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Abstract

The IAEA nuclear material safeguards system consists basically of two different parts. One is the data verification scheme: the operators of nuclear plants report all relevant data on nuclear material processed in the plant to the safeguards authority. These data are then verified by the safeguards authority with independent measurements. The other part is the material accountability scheme: in case there are no significant differences between the operator's and the safeguards authority's data, all of the operator's data are taken for the nuclear material balance establishment.

The purpose of this paper is to evaluate the overall probability of detection of this system in case someone tries to divert material. This evaluation takes into account the different diversion strategies available. It is complicated because the two decision functions on which the evaluation is based--the difference between operator's and inspector's data and the difference between book and physical inventory--are stochastically dependent. Exact formulas are derived and applied to a realistic case; it is shown that with a good approximation, one may neglect the correlation and thus, use simplified formulas.

Material Accountability and Its Verification: A Special Example of Multivariate Statistical Inference

Rudolf Avenhaus and Nebojsa Nakicenovic

1. Introduction

On March 5, 1970, the Treaty on the Non-Proliferation of Nuclear Weapons [1] was enforced after having been verified by forty-three nations. This treaty is aimed at preventing the proliferation of nuclear weapons; it was conceived by Great Britain, the U.S.A. and the U.S.S.R., and was signed on July 1, 1968. The exceptions are those nations which possessed nuclear weapons prior to the signing of the treaty. In order to achieve non-proliferation , the treaty has established international safeguards which guarantee that a diversion of significant amounts of nuclear material from the peaceful nuclear fuel cycle will be detected early. These safeguards are carried out by the International Atomic Energy Agency (IAEA) in Vienna, Austria.

At the time of the Treaty's conception there existed, at least in the U.S.A., 25 years of experience of handling and controlling nuclear material; it quickly became clear, however, that an international control of national industries would cause completely new problems. For this reason, various nations began intense research and development activities with the purpose of establishing a practicable and acceptable international safeguards system (see, e.g. [2,3,4]).

A significant step was made when the Safeguards Committee was able to establish a model agreement for an international Safeguards System [5]; this was conceived as a model for the Safeguards Treaties between the IAEA and those nations which signed the treaty. The Safeguards Committee was established by the Board of Governors at the IAEA, and represented more than forty nations. The agreement was negotiated from July 1970 to February 1971.

According to this model agreement, material accountability was established as the fundamental safeguards measure, with containment and surveillance as complementary measures. In this context, material accountability means the comparison between the book inventory, i.e. the added material inputs and outputs of a material balance area during the inventory period, and the physical inventory at the end of an inventory period. The reason for this structure of the Safeguards System was the fact that such a system can be formalized better and is more objective than any other possible system; this was a necessary condition for international acceptability.

Furthermore, in the IAEA Model Agreement, the rules were established according to the way in which nuclear material safeguards must be carried out: the operator of a nuclear plant collects all source data which are necessary for the material balance establishment. The safeguards authority verifies these data with the help of independent measurements on a random sampling basis. If there exist no significant differences between the operator's and the inspector's data, then the safeguards authority assumes all of the operator's data to be correct and establishes the material balance with the help of these data. If significant differences exist either in the data comparison or in the material balance, then a "second action level" is induced to clarify whether or not they indicate a diversion of nuclear material.

Due to the fact that only declared material is subject to international safeguards ("misuse" of nuclear plants is not the subject of IAEA safeguards), the nuclear plant operator who wants to divert nuclear material has two different possibilities or strategies:

1) Either he diverts nuclear material without falsifying any data which he reports to the safeguards authority and expects that the measurement uncertainties of the material balance to cover the diversion; or
2) he falsifies the data to be reported and diverts the corresponding amount of material in such a way that the material balance is correct and expects that either the measurement uncertainties or the random sampling procedure to cover the diversion.

Clearly, a combination of both strategies is also possible.

The evaluation scheme of the safeguards authority is based on two "decision functions": (1) the difference between the book and physical inventory MUF ("Material Unaccounted For"), and (2) the difference D between the operator's and inspection team's data. These decision functions are subject to significance tests of the following form: If the realized values of MUF resp. D are smaller than given significance thresholds s_1 resp. s_2, then it is stated that the operator behaved legally. If, on the contrary, at least one of these quantities is larger than the significance threshold, then the second action level is induced.

A measure for the efficiency of this procedure is the overall probability of detection for a given amount M of material to be diverted. The safeguards authority has to assume that the operator who intends to divert the amount M of material will do it in the most efficient way (from his point of view) and will choose that strategy which minimizes the probability of detection. On the contrary, the safeguards authority chooses that inspection strategy which maximizes the probability of detection, minimized by the operator. We call this the guaranteed probability of detection as it represents a lower limit of the probability of detection. These considerations have been discussed in an illuminating way by W. Häfele [6].

The determination of the overall probability of detection is complicated because the two decision functions MUF and D are stochastically dependent: The operator's data are used in both cases. The purpose of this paper is to show that in practical cases, the overall guaranteed probability of detection can be easily determined with simplified formulas as a good approximation.

In order to achieve this we will first develop the theory of the material balance establishment as well as the theory of data verification. Thereafter, we will determine the overall probability of detection and study its properties: we can show that the probability of detection is practically independent of the correlation between the two decision functions MUF and D, if the correlation is smaller than zero. Furthermore, it will be shown that under general assumptions the correlation is, in fact, smaller than zero.

The theoretical results obtained are illustrated by a realistic example (an irradiated nuclear fuel reprocessing plant) which was a subject of contract research between the IAEA and among others the authors of this paper [7].

2. Theoretical Considerations

2.1 The Material Balance Concept

Let us consider a "material balance area" which contains at a given time t_o, some material into which material enters, and from which material goes out during a given interval of time (t_o,t_1).

The material contained in the material balance area at time t_o is called the <u>physical inventory I_o.</u> The algebraic sum of the amounts of material which enter and leave the material balance area in the interval of time (t_o, t_1) is called the throughput D. The physical inventory at t_o plus the throughput in (t_o, t_1) give the book inventory B at t_1, i.e. the amount of material which should be contained in the material balance area at t_1:

$$B = I_o + D \quad . \tag{2-1}$$

The amount of material actually contained in the material balance area at t_1 is the physical inventory I_1.

If all material contained in and passing through the material balance area is carefully accounted for, and if no material has been diverted, then the difference between the book inventory B at t_1 and the physical inventory I_1 should be zero. This difference is called "Material Unaccounted For":

$$MUF = B - I_1 \quad . \tag{2-2}$$

Thus, we have the problem of finding out whether the nonzero difference is caused by measurement errors, or by the diversion of material.

In order to solve this problem, a significance test must be performed where the null hypothesis is given by the statement: the expectation value of MUF is zero,

$$E(MUF/H_o) = 0 \quad , \tag{2-3a}$$

and where the alternative hypothesis is given by the statement: the expectation value of MUF is $M_1 > 0$,

$$E(MUF/H_1) = M_1 > 0 \quad . \tag{2-3b}$$

The significance test is determined by the significance threshold s_1: if the realized value of MUF is smaller than or equal to s_1, then the inspector will state "H_o is correct"; but if MUF is larger than s_1, he will state "H_1 is correct" (which does not immediately mean that a diversion of material is stated):

$$\begin{aligned} MUF \leq s_1: &\quad H_o \text{ is true} \quad , \\ MUF > s_1: &\quad H_1 \text{ is true} \quad . \end{aligned} \tag{2-4}$$

This procedure may cause two kinds of false statements:

i) the inspector states "H_1 is true", when in fact H_o is true;

ii) the inspector states "H_o is true", when in fact H_1 is true.

The probabilities of committing these errors are called α_1 and β_1:

$$\alpha_1 := \text{prob}\{MUF > s_1/H_o\} \quad , \tag{2-5a}$$

$$\beta_1 := \text{prob}\{MUF \leq s_1/H_1\} \quad . \tag{2-5b}$$

It is assumed that it will be clarified at a "second action level" whether or not the "alarm" was justified at $MUF > s_1$. Here, α_1 is called false alarm probability, whereas $1 - \beta_1$ is called probability of detection.

Because of the random measurement errors, the quantities I_o, D, I_1 and, therefore, MUF are random variables. Let σ_{Io}^2, σ_D^2, and σ_{I1}^2 be the variances of these random variables. Then the variance of MUF is given by

$$\text{var (MUF)} = \sigma_{Io}^2 + \sigma_D^2 + \sigma_{I1}^2 =: \sigma^2 \tag{2-6}$$

independent of whether or not a diversion MUF would take place. If the random variables I_o, σ_D^2 and I_1^2 are normally distributed, then MUF is also normally distributed and one obtains from (2-5)

$$1 - \alpha_1 = \phi(\frac{s_1}{\sigma_{MUF}}) \quad , \tag{2-7a}$$

$$\beta_1 = \phi(\frac{s_1 - M_1}{\sigma_{MUF}}) \quad , \tag{2-7b}$$

where ϕ is the Gaussian distribution function:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})\,dt \quad .$$

If one eliminates the significance threshold s_1 in (2-7b), with the help of (2-7a), one obtains

$$1 - \beta_1 = \phi\left(\frac{M_1}{\sigma_{MUF}} - U_{1-\alpha_1}\right) \quad , \tag{2-8}$$

where U is the inverse of the Gaussian distribution function.

Up to now we have considered one inventory period. The treatment of a sequence of inventory periods poses special problems because of the question of how to choose the starting inventory: If at the end of an inventory period there are no significant differences between book and ending physical inventories, one can take one of these inventories or a linear combination of both as the starting inventory for the next period (see, e.g. [8],[9]). However, since the variance of the physical inventory is much smaller than the variance of the throughput, as in the example analyzed in the next chapters, we will take the ending physical inventory as the starting inventory for the next period. Thus, the correlation between different inventory periods may be neglected. If amounts M_1 and M_2 are diverted in two periods, the total probability of detection is simply given by

$$1 - \beta = 1 - \phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_{MUF}})\,\phi(U_{1-\alpha_1} - \frac{M_2}{\sigma_{MUF}}) \quad .$$

In the following, we will consider only one inventory period.

2.2 Data Verification

As described in the introduction, the safeguards system is constructed in such a way that the plant operator performs all measurements necessary for the establishment of the material balance; he then reports the measurement data to the inspector, who in turn verifies these data with the help of independent measurements. Among the many possibilities for the comparison of the operator's and the inspector's data, the use of the so-called D-statistics (see [10,11]) has proven most successful. Therefore, we will also use it here. In the following, we will describe the D-statistics with the help of a simplified model; the application to a realistic case will be given in the next chapter.

Let us assume that there are R classes of material, and that in the inventory period under consideration the i^{th} class ($i = 1,\ldots,R$) consists of N_i batches. Let X_{ij}, $j = 1,\ldots,N_i$, $i = 1,\ldots,R$, be the measurement result for the material content of the j^{th} batch of the i^{th} class reported by the operator. Let us furthermore assume that the inspector verifies n_i measurements in the i^{th} class with the help of independent measurements, and that his results are Y_{ij}, $j = 1,\ldots,n_i$, $i = 1,\ldots,R$. The variances of the random (r) and systematic (s) errors of the operator's (o) and inspector's (I) measurements are σ_{or}^2, σ_{os}^2, σ_{Ir}^2 and σ_{Is}^2 and are assumed to be known, where the errors themselves are assumed to be normally distributed.

In order to check whether or not the data of the operator are correct, the inspector forms the D-statistic which is defined by

$$D: = \sum_{i=1}^{R} \frac{N_i}{n_i} \sum_{j=1}^{n_i} (Y_{ij} - X_{ij}) \ . \tag{2-9}$$

It should be noted that this definition specifies that the inspector verifies only data from those batches reported by the operator which he has measured himself. The reason for this is that by means of this procedure, the influence of the variation of the true material contents of the batches within a class is eliminated.

Under the null hypothesis, i.e. under the assumption that no data reported by the operator are falsified, the expectation value and the variance of D are given by the following expressions:

$$E(D/H_o) = 0 \quad , \quad \text{var } (D/H_o) = \sigma_{D/H_o}^2 = \sum_{i=1}^{R} N_i^2 \left(\frac{\sigma_{ri}^2}{n_i} + \sigma_{si}^2\right) \ ,$$

$$\sigma_r^2 : = \sigma_{or}^2 + \sigma_{Ir}^2 \quad , \quad \sigma_s^2 : = \sigma_{os}^2 + \sigma_{Is}^2 \ . \tag{2-10a}$$

Under the alternative hypothesis H_1, i.e. under the assumption that r_i of the N_i batches of the i^{th} class are falsified by the amount $\mu_i, i = 1,\ldots,R$, one obtains

$$E(D/H_1) = \sum_{i=1}^{R} \mu_i r_i \tag{2-10b}$$

$$\text{var } (D/H_1) = : \sigma_{D/H_1}^2 = \sum_{i=1}^{R} N_i^2 \left(\frac{\sigma_{ri}^2}{n_i} + \sigma_{si}^2 + \mu_i^2 \cdot \frac{r_i}{N_i} \left(\frac{N_i - r_i}{N_i} \left(\frac{1}{n_i} - \frac{1}{n_i} \, \frac{n_i - 1}{N_i - 1} \right) \right) \right) .$$

According to this scheme, the maximum amount of material which can be diverted is given by $r_i = N_i$, $i = 1,\ldots,R$:

$$M_2^{max} = \sum_i \mu_i N_i \quad . \tag{2-11}$$

For the diversion without data falsification as described in the foregoing section such an upper limit does not exist.

If the measurement of one batch does not consist of a single measurement, but of several (e.g. weight and concentration determination), $\mu \cdot r$ is not the amount directly falsified. An example for this is given in the next section.

Let s_2 be the significance threshold of the inspector's test. Then we have as in (2-5)

$$\alpha_2 := \text{prob}\,\{D > s_2/H_o\} \tag{2-11a}$$

$$\beta_2 := \text{prob}\,\{D \leq s_2/H_1\} \quad . \tag{2-11b}$$

If we assume that D/H_o and D/H_1 are approximately normally distributed (see [11]), then we obtain (corresponding to (2-8)) the following expression for the probability of detection:

$$1 - \beta_2 = \phi\ \frac{E(D/H_1) - \sigma_{D/H_o} \cdot U_{1-\alpha_2}}{\sigma_{D/H_1}} \tag{2-12}$$

We will not go into the details of the question of how the inspector chooses the μ_i, and how the operator chooses the r_i, as this has been analyzed elsewhere (see [11]).
Here, only the results of an approximation procedure will be given. Let the inspector's effort for the measurement of one batch in the i^{th} class be E_i, and let the total effort available be C. Then a game theoretical treatment gives the following optimal values:

$$n_i^0 = \frac{C}{\sum_j \varepsilon_j \mu_j N_j} \cdot \mu_i N_i \quad , \tag{2-13a}$$

$$r_i^0 = \frac{M}{\sum_j \varepsilon_j \mu_j N_j} \cdot \varepsilon_i N_i \quad . \tag{2-13b}$$

2.3 Total Probability of Detection

As a measure for the efficiency of the entire test procedure described above--data verification and material-balance establishment with the help of the operator's data--we define the total probability of detection $1 - \beta$:

$$1 - \beta : = 1 - \text{prob}\{D \leq s_2 \wedge MUF \leq s_1 / H_1\} \quad , \tag{2-14a}$$

where H_1 means

$$E(D/H_1) = M_2 \wedge E(MUF/H_1) = M_1 \quad . \tag{2-14b}$$

In the same sense we define the total false alarm probability α by

$$1 - \alpha : = \text{prob}\{D \leq s_2 \wedge MUF \leq s_1 / H_0\} \quad , \tag{2-15a}$$

where H_o means

$$E(D/H_0) = E(MUF/H_0) = 0 \quad . \tag{2-15b}$$

As the operator's data are used both for the data verification procedure and for the material balance establishment, the random variables D and MUF are stochastically dependent, and one obtains

$$1 - \alpha = \frac{1}{2\pi\sqrt{1 - \rho^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}} dt_2 \exp\left(- \frac{t_1^2 - 2t_2 t_2 \rho + t_2^2}{2(1 - \rho^2)}\right) \tag{2-16a}$$

$$\beta = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}}} dt_1 \int_{-\infty}^{\frac{\sigma_{D/H_0} U_{1-\alpha_1} - M_1}{\sigma_{D/H_1}}} dt_2$$

$$\cdot \exp\left(-\frac{t_1^2 - 2t_1t_2\rho + t_2^2}{2(1-\rho^2)}\right) \quad . \tag{2-16b}$$

where

$$\rho : = \frac{\text{cov } (D_1 MUF)}{\sigma D/H_1 \cdot \sigma MUF} \tag{2-16c}$$

is the correlation coefficient.

For $\rho = 0$, one obtains from (2-16)

$$1 - \alpha = (1 - \alpha_1) \cdot (1 - \alpha_2) \quad , \tag{2-17a}$$

$$\beta = \phi\left(U_{1-\alpha_1} - \frac{M_1}{\sigma MUF}\right) \cdot \phi\left(\frac{\sigma_{D/H_0} \cdot U_{1-\alpha_1} - M_2}{\sigma D/M_1}\right)$$

$$= \beta_1\beta_2 \quad . \tag{2-17b}$$

Eq. (2-17a) is well known in the area of multivariate statistical inference. A discussion of this equation is given in the Annex. In Figures 1 and 2, the results of numerical calculations are presented: Figure 1 shows for $\alpha_1 = \alpha_2$, the dependence of α_1 from ρ, with α as parameter; Figure 2 shows the dependence of α_1 from α_2, with ρ as parameter, and for fixed $\alpha = 0.005$. The main result is that for $\rho < 0$ (which is the case in the example given in the next chapter) Eq. (2-16a) can be well approximated by Eq. (2-17a).

In order to achieve as high an efficiency of the safeguards procedures as possible, in other words, to achieve as high a total probability of detection as possible, the inspector will use those values for α_1 and α_2 which maximize $1 - \beta$. For obvious reasons, however, he cannot use values which are too high. Therefore, we assume that there is an agreed value of the total false alarm probability α, and that the inspector can choose only those values of α_1 and α_2 which satisfy the boundary condition (2-16a).

On the other hand, as the inspector does not know the values of M_1 and M_2 chosen by the operator, and as the inspector wants to optimize his system for a given value of a total amount $M = M_1 + M_2$ assumed to be diverted, he must take into account the best strategy from the operator's point of view; i.e. that choice of M_1 and M_2 which minimizes $1 - \beta$.

Therefore, the optimum strategy $(\alpha_1{}^*, \alpha_2{}^*; \alpha)$ of the inspector is defined as the result of the following optimization problem:

$$\max_{\substack{\alpha_1,\alpha_2: \\ \text{subject to eq.} \\ \text{(2-16a) for} \\ \text{given value of } \alpha}} \quad \min_{\substack{M_1,M_2: \\ M_1+M_2=M}} (1 - \beta) =: 1 - \beta^{**} \quad . \tag{2-18}$$

$1 - \beta^{**}$ is called the total guaranteed probability of detection.

It is clear that the optimization problem defined above cannot be carried out analytically. In addition, it is too complicated for practical purposes. Therefore, one might want to replace it with $\rho = 0$. As can be seen from Figures 1 and 2, at least the false alarm equation (2-16a) can be suitably replaced by the approximate equation (2-17a). It is the question of whether or not this approximation also holds for the probability of detection. In order to answer this question, a realistic example will be analyzed in the next chapter. It may be stated at this point that one can, in fact, approximate the probability of detection given by (2-16b) and by the simplified formula (2-17b). Furthermore, for practical purposes one might want to put

$$\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha} \quad . \tag{2-19}$$

Therefore, another purpose of the following numerical calculations is to see how far the guaranteed probability of detection (2-18) deviates from a probability of detection which has been determined on the basis of (2-19).

One general question may be raised concerning our procedure: As the variances of the measurement errors are assumed to be known, could one transform to the two independent random variables Y_1 and Y_2, and thus, avoid the complicated formulas (2-16)? In fact, such a scheme has been discussed recently by Bennet et al. [12]. The answer is that the safeguards authority would like to perform the two tests concerning material balance and data verification separately and see whether or not one of these tests indicates a significant difference; this would provide an immediate idea as to the source of the errors, losses, or diversion Therefore, a transformation to quantities which have no physical meaning is not of much help.

3. Application to a Realistic Case

3.1 Basic Data of the NFS Irradiated Fuel Processing Plant

In the following we consider as an example The Nuclear Fuel Services (NFS) plant near Buffalo, N.Y. This plant reprocesses irradiated fuel elements of reactors on the basis of the PUREX process. The numerical data are taken from Ref. [7].

We shall consider the case of one inventory period. As it is assumed that there are two inventory periods per year, this means a time period of 6 months. The campaign data and the batch data are given in Table 1 for plutonium; this is the only important material in this context and will be considered exclusively in the following.

3.2 Measurement Accuracies; Variance of the "Material Unaccounted For"

According to Section 2.1 the establishment of the material balance includes the establishment of the

i) inital physical inventory I_o;
ii) book inventory B (I_o + input - product - waste);
iii) ending physical inventory I_1.

3.2.1 Physical Inventories

We assume

$$EI_0 = EI_1 = 1[kg] \quad , \tag{3-1a}$$

and assume further that the variation of these inventories is of the same order of magnitude:

$$1 - 1[kg] \leq I_{0,1} \leq 1 + 1[kg] \quad . \tag{3-1b}$$

If we assume, in addition, that the physical inventories are equally distributed random variables with a range given by (3-1b), we obtain

$$\text{var } I_0 = \text{var } I_1 = 0.333[kg^2] \quad . \tag{3-1c}$$

3.2.2 Input

One measurement of the plutonium content G_{1j} of the j^{th} input batch consists of a

i) volume determination v_{ij} [l];
ii) drawing of a sample [g Pu/ℓ];
iii) concentration measurement c_{1j} of the sample.

Therefore, in the case of no data falsification the operator reports the data

$$G_{1j} = v_{1j} \cdot c_{1j}, \; J = 1 \cdots N_1 \tag{3-2a}$$

where

$$v_{1,j} = Ev_1 + e_{1,j}^{v,r} + e_1^{v,s} \tag{3-2b}$$

$$c_{1,j} = EC_1 + e_{1,j}^{c,r} + e_1^{c,s} + d_{1,j}^{o,c} \tag{3-2c}$$

and where $e_{1,j}^{v,r}$ and $e_1^{v,s}$ are the random and systematic errors of the volume determination; $e_{1,j}^{c,r}$ and $e_1^{c,s}$ are the random and systematic errors of the concentration determination; and $d_{1,j}^{o,c}$ is the sampling error in the operator's sample.

The variances of these errors are

$$\begin{aligned} \operatorname{var} e_{1,j}^{v,r} &= \sigma_{v,r,1}^2 \\ \operatorname{var} e_{1,j}^{v,s} &= \sigma_{v,s,1}^2 \\ \operatorname{var} e_{1,j}^{c,r} &= \sigma_{c,r,1}^2 \\ \operatorname{var} e_{1,j}^{c,s} &= \sigma_{c,s}^2 \\ \operatorname{var} d_{1,j}^{o,c} &= \sigma_{s,1}^2 \quad . \end{aligned} \tag{3-3}$$

If one assumes that one calibration per inventory period is performed both for the volume and for the concentration measurement, and if one neglects error terms of the second order, then the total input reported by the operator is given by

$$\text{Input} = N_1 \cdot Ev_1 \cdot Ec_1 + \sum_{j=1}^{N_1} \left[Ev_1 \left(e_{1,j}^{c,r} + e_1^{c,s} + d_{1,j}^{o,c} \right) + \right. \\ \left. + Ec_1 \left(e_{1,j}^{v,r} + e_{1,j}^{v,s} \right) \right] \tag{3-4}$$

and the variance is

$$\text{var [Input]} = E^2 v_1 (N_1 \cdot \sigma^2_{c,r,1} + N_1 \cdot \sigma^2_{s,1} + N_1 \cdot \sigma^2_{c,s,1}) + E^2 c_1 (N_1 \cdot \sigma^2_{v,r,1} + N_1^2 \cdot \sigma^2_{v,s,1}) \quad . \tag{3-5}$$

3.2.3 Waste

The situation in the case of waste is exactly the same as in the case of input except that all the characteristics quantities have different values. Thus, for waste--characterized by the index 3-- we have

$$\text{var [Waste]} = E^2 v_3 (N_3 \cdot \sigma^2_{c,r,3} + N_3 \cdot \sigma^2_{s,3} + N_3 \cdot \sigma^2_{c,s,3}) + E^2 c_3 (N_3 \cdot \sigma^2_{v,r,3} + N_3 \cdot \sigma^2_{v,s,3}) \quad . \tag{3-6}$$

3.2.4 Product

The situation in the case of the product is different, insofar as not the volume but the total weight of the batch is determined by taking the gross and the tare weight of the batch; thus, the systematic errors of these measurements are cancelled. Therefore, one has for the material content G_{2j} of the j^{th} product batch.

$$G_{2j} = v_{2j} \cdot c_{2j} \tag{3-7a}$$

$$v_{2j} = Ev_2 + e^{v,g}_{1,j} + e^{v,t}_{1,j} \,[\mathrm{kg}] \tag{3-7b}$$

$$C_{2j} = Ec_2 + e^{c,r}_{2,j} + e^{c,s}_{2} + d^{o,c}_{2,j} \left[\frac{\mathrm{g\ Pu}}{\mathrm{kg\ mat}}\right] \quad , \tag{3-7c}$$

where $e^{v,s}_{2,j}$ and $e^{v,t}_{2}$ are the random errors of the gross and tare weights of the weighing procedure; $e^{c,r}_{2,j}$ and $e^{c,s}_{2}$ are the random and systematic errors of the concentration measurement; and $d^{o,c}_{2,j}$ is the sampling error of the concentration measurement.

The variances of these errors are

$$
\begin{aligned}
\operatorname{var} e_{2,j}^{v,s} &= \operatorname{var} e_{2,j}^{v,t} = :\sigma_{v,2}^2 \\
\operatorname{var} e_{2,j}^{c,r} &= \sigma_{c,r,2}^2 \\
\operatorname{var} e_{2,j}^{c,s} &= \sigma_{c,s,2}^2 \\
\operatorname{var} d_{2,j}^{o,c} &= \sigma_{s,2}^2 \quad .
\end{aligned}
\tag{3-8}
$$

Therefore, the variance of the total product during the reference time is

$$
\begin{aligned}
\operatorname{var}\,[\text{Product}] &= \operatorname{var}\left[\sum_{j=1}^{N_2}\left(Ev_2\left(e_{c,j}^{c,r} + e_2^{c,s} + d_{2,j}^{o,c}\right) + \right.\right. \\
&\quad \left.\left. + Ec_2\left(e_{2,j}^{v,g} + e_{2,j}^{v,t}\right)\right)\right] = \\
&= \underline{E^2 v_2\left(N_2 \cdot \sigma_{c,r,2}^2 + N_2 \cdot \sigma_{s,2}^2 + N_2 \cdot \sigma_{c,s,2}^2\right) +} \\
&\quad \underline{+ E^2 c_2 \cdot 2N_2 \cdot \sigma_{v,2}^2} \quad .
\end{aligned}
\tag{3-9}
$$

3.2.5 Material Unaccounted For

According to Eq. (2-2) the Material Unaccounted For is defined as

$$
\text{MUF} : = I_0 + \text{Input} - \text{Product} - \text{Waste} - I_1 \quad . \tag{3-10}
$$

If the operator does not divert any material (null hypothesis H_0), the expectation value of MUF is zero; in case of diversion of the amount M, the expectation value of MUF is M (see Eqs. (2-4)). The variance of MUF is, in both cases, given by

$$\mathrm{var(MUF)} =: \sigma^2_{MUF} = 2\mathrm{var}\ I_0 + \mathrm{var\ (Input)} + \\ + \mathrm{var\ (Product)} + \mathrm{var\ (Waste)} \tag{3-11}$$

where the single expressions are given by Eqs. (3-1c, 5, 6, 9).

Numerical values for all variances (resp. relative standard deviations) are listed in Table 2. The results of the Material Unaccounted For are given in Table 3.

3.3 Verification Procedure

It is assumed that the inspector observes all of the measurements necessary for taking the physical inventory, and that he must not verify the volume and weight determinations or the sampling procedures, as they are automatized and therefore, tamperproof. It is further assumed, that the inspector verifies the concentration determinations on the basis of a random sampling scheme, and that both the operator and the inspector use the same measurement methods.

In case the operator wants to divert material by means of data falsification, he proceeds as follows: he dilutes r_1 of his samples in order to simulate a smaller amount of input. In this way he gains material which he can divert. Therefore, instead of (3-2c) we have

$$\left.\begin{array}{ll} c_{1,j} = Ec_1 + e^{c,r}_{1,j} + e^{c,s}_1 + d^{o,c}_{1,j} - \mu^c_1 & \text{for } j = 1,\ldots,r_1 \\ c_{1,j} = Ec_1 + e^{c,r}_{1,j} + e^{c,s}_1 + d^{o,c}_{1,j} & \text{for } j = 1,\ldots,N_1 - r_1 \end{array}\right\} . \tag{3-12}$$

The operator reports, however, $c_{ij} + \mu_1$, for $j = 1,\ldots,r_1$ in order to keep the material balance.

He proceeds in the same way for the product and the waste, except that in these two cases he concentrates the samples.

Remark: Clearly, the effects will be the same if the operator does not dilute or concentrate samples, but simplify reports wrong data.

Therefore, if $c_{i,j}^{0,I}$, $i = 1, 2, 3$, $j = 1,\ldots,n_i$, are the results of the concentration measurements reported by the operator and those of the inspection team, then the D-statistics according to eq. (2-8) are given by the following expression:

$$D = \frac{N_1}{n_1} \sum_1^{n_1} \left(c_{1,j}^I - c_{1,j}^0 \right) + \frac{N_2}{n_2} \sum_1^{n_2} \left(c_{2,j}^0 - c_{2,j}^I \right) + \frac{N_3}{n_3} \sum_1^{n_3} \left(c_{3,j}^0 - c_{3,j}^I \right) . \qquad (3\text{-}13)$$

The reason for this special choice of signs was explained above.

The expectation values of D under the null and alternative hypothesis are given by

$$E(D/H_0) = 0 \qquad (3\text{-}14a)$$

$$E(D/H_1) = \sum_i \mu_i^c \, r_i \qquad (3\text{-}14b)$$

where μ_i^c is the amount by which the concentration of a falsified batch of class i is falsified. The amount of material which can be diverted this way is given by

$$M_2 = \sum_{i=1}^{3} Ev_i \mu_i^c \, r_i = \sum_{i=1}^{3} \mu_i r_i \qquad (3\text{-}15)$$

where

$$\mu_i := Ev_i \mu_i^c \quad . \qquad (3\text{-}16)$$

As one can see from Eqs. (3-15) and (3-14), M_2 and $E(D/M_1)$ are not identical. Therefore, the optimization procedure sketched in Chapter 2 must be modified; instead of Eq. (2-13) we now have

$$n_i^0 = \frac{C}{\sum_j \frac{N}{E^2 v_j} \cdot \mu_j \varepsilon_j} \cdot \mu_i \cdot \frac{N_i}{E^2 v_i} , \tag{3-17a}$$

$$r_i^0 = \frac{M}{\sum_j \frac{N_j}{E^2 v_j} \cdot \mu_j \cdot \varepsilon_j} \cdot \varepsilon_i \cdot \frac{N_i}{E^2 v_i} . \tag{3-17b}$$

Under these conditions we have

$$E(D/H_1) = \sum_i \frac{\mu_i}{E v_i} \cdot r_i^0$$

or

$$E(D/H_1) = \frac{\sum_i \frac{N_i}{E^2 v_i} \cdot \varepsilon_i \cdot \mu_i}{\sum_i \frac{N_i}{E^2 v_i} \cdot \varepsilon_i \cdot \mu_i} \cdot M_2 . \tag{3-18}$$

The basic data for the verification scheme are collected in Table 4a. Because of the large difference of the amounts μ_i^c by which the data have to be falsified, practically all of the effort must go to the product stream. It does not mean, however, that the input and waste stream data must not be verified at all. The following procedure is proposed:

For small amounts of effort, only one batch is verified in the input and one in the waste stream; the rest goes to the product stream. If there is more effort available than for the verification of all product batches, then the remaining effort must be distributed between input and waste according to formula (3-17).

The optimal sample sizes n_i^0 are given in Table 4c as a function of the total effort C. The optimal numbers of falsified batches r_i^0 are given in Table 4c as a function of the total amount M_2 assumed to be diverted. The standard deviations of the D-statistics under the null and the alternative hypotheses as a function of the effort C and the amount M_2 assumed to be diverted are given in Table 5.

3.4 Determination of the Correlation Between Data Verification and Material Balance Establishment

It was previously mentioned, the random variables MUF and D are stochastically dependent because the data of the operator are used both for data verification and for material balance establishment. In case of the null hypothesis H_0, we have:

$$
\begin{aligned}
\mathrm{cov}\,(MUF, D/H_0) &= E[(MUF - EMUF) \cdot (D - ED)/H_0)] \\
&= E[MUF \cdot D/H_0] \\
&= E\Bigg[\Bigg[N_1 Ev_1 Ec_1 - N_2 Ev_2 Ec_2 - N_3 Ev_3 Ec_3 + \\
&+ \sum_{j=1}^{N_1} \left(Ev_1\left(e_{1,j}^{c,r} + e_1^{c,s} + d_{1,j}^{0,c}\right) + \right.\\
&\left. + Ec_1\left(e_{1,j}^{v,r} + e_{1,j}^{v,s}\right)\right) - \sum_{j=1}^{N_2} \left(Ev_2\left(e_{2,j}^{c,r} + e_2^{c,s} + d_{2,j}^{0,c}\right) + \right.\\
&\left. + Ec_2\left(e_{2,j}^{v,g} + e_{2,j}^{v,t}\right)\right) - \sum_{j=1}^{N_3} \left(Ev_3\left(e_{3,j}^{c,r} + e_{3,j}^{c,s} + d_{3,j}^{0,c}\right) + \right.\\
&\left. + Ec_3\left(e_{3,j}^{v,r} + e_{3,j}^{v,r} + e_{3,j}^{v,s}\right)\right)\Bigg] \times \\
&\times \Bigg[\frac{N_1}{n_1} \sum_j \left(f_{1,j}^{c,r} + f_1^{c,s} + d_{1,j}^{I,c} - e_{1,j}^{c,r} - e_1^{c,s} - d_{1,j}^{0,c}\right) + \\
&+ \frac{N_2}{n_2} \sum_j \left(e_{2,j}^{c,r} + e_2^{c,s} + d_{2,j}^{0,c} - f_{2,j}^{c,r} - f_2^{c,s} - d_{2,j}^{I,c}\right) + \\
&+ \frac{N_3}{n_3} \sum_j \left(e_{3,j}^{c,r} + e_3^{c,s} + d_{3,j}^{0,c} - f_{3,j}^{c,r} - f_3^{c,s} - d_{3,j}^{I,c}\right)\Bigg]\Bigg],
\end{aligned}
\tag{3-19}
$$

where f and d^I are the errors of the inspector corresponding to those of the operator.

If we omit the vanishing terms we obtain

$$\underline{\text{cov (MUF, } D/H_0)} =$$

$$= - E\Bigg[Ev_1 \cdot \frac{N_1}{n_1} \cdot \Bigg[\sum_j \left(\left(e_{1,j}^{c,r}\right)^2 + \left(d_{1,j}^{0,c}\right)^2 \right) + n_1^2 \cdot e_1^{c,s} \Bigg] +$$

$$+ Ev_2 \cdot \frac{N_2}{n_2} \cdot \Bigg[\sum_j \left(\left(e_{2,j}^{c,r}\right)^2 + \left(d_{2,j}^{0,c}\right)^2 \right) + n_2^2 \cdot e_2^{c,s} \Bigg] +$$

$$+ Ev_3 \cdot \frac{N_3}{n_3} \cdot \Bigg[\sum_j \left(\left(e_{3,j}^{c,r}\right)^2 + \left(d_{3,j}^{0,c}\right)^2 \right) + n_3^2 \cdot e_3^{c,s} \Bigg]\Bigg] =$$

$$= \underline{- \sum_{i=1}^{3} \left[Ev_i \cdot N_i \left(\sigma_{c,r,i}^2 + \sigma_{d,i}^2 + n_i^2 \cdot \sigma_{c,s,i}^2 \right) \right]} \quad . \qquad (3\text{-}20)$$

This means that MUF and D are negatively correlated.

From Eq. (3-20) we obtain the correlation coefficient for the null hypothesis H_0:

$$\rho_{H_0} := \frac{\text{cov (MUF}, D/H_0)}{\sqrt{\text{var (MUF)}} \cdot \sqrt{\text{var } (D/H_0)}} \quad . \qquad (3\text{-}21)$$

In case of the alternative hypothesis H_1 (diversion of the amounts M_1 and M_2 by means of the two strategies) we have, instead of Eq. (3-17),

$$\text{cov (MUF}, D/H_1) = E[(\text{MUF} - M_1)(D - ED)/H_1] \qquad (3\text{-}22)$$

where

$$ED = \sum_v \mu_v^c \cdot r_v \quad .$$

Here, E(MUF, D) is given by Eqs. (3-4) etc., (3-12) etc., and (3-19) by the following expression:

$$E\left[(MUF - M_1)(D - ED)\right] =$$

$$E\Bigg[\Bigg[N_1 \cdot Ev_1 \cdot Ec_1 - N_2 \cdot Ev_2 \cdot Ec_2 - N_3 \cdot Ev_3 \cdot Ec_3 +$$

$$+ \sum_{j=1}^{N_1} \left(Ev_1\left(e_{1,j}^{c,r} + e_1^{c,s} + d_{1,j}^{O,c}\right) + Ec_1\left(e_{1,j}^{v,r} + e_{1,j}^{v,s}\right)\right) +$$

$$- \sum_{j=1}^{N_2} \left(Ev_2\left(e_{2,j}^{c,r} + e_2^{c,s} + d_{2,j}^{O,c}\right) + Ec_2\left(e_{2,j}^{v,g} + e_{2,j}^{v,t}\right)\right) +$$

$$- \sum_{j=1}^{N_3} \left(Ev_3\left(e_{3,j}^{c,r} + e_3^{c,s} + d_{3,j}^{O,c}\right) + Ec_3\left(e_{3,j}^{v,r} + e_{3,j}^{v,s}\right)\right) - M_1\Bigg] \times$$

$$\Bigg[\frac{N_1}{n_1} \sum_j \left(f_{1,j}^{c,r} + f_1^{c,s} + d_{1,j}^{I,c} - e_{1,j}^{c,r} - e_1^{c,s} - d_{1,j}^{O,c}\right) + \frac{N_1}{n_1} h_1 \mu_1^c +$$

$$+ \frac{N_2}{n_2} \sum_j \left(e_{2,j}^{c,r} + e_2^{c,s} + d_{2,j}^{O,c} - f_{2,j}^{c,r} - f_2^{c,s} - d_{2,j}^{I,c}\right) + \frac{N_2}{n_2} h_2 \mu_2^c +$$

$$+ \frac{N_3}{n_3} \sum_j \left(e_{3,j}^{c,r} + e_3^{c,s} + d_{3,j}^{O,c} - f_{3,j}^{c,r} - f_3^{c,s} - d_{3,j}^{I,c}\right) + \frac{N_3}{n_3} h_3 \mu_3^c +$$

$$- \sum_j \mu_j^c \cdot r_j\Bigg]\Bigg] \tag{3-23}$$

where h_v, v = 1, 2, 3 are the numbers of batch data falsified by the operator and contained in the samples of the inspection team.

With

$$N_1 Ev_1 Ec_1 - N_2 Ev_2 Ec_2 - N_3 Ev_3 Ec_3 = M_1$$

and because of the independence of the e, d, f on one hand and k_V on the other hand, we obtain

$$\text{cov (MUF, } D/H_1) = \text{cov (MUF, } D/H_0) \tag{3-24}$$

which also means that in this case we have $\rho < 0$. However, because of the difference of the variance of the D-statistics in case of H_0 and H_1 we have, instead of (3-21),

$$\boxed{\rho_{H_1} = \frac{\text{cov } (MUF_1 D/H_0)}{\sqrt{\text{var (MUF)}} \cdot \sqrt{\text{var } (D/H_1)}}} \quad . \tag{3-25}$$

The correlations ρ_{H_0} and ρ_{H_1} as a function of the effort C and amount M of diverted material are given in Table 6.

3.5 Overall Probability of Detection

In Figure 3, the results of the numerical calculations for the overall probability of detection $1 - \beta$ according to Eqs. (2-16b) and (2-16a) are presented for one inventory period (i.e. 6 months) for the parameters $M = M_1 + M_2 = 10$kg Pu, $\alpha = 0.05$, $\alpha_1 = \alpha_2$, and for varying M_1 (resp. M_2) and effort C. The corresponding probabilities of detection for $\rho = 0$ which have been calculated according to (2-17b and (2-17a) are almost the same as those for $\rho < 0$; this is not surprising because for $\rho < 0$, the false alarm relation Eq. (2-16a) is practically the same as that for $\rho = 0$, i.e. Eq. (2-17a).

As can be checked numerically, the minimum of the probability of detection is given approximately for those values of M_1 and M_2 for which the following relation holds.

$$\frac{M_1}{\sigma_{MUF}} = \frac{E(D/H_1)}{\sigma_{D/H_1}} \quad . \tag{3-26}$$

The relation is intuitive because of the symmetry of the formulas, at least for $\rho = 0$. Accordingly, the maximum of the probability of detection with respect to the inspector's strategies (for an optimal operator's strategy) is approximately given for $\alpha_1 = \alpha_2$. This can be seen in Figures 4 and 5 where the values of α_1 and α_2 are different.

At first sight it seems strange that for a certain range of the M_1 (resp. M_2) values, the probability of detection decreases with increasing effort C. However, the explanation is given easily. As shown in Table 5, the variance $\mathrm{var}(D/M_1)$ decreases monotonously with increasing effort C, which is intuitive. This means that the probability of detection

$$\phi\left(\frac{E(D/H_1) - \sigma_{D/H_0} \cdot U_{1-\alpha_2}}{\sigma_{D/H_1}}\right)$$

increases with increasing effort if the argument of the ϕ function is positive, and decreases if the argument is negative. As can be seen from the numerical data, the change in direction of effort C's influence is given at that place where the argument of the ϕ-function changes its sign.

The numerical calculations may be summarized by stating that the overall guaranteed probability of detection for a given effort C, and a total amount M of material to be diverted for one inventory period is simply calculated according to formulas (2-17b) and (2-17a) for $\alpha_1 = \alpha_2$; M_1 amd M_2 are chosen according to (3-26).

4. Conclusion

The purpose of this paper was to evaluate the efficiency of the international nuclear material safeguards system which is based on material accountability and its verification at the hand of a realistic numerical example. The problem was complicated because the two statistics on which the inspector's statements are based are stochastically dependent. It was shown that this depencence may be neglected in the practical situation. Therefore, rather simple formulas may be used for the determination of the system efficiency, i.e. the total guaranteed probability of detection.

All considerations were based on the case of one material balance area which was one plant. If one considers more than one material balance area, then new correlations arise; in some cases, these may be important for the reduction of inspection effort is kept constant. An example is the shipper-receiver-correlations between two different nuclear plants; they may be used either to replace the measurements at both sites by simple sealing measures, or as an additional check if both measurements are kept. Therefore, the consideration of a nuclear fuel cycle as a whole which includes many material balance areas, raises questions which go beyond the scope of this work.

Table 1. NFS campaign and batch data for the reference time T (6 months) for the plutonium throughput.

Pu throughput/T [kg]	1750
Liquid waste [% of input]	0.9
Hull losses [% of input]	0.1
Number of campaigns/T	5
Number of working days/T	125
Input	
Input/campaign [kg]	175
Number of batches/campaign	25
Batch volume [l]	4000
Pu content/batch [kg]	7
Batch-to-batch variation [%]	10
Product	
Number of batches/campaign	76
Weight of batch [kg]	15
Pu content/batch [kg]	2.28
Batch-to-batch variation [%]	10
Liquid Waste	
Number of batches/campaign	90
Batch volume [l]	5000
Pu content/batch [kg]	0.019
Batch-to-batch variation [%]	10

Table 2. Pu measurement system for the NFS plant (source: [7]).

Class	Measurement	Standard deviation per single measurement		Effort per single measurement	
				Man-hours	Cost
		%	%	[h]	[US$]
Input	Volume determination (diptube system)	.35	.1	.7	-
	Sampling	1	-	1.5	-
	Concentration determination (isotopic dilution)	.6	.3	-	400
Product	Weighing	.02	-	3	-
	Sampling	.5	-	2.25	-
	Concentration determination (amperometric titration and isotopic analysis)	.4	.3	-	200
Liquid Waste	Volume determination (level indicator)	5	5	.1	-
	Sampling	50	-	.5	-
	Concentration determination (TTA extraction and counting)	15	10	2	40
Physical Inventory	Washout	.577 [kg]		-	

Table 3. Variance of the material unaccounted for (MUF) for one inventory period.

	Variance [kg]		Standard deviation [kg]
Input	8.564	1)	2.926
Product	6.837	2)	2.615
Waste	0.958	3)	0.979
Inventory	0.333	4)	0.577
MUF	17.026	5)	4.126

1) Eq. (3-5)

2) Eq. (3-9)

3) Eq. (3-6)

4) Eq. (3-1c)

5) Eq. (3-11)

Table 4a. Input data for the concentration measurement verification.

	Class i	Total number of batches N_i	Batch size Ev_i	Pu content per batch [kg]	Effort E_i (US$) per verification	Amount μ_i [kg] per batch to be diverted
Input	1	125	4000[l]	7	400	.147
Product	2	380	15[kg]	2.28	200	.0342
Waste	3	450	5000[l]	.019	40	.0114

Table 4b. Optimal sample sizes of the inspector (1): Here, the application of (3-23a) gave $n_i^o > N_i$; therefore in this class $n_i^o = N_i$ was taken and the remaining effort $C - E_i n_i$ was distributed according to (2-23a).

C [% of max effort]	100	80	60	50	30	20	10	5	1
n_1^o	125	96	26	1	1	1	1	1	1
n_2^o	380	380[1)]	380[1)]	358	214	142	70	34	5
n_3^o	450	17	5	1	1	1	1	1	1

Table 4c. Optimal sample sizes of the operator.

Amount M to be diverted [kg]	.1	.5	1	2	3	4	5	6	7	8	9	10
r_1^o	0	1	1	1	1	1	1	1	1	1	1	1
r_2^o	3	10	25	54	83	112	142	171	200	229	258	288
r_3^o	1	1	1	1	1	1	1	1	1	1	1	1

Table 5. Standard deviations $\sqrt{\sigma^2_{D/H_1}}$ under the alternative hypothesis (M > 0) and $\sqrt{\sigma^2_{D/H_0}}$ under the null hypothesis (M = 0) as a function of amount M [kg] to be diverted and inspection effort C [% of maximun effort].

C \ M	0	.1	.5	1	2	3	4	5	6	7	8	9	10
100	.247	.247	.247	.247	.247	.247	.247	.247	.247	.247	.247	.247	.247
80	.247	.248	.249	.249	.249	.249	.249	.249	.249	.249	.249	.249	.249
60	.247	.252	.259	.259	.249	.249	.249	.249	.249	.249	.249	.249	.259
50	.247	.272	.430	.430	.430	.430	.431	.431	.431	.431	.431	.431	.430
30	.248	.274	.431	.433	.435	.436	.438	.438	.439	.439	.439	.438	.437
20	.249	.275	.433	.436	.440	.444	.446	.448	.449	.449	.449	.447	.447
10	.253	.280	.439	.445	.457	.446	.472	.479	.479	.480	.479	.474	.469
5	.261	.291	.450	.465	.489	.508	.522	.532	.537	.537	.534	.526	.513
1	.339	.388	.558	.643	.766	.852	.912	.952	.972	.974	.960	.927	.873

Table 6. Correlation g under the alternative hypothesis ($M > 0$) and under the null hypothesis ($M = 0$) as a function of amount M[kg] to be diverted, and inspection effort C [% of maximum effort].

C \ M	0	.1	.5	1	2	3	4	5	6	7	8	9	10
100	-.445	-.445	-.445	-.445	-.445	-.445	-.445	-.445	-.445	-.445	-.445	-.445	-.445
80	-.444	-.442	-.441	-.441	-.441	-.441	-.441	-.441	-.441	-.441	-.441	-.441	-.441
60	-.443	-.444	-.423	-.423	-.423	-.423	-.423	-.423	-.423	-.423	-.423	-.423	-.423
50	-.417	-.378	-.239	-.239	-.239	-.239	-.239	-.239	-.239	-.239	-.239	-.239	-.239
30	-.248	-.245	-.143	-.142	-.142	-.141	-.141	-.140	-.140	-.140	-.140	-.141	-.141
20	-.164	-.148	-.094	-.094	-.093	-.092	-.092	-.091	-.091	-.091	-.091	-.091	-.092
10	-.080	-.072	-.046	-.045	-.044	-.043	-.043	-.042	-.042	-.042	-.042	-.043	-.043
5	-.038	-.034	-.022	-.021	-.020	-.019	-.019	-.019	-.018	-.018	-.019	-.019	-.019
1	-.005	-.005	-.003	-.003	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002	-.002

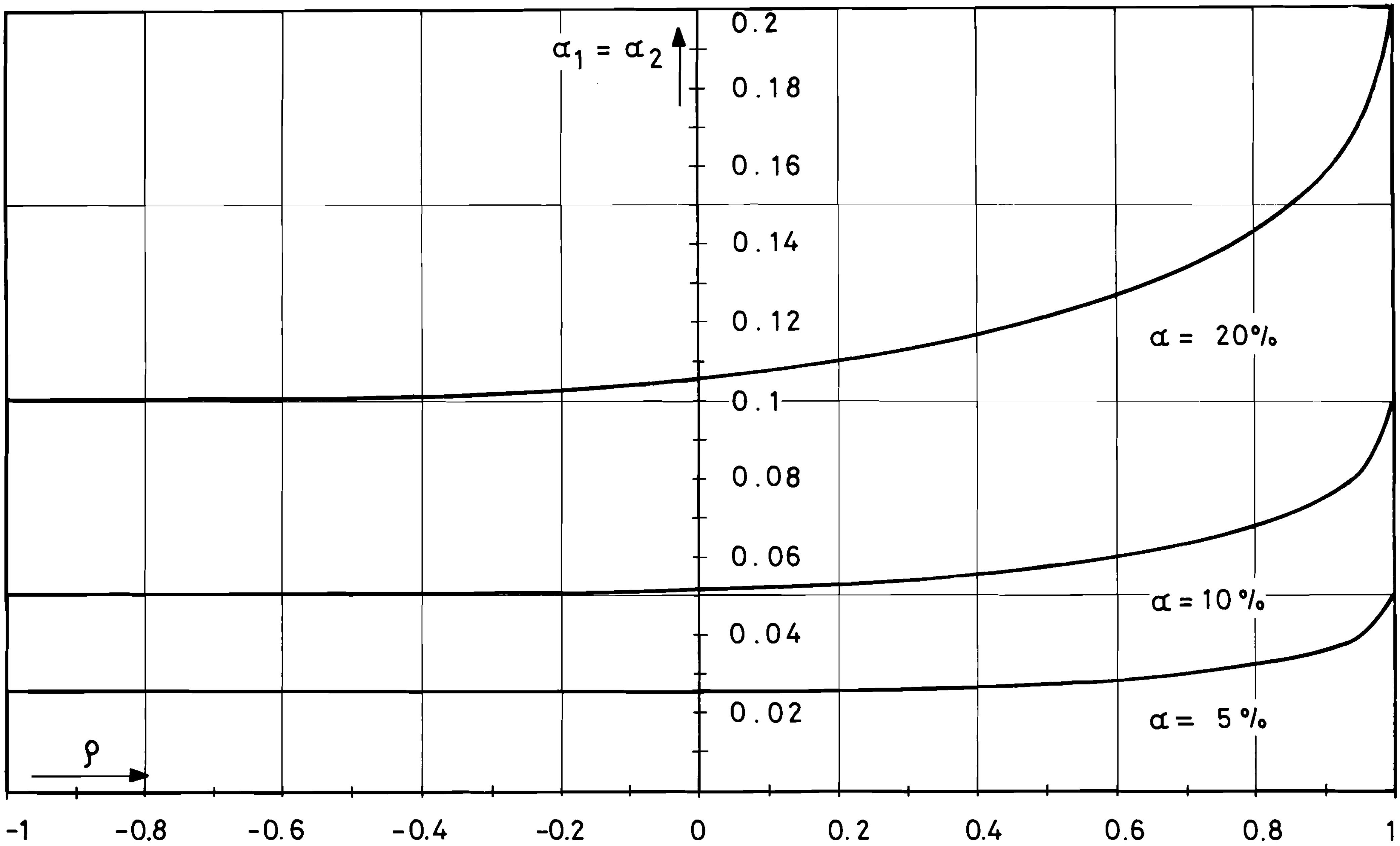

FIG. 1 : SINGLE TEST FALSE ALARM PROBABILITIES α_1 AND α_2 FOR $\alpha_1 = \alpha_2$ AS A FUNCTION OF CORRELATION ρ WITH TOTAL FALSE ALARM PROBABILITY α AS PARAMETER.

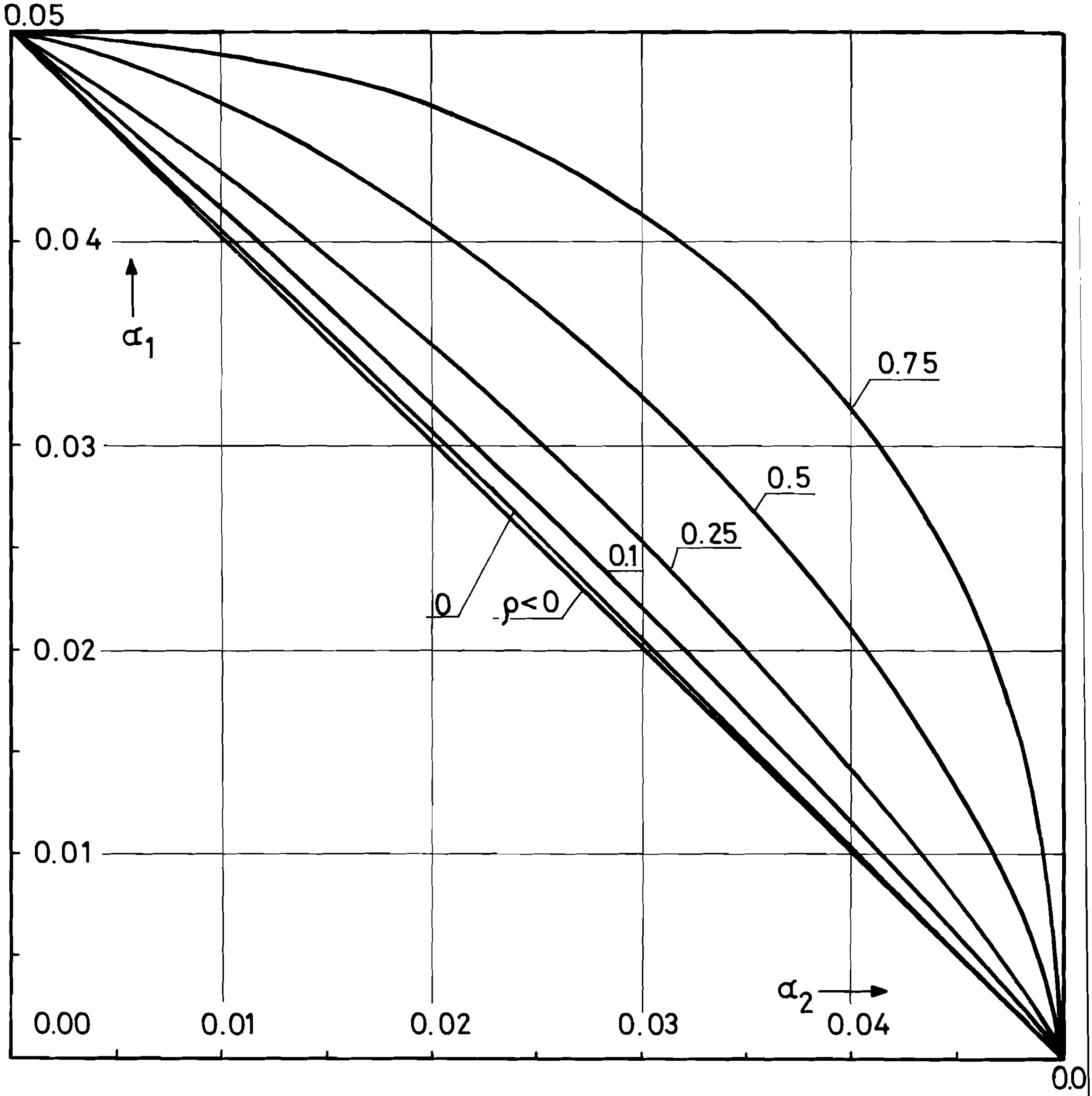

FIG. 2: MUTUAL DEPENDENCE OF THE SINGLE TEST FALSE ALARM PROBABILITIES α_1 AND α_2 WITH CORRELATION ρ AS PARAMETER FOR TOTAL FALSE ALARM PROBABILITY $\alpha = 0.05$

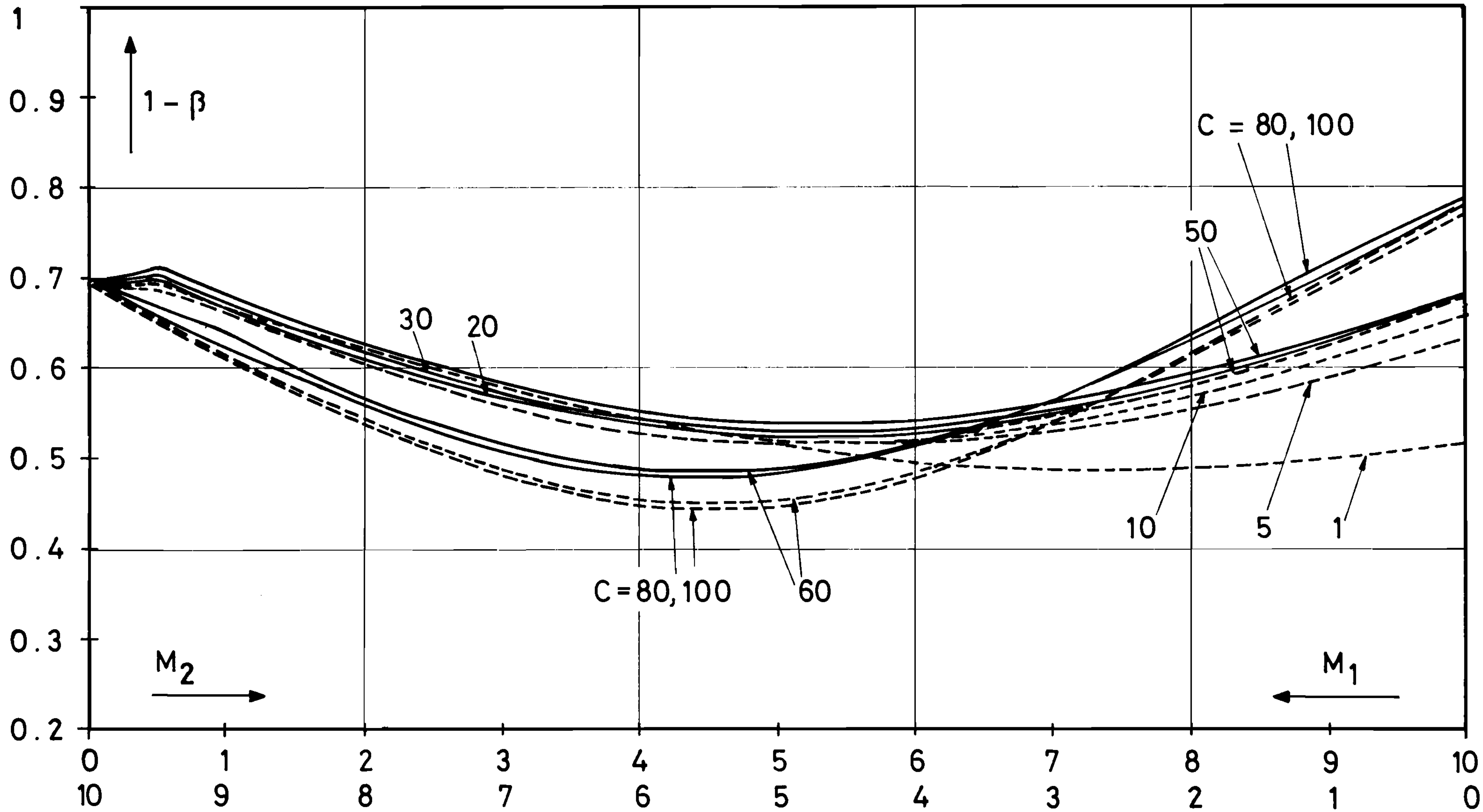

FIG. 3 : TOTAL PROBABILITY OF DETECTION AS FUNCTION OF AMOUNT M_1 OF MATERIAL DIVERTED, WITH EFFORT C [% OF MAXIMUM EFFORT] AS PARAMETER, AND $M_1 + M_2 = 10$ [kg], $\alpha_1 = \alpha_2$, $\alpha = 0.05$. DASHED LINES: $\rho = 0$. FOR C = 10, 5, 1 DASHED AND CONTINUOUS LINES COINCIDE.

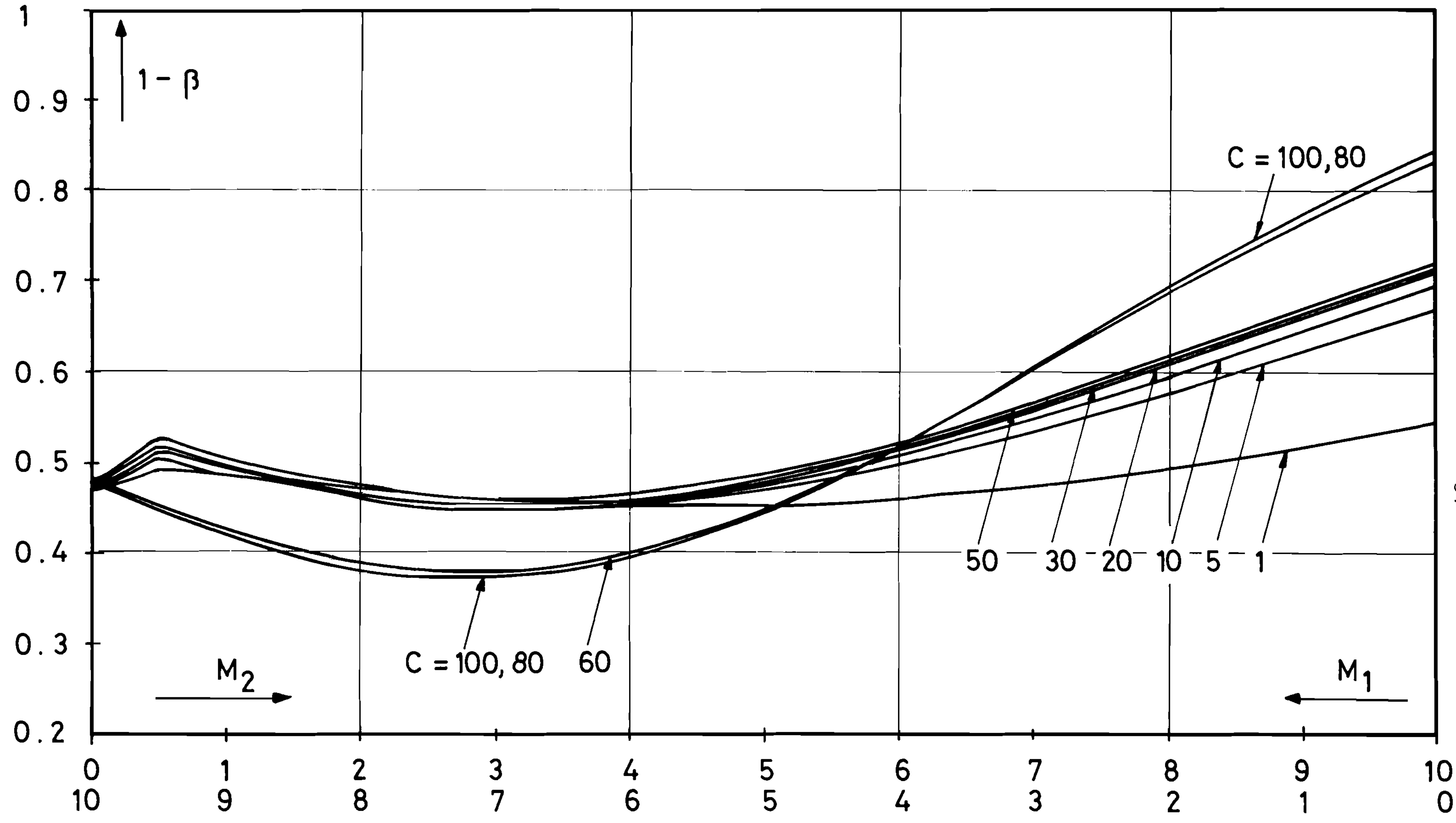

FIG. 4 : TOTAL PROBABILITY OF DETECTION AS FUNCTION OF AMOUNT M_1 OF MATERIAL DIVERTED, WITH EFFORT OF C [% OF MAXIMUM EFFORT] AS PARAMETER AND $M_1 + M_2 = 10$ [kg], $\alpha_2 = 0.005$, $\alpha = 0.05$.

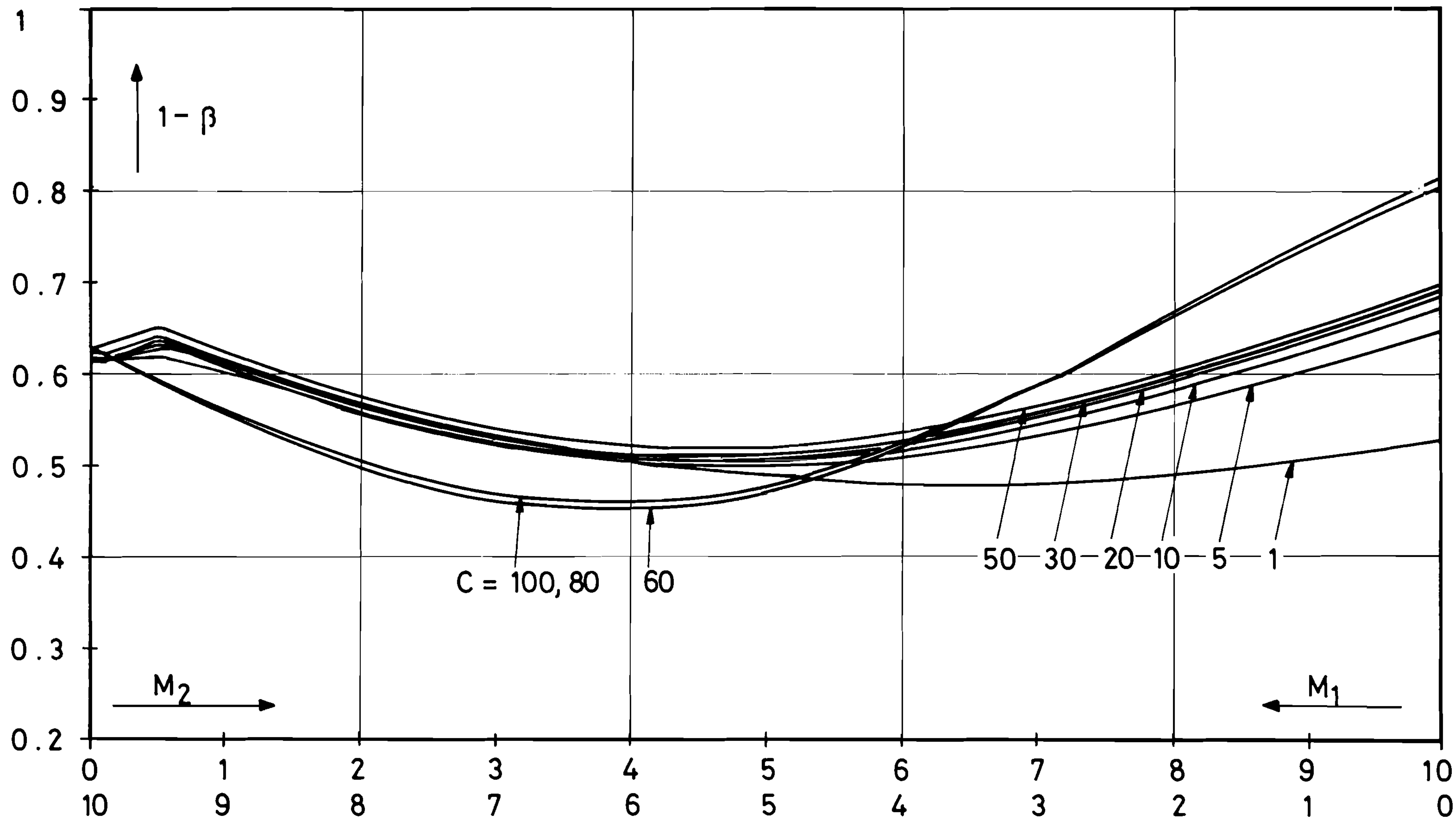

FIG. 5 : TOTAL PROBABILITY OF DETECTION AS FUNCTION OF AMOUNT M_1 OF MATERIAL DIVERTED, WITH EFFORT C [% MAXIMUM EFFORT] AS PARAMETER, AND $M_1 + M_2 = 10$ [kg], $\alpha_2 = 0.015$, $\alpha = 0.05$.

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Annex
Discussion of the False Alarm Equation

A.1 Formulation of the Problem

According to (2-15a), the false alarm equation is given by the following expression:

$$1-\alpha = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_7}} dt_2 \exp\left[-\frac{(t_1^2 - 2t_1t_2\rho + t_2^2)}{2(1-\rho^2)}\right] , \qquad \text{(A-1)}$$

where U is the inverse of the normal distribution function ϕ

$$u(x) = \phi^{-1}(x) \quad ; \quad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-t^2}{2}\right) dt \ . \qquad \text{(A-2)}$$

As one can see immediately, Eq. (A-1) reduces to the following form for $\rho = 0$:

$$1 - \alpha = (1 - \alpha_1)\cdot(1 - \alpha_2) \quad , \qquad \text{(A-3)}$$

which is well known in the field of multivariate statistical inference (see, e.g. [A-1]). Therefore, (A-1) may be considered as a generalization of (A-3) for the case of stochastically dependent random variables.

In the following, we will discuss the analytic properties of the false alarm equation, as well as graphical and numerical methods for the tabulation of the relation between α_1 and α_2 for given values of the parameters α and ρ.

A.2 Bonferroni's Inequality

Let X and Y be Gaussian distributed random variables with expectation values 0 and variances 1. Then (A-1) is equivalent to the following form:

$$1 - \alpha = \mathrm{pr}\left\{X \le U_{1-\alpha_1} \cap Y \le U_{1-\alpha_2}\right\} \ . \qquad \text{(A-4)}$$

Now, Bonferroni's inequality [A-1] generally states

$$\text{pr}\ \{A \cup B\} = \text{pr}\ \{A\} + \text{pr}\ \{B\} - \text{pr}\ \{A \cap B\} \leq \text{pr}\ \{A\} + \text{pr}\ \{B\} \tag{A-4}$$

or, with the duality theorem

$$\text{pr}\ \{A \cup B\} = \text{pr}\ \{\overline{\bar{A} \cap \bar{B}}\} = 1 - \text{pr}\ \{\bar{A} \cap \bar{B}\} \leq \text{pr}\ \{A\} + \text{pr}\ \{B\}\ .$$

Therefore, with $\bar{A} \rightarrow \bar{C}$, $\bar{B} \rightarrow \bar{D}$, we obtain

$$\text{pr}\ \{C \cap D\} > \text{pr}\ \{C\} + \text{pr}\ \{D\} - 1\ .$$

Application to Eq. (A-4) gives with Eqs. (A-2)

$$\alpha \leq \alpha_1 + \alpha_2 \tag{A-5}$$

for any value of ρ. (The complementary inequality which can be derived from (A-4),

$$\alpha \geq \alpha_1 + \alpha_2 - 1$$

is without practical application in this text.)

A.3 The Bivariate Normal Distribution Function

The random variables X and Y are said to be distributed as a bivariate normal distribution with means and variances (0,0) and (1,1) and correlation ρ, if the joint probability that X is less than or equal to h and Y is less than or equal to k is given by

$$\text{pr}\ \{X \leq h, Y \leq k\} = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{h} ds \int_{-\infty}^{k} dt\ \exp\left[-\ \frac{\rho^2 - 2\rho st + t^2}{2(1-\rho^2)}\right]$$

$$= : L(-h,-k,\rho)\ . \tag{A-6}$$

The following properties are important for the discussion of (A-1):

$$L(h,k,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_h^\infty ds \int_k^\infty dt \, \exp\left[-\frac{\rho^2 - 2\rho st + t^2}{2(1-\rho^2)}\right] \qquad \text{(A-7)}$$

$$L(h,k,1) = \begin{cases} 1 - \phi(h) \quad , & \text{for } k \leq h \\ 1 - \phi(k) \quad , & \text{for } k \geq h \end{cases} \qquad \text{(A-8)}$$

$$L(h,k,-1) = \begin{cases} 0 \quad , & \text{for } h + k \geq 0 \\ 1 - \phi(h) - \phi(k) \quad , & \text{for } h + k \leq 0 \end{cases} \qquad \text{(A-9)}$$

With the help of (A-6), (A-1) can be expressed in the following way

$$1 - \alpha = L(-U_{1-\alpha_1} - U_{1-\alpha_2}, \rho)$$

Or, if we use the relation

$$-U_x = U_{1-x} \quad ,$$

we obtain

$$1 - \alpha = L(U_{\alpha_1}, U_{\alpha_2}, \rho) \quad . \qquad \text{(A-10)}$$

A.4 Extreme Values for the False Alarm Equation

For $\alpha_2 = 0$ we obtain, using $\lim_{\alpha_2 \to 0} U_{1-\alpha_2} = \infty$, from Eq. (A-1)

$$1 - \alpha = \lim_{U_{1-\alpha_2} \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \exp\left[-\frac{t_2^2}{2}\right] \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}$$

$$\cdot \int_{-\infty}^{U_{1-\alpha_2}} \exp\left[-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}\right]$$

$$= \lim_{U_{1-\alpha_2}\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \exp\left[-\frac{t_2^2}{2}\right] \phi\left(\frac{U_{1-\alpha_2} - \rho t_2}{\sqrt{1-\rho^2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \exp\left[-\frac{t_2^2}{2}\right] = 1 - \alpha_1 \quad .$$

For $\alpha_1 = 0$, we obtain the same result for reasons of symmetry, i.e.

$$\alpha = \begin{cases} \alpha_1 \quad , & \text{for } \alpha_2 = 0 \\ \alpha_2 \quad , & \text{for } \alpha_1 = 0 \end{cases} \quad . \tag{A-11}$$

For $\rho = 1$ we obtain, from (A-8) and (A-10),

$$1 - \alpha = L(U_{\alpha_1}, U_{\alpha_2}, 1) = \begin{cases} 1 - \phi(U_{\alpha_1}) \; , & \text{for } U_{\alpha_2} \leq U_{\alpha_1} \\ 1 - \phi(U_{\alpha_2}) \; , & \text{for } U_{\alpha_2} \geq U_{\alpha_1} \end{cases} \quad .$$

Therefore,

$$\alpha = \begin{cases} \alpha_1 \quad , & \text{for } \rho = 1, \text{ and } \; \alpha_2 \leq \alpha_1 \\ \alpha_2 \quad , & \qquad\qquad\qquad\;\; \alpha_2 \geq \alpha_1 \end{cases} \quad . \tag{A-12}$$

For $\rho = -1$ we obtain, from (A-9(and (A-10),

$$1 - \alpha = L(U_{\alpha_1}, U_{\alpha_2}, -1) = \begin{cases} 0 & , \text{ for } U_{\alpha_1} + U_{\alpha_2} \geq 0 \\ 1 - \phi(U_{\alpha_1}) - \phi(U_{\alpha_2}) & , \text{ for } U_{\alpha_1} + U_{\alpha_2} \leq 0 \end{cases} \quad .$$

As the case

$$U_{\alpha_1} + U_{\alpha_2} \geq 0 \iff \alpha_1 + \alpha_2 \geq 1$$

is not interesting here, we have

$$\alpha = \alpha_1 + \alpha_2 \quad , \qquad \text{for } \rho = -1 \text{ and } \alpha_1 + \alpha_2 \leq 1 \tag{A-13}$$

which is the limiting case in Bonferroni's Equation (A-5).

A.4 Monotony of the Function $\alpha_1(\rho)$ for $\alpha_1 = \alpha_2$ and α given

In this section we show that for $\alpha_1 = \alpha_2$ and α given, the function $\alpha_1(\rho)$ as defined implicitly by (A-1), is monotonously increasing for $-1 \leq \rho \leq 1$.

We start by performing the second integration in (A-1) which immediately gives

$$1 - \alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_{1-\alpha_2}} dx \exp\left(-\frac{x^2}{2}\right) \phi\left(\frac{U_{1-\alpha_1} - x \cdot \rho}{\sqrt{1-\rho^2}}\right) \tag{A-14}$$

For $\alpha_1 = \alpha_2$, we obtain the implicit representation of the function $\alpha_1(\rho)$ we are interested in:

$$1 - \alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_{1-\alpha_1}} dx \exp\left(-\frac{x^2}{2}\right) \phi\left(\frac{U_{1-\alpha_1} - x \cdot \rho}{\sqrt{1-\rho^2}}\right) \tag{A-15}$$

We want to show that the derivative $\frac{d\alpha_1}{d\rho}$ does not change its sign. As $U_{1-\alpha_1} = -U_{\alpha_1}$, and furthermore,

$$\frac{dU_{\alpha_1}}{d\rho} = \frac{dU_{\alpha_1}}{d\alpha_1} \cdot \frac{d\alpha_1}{d\rho} \tag{A-16}$$

we may simply consider the derivative $\frac{dU_{\alpha_1}}{d\rho}$ because $\frac{dU_{\alpha_1}}{d\alpha_1}$ does not change its sign.

Partial derivation of (A-15) gives

$$0 = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{U^2_{1-\alpha_1}}{2}\right) \cdot \phi\left(\frac{U_{1-\alpha_1} - U_{1-\alpha_1} \cdot \rho}{\sqrt{1-\rho^2}}\right) \cdot \frac{dU_{1-\alpha_1}}{d\rho}$$

$$+ \frac{1}{2\pi} \int_{\infty}^{U_{1-\alpha_1}} dx \exp\left(-\frac{x^2}{2}\right) \exp\left(\frac{U_{1-\alpha_1} - x \cdot \rho^2}{2(1-\rho^2)}\right) \frac{d}{d\rho}\left(\frac{U_{1-\alpha_1} - x \cdot \rho}{\sqrt{1-\rho^2}}\right) .$$

(A-17)

In the following we simply write α instead of α_1. We then obtain with the following relation

$$\frac{d}{d\rho}\left(\frac{-U_\alpha - x \cdot \rho}{\sqrt{1-\rho^2}}\right) = -\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\rho - U_\alpha + x \cdot \rho^2}{\sqrt{1-\rho^2}} + \frac{dU_\alpha}{d\rho} + x\right)$$

$$x^2 + \frac{(U_\alpha + x \cdot \rho)^2}{1-\rho^2} = \frac{(x + \rho \cdot U_\alpha)^2}{1-\rho^2} + U^2_\alpha$$

from (A-17)

$$0 = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{U^2_\alpha}{2}\right) \cdot \phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}}\right) \cdot \frac{dU_\alpha}{d\rho} +$$

$$-\frac{1}{2\pi} \exp\left(\frac{U^2_\alpha}{2}\right) \int_{-\infty}^{-U_\alpha} dx \exp\left(-\frac{(x+\rho U_\alpha)^2}{2(1-\rho^2)}\right)$$

$$\cdot \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\rho \cdot U_\alpha}{1-\rho^2} + \frac{dU_\alpha}{d\rho} + \frac{x}{1-\rho^2}\right) .$$

With

$$\frac{x + \rho U_\alpha}{1-\rho^2} = z \qquad \text{or} \qquad x = z \cdot \sqrt{1-\rho^2} - \rho \cdot U_\alpha$$

we obtain

$$0 = + \phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}}\right) \cdot \frac{dU_\alpha}{d\rho} + \frac{1}{\sqrt{2\pi}} \int_{\infty}^{(-U_\alpha+\rho U_\alpha)/\sqrt{1-\rho^2}} dz \exp\left[-\frac{z^2}{2}\right] \cdot \left(\frac{dU_\alpha}{d\rho} + \frac{z}{\sqrt{1-\rho^2}}\right) .$$

Or, by use of

$$\int_{-\infty}^{a} dz \cdot z \cdot \exp\left(-\frac{z^2}{2}\right) = -\exp\left(-\frac{a^2}{2}\right) ,$$

we finally obtain

$$0 = \phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}}\right) \frac{dU_\alpha}{d\rho} + \phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}} \cdot \frac{dU_\alpha}{d\rho}\right) +$$

$$- \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{U_\alpha^2}{2} \cdot \frac{(1-\rho^2)}{1-\rho^2}\right) .$$

This can be solved for $\frac{dU_\alpha}{d\rho}$:

$$\phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}}\right) \cdot \frac{dU_\alpha}{d\rho} = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{U_\alpha^2}{2} \cdot \frac{1-\rho}{1+\rho}\right) \qquad \text{(A-18)}$$

As the term on the right hand side of (A-18), as well as the factor of $\frac{dU_\alpha}{d\rho}$ are greater than zero, we have shown that $\frac{dU_\alpha}{d\rho}$ and therefore, that $\frac{d\alpha}{d\rho}$ is greater than zero of $-1 \leq \rho \leq 1$.

We will show, in addition, that the function $\alpha(\rho)$ has no inflection points. For this purpose it is again sufficient to consider $\frac{d^2U_\alpha}{d\rho^2}$ as, according to (A-16), we have

$$\frac{d^2U_\alpha}{d\rho^2} = \frac{dU_\alpha}{d\rho} \cdot \frac{d\alpha^2}{d\rho^2} \quad . \qquad \text{(A-19)}$$

From (A-18) we get

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{U_\alpha^2}{2} \cdot \frac{(1-\rho)^2}{1-\rho^2}\right) \cdot U_\alpha \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{1}{1+\rho} + \phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}}\right) \cdot \frac{d^2U_\alpha}{d\rho^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{2}\right) \cdot (1-\rho^2)^{-\frac{3}{2}} \cdot (-2\rho) \cdot \exp\left(-\frac{U_\alpha^2}{2} \cdot \frac{1-\rho}{1+\rho}\right) +$$

$$+ \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{U_\alpha^2}{2} \cdot \frac{1-\rho}{1+\rho}\right) \cdot \left(-\frac{U_\alpha^2}{2}\right) \cdot \left(-\frac{2}{(1+\rho)^2}\right)$$

which gives

$$\phi\left(U_\alpha \cdot \frac{\rho - 1}{\sqrt{1-\rho^2}}\right) \cdot \frac{d^2 U_\alpha}{d\rho^2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{U_\alpha^2}{2} \cdot \frac{1-\rho}{1+\rho}\right)\left(-\frac{U_\alpha}{1+\rho} + \frac{\rho}{2} \cdot \frac{1}{1-\rho^2} + \frac{U_\alpha^2}{2} \cdot \frac{1}{(1+\rho)^2}\right)$$

The right hand side is greater than zero if and only if

$$-2 \cdot U_\alpha + \frac{\rho}{1-\rho} + U_\alpha^2 \cdot \frac{1}{1+\rho} > 0$$

or equivalently, if and only if

$$f(\rho) := \rho^2 (1 + 2U_\alpha) + \rho(1 - U_\alpha^2) - 2U_\alpha + U_\alpha^2 > 0 \quad . \qquad \text{(A-20)}$$

For $1 + 2U_\alpha < 0$, or $\alpha < 0.31$, this is true for all ρ with $\rho_1 \leq \rho \leq \rho_2$, where

$$\rho_{1,2}^2 \cdot (1 - 2U_\alpha) + \rho_{1,2} \cdot (1 - U_\alpha^2) - 2U_\alpha + U_\alpha^2 = 0 \quad .$$

As can be seen easily, for $U_\alpha < -1$ (or $\alpha < 0.16$), the inequality (A-10) is fulfilled for any ρ with $-1 \leq \rho \leq 1$.

A.5 Monotony of the Function $\alpha_2(\alpha_1)$ for given α and ρ

In order to determine the derivative $\frac{d\alpha_2}{d\alpha_1}$, for given α and ρ of the function $\alpha_2(\alpha_1)$, which is given implicitly by (A-1), we start again from (A-14).

Partial derivation gives

$$0 = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{U_{\alpha_2}^2}{2}\right) \cdot \phi\left(-\frac{U_{\alpha_1} + U_{\alpha_2} \cdot \rho}{\sqrt{1-\rho^2}}\right) \cdot (-1) \cdot \frac{dU_{\alpha_2}}{d\alpha_1}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{-U_{\alpha_2}} dx \exp\left(-\frac{x^2}{x} - \frac{\left(-U_{\alpha_1} - x \cdot \rho\right)^2}{2(1-\rho)^2}\right) \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \left(-\frac{dU_{\alpha 1}}{d\alpha_1}\right)$$

or

$$0 = \exp\left(-\frac{U_{\alpha_2}^2}{2}\right) \cdot \phi\left(\frac{U_{\alpha_2} \cdot \rho - U_{\alpha_1}}{\sqrt{1-\rho^2}}\right) \cdot \frac{dU_{\alpha_2}}{d\alpha_1}$$

$$+ \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{U_{\alpha_1}^2}{2}\right) \cdot \int_{-\infty}^{-U_{\alpha_2}} dx \exp\left(\frac{\left(x + U_{\alpha_1} \cdot \rho\right)^2}{2(1-\rho^2)}\right) \cdot \frac{dU_{\alpha_1}}{d\alpha_1} \quad . \tag{A-21}$$

Therefore we obtain the result $\frac{dU_{\alpha_2}}{d\alpha_1} < 0$, and with

$$\frac{dU_{\alpha_2}}{d\alpha_1} = \frac{dU_{\alpha_2}}{d\alpha_2} \cdot \frac{d\alpha_2}{d\alpha_1} \quad , \tag{A-22}$$

that

$$\frac{d\alpha_2}{d\alpha_1} < 0 \quad , \quad \text{as} \quad \frac{dU_{\alpha_2}}{d\alpha_2} > 0 \quad .$$

The question arises whether or not the function $\alpha_2(\alpha_1)$ has inflection points. In order to analyze this we write (A-21) in the following form

$$0 = \exp\left(-\frac{U_{\alpha_2}^2}{2}\right) \cdot \phi\left(\frac{U_{\alpha_2} \cdot \rho - U_{\alpha_1}}{\sqrt{1-\rho^2}}\right) \cdot \frac{d\alpha_2}{d\alpha_1} + \exp\left(-\frac{U_{\alpha_1}^2}{2}\right) \cdot \phi\left(\frac{U_{\alpha_1} \cdot \rho - U_{\alpha_2}}{\sqrt{1-\rho^2}}\right) \quad .$$

Derivation after α_1 gives

$$0 = \exp\left(-\frac{U_{\alpha_2}^2}{2}\right)\cdot\phi\left(\frac{U_{\alpha_2}\cdot\rho - U_{\alpha_1}}{\sqrt{1-\rho^2}}\right)\cdot\left(-U_{\alpha_2}\right)\cdot\frac{dU_{\alpha_2}}{d\alpha_1}\cdot\frac{d\alpha_2}{d\alpha_1} + \exp\left(-\frac{U_{\alpha_2}^2}{2}\right)$$

$$\otimes\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{U_{\alpha_2}\cdot\rho - U_{\alpha_1}}{2(1-\rho^2)}\right)\cdot\frac{1}{\sqrt{1-\rho^2}}\left(\rho\cdot\frac{dU_{\alpha_2}}{d\alpha_1} - \frac{dU_{\alpha_1}}{d\alpha_1}\right)\cdot\frac{d\alpha_2}{d\alpha_1}$$

$$+ \exp\left(-\frac{U_{\alpha_2}^2}{2}\right)\cdot\phi\left(\frac{U_{\alpha_1}\cdot\rho - U_{\alpha_1}}{\sqrt{1-\rho^2}}\right)\cdot\frac{d^2\alpha_2}{d\alpha_1^2} + \exp\left(-\frac{U_{\alpha_1}^2}{2}\right)$$

$$\cdot\phi\left(\frac{U_{\alpha_1}\cdot\rho - U_{\alpha_2}}{\sqrt{1-\rho^2}}\right)\left(-U_{\alpha_1}\right)\cdot\frac{dU_{\alpha_1}}{d\alpha_1}$$

$$+ \exp\left(-\frac{U_{\alpha_1}^2}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(U_{\alpha_1}\cdot\rho - U_{\alpha_2}\right)^2}{2(1-\rho^2)}\right)\cdot\frac{1}{\sqrt{1-\rho^2}}\left(\phi\cdot\frac{dU_{\alpha_1}}{d\alpha_1} - \frac{dU_{\alpha_2}}{d\alpha_1}\right).$$

As the factor of $\frac{d^2\alpha_2}{d\alpha_1^2}$ is greater than zero, we obtain

$$\operatorname{sgn}\left(\frac{d^2\alpha_2}{d\alpha_1^2}\right) = \exp\left(-\frac{U_{\alpha_2}^2}{2}\right)\Bigg[\phi\left(\frac{U_{\alpha_2}\cdot\rho - U_{\alpha_1}}{\sqrt{1-\rho^2}}\right)\cdot U_{\alpha_2}\cdot\frac{dU_{\alpha_2}}{d\alpha_2}\cdot\left(\frac{d\alpha_2}{d\alpha_1}\right)^2$$

$$+ \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(U_{\alpha_2}\cdot\rho - U_{\alpha_1}\right)^2}{2(1-\rho)^2}\right)\cdot\frac{1}{\sqrt{1-\rho^2}}\left(\frac{dU_{\alpha_1}}{d\alpha_1} - \rho\,\frac{dU_{\alpha_2}}{d\alpha_2}\cdot\frac{d\alpha_2}{d\alpha_1}\right)\cdot\frac{d\alpha_2}{d\alpha_1}\Bigg]$$

$$+ \exp\left(-\frac{U_{\alpha_1}^2}{\sqrt{2\pi}}\right)\Bigg[\phi\left(\frac{U_{\alpha_1}\cdot\rho - U_{\alpha_2}}{\sqrt{1-\rho^2}}\right)\cdot U_{\alpha_1}\cdot\frac{dU_{\alpha_1}}{d\alpha_1}$$

$$+ \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(U_{\alpha_1}\cdot\rho - U_{\alpha_2}\right)^2}{2(1-\rho^2)}\right)\cdot\frac{1}{\sqrt{1-\rho^2}}\left(\frac{dU_{\alpha_2}}{d\alpha_2}\cdot\frac{d\alpha_2}{d\alpha_1} - \rho\,\frac{dU_{\alpha}}{d\alpha_1}\right)\Bigg]$$

for the sign of $\frac{d^2\alpha_2}{d\alpha_1^2}$.

For $\alpha_1, \alpha_2 < 0.5$, we obtain

$$\operatorname{sgn} \frac{d^2\alpha_2}{d\alpha_1^2} < 0$$

if $\rho = 0$, or if

$$\frac{dU_{\alpha_1}}{d\alpha_1} - \rho \cdot \frac{dU_{\alpha_2}}{d\alpha_2} \cdot \frac{d\alpha_2}{d\alpha_1} > 0 \quad \text{and} \quad \frac{dU_{\alpha_2}}{d\alpha_2} \cdot \frac{d\alpha_2}{d\alpha_1} - \rho \cdot \frac{dU_{\alpha_1}}{d\alpha_1} < 0 \quad .$$

This is true for $\rho > 0$. It cannot be shown in this way that the sign of $\frac{d^2\alpha_2}{d\alpha_1^2}$ does not change for $\rho < 0$.

A.6 Graphical Representation of the False Alarm Equation

In the following, we want to represent Equation (A-1) graphically: we plot α_1 as a function of α_2 with ρ as a parameter for a given value of α. For $\rho = -1, 0, 1$ we already know the analytical form (Eqs. (A-13, 17 and 3)); we also know the form for $\alpha_1 = 0$ and $\alpha_2 = 0$ for arbitrary values of ρ. Bonferroni's inequality (A-5) and the conditions

$$0 \leq \alpha_1 \ , \quad \alpha_2 \leq \alpha \qquad \text{(A-23)}$$

(which follow from (A-11) and from symmetry considerations) limit the possible values in the $(\alpha_1 - \alpha_2)$ plane.

In the following, three different methods for the tabulation of the false alarm equation are discussed

i) Graphical method;
ii) Simulation method;
iii) Use of approximate formulas for L (h,k,ρ).

A.6.1 Graphical Method

In Ref. [A-2], the function

$$L\left(h,0,\frac{(\rho h-k)\cdot \operatorname{sgn} h}{\sqrt{h^2-2hk\rho+k^2}}\right) \tag{A-24}$$

is tabulated for $-1 \leq \rho \leq 1$ and $0 \leq h \leq 2.5$. With the help of the relation

$$L(h,0,-\rho) = \frac{1}{2} - L(-h,0,\rho) \quad ,$$

function (A-24) can be tabulated also for negative values of h. In addition, we have

$$L(h,k,\rho) = L\left(h,0,\frac{(\rho h-k)\ \operatorname{sgn} h}{\sqrt{h^2-2hk\rho+k^2}}\right) + L\left(k,0,\frac{(\rho h-h)\ \operatorname{sgn} k}{\sqrt{h^2-2hk\rho+k^2}}\right) + $$

$$- \begin{cases} 0 \ , & \text{if } hk \geq 0 \text{ and } h+k \geq 0 \\ \frac{1}{2} \ , & \text{otherwise} \end{cases} \quad . \tag{A-25}$$

Therefore, the false alarm equation can be represented in the following form:

$$1-\alpha = L\left(U_{\alpha_1},0,\frac{\left(\rho U_{\alpha_1}-U_{\alpha_2}\right)\operatorname{sgn} U_{\alpha_1}}{\left(U_{\alpha_1}^2-2U_{\alpha_1}\rho+U_{\alpha_2}^2\right)^{1/2}}\right) + L\left(U_{\alpha_1},0,\frac{\left(\rho U_{\alpha_2}-U_{\alpha_1}\right)\operatorname{sgn} U_{\alpha_2}}{\left(U_{\alpha_1}^2-2U_{\alpha_1}U_{\alpha_2}+U_{\alpha_2}^2\right)^{1/2}}\right)$$

$$- \begin{cases} 0 \ , & \text{if } U_{1-\alpha_1}\cdot U_{1-\alpha_2} \geq 0 \text{ and } \alpha_1+\alpha_2 \geq 1 \\ \frac{1}{2} \ , & \text{otherwise} \end{cases} \quad .$$

This relation has been used to tabulate α_1 as a function of α_2 with ρ and α as parameters on the basis of the graphical representation of the function (A-24) in Ref. [A-2].

As the accuracy of the (A-24) representation is not better than 0.01, it has not been possible to obtain a satisfying accuracy for values of (α_1, α_2) approaching $(0, \alpha)$ and $(\alpha, 0)$ therefore, different methods had to be used in these critical regions.

A.6.2 Simulation Method

In order to tabulate Eq. (A-1) with the help of a simulation method, the following procedure is used: Let A, B, and C be normally distributed random variables, with

$$EA = EB = EC = 0 \quad , \quad \text{var } A = \sigma_A^2 \quad , \quad \text{var } B = \sigma_B^2 \quad , \quad \text{var } C = \sigma_C^2 \quad .$$

Then we can tabulate (A-1) by means of the following form:

$$1 - \alpha = \left\{ \text{prob} \quad A + B \leq U_{1-\alpha_1} , \quad \pm A + C \leq U_{1-\alpha_2} \right\} \quad ,$$

where the variances are determined in such a way that the variances of A + B and $\pm$A + C are 1:

$$\sigma_A^2 + \sigma_B^2 = 1 \quad , \quad \sigma_A^2 + \sigma_C^2 = 1$$

and where the correlation takes the value $\pm\rho$:

$$\text{cor } (A + B, \pm A + C) = \frac{\pm \sigma_A^2}{\sqrt{\sigma_A^2 + \sigma_B^2} \sqrt{\sigma_A^2 + \sigma_C^2}} = \pm\sigma_A^2 = \rho \quad .$$

The disadvantage of this method is that it provides no direct method of calculation of α_1 as a function of α_2 for given values of α and ρ; one has to fix α_1, α_2, and ρ and determine α, which means that one must iterate until one has reached the previously chosen value of α chosen below.

A.6.3 Use of Approximate Formulas for L (h,k,ρ)

The method which has proven most successful for the numerical calculations uses approximate formulas for the bivariate normal distribution function given by Owen [A-3].

Let us define

$$B(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{h} dx \int_{-\infty}^{k} dy \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right] \qquad \text{(A-26)}$$

and furthermore,

$$T(h,a) = \frac{1}{2\pi} \int_{0}^{a} \frac{dx}{1+x^2} \exp\left(-\frac{h^2(1+x^2)}{2}\right) . \qquad \text{(A-27)}$$

Then we have according to Owen

$$B(h,k,\rho) = \begin{cases} \frac{1}{2}\phi(h) - T\left(h, \frac{k-\rho h}{h\sqrt{1-\rho^2}}\right) + \frac{1}{2}\phi(k) - T\left(k, \frac{h-\rho k}{k\sqrt{1-\rho^2}}\right) \\ \qquad \text{if } hk > 0 \text{ or if } hk = 0 \text{ , } h \text{ or } k \geq 0 \\ \frac{1}{2}\phi(h) - T\left(h, \frac{k-\rho h}{h\sqrt{1-\rho^2}}\right) + \frac{1}{2}\phi(k) - T\left(k, \frac{h-\rho k}{k\sqrt{1-\rho^2}}\right) - \frac{1}{2} \\ \qquad \text{if } hk < 0 \text{ or if } hk = 0 \text{ , } h \text{ or } k < 0 \end{cases} . \qquad \text{(A-28)}$$

Furthermore, we have

$$T(h,a) = \frac{\text{arctg } a}{2\pi} - \frac{1}{2\pi} \sum_{j=0}^{\infty} c_j \quad a^{2j+1} \qquad \text{(A-29)}$$

where

$$C_j = (-1)^j \cdot \frac{1}{2j+1} \left(1 - \exp\left(-\frac{h^2}{2}\right) \cdot \sum_{i=0}^{j} \frac{h^{2i}}{2^i j!}\right)$$

converge rapidly for small values of a and h.

On the basis of these formulas, the false alarm relation (A-1) has been determined numerically for fixed values of M.

In Figure 1 (in the main part of this paper), for $\alpha_1 = \alpha_2$, the values of α_1 as a function of ρ have been represented for different values of α. As can be seen directly, the value of ρ is practically independent of the value of α for $\rho < 0$. Therefore, for $\rho < 0$ (A-3) is favorable instead of (A-1), $\alpha_1 = \alpha_2 = 1 - \sqrt{1-\alpha}$.

In Figure 2, for a fixed value of $\alpha = 0.05$, the values of α_1 as a function of α_2 with ρ as a parameter have been represented. Again, for $\rho < 0$, (A-3) is favorable instead of (A-1).

A.7 False Alarm Equation for Symmetric Tests

All the considerations of this paper have been based on one-sided tests. For completeness we give the false alarm equation for symmetric tests, i.e. for tests where the null hypothesis is given by

$$H_0: \quad -\rho_1 < ED \leq \rho_2 \ , \quad -\rho_2 < EMUF \leq \rho_2 \quad .$$

As can be seen easily, in this case, the false alarm equation is given by the following formula:

$$1 - \alpha = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \int_{-U_{1-\frac{\alpha_1}{2}}}^{U_{1-\frac{\alpha_1}{2}}} dt_1 \int_{-U_{1-\frac{\alpha_2}{2}}}^{U_{1-\frac{\alpha_2}{2}}} dt_2 \cdot \exp\left(-\frac{(t_1^2 - 2t_1t_2\rho + t_2^2)}{2(1-\rho^2)}\right) .$$

Without going into a thorough discussion of this formula, it should be stated only that it is invariant to the change of the sign of ρ. So for $\alpha_1 = \alpha_2$ we obtain

$$\alpha_1 = \alpha_2 = \alpha \quad \text{for} \quad \rho = \pm 1 \quad .$$

We lose the nice property of the one-sided test that for $\rho < 0$, the false alarm relation is practically independent of the value of α.

References of the Annex

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