International Institute for Applied Systems Analysis
IIASA
www.iiasa.ac.at

A Group Utility Function by Optimal Allocation

Bell, D.E.

IIASA Working Paper

WP-74-055

1974

A GROUP UTILITY FUNCTION BY OPTIMAL ALLOCATION

David E. Bell

October 1974 WP-74-55

Consider an imaginary research institute where the director has to make decisions under uncertainty concerning the institute's research funds which when received, must be allocated by him between several projects. The project leaders have supplied him with their utility functions $u_1, u_2, \ldots u_n$ for their own project funding, that is, project leader k uses utility function u_k in matters concerning funding for his own project.

Now the director decides that, given an amount of research funding x, he will allocate an amount x_k to project k so as to maximize

$$u_1(x_1) + u_2(x_2) + \ldots + u_n(x_n)$$

subject to the constraint $x_1 + \ldots + x_n = x$.

The question considered here, posed to me by Ralph L. Keeney, is, what is the implied utility function of the director?.

Note that what follows applies also to the case where the director chooses to maximize

$$k_1 u_1(x_1) + \ldots \quad k_n u_n(x_n)$$

for some positive weights k_i since utility function u_i may be rescaled without loss to $k_i u_i$.

Let

$$u(x) = \text{Max } [u_1(x_1) + \ldots + u_n(x_n)] \quad .$$
$$\text{s.t.} \quad x_1 + \ldots + x_n = x$$

<u>Result 1</u> It is sufficient to consider the case $n = 2$ since a general case may be obtained by repeated application of the $n = 2$ case.

<u>Proof</u>.

We require

$$u(x) = \max_{x_1+x_2+x_3 = x} \left[u_1(x_1) + u_2(x_2) + u_3(x_3)\right]$$

but let us consider first the function

$$v(y) = \max_{y_2+y_3 = y} \left[u_2(y_2) + u_2(y_3)\right] \quad .$$

It will be sufficient for the result to show that

$$u(x) = \max_{x_1+y = x} \left[u_1(x_1) + v(y)\right] \quad .$$

But

$$\max_{x_1+y = x} \left[u_1(x_1) + \max_{y_2+y_3 = y} (u_2(y_2) + u_3(y_3))\right] \quad .$$

$$= \max_{x_1+y_2+y_3 = x} \left[u_1(x_1) + u_2(y_2) + u_3(y_3)\right] = u(x) \quad . ||$$

Hence until further notice this paper will consider the problem

$$u(x) = \max_{y} \; u_1(y) + u_2(x - y) \quad . \qquad (1)$$

For clarification of notation

$$f'(x) = \frac{d}{dt} f(t) \Big|_{t=x}$$

so, for example,

$$\frac{d}{dx}u(y-x) = u'(y-x)(\frac{dy}{dx}-1)$$

where

$$u'(y-x) = \frac{du(t)}{dt}\bigg|_{t=y-x}$$

<u>Result 2</u> An optimal y exists and is unique for (1) if u_1 and u_2 are strictly concave.

<u>Proof</u> First note that $u_1(y) + u_2(x - y)$ is strictly concave in y since u_1 and u_2 are, and the sum of two concave functions is concave.

Hence a maximum is attained and is unique. ||

Define y(x) by the relation

$u(x) = u_1(y(x)) + u_2(x - y(x))$

then result 2 shows that y(x) is well defined on x.

<u>Result 3</u> If u_1 and u_2 are twice differentiable and strictly concave then

$$u'(x) = u_1'(y(x)) = u_2'(x - y(x)) \quad .$$

<u>Proof</u>

Let

$$\theta(y) = u_1(y) + u_2(x - y)$$

then

$$\frac{d\theta}{dy} = u_1'(y) - u_2'(x - y) \quad .$$

Hence

$$\frac{d\theta}{dy} = 0 \quad \text{where } u_1'(y) = u_2'(x - y) \quad . \qquad (2)$$

This is a maximum since $\Theta(y)$ is strictly concave in y. Equation (2) will always be satisfied if it is assumed that the ranges of u_1' and u_2' are equal. For most utility functions this range is $(-\infty, 0)$, and thus (2) will hold.

Using a Taylor expansion

$$u(x) + \delta x\, u'(x) = u_1(y) + \delta y\, u_1'(y) + u_2(x-y) + (\delta x - \delta y)\, u_2'(x-y). \tag{3}$$

and (1) and (2) give that

$$\delta x u'(x) = \delta y(u_1'(y)) + (\delta x - \delta y)\, u_2'(x-y) = \delta x u_1'(y) \quad . \qquad ||$$

<u>Result 4</u> If the total funding x is increased then each group receives an increased allocation.

<u>Proof</u> Using a Taylor Expansion for the optimal $y = y(x)$,

$$u(x + \delta x) = u_1(y + \delta y) + u_2(x + y + \delta x - \delta y)$$

so that

$$u(x + \delta x) = u_1(y) + \delta y\, u_1'(y) + \frac{\delta y^2}{2}\, u_1''(y)$$

$$+ u_2(x-y) + (\delta x - \delta y)\, u_2'(x-y) + \frac{(\delta x - \delta y)^2}{2}\, u_2''(x-y)$$

using (1) and (2) gives

$$u(x+\delta x) = u(x) + \delta x\, u'(x) + \frac{\delta y^2}{2}\, u_1''(y) + \frac{(\delta x - \delta y)^2}{2}\, u_2''(x-y). \tag{4}$$

Maximizing (4) with respect to δy implies that

$$\delta y\, u_1''(y) - (\delta x - \delta y)\, u_2''(x - y) = 0$$

that is

$$\frac{\delta y}{\delta x} = \frac{u_2''(x-y)}{u_1''(y) + u_2''(x-y)} > 0$$

since u_1 and u_2 are strictly concave.

Also

$$1 - \frac{\delta y}{\delta x} = \frac{u_1''(y)}{u_1''(y) + u_2''(x-y)} > 0$$

also so that each group receives a strict increase in allocation. ||

<u>Theorem 1</u> Assuming

(i) u_1 and u_2 are strictly increasing, strictly concave with continuous second derivatives

(ii) $u_i'(x) \to 0$ as $x \to \infty$, $\to -\infty$ as $x \to -\infty$, $i = 1,2$,

then

$$u(x) = (u_1 + u_2\, u_2'^{-1}\, u_1')\,[(1 + u_2'^{-1}\, u_1')^{-1}(x)] \quad .$$

<u>Proof</u> We have from (2) that

$$u_1'(y) = u_2'(x-y)$$

and since u_2' is strictly decreasing and continuous it has an inverse $u_2'^{-1}$ which is valid over the range of u_1', so that

$$x - y = u_2'^{-1}\, u_1'(y) \tag{5}$$

or

$$x = y + u_2'^{-1}\, u_1'(y) = (1 + u_2'^{-1}\, u_1')(y) \quad .$$

Since $\frac{dy}{dx} > 0$ and $u_2'^{-1}$ is continuous we have that $1 + u_2'^{-1} u_1'$ is strictly increasing and continuous so that it too has an inverse, hence

$$y = (1 + u_2'^{-1} u_1')^{-1} (x) \quad . \qquad (6)$$

Since

$$u(x) = u_1(y) + u_2(x - y)$$

from (5) and (6) we have that

$$u(x) = (u_1 + u_2 u_2'^{-1} u_1')(y)$$

$$= (u_1 + u_2 u_2'^{-1} u_1') [(1 + u_2'^{-1} u_1')^{-1} (x)] \quad . || \qquad (7)$$

Note that the formula is entirely in terms of the known functions u_1 and u_2.

Examples

Take $u_1(x) = \log (x + a) \quad u_2(x) = \log (x + b)$

so the problem is, find $u(x)$ such that

$$u(x) = \max_y [\log (y + a) + \log (x - y + b)]$$

By differentiating, setting to to zero we have

$$\frac{1}{y + a} - \frac{1}{x - y + b} = 0$$

$$\therefore \quad x - y + b = y + a$$

$$\therefore \quad y = \frac{x - a + b}{2}$$

Hence

$$u(x) = \log(\frac{x - a + b}{2} + a) + \log(\frac{x - b + a}{2} + b)$$

$$= 2 \log(\frac{x + a + b}{2})$$

Now to demonstrate that the same result may be had using (7).

$$u_1'(x) = \frac{1}{x + a} \qquad u_2'(x) = \frac{1}{x + b}$$

Thus

$$u_2'^{-1}(x) = \frac{1}{x} - b$$

$$\therefore \quad y(x) = (1 + u_2'^{-1} u_1')(x) = x + \frac{1}{u_1'(x)} - b \quad .$$

$$= 2x + a - b$$

$$\therefore \qquad (1 + u_2'^{-1} u_1')^{-1}(x) = \frac{x - a + b}{2} \quad .$$

Now

$$(u_1 + u_2 u_2'^{-1} u_1')(x) = \log(x + a) + \log(x + a - b + b)$$

Hence

$$(u_1 + u_2 u_2'^{-1} u_1') \, [(1 + u_2'^{-1} u_1')^{-1}(x)]$$

$$= 2 \log(\frac{x - a + b}{2} + a)$$

$$= 2 \log(\frac{x + a + b}{2}) \text{ as required.}$$

Hence $u_1 = \log(x + a)$ $u_2 = \log(x + b)$ yields a utility function of $\log(x + a + b)$ for the director.

As an exercise for the reader, it may be shown either by direct route or via (7) that if

$$u_1(x) = -e^{-cx} \quad \text{and} \quad u_2(x) = -e^{-dx}$$

then

$$u(x) = -e^{-\frac{cd}{c+d}x\left((\frac{c}{d})^{-\frac{c}{c+d}} + (\frac{d}{c})^{-\frac{d}{c+d}}\right)}$$

Hence $u_1(x) = -e^{-cx}$, $u_2(x) = -e^{-dx}$ yields a utility function of $-e^{-\frac{cd}{c+d}x}$ for the director.

In these two standard cases $u(x)$ is the same form as the functions u_1 and u_2.

Indeed if

$$u_1(x) = -me^{-cx} \qquad u_2(x) = -ne^{-dx}$$

then

$$u(x) \sim -e^{-\frac{cd}{c+d}x}$$

that is, <u>the directors weighting of the projects does not affect his decisions</u>.

<u>Theorem 2</u>

If each member of the group has

(i) a utility function $-e^{-c_k x}$ $\quad k = 1,\ldots,n$

(ii) a utility function $\log(x + a_k)$ $\quad k = 1,\ldots,n$

or (iii) a utility function $-(x + b_k)^{-p}$ $\quad p > 0, \quad k = 1,\ldots,n$

then the group utility function u is independent of the weightings given to the individual members utilities.

<u>Proof</u> Since it is merely a matter of solution the proof will be omitted. However the statement of the theorem will be amplified.

(i) If

$$u(x) = \max w_1u_1 + w_2u_2 + \dots + w_nu_n$$

where

$$u_k(x) = -e^{-c_kx}$$

then

$$u(x) = -f(w_1,\dots w_n, c_1,\dots,c_n)\, e^{-cx}$$

where

$$f(\cdot) > 0 \quad \text{and}$$

$$\frac{1}{c} = \frac{1}{c_1} + \dots + \frac{1}{c_n} \quad .$$

(ii) If

$$u_k(x) = \log(x + a_k)$$

then

$$u(x) = g(a_1,\dots,a_n, w_1,\dots,w_k) \log (x + a)$$

where

$$a = a_1 + \dots + a_n \qquad g(\cdot) > 0 \quad .$$

(iii) If

$$u_k(x) = -(x + b_k)^{-p} \text{ for some constant } p > 0$$

then

$$u(x) = -h(w_1,\dots,w_n, p)\,(x + b)^{-p} \quad h(\cdot) > 0 \quad .$$

where $b = b_1 + \dots + b_n$. ||

The importance of this result lies in the observation that the group decision maker need only know the group members' utilities before making decisions on group funds. He need not decide beforehand how he will weight the importance of the group members. In particular the role of group decision maker and weighting decision maker may be divided between two different people (or groups) who need not even communicate with each other.

The interestingly simple expression for the exponential coefficient in Theorem 2 (i) can be generalized.

Theorem 3 If $r(x)$ is the coefficient of risk for the group utility function and $r_k(x)$ that for the individual members then

$$\frac{1}{r(x)} = \frac{1}{r_1(x_1)} + \frac{1}{r_2(x_2)} + \ldots + \frac{1}{r_n(x_n)}$$

where x_k is the optimal allocation to group member k.

Proof Recall that $r(x) = -\frac{u''(x)}{u'(x)}$.

Note that this result is true whatever the forms of the u_k but that this does not imply the proof of a general form of Theorem 2 because the optimal x_k will depend on the weightings.

For simplicity's sake we will prove it for the case $n = 2$.

Now

$$u'(x) = u_1'(y) = u_2'(x - y) \qquad (8)$$

for an optimal y.

Thus, differentiating with respect to x,

$$u''(x) = u_1''(y)\,\frac{dy}{dx}$$

From the proof of Result 4

$$\frac{dy}{dx} = \frac{u_2''(x-y)}{u_1''(y) + u_2''(x-y)}$$

$$\therefore \qquad u''(x) = \frac{u_1''(y)\; u_2''(x-y)}{u_1''(y) + u_2''(x-y)}$$

that is

$$\frac{1}{u''(x)} = \frac{1}{u_1''(y)} + \frac{1}{u_2''(x-y)}$$

Using (8) once more we have that

$$-\frac{u'(x)}{u''(x)} = -\frac{u_1'(y)}{u_1''(y)} - \frac{u_2'(x-y)}{u_2''(x-y)}$$

which gives the result. The extension to a general n is trivial using Result 1. ||

The following is a characterization (perhaps not complete) of functions having the property of Theorem 2.

Theorem 4 If all members of the group have a utility function

$$u_k(x) = v(a_k x + b_k)$$

then u(x) the group utility function, is independent of the weightings assigned to the group members if

$$\text{(i)} \quad \theta v(x) = v(x + f(\theta)) \quad \text{(where } f^{-1} \text{ exists)}$$

or if $$\text{(ii)} \quad v(\theta x) = v(x) + g(\theta) \quad .$$

Proof Note that it is sufficient for (i) to take $b_k = 0$ and for (ii) to take $a_k = 1$.

We will also just prove it for the case $n = 2$.

(i) At optimality

$$a_1 \theta v'(a_1 y) = a_1 v'(a_2(x - y))$$

or using property (i)

$$v'(a_1 y + f(\theta)) = v'(a_2(x - y)) \quad .$$

Since v'^{-1} exists

$$a_1 y + f(\theta) = a_2(x - y)$$

and

$$y = \frac{a_2 x - f(\theta)}{a_1 + a_2}$$

$$\therefore \quad u(x) = \theta v\left(\frac{a_1 a_2 x - a_1 f(\theta)}{a_1 + a_2}\right) + v\left(\frac{a_1 a_2 x + a_2 f(\theta)}{a_1 + a_2}\right)$$

$$= v\left(\frac{a_1 a_2 x}{a_1 + a_2} - \frac{a_1 f(\theta)}{a_1 + a_2} + f(\theta)\right) + v\left(\frac{a_1 a_2 x}{a_1 + a_2} + \frac{a_2 f(\theta)}{a_1 + a_2}\right)$$

$$= 2v\left(\frac{a_1 a_2 x}{a_2 + a_2} + \frac{a_2 f(\theta)}{a_1 + a_2}\right)$$

$$= 2f^{-1}\left(\frac{a_2 f(\theta)}{a_1 + a_2}\right)\left(v \; \frac{a_1 a_2 x}{a_1 + a_2}\right)$$

which is independent of the weighting θ.

(ii) At optimality

$$v'(y + b_1) = v'(b_2 + x - y)$$

Now $\quad \theta v'(x) = v'(\frac{x}{\theta}) \quad$ from (ii)

hence

$$\frac{y + b_1}{\theta} = b_2 + x - y$$

or

$$y = \frac{\theta(b_2 + x) - b_1}{1 + \theta}$$

$$\therefore \quad u(x) = \theta v\left(\frac{\theta(b_1 + b_2 + x)}{1 + \theta}\right) + v\left(\frac{b_1 + b_2 + x}{1 + \theta}\right)$$

$$= (1 + \theta)\; v\; (b_1 + b_2 + x) + g(\frac{\theta}{1+\theta}) + g(\frac{1}{1+\theta})$$

which again is independent of the weightings. ||

Concluding Remarks Results have been presented when a group decision maker allocates resources using the criterion of maximizing total utilities. This criterion has not been justified, although Theorem 2 suggests to me that it may well be reasonable and certainly simple to use.

Additivity is not essential for many of the results. For example if the criterion used is

$$u(x) = \max_{x_1 + \ldots + x_n = x} u_1(x_1) u_2(x_2) \ldots u_n(x_n) \quad .$$

Many of the results are actually simpler because the problem of weightings does not arise. The results of this paper may be applied very straightforwardly to the situation

$$\log u(x) = \max_{x_1 + \ldots + x_n = x} \log u_1(x_1) + \ldots + \log u_n(x_n) \quad .$$