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David E. Bell

1. Outline

Consider the linear program

$$\max_{x \in X} cx \qquad (1)$$

where $X = \{x | Ax = b,\ x \geq 0\}$ and where the objective function represents the preferences of a decision maker over the set of alternatives. We assume that some information I has been obtained about his preferences which is insufficient to define the objective coefficients exactly. Hence, there is a set C(I) of possible vectors each of which is consistent with the known information. One element of C(I) is "correct", but we, the optimizers, do not know which.

It is evident that if for some $\hat{x} \in X$, $\hat{x}$ is optimal in (1) for all $c \in C(I)$, then it is unnecessary to know which element of C(I) is correct since they all give the same answer. The aim here is to obtain the minimum amount of information required from a decision maker in order to find his best alternative in a given problem.

2. Obtaining Correct Information

A criticism of modern decision analysis is that often a decision maker is required to give sophisticated answers in matters he may not understand. Few people have a good feeling for probability (how "likely" is one chance in fifty?), and much of the income of analysts arises precisely because people are not generally good at coming to precise conclusions.

Consider the following three questions dealing with a four attribute problem:

a) Which state do you prefer, (0,2,9,3) or (5,1,6,1)?

b) If the value of (0,0,0,0) is considered to be zero and that of (1,1,1,1) to be one, what would you consider to be the value of (1,2,3,4)?

c) For what value of β does (2,3,4,5) become indifferent to (β,0,0,6)?

Of these I would consider b) to require the most judgement on the part of the decision maker and a) the least. Question c) requires repeated applications of a) until β converges to some answer. Note that the "answer space" for b) and c) is infinite but for a) it is finite (of size two).

Of course the answer to question b) will do most to restrict the size of the set C(I) and a) the least. If the decision maker can make sophisticated judgements, all well and good. In what follows he will only be required to give a preference ordering

amongst two states, that is, answer question a). It will be assumed that when given a choice between two states, he will either

(i) prefer one to the other,

(ii) be indifferent,

(iii) not be sure,

but, for the time being, we will exclude (iii).

3. The Approach

We will start by assuming an initial quantity of information I is already known. For example, by making pairwise comparisons, the decision maker could order a list of states from best to worst. Preferences $x > y$, $x \sim y$ are imposed as restrictions on the coefficients of c by imposing the constraints $cx > cy$ and $cx = cy$ as appropriate.

Lemma 1 C(I) is a cone.

Proof Substitution for c by λc for any $\lambda > 0$ does not affect the validity of ordering relations. //

It will be useful later to have C(I) as a bounded set, which may be achieved by constraining

$$\Sigma c_i^2 = 1$$

but for the linear programming example with which most of this paper deals, it is better to have a linear constraint. By pairwise

comparisons, it is possible to determine for each i whether $c_i \geq 0$ or $c_i < 0$ (is an attribute good or bad?) and let $\delta_i = +1$ if $c_i \geq 0$, $\delta_i = -1$ if $c_i < 0$. The constraints

$$\sum_{i=1}^{n} \delta_i c_i = 1 \quad , \qquad \delta_i c_i \geq 0$$

are thus linear and make C(I) bounded. We will assume that $\delta_i = +1$ for all i since re-definition of the variables can make this so.

Now consider the space of solutions X. Let S be the set of possible solutions to problem (1) for all possible c's. In this case we may take S to be the set of extreme points of X. Now define the map

$$\phi : C(I) \to S$$

where $\phi(c)$ is the optimal basic solution to

$$\max_{x \in X} cx$$

assumed to be unique.

Lemma 2 The function $z(c) = c\phi(c)$ is convex.

Proof $(\frac{1}{2}c_1 + \frac{1}{2}c_2)\ \phi\ (\frac{1}{2}c_1 + \frac{1}{2}c_2) = \max_{x\in X} (\frac{1}{2}c_1 + \frac{1}{2}c_2)x$

$\leq \frac{1}{2} \max_{x\in X} c_1x + \frac{1}{2} \max_{x\in X} c_2x$

$= \frac{1}{2}c_1\ \phi\ (c_1) + \frac{1}{2}c_2\ \phi\ (c_2)$. //

There is only a problem if

$$|\phi(C(I))| > 1 \ ,$$

in which case, alternative optima exist. We seek a "minimal" set of information J for which

$$|\phi(C(I\cup J))| = 1 \ .$$

The set J will be regarded as minimal if it requires the fewest subsequent pairwise comparisons to be made by the decision maker.

4. Details

The line of attack taken here will be to find any two elements of $\phi(C(I))$, if two exist, and ask the decision maker to compare them.

Lemma 3 If $x^1, x^2 \in \phi(C(I))$ and the decision maker subsequently decides that $x^1 > x^2$, the constraint $cx^1 > cx^2$ is added to I, then $x^2 \notin \phi(C(I))$.

Proof If $x^2 \in \phi(C(I))$, then there exists some $\bar{c} \in C(I)$ such that

$$\bar{c}x^2 \geq \bar{c}x \quad \text{for all } x \in X$$

in particular $\bar{c}x^2 \geq \bar{c}x^1$.

This contradicts the new constraint. //

The point of lemma 3 is that the new cut removes all those $c \in C(I)$ which gave x^2 as the optimum. This cut may well remove many other elements of C(I) as well as $\phi^{-1}(x^2)$. This paper will not throw any light on the question of which two elements of $\phi(C(I))$ it is "best" to present to the decision maker. Remember that beforehand it is not known whether he will answer $x^1 > x^2$, $x^2 > x^1$, or $x^1 \sim x^2$. The problem that is addressed here is that of finding <u>any</u> two different elements of $\phi(C(I))$.

Finding one is simple: choose any $c \in C(I)$ (phase 1 L.P. procedure perhaps), then solve

$$\max_{x \in X} cx$$

to find $\phi(c)$. The question now is, how to find another.

One sure method, which it seems will have to be used when testing for optimality, is to take the nonbasic price coefficients

$$c_N - c_B B^{-1} N$$

which are non-positive on the set $\phi^{-1}(\phi(c))$ and test to see if any may be made positive on C(I). That is, solve up to n - m of the L.P. problems

$$\max_{c \in C(I)} c_{N_i} - c_B B^{-1} N_i \tag{2}$$

If any of these have positive objective values, then that solution $c^* \notin \phi^{-1}(\phi(c))$ and the solution of

$$\max_{x \in X} c^* x$$

will produce a second element of $\phi(C(I))$. If none of (2) produce positive solutions, then $\phi(c)$ is the required optimal solution.

Whilst solving (n - m) versions of (2) may not be impossible, faster methods may be available, though not assured of success.

Consider the following two problems, where $z(c) = c\phi(c)$

$$z(c^*) = \max_{c \in C(I)} z(c) \tag{3}$$

$$z(c_*) = \min_{c \in C(I)} z(c) \tag{4}$$

The idea is that these two solutions have intuitively a good chance of satisfying $\phi(c^*) \neq \phi(c_*)$.

Lemma 4 Two elements of $\phi(C(I))$ may be found after solving three linear programs or it is shown that $\phi(c_*) = \phi(c^*)$ after solving only two linear programs.

Proof Assume, for the moment, that $C(I)$ may be defined by

$$Dc \geq 0$$
$$\Sigma c_i = 1$$
$$c_i \geq 0 \quad .$$

The only subterfuge here is that some of the strict inequalities have been replaced by non-strict inequalities. This point will be revived later.

Now $$z(c) = \max_{x \in X} cx \quad .$$

or taking the dual,

$$z(c) = \min \ \pi b$$
$$\text{s.t. } \pi A \geq c$$

so that
$$z(c_*) = \min_{c \in C(I)} z(c) = \min \ \pi b$$
$$\text{s.t. } \pi A \geq c$$
$$Dc \geq 0 \qquad (5)$$
$$\Sigma c_i = 1$$
$$c \geq 0 \quad .$$

Hence (4) may be solved by one linear program.

Now consider the solution, c^*, of

$$\max_{c \in C(I)} \; c \; \phi(c_*) \quad .$$

If $\phi(c_*) = \phi(c^*)$, then this will be clear since

$$c^*_N - c^*_B B^{-1} N \tag{6}$$

will be non-positive. If (6) has some positive elements, then

$$\max_{x \in X} \; c^* x$$

will provide $\phi(c^*)$. //

Even if $\phi(c_*) = \phi(c^*)$, there remains another chance.

Lemma 5 If $c(\lambda) = c^* + \lambda(c_* - c^*)$ is feasible for some $\lambda > 1$, then

$$\phi(c_*) = \phi(c^*) \neq \phi(c(\lambda)) \quad .$$

Proof By definition of c_*,

$$z(c(\lambda)) \geq z(c_*) \quad ,$$

but $(c^* + \lambda(c_* - c^*))\phi(c) < c_*\phi(c_*)$ for $\lambda > 1$

and hence $\phi(c_*) \neq \phi(c(\lambda))$. //

To summarize, the only case which fails is if for some $\bar{x} \in X$

$$c^*\bar{x} \geq c\phi(c) \geq c_*\bar{x} \quad \text{for all } c \in C(I) \ ,$$

and c_* is extreme in $C(I)$. Of course, there are other arbitrary ways of finding a $c \in C(I)$ to generate a $\phi(c)$. Computational experience may suggest a better approach.

5. Variations

This section just mentions the problems of converting constraints $cx > cy$ into $cx \geq cy$ and and answers such as "I'm not sure".

By introducing a small coefficient ε, $cx > cy$ may be written as $cx \geq cy + \varepsilon$ and "I'm not sure" may be expressed as

$$cx + \varepsilon \geq cy \geq cx - \varepsilon$$

where in this case ε is large enough to be accurate. An interesting choice of $c \in C(I)$ might be

$$\begin{aligned} \max \quad & \epsilon \\ \text{s.t.} \quad & Dc \geq \epsilon \underline{1} \\ & \Sigma c_i = 1 \\ & c \geq 0 \quad . \end{aligned}$$

which maximizes the minimum "gap" between states.

6. Generalizations

This general approach of not tying down the objective function is only suitable for finding exact solutions if the solution space S is finite.

The general problem

$$\begin{aligned} \max \quad & u(x_1, \cdots, x_n) \\ & g(x) \leq 0 \end{aligned}$$

is different in that respect. However, this technique is still useful when the solution space is {Reject, Accept} for some proposal. If the set of possible u's is U(I), then when

$$\min_{u \in U(I)} u(\text{Proposal}) > 0$$

the proposal may be accepted, or when

$$\max_{u \in U(I)} u(\text{Proposal}) < 0$$

it may be rejected.

7. Summary

This paper has considered a situation in which

1) The decision maker can make preference orderings but not accurate value judgements,

2) Computer time is less valuable than the decision makers time.

Variations exist for all other combinations of circumstances; the object here was merely to give an example.