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Equilibrium and Linear Complementarity - An Economy with Institutional Constraints on Prices

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1. Introduction and Summary

In the theory of perfect competition, it is supposed that there are no institutional restrictions upon prices. Much the same assumption is built into mathematical programming models. The presence of such constraints implies, for example, that the market price and the marginal productivity (shadow price) of the factors of production will not necessarily coincide. Unless such constraints are introduced, models cannot explain the simultaneous existence of excess supply of an item and yet a positive market price.

If there is a gap between market and shadow prices, this raises a question. By what set of prices are the economic agents' actions guided? In this paper, we assume that one sector of the economy, the private sector, is guided by market prices. The other, the public sector, is guided by shadow prices.

With conventional optimization techniques, it is awkward --and sometimes inpossible--to handle this type of problem.

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Here we shall show that some of these features can be introduced through linear complementarity. This approach permits us to introduce institutional constraints upon prices--in addition to the technological constraints that are normally generated through the coefficients of each activity in a linear programming model.

2. Competitive Equilibrium, Linear Complementarity and Linear Programming

In this section we review the connection between competitive equilibrium, linear complementarity and linear programming for a special case--a "small" economy that can sell unlimited amounts of its outputs upon world markets.

Consider such an economy with n productive processes. The matrix

$$A = (a_{ij})$$

describes the technology available to the economy. If all coefficients $a_{ij} \geq 0$ $(i=1,\ldots,m;\ j=1,\ldots,n)$, then these denote the amount of item i required to operate activity j at unit level.

Let the column vector[2] $B = (b_1, b_2, \ldots, b_m)'$. Then if $B \geq 0$, this vector denotes the resources available to the economy. (If a component were negative, this would denote a

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[2]Capital letters are used for matrices and vectors, and $'$ denotes transpose.

delivery requirement rather than a resource available).

We shall make the assumption that the payoff from each activity is determined by world market prices, and that these are independent of the activity levels in the economy. Let $C = (c_1,\dots,c_n)'$, where c_j denotes the value of output of activity j when it is operated at the unit level. Finally, let $X = (x_1,\dots,x_n)'$ denote the vector of activity levels and let $Z = (z_1,\dots,z_m)'$ denote the vector of item prices.

A competitive equilibrium is characterized by a nonnegative vector of activity levels, $\hat{X}$, and a nonnegative vector of prices, $\hat{Z}$, such that:

2.1. <u>The production plan is feasible, i.e.</u>

$$\hat{W} = B - A\hat{X} \geq 0 \quad .$$

2.2. <u>No activity makes a positive profit, i.e.</u>

$$\hat{U} = -C + A'\hat{Z} \geq 0 \quad .$$

2.3. <u>An item in excess supply has a zero price, i.e.</u>

$$\hat{Z}'\hat{W} = 0 \quad .$$

2.4. <u>No activity that makes a negative profit is operated at a positive level, i.e.</u>

$$\hat{X}'\,\hat{U} = 0 \quad .$$

Hence the problem of computing a competitive equilibrium is equivalent to solving the following linear complementarity problem.

Find vectors $(\hat{X}, \hat{Z}, \hat{U}, \hat{W})$ that satisfy:

2.5.

X	Z	U	W	=
O	-A'	I_n	O	-C
A	O	O	I_m	B

,

2.6. $X'U + Z'W = 0$,

2.7. $X, U \geq 0, Z, W \geq 0$,

where I_n and I_m are identity matrices of order n x n and m x m respectively.

As is well known (see Simonnard [2]), relations (2.5.)-(2.7.) are equivalent to solving the linear programming problem:

$$\begin{array}{ll} \text{Maximize} & C'X \ , \\ \text{subject to} & A X \leq B \ , \\ & \quad X \geq 0 \ . \end{array}$$

3. An Equilibrium Problem with Institutional Constraints on Market Prices

In the economy described in the preceding section, there were no institutional constraints upon prices. In this section we shall introduce two such types of constraints:

1) Lower bounds upon individual prices, and
2) Upper bounds upon individual prices.

The presence of such constraints implies, for example, that the factors of production are not necessarily paid according to their marginal productivity. In the case of a minimum price, marginal productivity will coincide with the market price only if the marginal productivity of that item exceeds or equals the minimum price. The reverse holds for an upper bound upon a price. In the subsequent discussion, we shall use marginal productivity and shadow price interchangeably. Figures 1 and 2 illustrate the relationship between market and shadow price in the two cases.

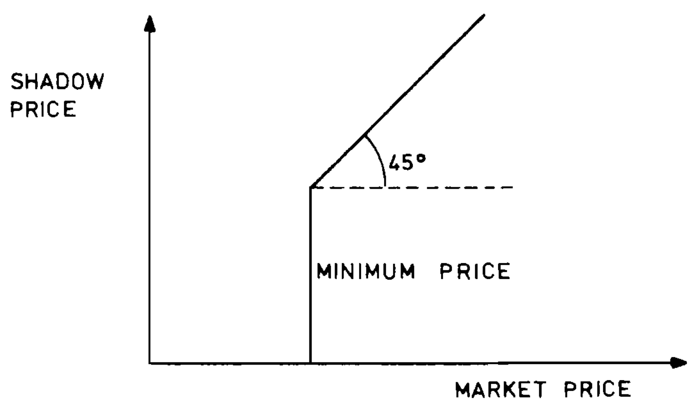

FIGURE 1.

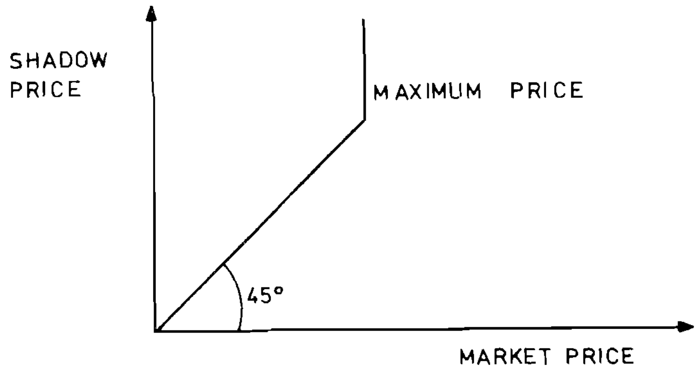

FIGURE 2.

In section 5, more general institutional constraints on prices will be discussed.

Suppose that there is a divergence between market prices and shadow prices. Which of these prices guide the actions of economic agents? In this paper, we assume that one sector of the economy (which may be interpreted as the private sector) is guided by market prices. The other (the public sector) is guided by shadow prices.

Suppose that the first n_1 activities refer to the private sector, whereas the remaining n_2 activities are publicly controlled. Partition X and C so that:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} , \qquad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} .$$

Then X_1 and C_1 refer to private activities whereas X_2 and C_2 refer to public activities. Let us next suppose that there is a maximum price constraint on the first m_1 items, whereas the remaining m_2 items have a minimum price constraint. The latter also include those items where the only price constraint is nonnegativity. Partition the matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is $m_1 \times n_1$, A_{12} is $m_1 \times n_2$, A_{21} is $m_2 \times n_1$, and A_{22} is $m_2 \times n_2$.

The vector $Z = (z_1,\ldots,z_m)'$ will refer to shadow prices, whereas the vector $Y = (y_1,\ldots,y_m)'$ will refer to market prices. Partition Z and Y

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

such that Z_1 and Y_1 refer to factors with a maximum price constraint and Z_2 and Y_2 refer to items with a minimum price constraint. Let the known nonnegative vectors Q_1 and Q_2 denote maximum and minimum prices respectively, i.e.

and
$$\begin{aligned} Y_1 &\leq Q_1 \\ Y_2 &\geq Q_2 \end{aligned} .$$

A price-constrained equilibrium, if it exists, is characterized by a vector of activity levels, $\hat{X}$, market prices, $\hat{Y}$, and shadow prices $\hat{Z}$ such that:

3.1. The production plan is feasible, i.e.

$$\hat{W} = B - A\hat{X} \geq 0 \quad .$$

3.2. No activity makes a positive profit, i.e.

3.2.1. $\hat{U}_1 = -C_1 + [A'_{11} \ A'_{21}] \ \hat{Y} \geq 0 \quad ,$

3.2.2. $\hat{U}_2 = -C_2 + [A'_{12} \ A'_{22}] \ \hat{Z} \geq 0 \quad .$

3.3. The market prices satisfy the institutional constraints, i.e.

$$\hat{V}_1 = Q_1 - \hat{Y}_1 \geq 0 \quad ,$$

$$\hat{V}_2 = \hat{Y}_2 - Q_2 \geq 0 \quad .$$

3.4. There is a nonnegative wedge between market and shadow prices, i.e.

3.4.1. $\hat{T}_1 = \hat{Z}_1 - \hat{Y}_1 \geq 0 \quad ,$

3.4.2. $\hat{T}_2 = \hat{Y}_2 - \hat{Z}_2 \geq 0 \quad .$

3.5. <u>If a factor is in excess supply, it has a zero shadow price, i.e.</u>

$$\hat{Z}'\hat{W} = 0 \quad .$$

3.6. <u>If an activity makes a negative profit, it is operated at a zero level, i.e.</u>

$$\hat{X}'\hat{U} = 0 \quad .$$

3.7. <u>The shadow price equals the market price if the institutional constraint is not binding, i.e.</u>

$$\hat{T}'\hat{V} = 0 \quad .$$

Let us substitute from (3.3.) into (3.2.1.) and (3.4.). We then get:

$$\hat{U}_1 = -C_1 + A_{11}'Q_1 + A_{21}'Q_2 - A_{11}'\hat{V}_1 + A_{21}'\hat{V}_2 \quad ,$$

$$\hat{T}_1 = \hat{Z}_1 + \hat{V}_1 - Q_1 \quad ,$$

$$\hat{T}_2 = \hat{V}_2 - \hat{Z}_2 + Q_2 \quad .$$

Hence the problem of computing an equilibrium is equivalent to solving the following linear complementarity problem.

Find vectors $(\hat{X}, \hat{Z}, \hat{V}, \hat{U}, \hat{W}, \hat{T})$ that satisfy:[3]

3.8.

X_1	X_2	Z_1	Z_2	V_1	V_2	U_1	U_2	W_1	W_2	T_1	T_2	=
0	0	0	0	A'_{11}	$-A'_{21}$	I_{n_1}	0	0	0	0	0	$-C_1 + A'_{11} Q_1 + A'_{21} Q_2$
0	0	$-A'_{12}$	$-A'_{22}$	0	0	0	I_{n_2}	0	0	0	0	$-C_2$
A_{11}	A_{12}	0	0	0	0	0	0	I_{m_1}	0	0	0	B_1
A_{21}	A_{22}	0	0	0	0	0	0	0	I_{m_2}	0	0	B_2
0	0	$-I_{m_1}$	0	$-I_{m_1}$	0	0	0	0	0	I_{m_1}	0	$-Q_1$
0	0	0	I_{m_2}	0	$-I_{m_2}$	0	0	0	0	0	I_{m_2}	Q_2

,

3.9. $\quad X'U + Z'W + V'T = 0 \quad ,$

3.10. $\quad X,\ U \geq 0, \quad Z,\ W \geq 0, \quad V, T \geq 0 \quad ,$

where I_s $(s=n_1,n_2,m_1,m_2)$ is an identity matrix of order $s \times s$.

The above problem [(3.8.) - (3. 10.)] does not correspond to a linear program for here there are institutional constraints upon prices. Hence, an equilibrium solution may not be ob-

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[3]The nonnegativity constraints $\hat{Y}_1 = Q_1 - \hat{V}_1 \geq 0$ and $\hat{Y}_2 = Q_2 + \hat{V}_2$ are dropped because they will be satisfied by any solution to (3.8.) - (3.10.).

tained by solving this as a linear programming problem. Moreover, there is no a priori reason why a solution (3.8.) - (3.10.) need be unique or even exist at all. This makes it worthwhile to devise an algorithm that generates multiple equilibrium solutions.

Hansen and Mathiesen [1] have developed and tested an algorithm that generates multiple complementary solutions to the problem:

3.11. $(M \; I_\ell) \begin{bmatrix} S \\ R \end{bmatrix} = D$,

3.12. $S' R = 0$,

3.13. $S, R \geq 0$,

where M is a matrix of order $\ell \times \ell$, I_ℓ is an identity matrix of order $\ell \times \ell$ and S, R and D are column vectors with ℓ components. No specific assumptions are made on the matrix M. The algorithm is not guaranteed to generate <u>all</u> complementary solutions to (3.11.) - (3.13.). In fifty-six small scale experiments, however, the algorithm generated all complementary solutions in all but one experiment. This algorithm may be applied to solve (3.8.) - (3.10.), each complmentary solution representing a different equilibrium.

It may be instructive to give an interpretation of the character of the equilibrium solution. Suppose we decompose the resource availability vector B so that

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \hat{X}_1$$

is made available to the private sector, and the remaining

$$B - \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \hat{X}_1$$

is made available to the public sector. Suppose that each sector maximizes the value of its output, i.e. we solve the linear programming problems:

3.14. Maximize $C_1' X_1$,
subject to

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} X_1 \leq \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \hat{X}_1 \quad ,$$

$$X_1 \geq 0 \quad .$$

3.15. Maximize $C_2' X_2$
subject to

$$\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} X_2 \leq B - \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \hat{X}_1 \quad , \qquad X_2 \geq 0 \quad .$$

$\hat{X}_1$ with associated price vector $\hat{Y}$, will then be a solution to (3.14.) whereas $\hat{X}_2$ with associated price vector $\hat{Z}$, will be a solution to (3.15.).

4. A Numerical Example

The following numerical example may be illustrative of the preceding discussion. We have an economy with four activities and three resources. Moreover,

$n_1 = 2$ (number of private activities),
$n_2 = 2$ (number of public activities),
$m_1 = 1$ (number of items with a maximum price, e.g. a ceiling on capital charges),
$m_2 = 2$ (number of items with a minimum price, e.g. a floor on wages or a "reservation wage").

$$C = \begin{bmatrix} C_1 \\ \hline C_2 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \\ \hline 1.6 \\ 2.6 \end{bmatrix},$$

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 2. & 1. & 2. & 1. \\ \hline .2 & .5 & .2 & .5 \\ 1. & 1. & 0. & 0. \end{array}\right],$$

$$B = \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} = \begin{bmatrix} 30 \\ \hline 20 \\ 10 \end{bmatrix},$$

$$Q_1 = .1 \quad \text{(maximum price)}$$
$$Q_2 = \begin{bmatrix} 2. \\ 0. \end{bmatrix} \quad \text{(minimum price)} \quad .$$

We have

$$-C_1 + A'_{11}Q_1 + A'_{21}Q_2 = -\begin{bmatrix} 1.2 \\ 1.6 \end{bmatrix} + \begin{bmatrix} 2. \\ 1. \end{bmatrix} (.1) + \begin{bmatrix} .2 & 1. \\ .5 & 1. \end{bmatrix} \begin{bmatrix} 2. \\ 0. \end{bmatrix} = \begin{bmatrix} - & .6 \\ - & .5 \end{bmatrix} .$$

Then an equilibrium solution is obtained by solving the following linear complementarity problem: Find vectors $(\hat{X}, \hat{Z}, \hat{V}, \hat{U}, \hat{W}, \hat{T})$ that satisfy:

4.1.

x_1	x_2	x_3	x_4	z_1	z_2	z_3	v_1	v_2	v_3	u_1	u_2	u_3	u_4	w_1	w_2	w_3	t_1	t_2	t_3	=
0	0	0	0	0	0	0	2.	-.2	-1.	1	0	0	0	0	0	0	0	0	0	-.6
0	0	0	0	0	0	0	1.	-.5	-1.	0	1	0	0	0	0	0	0	0	0	-.5
0	0	0	0	-2.	-.2	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1.6
0	0	0	0	-1.	-.5	0	0	0	0	0	0	0	1	0	0	0	0	0	0	-2.6
2.	1.	2.	1.	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	30
.2	.5	.2	.5	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	20
1.	1.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	10
0	0	0	0	-1	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	0	-.1
0	0	0	0	0	1	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	2
0	0	0	0	0	0	1	0	0	-1	0	0	0	0	0	0	0	0	0	1	0

,

4.2. $X'U + Z'W + V'T = 0$,

4.3. $X, U \geq 0, \quad Z, W \geq 0, \quad V, T \geq 0$.

In (4.1.) - (4.3.) have the unique solution

$$\hat{X} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 10 \end{bmatrix}, \qquad \hat{U} = \begin{bmatrix} 0 \\ 0.1 \\ 3.6 \\ 0 \end{bmatrix},$$

$$\hat{Z} = \begin{bmatrix} 2.6 \\ 0 \\ .6 \end{bmatrix}, \qquad \hat{W} = \begin{bmatrix} 0 \\ 13 \\ 0 \end{bmatrix},$$

$$\hat{V} = \begin{bmatrix} 0 \\ 0 \\ 0.6 \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} 2.5 \\ 2.0 \\ 0 \end{bmatrix}.$$

Moreover

$$\hat{Y} = \begin{bmatrix} .1 \\ 2. \\ .6 \end{bmatrix} \quad \text{and } C'\hat{X} = 38.$$

If there were no institutional constraints upon prices, equilibrium activity levels and prices would be given by

$$\bar{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 30 \end{bmatrix}, \qquad \bar{Z} = \begin{bmatrix} 2.6 \\ 0 \\ 0 \end{bmatrix}.$$

Moreover we would have

$$\bar{W} = \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} \quad \text{and } C'\bar{X} = 78.$$

For illustrative purposes suppose that the first, second and third resource are capital, labor and land respectively.

We may then conclude that the imposition of a ceiling on capital charges and a floor on wages have led to a considerable reduction in the value of the economy's output as well as a considerable increase in unemployment. Moreover land has become a scarce resource. The example illustrates how models of this kind may be used to study the consequence of institutional constraints on market prices.

5. General Institutional Constraints on Market Prices

Suppose that in addition to maximum and minimum constraints, there are p linear constraints on market prices given by

5.1. $$FY \leq G \quad .$$

An example will illustrate the character of these constraints. Suppose factor 1 is skilled labor and factor 2 is unskilled labor, and the two first constraints are:

$$y_1 - 1.2y_2 \leq 0 \quad ,$$

$$-y_1 + y_2 \leq 0 \quad .$$

These two constraints together imply that the wages of skilled labor shall be at least as great as the wages of unskilled labor, but shall not exceed the wages of unskilled labor by more than 20%. Constraints of this type may arise for several reasons. They may, for example, arise from the need for fairness as perceived within trade unions.

Let us partition the matrix F such that the first m_1 columns refer to items with a maximum price constraint and the remaining m_2 to those with a minimum price constraint, i.e.

$$F = (F_1 \; F_2) \quad .$$

We then have:

$$5.2. \qquad H = G - F_1Y_1 - F_2Y_2 \geq 0 \quad .$$

An equilibrium, if it exists, is therefore characterized by a vector of activity levels $\hat{X}$, market prices $\hat{Y}$, and shadow prices $\hat{Z}$, such that (3.1.) - (3.7.) and (5.2.) are satisfied.

Substitute from (3.3.) into (5.2.).

$$5.3. \qquad H = G - F_1Q_1 - F_2Q_2 + F_1V_1 - F_2V_2 \geq 0 \quad .$$

In order to simplify notation, let us rewrite (3.8.) in the following compact form:

3.8.

X	Z	V_1	V_2	U	W	T_1	T_2	$=$
				I_n	0	0	0	D_1
M_1	M_2	M_3	M_4	0	I_m	0	0	D_2
				0	0	I_{m_1}	0	D_3
				0	0	0	I_{m_2}	D_4

.

Let us introduce a vector of p dummy variables, $E = (e_1, \ldots, e_p)'$. The problem of computing an equilibrium is then equivalent to solving the following linear complementarity problem.

Find vectors $(\hat{X}, \hat{Z}, \hat{V}, \hat{E}, \hat{U}, \hat{W}, \hat{T}, \hat{H})$ that satisfy:

5.4.

X	Z	V_1	V_2	E	U	W	T_1	T_2	H	$=$
					I_n	0	0	0	0	D_1
					0	I_m	0	0	0	D_2
M_1	M_2	M_3	M_4	0	0	0	I_{m_1}	0	0	D_3
					0	0	0	I_{m_2}	0	D_4
0	0	$-F_1$	F_2	0	0	0	0	0	I_p	$G-F_1Q_1-F_2Q_2$

,

5.5. $X'U + Z'W + V'T + E'H = O$,

5.6. $X,U \geq O, \quad Z,W \geq O, \quad U,T \geq O, \quad E,H \geq O$.

The interpretation of equilibrium given in section three also applies to the equilibrium solution represented by (5.4.) - (5.6.). Again the Hansen-Mathiesen algorithm may be applied to obtain numerical solutions to this problem.

References

[1] Hansen, Terje, and Mathiesen, Lars. "Generating Stationary Points for a Non-Concave Quadratic Program by Lemke's almost Complementary Pivot Algorithm," Discussion Papers 11/73, Norwegian School of Economics and Business Administration, 5000 Bergen, Norway.

[2] Simonnard, M. Linear Programming. Englewood Cliffs, N. J., Prentice-Hall, 1966.