corresponding reservoir operational volume during the operational time period $(t_k, t_k + T)$. So if R is the absolute reservoir volume, then the remaining $R - r_k$ represents the "flood catchment" capacity which helps to reduce the damage cost from $f_k(\eta,\Delta)$ to $f_k(\eta,\delta)$, where

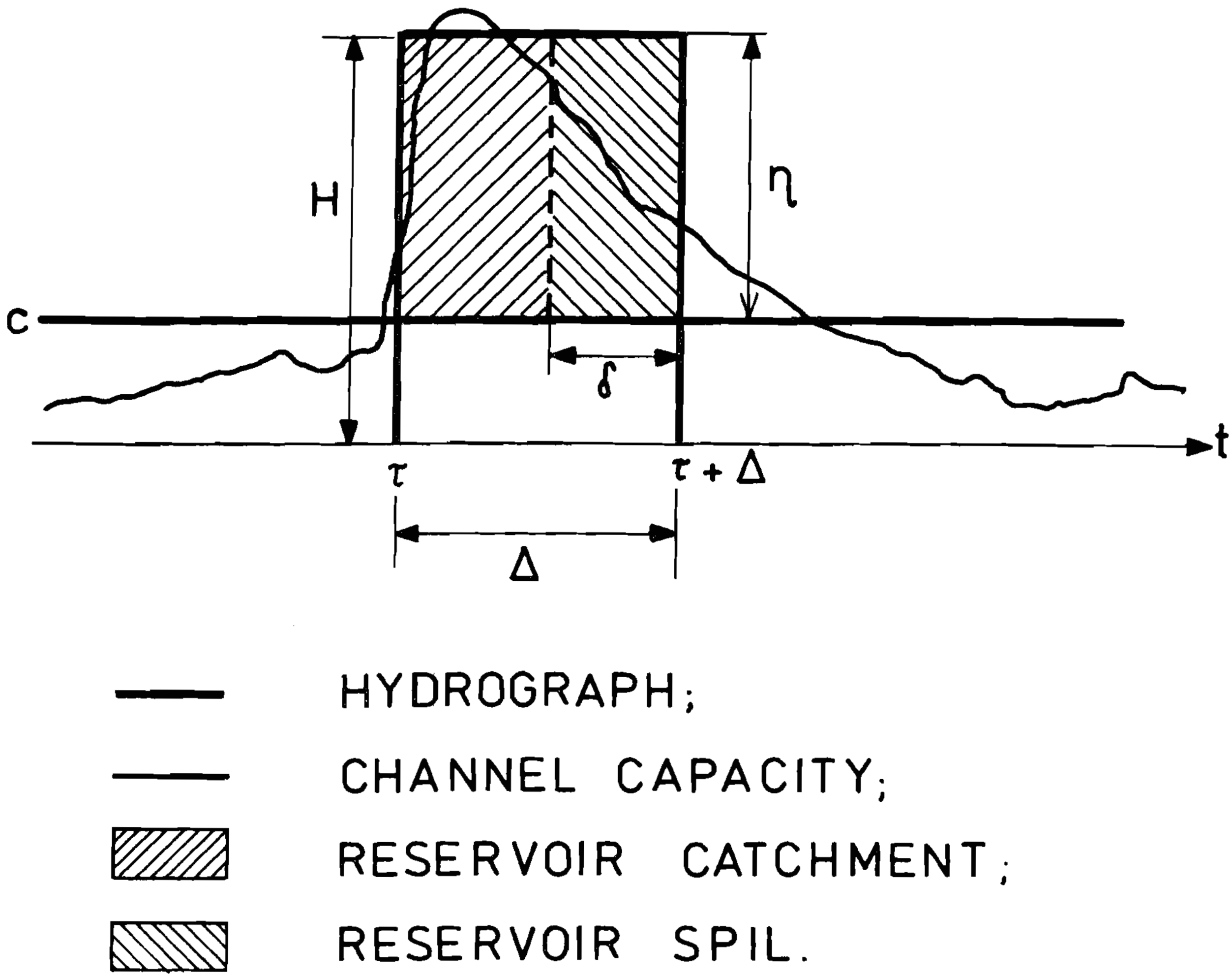

H
η
c
δ
t
τ
τ + Δ
Δ
HYDROGRAPH;
CHANNEL CAPACITY;
RESERVOIR CATCHMENT;
RESERVOIR SPIL.

FIGURE 3.

$$\eta = H - c, \quad \delta = \Delta - \frac{R - r}{H - c}$$

(see Figure 3). The proper operational volume $r = r_k$ may be determined, for example, from the condition that the damage cost may exceed some upper crucial level C only with a small probability ε:

$$P\{f_k(\eta,\delta) > C\} \leq \varepsilon \quad .$$

With regard to water demands, one can be careful about a water deficit only up to the first "wet" period τ when excess water (not necessarily a flood) enters the reservoir. So the problem is to determine operation functions $z_k = z_k(x_k)$ in such a way so as to minimize the total water deficit loss

$$\sum_{k=0}^{\tau-1} f_k(z_k) \rightarrow \min \quad . \qquad (2)$$

One may immediately notice the main difficulty that arises here: the water deficit loss defined above depends on the reservoir inflow $\xi_0, \xi_1, \ldots$, which is uncertain and must be treated as a random process.

One of the possible approaches to such minimization problems traditionally offered by optimal control theory is minimization of expected loss:

$$E \sum_{k=0}^{\tau-1} f_k(x_k) \rightarrow \min \quad ,$$

over all possible control functions

$$z_k = z_k(x_k, \omega)$$

which in our case depend not only on available water $x = x_k$, but also on river basin data "ω" up to the current time period.

The corresponding optimization technique involves (conditional) probabilities distribution of the random variables $\xi_0, \xi_1, \ldots$; this can hardly be used in practice, as there are usually no reliable data available to ensure that this or that sophisticated probabilistic model fits the reality.

We will next consider a rather simple probabilistic model of the inflow process $\xi_0, \xi_1, \ldots$, and a proper reservoir control which is optimal in the sense of some kind of minimax principle; this seems to be reliable from a practical point of view.

II. The reservoir inflow process

$$\xi_0, \xi_1, \ldots,$$

arises as a result of basic river flow, rainfall in different river basin areas, etc.; a mechanism of the random variables $\xi_0, \xi_1, \ldots$, formation is rather complicated to discuss here in detail.

Let us call a series of the considered periods

$$(t_k, t_k + T); \quad k = 0, 1, \ldots, \tau-1$$

up to the first "wet" period τ, $\tau > 0$, regular season, and a series of the inflow variables

$$\xi_k \quad ; \quad k = 0, 1, \ldots, \tau-1 \quad ,$$

regular process. It seems reasonable to assume that the inflow variables $\xi_0,\ldots,\xi_n$ associated with the regular season are independent (under condition $\tau > n$) on the beginning of the future "wet" period τ; furthermore, probabilities distributions of $\xi_0,\ldots,\xi_n$ with respect to the condition $\tau > n + k$ are the same for all future periods $n + k$; $k = 0,1,\ldots$.

During the regular season there are comparitively minor random fluctuations in the regular inflow process $\xi_0,\xi_1,\ldots$, mainly due to such random events as rainfall in different river basin areas. Water requires some transient time to flow from the reservoir to an area. We believe that if such a transient time for any area is comparatively small with respect to the chosen reservoir period T, then, for purposes of possible future water deficit estimation and sensible water supply during the regular season, one may treat the regular inflows $\xi_0,\xi_1,\ldots$, as independent (random) variables.

Actually, our minimization problem (see (2)) concerns an optimal water supply during the regular time interval $(t_0,t_0 + T)$; the main difficulty is estimating possible future water deficits up to the wet τ-period.

Let us set

$$P_n = P\{\tau>n/_{0<\tau\leq N}\} \quad ; \quad n = 0,1,\ldots, \quad .$$

For example, one may assume (not unreasonably) that the "waiting time" for the wet period has a probabilities distribution of the exponential type:

$$P_0 = P\{\tau > 0\} = 1 \quad ,$$

$$P_n = P\{\tau > n\} = q^n/1 - q^N, \quad n = 1,\dots,N \quad ,$$

where a parameter q may be interpreted as the probability of being regular for each of the considered (independent) time-periods $(t_k, t_k + T)$; a parameter N arises in a case when the wet period certainly occurs during the annual cycle (i.e., the melting of snow certainly occurs before the summer season, etc.).

Let us fix all inflow variables ξ_k in the regular part of the inflow process and consider the expected water deficit loss

$$\Phi(z,\xi) = E\left\{\sum_{k=0}^{\tau-1} f_k(z_k) \Big/ \xi_0,\dots,\xi_{\tau-1}\right\}_{0<\tau\leq N} \qquad (3)$$

as a function of the random process

$$\xi = \{\xi_0,\dots,\xi_{N-1}\} \quad ,$$

and the control parameters

$$z = \{z_1,\dots,z_{N-1}\} \quad .$$

The optimal control parameters

$$z_k^0 = z_k^0(x;\xi) \quad ; \quad k = 0,1,\dots,N-1$$

which minimize the loss function $\Phi(x,\xi)$:

$$\Phi(z^0,\xi) = \min_z \Phi(z,\xi)$$

can be easily determined by standard dynamic programming, i.e.

$$z^o_{N-1}: \quad f_{N-1}(z) \to \min_{0 \le z \le x} f_{N-1}(z) = \Phi_{N-1}(x) \quad ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z^o_k: \quad f_k(z) + \frac{P_{k+1}}{P_k} \Phi_{k+1}(x - z + \xi_{k+1}) \to \tag{4}$$

$$\min \{f_k(z) + \frac{P_{k+1}}{P_k} \Phi(x - z + \xi_{k+1})\} = 0 \le z \le x = \Phi_k(x)$$

$$k = N - 2, N - 3, \ldots, 0 \quad .$$

At each step of the program (4) one can use the following preposition concerning the minimization procedure

$$\sum_{i=1}^{m} F_i(z_i) \to \text{ subject to } \min \sum z_i \le x (z_i \ge 0)$$

for <u>convex</u> functions F_i.* Here parameter x may be interpreted as a resource distributed amongst consumers; let us suppose it is distributed in some units Δx.

<u>Lemma</u>. If $z^o_i(x)$; $i = 1,\ldots,m$ is the optimal distribution with respect to the parameter x, then

$$z^o_i(x + \Delta x) = z^o_i(x) + \delta_{ij}\Delta x \quad ; \quad i = 1,\ldots, \tag{5}$$

where δ_{ij} is a Kronecker symbol and the corresponding preferable "j" is determined by the condition

$$F_j(z^o_j + \Delta x) - F_j(z^o_j) = \min_{1 \le i \le m} F_i(z^o_i + \Delta x) - F_i(z^o_i) \quad .$$

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*At each k^{th} step of the program (4) we have $m = 2$, $z_1 = z$, $z_2 = x - z$, $F_1(z_1) = f_k(z)$, $F_2(z_2) = \frac{P_{k+1}}{P_k} \Phi_{k+1}(x - z + \xi_{k+1})$.

The function

$$F(x) = \min \sum_{i=1}^{m} F_i(x)$$

is convex and

$$F(x + \Delta x) - F(x) = F_j(z_j^o + \Delta x) - F_j(z_j^o) \quad ;$$

the recurrent equation (5), lets us determine $z_i^o(x)$; $x = 0, \Delta x, 2\Delta x, \ldots,$. Note that this very simple minimization procedure is not valid in a case of non-convex functions. Suppose that we consider the water distribution problem with loss functions $F_1(z_1)$ and $F_2(z_2)$, which has arisen because of current and future water deficits. Further suppose that the future loss can be reduced only by a significant water supply $z_2 > \Delta x$, but that the current loss F_1 becomes less even with a minor water supply Δx (see Figure 4). Then, according to Equation (5), one must meet current water demands using <u>all</u> available water:

$$z_1^o(x) = x \quad , \quad z_2^o(x) = 0$$

for any x. Obviously, this procedure is wrong in a case represented in Figure 4, where

$$z_1^o(x) = 0 \quad , \quad z_2^o(x) = x$$

for $x = 2\Delta x$.

It is worth noting that program (4) gives us the optimal control functions $z_k^o = z_k^o(x;\xi)$ which minimize <u>all</u> expected values

$$E\left\{\sum_{j=n}^{\tau-1} f_k(z_k) \Big/ x_n = x \quad ; \quad \xi_{n+1},\ldots,\xi_\tau \quad , \quad n < \tau \leq N\right\}$$

$$= \frac{1}{P_n} \sum_{k=n}^{N-1} P_k f_k(z_k) \quad .$$

Thus if we set

$$z_k = z_k^o(x_k,\xi)$$

in the water balance Equation (1), then the corresponding reservoir process

$$(x_k,y_k,z_k) \quad ; \quad k = 0,1,\ldots,$$

will be optimal (i.e. water deficit loss will be minimal).

But one can not implement this optimal process because at each current n-period, the future inflow variables $\xi_{n+1},\ldots,$ remain unknown.

Our suggestion is to substitute uncertain variables $\xi_o,\xi_1,\ldots,$ with some reliable <u>lower</u> estimates, $\underline{\xi}_o,\underline{\xi}_1,\ldots,$ such that

$$P\{\xi_k \geq \underline{\xi}_k\} \sim 1 - \alpha \quad , \tag{11}$$

where $1 - \alpha$ is the proper confidence level, $(0 < \alpha < 1)$

The corresponding control functions $z_k - z_k^o(x,\underline{\xi})$; $k = 0,1,\ldots,$ appear to be optimal with respect to some kind of minimax criterion when we consider that only $100 \cdot (1 - \alpha)$ per cent of possible inflows ξ_k satisfy the condition (11).

Let us consider an arbitrary reservoir control based on operation graphs:

$$z_k = z_k(x) \quad ; \quad k = 0,1,\ldots.$$

We believe there is no sense in a control for which the water deficit loss $\Phi(z,\xi)$ can increase when the reservoir inflows are increasing. So let us consider the regular control in which the water deficit loss $\Phi(z,\xi)$ is a monotone decreasing function of each inflow variable ξ_k, $k = 0,1,\ldots.$

The control suggested above, i.e.

$$z_k^o = z_k^o(x,\xi) \quad ; \quad k = 0,1,\ldots, \tag{12}$$

is regular for any $\xi = \{\xi_x\}$.

Indeed, according to the general Equation (5), all parameters

$$z_k^o(x,\xi) \quad , \quad y_{k+1}^o = x - z_k^o(x,\xi)$$

are monotone increasing functions of $x = x_k$ and, in a case where the inflow ξ_k is increasing, we deal with the increased variables

$$x_k = y_k^o + \xi_k \quad , \quad z_k^o \text{ and } y_{k+1}^o \quad ,$$

$$x_{k+1} = y_{k+1}^o + \xi_{k+1} \quad , \quad z_{k+1}^o \text{ and } y_{k+2}^o \quad ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{N-1} = y_{N-1}^o + \xi_{N-1} \quad , \quad z_{N-1}^o \quad .$$

Hence the water deficit loss

$$\Phi(z^{o},\xi) = \sum_{n=0}^{N-1} P_n f_n(z_n^{o})$$

will be reduced because each local loss $f_n(z_n^{o})$ is a monotone decreasing function of the water supply z_n^{o}.

For any regular control $z = \{z_k(x)\}$ we have

$$\Phi(\overset{*}{z}) = \max_{\xi \geq \underline{\xi}} \Phi(z,\xi) = \Phi(z,\underline{\xi}) \quad ,$$

and the reservoir control $z^{o} = \{z_k^{o}(x,\underline{\xi})\}$ is optimal, in the sense that it gives the minimum of the maximum loss $\Phi^{*}(z)$ over all reservoir control policies $z = \{z_k(x)\}$,

$$\min_{z} \Phi^{*}(z) = \Phi^{*}(z^{o}) \quad .$$

Moreover, the pair $(z^{o},\underline{\xi})$ is a saddle point in our reservoir game against nature with its strategy $\xi = \{\xi_k\}$:

$$\min_{z} \max_{\xi \geq \underline{\xi}} \Phi(z,\xi) = \max_{\xi \geq \underline{\xi}} \min_{z} \Phi(z,\xi) = \Phi(z^{o},\underline{\xi}) \quad . \qquad (13)$$

Indeed,

$$\min_{z} \Phi(z,\xi) \geq \Phi[z^{o}(\cdot,\xi),\xi]$$

and

$$\max_{\xi \geq \underline{\xi}} \min_{z} \Phi(z,\xi) \geq \max_{\xi \geq \underline{\xi}} \Phi[z^{o}(\cdot,\xi),\xi] = \Phi[z^{o}(\cdot,\underline{\xi})\underline{\xi}] = \Phi(z^{o},\underline{\xi})$$

III. Most practical applications of stochastic reservoir models are based on the assumption that the random process

$$(x_k, y_x, z_k) \quad ; \quad k = 0, 1, \ldots,$$

in the reservoir system (see (1)) eventually reaches a so-called statistical equilibrium; this actually means that the probabilities distribution P_n of

$$(x_{k+n}, y_{k+n}, z_{k+n}) \quad ; \quad k = 0, 1, \ldots,$$

tends to some limit:

$$\lim_{N-\infty} P_n = P \quad , \tag{14}$$

and this limit probability distribution P is invariant with respect to the annual time shift transformation

$$(x_k, y_k, z_k) \rightarrow (x_{k+\Delta}, y_{k+\Delta}, z_{k+\Delta}) \quad ,$$

where Δ means the entire year period; moreover, the frequency of any annual event A during a series of years N also tends toward the corresponding probability P(A):

$$\lim_{N\to\infty} \frac{\nu_N(A)}{N} = P(A) \quad , \tag{15}$$

where $\nu_N(A)$ is the number of years in which the event A occurs.

Let us consider the arbitrary water release policy; the only assumption is that the current release $z_k = z_k(x)$ does not exceed the water demands w_k, if there is no water excess:

$$z_k \leq w_k \quad \text{if} \quad x_k \leq r_k + w_k$$

(remember that r_k is the upper operational reservoir volume--see Figure 1).

We believe that the current water demands w_k, as well as the river basin inflows ξ_k, do not physically depend on the reservoir existence; further, the random process (ξ_k, w_k); $k = 0,1,\ldots,$ can be considered as a part of the process (ξ_k, w_k), $-\infty < k < \infty$, which is stationary with respect to the annual time shift transformation. One may treat (ξ_k, w_k) as a component of general climatological process $\omega = \omega_t$, $-\infty < t < \infty$, (in the considered river basin) assuming that the annual time shift transformation does not change the probabilities distribution.

One can treat the operational upper level r_k in the same way, because it depends only on the actual reservoir capacity R and ω_t, $t \leq t_k$.

Naturally, we can expect statistical equilibrium (see (15)) only under some ergodicity conditions for the process $\omega = \omega_t$, $-\infty < t < \infty$, and under such conditions, the following result holds true: suppose that during a long range operation, the total reservoir inflow $\sum_{k=0}^{n} \xi_k$ sometimes becomes comparatively high with respect to the total water demands $\sum_{k=0}^{n} w_k$; suppose more precisely that the sequence

$$\eta_n = \sum_{k=0}^{n} (\xi_k - w_k) \quad ; \quad n = 0,1,\ldots,$$

with non-zero probability may exceed the reservoir level R (or at least the operational capacity $r = r_n$; $n = 0,1,\ldots,$) at some random time period $n = \tau$. Then the statistical equilibrium phenomenon holds true; furthermore, we find the ergodic process (x_k^*, y_k^*, z_k^*); $-\infty < k < \infty$, stationary concerning the annual cycle such that

$$(x_k, y_k, z_k) = (x_k^*, y_k^*, z_k^*) \quad , \quad k \geq \tau \quad . \tag{16}$$

The limit in (15) coincides with the probabilities distribution P of the process

$$(x_k^*, y_k^*, z_k^*) \quad ; \quad k = 0, 1, \ldots,$$

and

$$\text{Var } (P_n - P) \leq 2P\{\tau > n\} \quad . \tag{17}$$

Note that in most interesting cases the τ distribution is of an exponential type so the convergence rate (accounting to (17)) is very high.

All of the results presented above can be obtained by obvious modification of the "imbedded stationary processes" methods developed in (4), where the specific z-shape reservoir policy was analyzed. The main idea is based on the phenomenon whereby all possible trajectories of the reservoir process will be at the same point (the reservoir will be full) at the considered τ-period, no matter what the initial reservoir conditions.

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