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DECENTRALIZED MANAGEMENT AND OPTIMIZATION OF DEVELOPMENT IN LARGE PRODUCTION ORGANIZATIONS

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1. Introduction

The paper deals with the model of large production organizations which consists of n sectors S_i, $i = 1 \ldots n$, (Fig. 1.a). Each sector produces X_{ii} goods per year and purchases X_{ji}, $j = 1 \ldots n$, goods from sectors S_j. It employs X_{oi} labor per year and receives Z_i funds for capital investment from the central management S^*. There is a decentralized management system being used. Each sector maximizes the net profit by choosing the best X_{ji}, $j = 1 \ldots n$, mix and using the development resources (i.e. the investment and labor) allotted by central management S^*. The objective of S^* is to maximize the long range profit (development) by the best allocation of global available resources (capital and labor). However, S^* does not pay attention to the inter-sector flow of goods, X_{ij}, $i, j = 1 \ldots n$.

The organization structure of each sector S_i (see Fig. 1.b) is similar to S. It consists of S_{ik} subsectors, $k = 1 \ldots n_i$, which exchange the final subsector goods $X_{ik\ell}$, $k, \ell = 1 \ldots n_i$ and maximize the net subsectors profits.

The subsector management centers S_i^*, $i = 1 \ldots n$, allocate in an optimum manner the resources (allotted by S^*) among the S_{ik}, $k = 1 \ldots n_i$, subsectors. The sum of goods leaving sector S_{ik}, $k = 1 \ldots n_i$ and directed to S_j is labeled X_{ij}, $j = 1 \ldots n$. In the same manner the sum of goods received by X_{ik}, $k = 1 \ldots n_i$, from S_j is labeled X_{ji}, $j = 1 \ldots n$.

Each subsector S_{ik}, $k = 1 \ldots n_i$, can be represented in the expanded form of subsectors of lower order etc.

Dealing with such a complex, hierarchically organized structure, it is convenient to decompose it in such a form that the intersector flows

X_{ij}, $X_{i\ell k}$, do not interfere with the S^*, S_i^* allocation strategies. That constitutes the first task undertaken in the paper. Then the problem of best allocation of development resources is investigated.

The third part of the paper consists of investigation of prices on the long-range development strategy.

As a concrete example the Cobb-Douglas production functions have been used to describe the sector input-output relations.

Using the present model it is possible to avoid the gap which exists between the micro-production and macro-economic models.

2. Decomposition

Consider the system shown in Fig. 1.a, which will be called the normative n-sector production model. Let the input-output production functions of S_i, i = 1 ... n, be given in the form:

$$X_{ii} = F_i^{q_i} \prod_{\substack{j=0 \\ j\neq i}}^{n} X_{ji}^{\alpha_{ji}}, \quad i = 1 \dots n \tag{1}$$

where F_i, α_{ji} - given positive numbers, $0 \leq \alpha_{ji} < 1$,

$$q_i = 1 - \sum_{\substack{j=0 \\ j\neq i}}^{n} \alpha_{ji} > 0, \quad i = 1 \dots n, \quad j = 0, 1 \dots n$$

$$X_{oi} = \text{employment at } S_i \ .$$

Assume the prices p_i of goods X_{ii} be given, i = 1 ... n, so that $Y_{ii} = p_i X_{ii}$, $Y_{ji} = p_j X_{ji}$, i, j = 1 ... n, p_o = average wage, so that:

$$Y_{ii} = K_i \prod_{\substack{j=0 \\ j \neq i}}^{n} Y_{ij}^{\alpha_{ji}} \quad , \tag{2}$$

where

$$K_i = p_i F_i^{q_i} \prod_{\substack{j=0 \\ j \neq i}}^{n} p_j^{-\alpha_{ji}} \quad .$$

Assume also that the local objective functions are the net profits:

$$P_i = Y_{ii} - \sum_{\substack{j=0 \\ j \neq i}}^{n} Y_{ji}, \quad i = 1 \dots n \quad . \tag{3}$$

The sector S_i strategy consists in maximization of (3), where Y_{ii} is expressed by (2), subject to the limitation of input cost i.e.:

$$\sum_{\substack{j=0 \\ j \neq i}}^{n} Y_{ji} \leq Y_i^* \; , \quad i = 1 \dots n \quad , \quad j = 0, 1 \dots n \quad , \tag{4}$$

$$\text{and } Y_{ji} \geq 0 \quad , \tag{5}$$

where Y_i^* is assumed to be given.

Since $P_i(Y_{oi} \dots Y_{ni})$ is a strictly concave function in the compact set Ω defined by (4) and (5) the unique values $Y_{ji} = \hat{Y}_{ji}(Y_i^*)$, $j = 0, 1 \dots n$, exist; such that

$$\max_{Y_{ij} \in \Omega} P_i(Y_{oi} \dots Y_{ni}) = P_i(\hat{Y}_{oi} \dots \hat{Y}_{ni}) \quad .$$

These values can be easily derived by standard Lagrange multiplier technique yielding:

$$\hat{Y}_{ji} = \frac{\alpha_{ji}}{\alpha_i} Y_i^*, \quad j = 0, 1 \dots n, \quad i = 1 \dots n \quad , \tag{6}$$

where

$$\alpha_i = \sum_{\substack{j=0 \\ j \neq i}}^{n} \alpha_{ji} \quad .$$

One also obtains

$$\hat{Y}_{ii} = Y_{ii}(\hat{Y}_{ji}) = M_i Y_i^{*\alpha_i}, \quad i = 1 \dots n \quad , \tag{7}$$

where

$$M_i = K_i \alpha_i^{-\alpha_i} \prod_{\substack{j=0 \\ j \neq i}}^{n} \alpha_{ji}^{\alpha_{ji}} = F_i^{q_i} \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{\alpha_{ji}}{p_j \alpha_i} \right)^{\alpha_{ji}} \quad ,$$

$$(\alpha_{ii} = -1), \quad i = 1 \dots n \quad .$$

Now it is possible to choose the optimum input cost level Y_i^*, in such a way that the profits

$$P_i = M_i Y_i^{*\alpha_i} - Y_i^*, \quad i = 1 \dots n \quad , \tag{8}$$

attain the maximum value. Since (8) is strictly concave function a unique optimum value $Y_i^* = \overline{Y}_i$, $i = 1 \dots n$, exists, such that

$$\max P_i(Y_i^*) = P_i(\overline{Y}_i) \quad .$$

That value becomes

$$Y_i^* = \overline{Y}_i = (\alpha_i M_i)^{1/q_i}, \quad i = 1 \dots n \quad . \tag{9}$$

Then setting $\overline{Y}_i$, into (7), (6), (8) one gets:

$$\hat{Y}_{ii} = M_i \overline{Y}_i^{\alpha_i} = M_i^{1/q_i} \alpha_i^{\alpha_i/q_i} = F_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{\alpha_{ji}}{p_j}\right)^{\alpha_{ji}/q_i} p_i^{1/q_i} \quad , \tag{10}$$

$$\hat{Y}_{ji} = \frac{\alpha_{ji}}{\alpha_i} \overline{Y}_i = \alpha_{ji} M_i^{1/q_i} \alpha_i^{\alpha_i/q_i} = \alpha_{ji} \hat{Y}_{ii} \quad , \tag{11}$$

$$\hat{P}_i = P_i(\overline{Y}_i) = M_i^{1/q_i} \alpha_i^{\alpha_i/q_i} (1 - \alpha_i) = q_i \hat{Y}_{ii} \quad , \tag{12}$$

$i = 1 \ldots n,\ j = 0, 1 \ldots n$.

One should observe that the global net profit becomes:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} \hat{Y}_i = \sum_{i=1}^{n} q_i F_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{\alpha_{ji}}{p_j}\right)^{\alpha_{ij}/q_i} p_i^{1/q_i} \tag{13}$$

where

$$\hat{Y}_i = \hat{Y}_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \hat{Y}_{ij} = \hat{Y}_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} \hat{Y}_{jj} \quad ,$$

is the net output of S_i under optimum decision strategies.

The main result (see also Ref. [3]) can be formulated in the form of a theorem:

Theorem 1

The optimum input output share $\hat{Y}_{ji}/\hat{Y}_{ii}$, $i = 1 \ldots n$, $j = 0, 1 \ldots n$, in the normative n sector Cobb-Douglas production model is equal to the

the production function elasticties: $dY_{ii}/Y_{ii} : dY_{ij}/Y_{ij} = \alpha_{ji}$, $j = 0, 1 \ldots n$, $i = 1 \ldots n$.

In other words, Theorem 1 states that the normative n-sector Cobb-Douglas production system behaves under optimum strategies in the same way as the Leontief model with the technological coefficients $Y_{ji}/Y_{ii} = \alpha_{ji}$, $j = 0, 1 \ldots n$, $i = 1 \ldots n$, $j \neq i$.

Remark 1

In the case when the labor supply L_{oi} is less than the optimum demand $X_{oi} = p_o^{-1}\alpha_{oi}\hat{Y}_{ii}$, $i = 1 \ldots n$, one should consider $X_{oi} = L_{oi}$ as constant in (1) and maximize (3) subject to (4) (5) with $j = 1 \ldots n$. That is equivalent to problem with production functions:

$$X_{ii} = \bar{F}_i^{q_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} X_{ji}^{\alpha_{ji}} ,$$

where

$$\bar{F}_i = F_i L_{oi}^{\alpha_{oi}/q_i} , \quad i = 1 \ldots n .$$

Remark 2

The relations (10) ÷ (13) can be easily extended to the case of sectors described by C.E.S. production functions

$$Y_{ii} = F_i^{q_i} \left\{ \sum_{j=0}^{n} \vartheta_{ji} Y_{ji}^{-\nu} \right\}^{-\frac{\alpha_i}{\nu}} , \tag{14}$$

where $\nu \in [-1, 0]$, $\sum^{n} \vartheta_{ji} = 1$, $\vartheta_{ji} > 0$, $j = 0, 1 \ldots n$, $i = 1 \ldots n$.

Since the solution of problems (3), (4) and (5) with production function (14) yields:

$$\hat{Y}_{ji} = \frac{\bar{\vartheta}_{ji}}{\bar{\vartheta}_i} Y_i^* , \quad j = 0, 1 \dots n, \quad i = 1 \dots n,$$

where

$$\bar{\vartheta}_{ji} = \vartheta_{ji}^{\frac{1}{(1 + \nu)}}, \qquad \bar{\vartheta}_i = \sum_{j=0}^{n} \bar{\vartheta}_{ji} \quad ,$$

and

$$\hat{Y}_{ii} = \bar{M}_i Y_i^{*\alpha_i} \quad ,$$

where

$$\bar{M}_i = F_i^{q_i} \bar{\vartheta}_i^{\frac{(1 + \nu)}{\nu} \alpha_i} \quad ,$$

one gets:

$$\hat{Y}_{ii} = \bar{M}_i^{1/q_i} \alpha_i^{\alpha_i/q_i} = F_i \left[\bar{\vartheta}_i^{\frac{(1 + \nu)}{\nu}} \alpha_i \right]^{\alpha_i/q_i} \quad , \tag{15}$$

$$\hat{Y}_{ji} = \frac{\bar{\vartheta}_{ji}}{\bar{\vartheta}_i} \bar{Y}_i = \frac{\bar{\vartheta}_{ji}}{\bar{\vartheta}_i} (\alpha_i \bar{M}_i)^{1/q_i} = \frac{\bar{\vartheta}_{ji} \alpha_i}{\bar{\vartheta}_i} \hat{Y}_{ii} \quad , \tag{16}$$

$$\hat{P}_i = (M_i \alpha_i^{\alpha_i})^{\frac{1}{q_i}} (1 - \alpha_i) = q_i \hat{Y}_{ii} \quad , \tag{17}$$

$$i = 1 \dots n, \quad j = 0, 1 \dots n \quad .$$

Remark 3

It is possible to extend the results (10) ÷ (13) to the case when Y_{ij}, α_{ij}, $i = 1 \ldots n$, $j = 0, 1 \ldots n$ are changing continuously in time. In that case relations (10) ÷ (13) remain valid.

It should also be observed that the sector output (10) has been entirely decomposed, so it depends only on S_i production function parameters and prices. When prices are fixed the changes in S_j, $j \neq i$ parameters will have no effect on the S_i production. The supply of goods on the market, i.e. Y_i, may change, however, when S_j change. In order to change $\hat{Y}_{ii}$ or profit (12) one has to change the technology (i.e. α_{ij} coefficients) or F_i--what can be done by reallocation of investments--or labor (in the case when it is in short supply as shown in Remark 1).

Assuming that α_{ji} and p_j, $i, j = 1 \ldots n$, are given one can consider the output production (10) (where F_i depends on the investment Z_i) as a nonlinear, dynamic operator A_i of the investment strategy, i.e.

$$\hat{Y}_{ii} = A_i(Z_i), \quad i = 1 \ldots n \quad . \tag{18}$$

The central management center allocates the given amount of investment resources Z among the sectors S_i, $i = 1 \ldots n$, in such a manner that the maximum production

$$Y = \sum_{i=1}^{n} A_i(Z_i) \quad ,$$

or the optimum system development follows. The sector management centers S_i^*, $i = 1 \ldots n$, allocate the resources Z_i, $i = 1 \ldots n$, received from S^*, among the lower level subsystems S_{ik}^*, $k = 1 \ldots n_i$. As a result a multilevel structure of decision centers follows (Fig. 2). The production plants are grouped generally at the bottom of that structure.

Besides the investments Z_i the employment (X_{oi}) and other resources, which are in short supply, can be allocated using the decomposition technique described by (10) ÷ (13) or (15) ÷ (17). One should observe that the the present model is interesting first of all for the centrally planned economies, where the hierarchical system of development planning is commonly used.

3. Optimization of Development

Instead of dealing with the aggregated (within 1 year) variables $\hat{Y}_{ii}$, Z_i, X_{oi}, i = 1 ... n, we shall introduce the resources intensity, i.e. the rates of resources flow in unit time. We shall denote these new variables by $y_i(t)$, $z_i(t)$ $x_i(t)$ respectively. Then the relation which relates $x_i(t)$, $z_i(t)$ to $y_i(t)$ can be written in the form of an operator: $A_i : X \times Z \to Y$, or explicitly:

$$y_i(t) = A_i(x_i(\cdot),\ z_i(\cdot)), \quad i = 1 \ldots n \quad , \qquad (19)$$

where X, Z, Y are, generally speaking, the given Banach spaces.

It should be noted that the relation between the investment intensity z(t) and the productive capital (or the so called plant capacity) c(t) usually is written in the form of a differential equation:

$$\frac{dc}{dt} = Kz(t) - \delta c(t) \quad ,$$

where δ = depreciation of capital, K = positive constant, $c(0) = \bar{c}$ = given, Integrating that equation one gets:

$$c(t) = A(z(\cdot)) = e^{-\delta t}\left[\bar{c} + \int_0^t Ke^{\delta\tau} z(\tau)d\tau\right] \quad . \tag{20}$$

In our approach it is proposed to describe the A_i operators by the more general than (20) expression:

$$y_i = \left[c_i(z_i)\right]^{\beta} M(x_i), \quad i = 1 \ldots n \quad , \tag{21}$$

where

$$c_i(t) = \int_0^t k_i(t, \tau) \left[z_i(\tau)\right]^{\alpha} d\tau \quad , \tag{22}$$

$$M(x_i) = \left[x_i(t)\right]^{1-\beta} \quad , \tag{23}$$

α, β = given numbers, $0 < \alpha < 1$, $0 < \beta < 1$,

$k_i(t, \tau)$ = given continuous function, $k_i(t, \tau) = 0$, for $t < \tau$.

In the case where $k_i(t, \tau) = e^{-\delta(t - \tau)}$ for $t - \tau > 0$ and $\alpha = 1$, (22) is equivalent to (20). There exist however cases when using (22) one can describe better the real investment processes. First of all it is possible to take into account the plant construction delay, T_{oi}. Besides, the capacity increases usually in a gradual manner rather as shown in Fig. 3 for the case of $z_i(t) = 1$, $t > 0$. The α, β coefficients take into account the nonlinear effects of the investment processes. It is assumed that within the range of planned plant capacities no increased return to scale can be achieved (α, $\beta < 1$).

The operator (23) represents the employment or generally the aggregated operation, repairs and maintenance (ORM) costs. The expected production output:

$$Y = \sum_{i=1}^{n} Y_i(x_i, z_i) = \sum_{i=1}^{n} \int_o^T w_1(t) y_i(t) dt \quad , \tag{24}$$

where $w_1(t)$ = given discount function,

T = given planning horizon,

depends on the strategies x_i, y_i, $i = 1 \dots n$ which are bounded by the given cumulative investment Z and ORM cost X:

$$\sum_{i=1}^{n} \int_o^T w_2(t) x_i(t) dt \leq X \quad , \tag{25}$$

$$\sum_{i=1}^{n} \int_o^T w_3(t) z_i(t) dt \leq Z \quad , \tag{26}$$

where $w_2(t)$, $w_3(t)$ - given discount functions.

As a discount function one can take

$$w_i(t) = (1 + \varepsilon)^{-t}, \qquad i = 1, 2, 3$$

or $$w_i(t) = (1 + \varepsilon)^{T - t}, \quad i = 2, 3$$

where ε = given discount rate. The last form is used when the investment is financed by a bank and it is necessary to pay the interests back at the end of the T interval.

Now it is possible to formulate the development optimization problem: Find the non-negative strategies $x_i(t) = \hat{x}_i(t)$, $z_i(t) = \hat{z}_i(t)$, $i = 1 \dots n$, $t \in [0, T]$, such that

$$\max_{x_i,\, z_i \in \Omega} Y(x, z) = Y(\hat{x}, \hat{z}) = \hat{Y} \quad ,$$

where Ω is the set of all non-negative functions which satisfy (25) and (26).

As shown in Ref [2] for n = 1, there exists a unique solution to the present problem and*

$$\hat{z}(t) = \frac{g(t)}{\int_o^T w_3(t)g(t)dt} Z \quad , \qquad (27)$$

$$\hat{x}(t) = \frac{h(t)}{\int_o^T w_2(t)h(t)dt} X \quad , \qquad (28)$$

where

$$h(t) = \left[\frac{w_1(t)}{w_2(t)}\right]^{\frac{1}{\beta}} C[\hat{z}] \quad ,$$

$$g(t) = \left[w_3^{-1}(\tau) \int_\tau^T w_1(t)^{\frac{1}{\beta}} w_2(t)^{-1} k(t, \tau)dt\right]^{\frac{1}{1-\alpha}} \quad ,$$

One gets also

$$Y(\hat{x}, \hat{z}) = F^{\beta(1 - \alpha)} Z^{\alpha\beta} X^{(1 - \beta)} \quad , \qquad (29)$$

where

$$F = \int_o^T g(t)[w_3(t)]^{\alpha} dt \quad .$$

Example. Let $w_1(t) = w_2(t) = w_3(t) = 1$, $\alpha = \frac{1}{2}$, $k(t, \tau) = \exp[-\delta(t - \tau)]$.

One gets

$$g(\tau) = \left[e^{\delta\tau} \int_\tau^T e^{-\delta t} dt\right]^2 = \frac{1}{\delta^2}\left[1 - e^{-\delta(t - \tau)}\right]^2 \quad ,$$

$$h(t) = e^{-\delta t} \int_o^t e^{\delta\tau}\left[1 - e^{-\delta(T - \tau)}\right]\frac{1}{\delta}\left(\frac{Z}{G}\right)^{\frac{1}{2}} d\tau$$

$$= H\left[1 - e^{-\delta t} + \delta t e^{-\delta(t + T)}\right] ,$$

$$H = \frac{1}{\delta^2}\left(\frac{Z}{G}\right)^{\frac{1}{2}} , \quad G = \int_o^T g(\tau)d\tau .$$

The plots of $\hat{x}(t)$, $\hat{z}(t)$, for $\delta T = 4$, have been shown in Fig. 4. The optimum investment strategy $\hat{z}(t)$ decreases monotonously for $t \to T$ while the ORM cost intensity increases for $t \to T$ to the maximum value (the explanation is that it does not pay to spend resources on ORM cost when the plant contruction is not finished yet).

It should also be observed that the expected income under optimum strategies (29) is an increasing function of the planning interval T: i.e.:

$$Y(\hat{x}, \hat{z}) = \hat{Y}(T) = F(T)^{\beta(1 - \alpha)} Z^{\alpha\beta} X^{1 - \beta} ,$$

where $F(\tau)$ increases along with T.

In our example for instance

$$F(t) = \delta^{-2} \int_o^T \{1 - \exp[-\delta(T - \tau)]\}^2 d\tau \cong \frac{1}{\delta^2} \text{ for large } \delta T.$$

Then $\hat{Y}(t)$ for large δT increases as fast as $(\delta T)^{\gamma/2}$. There exist then such point $T = T_m$ that $\hat{Y}(T_m) = X + Z$. At that time instant a return of input cost $X + Z$ can be achieved.

It should be also observed that the ratio

$$\eta(T) = \frac{X + Z}{Y(T)}$$

is a convenient measure of investment effectiveness and is being used in the standard practice of investment planning. Namely one chooses from the set o possible investment projects, characterized by different $\eta_i(T)$, those which have the smallest values of η_i.

Consider now the solution of the general optimization problem (24) ÷ (26). One can use a decomposition approach starting with the local solutions of n sub-problem:

$$\max_{x_i,\ y_i \epsilon \Omega_i} \int_0^T w_1(t) y_i(t) dt \quad , \qquad (30)$$

where

$$\Omega_i = \left\{ \begin{array}{l} x_i : \int_0^T w_2(t) x_i(t) dt \leq X_i,\ x_i(t) \geq 0,\ t \epsilon [0, T] \\ \\ z_i : \int_0^T w_3(t) z_i(t) dt \leq Z_i,\ z_i(t) \geq 0 \quad t \epsilon [0, T] \end{array} \right\}$$

X_i, Z_i = given numbers.

Using formulae (27) and (28) one can write down the explicit form of these solutions and by (29) one gets

$$Y_i(\hat{x}_i, \hat{z}_i) = F_i^{\beta(1 - \alpha)} Z_i^{\alpha\beta} X_i^{(1 - \beta)}, \quad i = 1 \dots n \quad , \qquad (31)$$

where

$$F_i = \int_0^T \left\{ w_3^{-\alpha}(\tau) \int_\tau^T [w_1(t)]^{\frac{1}{\beta}} w_2(t)^{-1} k_i(t, \tau) dt \right\}^{\frac{1}{1 - \alpha}}$$

The solution of the coordinating (or global) problem can be formulated as follows. Find the strategies X_i, Z_i ϵ Ω such that

$$Y = \sum_{i=1}^{n} F_i^{\beta(1 - \alpha)} Z_i^{\alpha\beta} X_i^{1 - \beta} \qquad (32)$$

attains maximum in the set:

$$\Omega = \left\{ \begin{array}{l} X_i : \sum_{i=1}^{n} X_i \leq X, \quad X_i \geq 0, \quad i = 1 \dots n \\ \\ Z_i : \sum_{i=1}^{n} Z_i \leq Z, \quad Z_i \geq 0, \quad i = 1 \dots n \end{array} \right\}$$

Since Y is a strictly concave continuous function in the compact set Ω, it attains, according to the Weierstrass theorem, the upper bound which is on the border of Ω. Then using the standard Lagrange multiplier technique one can derive the optimum solution which becomes

$$\hat{X}_i = \frac{F_i}{F} X, \quad i = 1 \dots n \tag{33}$$

$$\hat{Z}_i = \frac{F_i}{F} Z, \quad i = 1 \dots n \tag{34}$$

Then one can derive

$$Y(\hat{X}_1 \dots \hat{X}_n, \hat{Y}_1 \dots \hat{Y}_n) = F^{\beta(1 - \alpha)} Z^{\alpha\beta} X^{1 - \beta}$$

When the values X_i, Z_i, $i = 1 \dots n$, are known it is possible to solve all the local subproblems explicitly.

The result obtained can be formulated in the form of a theorem.

<u>Theorem 2</u>

The unique optimum strategy for the problem (24) ÷ (26) exists:

$$\hat{z}_i(t) = g_i(t) \frac{Y}{F}, \quad i = 1 \dots n \ ,$$

$$\hat{x}_i(t) = h_i(t) \frac{X}{\Phi}, \quad i = 1 \dots n \ ,$$

where

$$g_i(t) = \left[w_3^{-1}(\tau) \int_\tau^T w_1(t)^{\frac{1}{\beta}} w_2(t)^{-1} k_i(t, \tau) dt \right]^{\frac{1}{1 - \alpha}} ,$$

$$h_i(t) = \left\{ \frac{w_1(t)}{w_2(t)} \quad C_i[Z_i] \right\}^{\frac{1}{\beta}} ,$$

$$\Phi = \sum_{i=1}^{n} \int_o^T w_2(t) h_i(t) dt; \quad F = \sum_{i=1}^{n} \int_o^T w_3(t) g_i(t) dt;$$

such that

$$\hat{Y} = \max_{x_i y_i \epsilon \Omega} Y(x, y) = F^{\beta(1 - \alpha)} Z^{\alpha\beta} X^{(1 - \beta)} . \tag{35}$$

Since the resulting output (i.e. the resulting production function) (35) is of the identical analytic form as the subsystems production functions (31) Theorem 2 can be regarded as an aggregation principle.

According to that principle one can aggregate the production functions in the decomposed hierarchic system shown in Fig. 2, starting with the lowest level, and getting the function of the type (35) at each decision level. The global production function of the entire system of Fig. 2 assumes the well-known macro-economic Cobb-Douglas function. In that way it is possible to obtain the macro-economic production function as a result of aggregation performed on the micro-production functions.

Two more remarks should be formulated:

<u>Remark 1</u>

Since the statistical information, regarding the input-output relations, is usually given in the quantitative form one can replace the time functions: $x_i(t)$, $z_i(t)$, $y_i(t)$, $i = 1 \ldots n$, $t \in [0, T]$ by vectors with components x_{ij}, z_{ij}, y_{ij}, $j = 0, 1 \ldots T$. Consequently, the integrals in (24) ÷ (26) should be replaced by sums etc.

Remark 2

If it is necessary to consider separately the existing and the planned production resources (i.e. labor and capital) one can write instead of (21)

$$y_i = \bar{y}_i + \overset{+}{y}_i \quad ,$$

where

$$\bar{y}_i(t) = \left\{\int_{-\infty}^{0} k_i(t, \tau)\left[\bar{z}_i(\tau)\right]^{\alpha} d\tau\right\}^{\beta}\left[\bar{x}_i(\tau)\right]^{1 - \beta} \quad ,$$

$$\overset{+}{y}_i(t) = \left\{\int_{0}^{t} k_i(t, \tau)\left[\overset{+}{z}_i(\tau)\right]^{\alpha} d\tau\right\}^{\beta}\left[\overset{+}{x}_i(\tau)\right]^{1 - \beta} \quad ,$$

$\bar{z}_i(t)$, $\bar{x}_i(t)$ = investment and labor in already existing economy,

$\overset{+}{z}_i(t)$, $\overset{+}{x}_i(t)$ = investment and labor in the planned economy.

These and other details of the model have been studied extensively in Ref [4].

3. The Influence of Prices

In the model studied in Section 2, the prices were treated as given egzogeneous factors. This will not be true if the model final production*

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*The influence of the egzogeneous labor will be neglected in the present section, so $j = 1 \ldots n$.

$$Y_i = \hat{Y}_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \hat{Y}_{ij}, \qquad i = 1 \ldots n \quad , \tag{36}$$

is sold on the monopolistic market.

In order to investigate the last case let us find first of all the numerical values of prices which will ensure the given values of final products, say $Y_i = Q_i$, $i = 1 \ldots n$.

Taking into account that

$$\hat{Y}_{ij} = \alpha_{ij} Y_{jj}, \qquad i = 1 \ldots n, \quad j = 1 \ldots n$$

the equations (36) can be written in the following matrix form

$$(\underline{I} - \underline{A})\underline{\hat{Y}} = \underline{Q} \quad , \tag{37}$$

Since $\underline{A}$ is a matrix composed of technological coefficients (the Leontief model) it is reasonable to assume that the inverse $\underline{B} = [I - A]^{-1}$ exists and $\underline{B} > 0$. Then for a given vector $\underline{Q}$ there exists a unique solution $\underline{\hat{Y}}(\underline{Q}) > 0$.

Then there exist positive numbers:

$$L_i(\underline{Q}) = \left[\frac{\hat{Y}_{ii}(\underline{Q})}{F_i \prod\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ji}^{\alpha_{ji}/q_i}} \right]^{q_i}, \quad i = 1 \ldots n$$

and by (2) one gets the following set of equations

$$p_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (p_j)^{-\alpha_{ji}} = L_i(\underline{Q}), \quad i = 1 \ldots n \quad ,$$

or (by taking logarithms from both sides):

$$\ln p_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ji} \ln p_j = \ln L_i(\underline{Q}), \quad i = 1 \ldots n \tag{38}$$

The result obtained can be formulated in the form of a theorem.

Theorem 3

In the normative decentralized production system, described by equations (1) ÷ (12) with the determinant

$$D = \begin{vmatrix} 1, & -\alpha_{21}, & \ldots, & -\alpha_{n1} \\ -\alpha_{12}, & 1, & \ldots, & -\alpha_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ -\alpha_{1n} & \ldots & \ldots & 1 \end{vmatrix} \neq 0 ,$$

there exists, for each positive vector $\underline{Q}$, a unique set of positive prices which can be derived by (38).

Now the problem can be approached from the point of view of welfare economics. On the supply side we have the production system which tries to maximize the output $\underline{Y}$. On the demand side we have consumers with the given utility function:

$$U(X_1 \ldots . X_n) ,$$

where X_i - the goods consumed (in natural units).

As an example one can consider the following utility function:

$$U = a \prod_{i=1}^{n} X_i^{\gamma_i} , \quad \sum_{i=1}^{n} \gamma_i = 1 ,$$

a, γ = positive numbers

Introducing prices p_i, $i = 1 \ldots n$, one can write

$$U = A \prod_{i=1}^{n} Y_i^{\gamma_i}, \quad A = Q \prod_{i=1}^{n} p_i^{-\gamma_i}$$

Then if the total consumer's budget is B he will spend on the good "i" the $\gamma_i B$ fraction of B. Then it is possible to set $Q_i = \gamma_i B$, $i = 1 \ldots n$, into the formula (38) and investigate the change of prices in terms of the utility parameters γ_i, $i = 1 \ldots n$.

It is possible also to take into account the balance of payment between the selected sectors. Suppose, for example, that S_n represent the foreign trade, and one would like to have

$$\sum_{i=1}^{n-1} Y_{in} - \sum_{i=1}^{n-1} Y_{ni} = 0 \quad .$$

That equation can be written in the form

$$\hat{Y}_{nn} \sum_{i=1}^{n-1} \alpha_{in} - \sum_{i=1}^{n-1} \alpha_{ni} \hat{Y}_{ii} = 0 \quad , \tag{40}$$

and should be considered as another constraint to the set (37). Then in order to observe the balance of payment type of constraints (40) it is necessary to resign generally speaking, with some of the utility constraints $Q_i = \gamma_i B$, $i = 1 \ldots n$.

R e f e r e n c e s

[1] Kulikowski, R., "Control in Large Scale Systems," (in Polish) WNT (1970).

[2] Kulikowski, R., "Optimization of Complex Investment and Production Processes," Bull. Pol. Acad. Sci. Vol. XXII No. 6 (1974).

[3] Kulikowski, R., "Modelling and Optimization of Complex Development" Bull. Pol. Acad. Sci. (in press).

[4] Kulikowski, R., "System Analysis and Modelling of Complex Development" (Book in Polish, to appear in 1975).

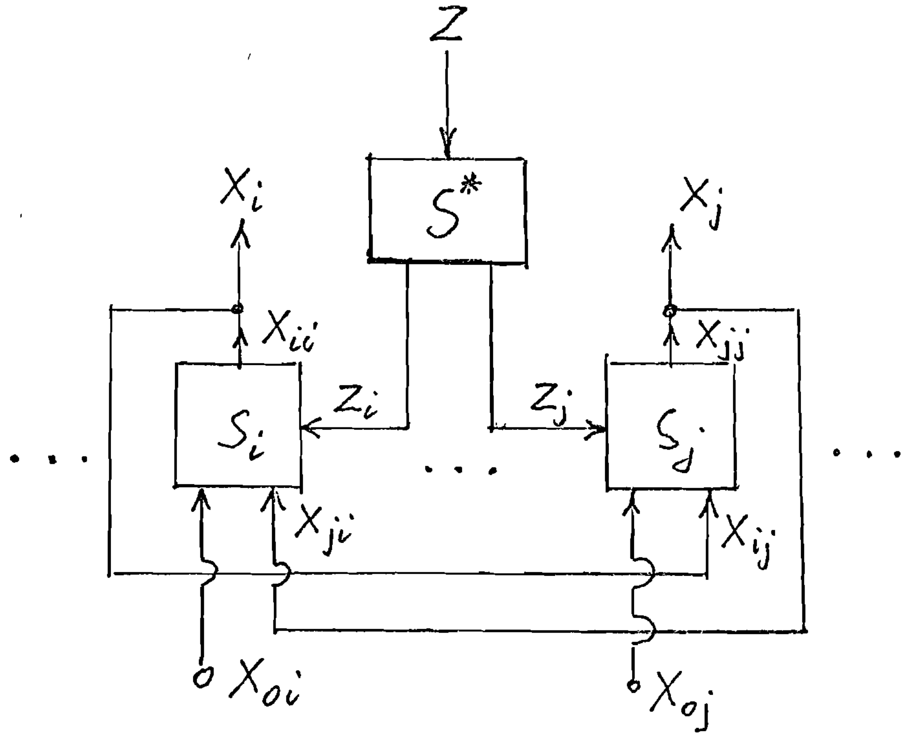
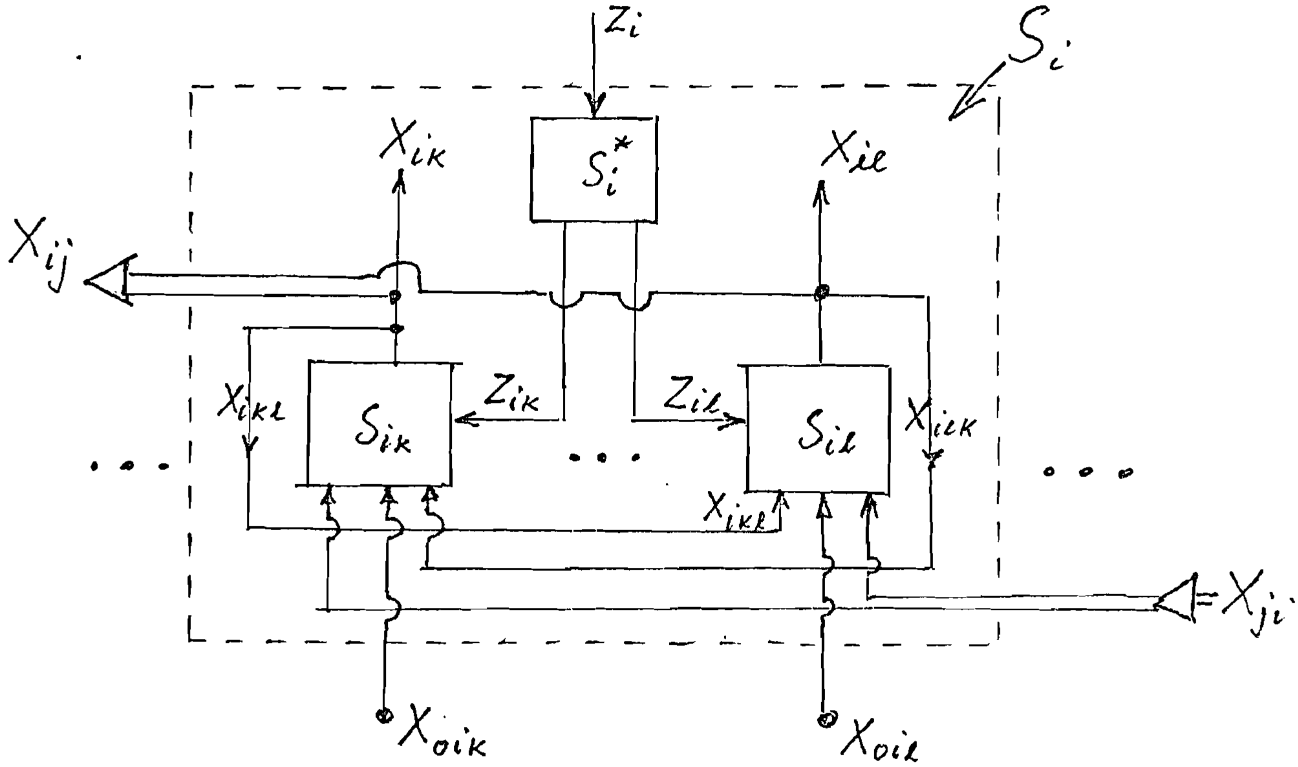

Z
S*
X_i
X_j
X_ii
X_jj
Z_i
Z_j
S_i
S_j
X_ji
X_ij
X_oi
X_oj

Fig. 1a.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S_i
X_ik
S_i*
X_il
X_ij
X_ikl
Z_ik
Z_il
S_ik
S_il
X_ilk
X_ikl
X_ji
X_oik
X_oil

Fig. 1. b.

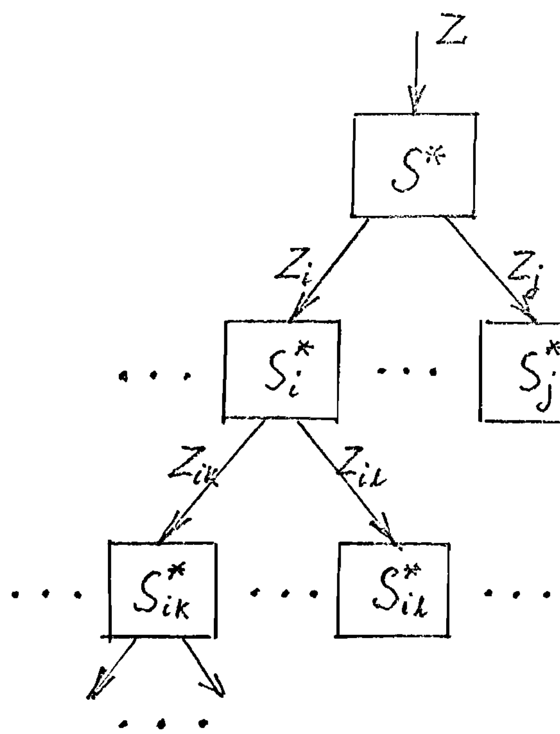


Fig. 2.

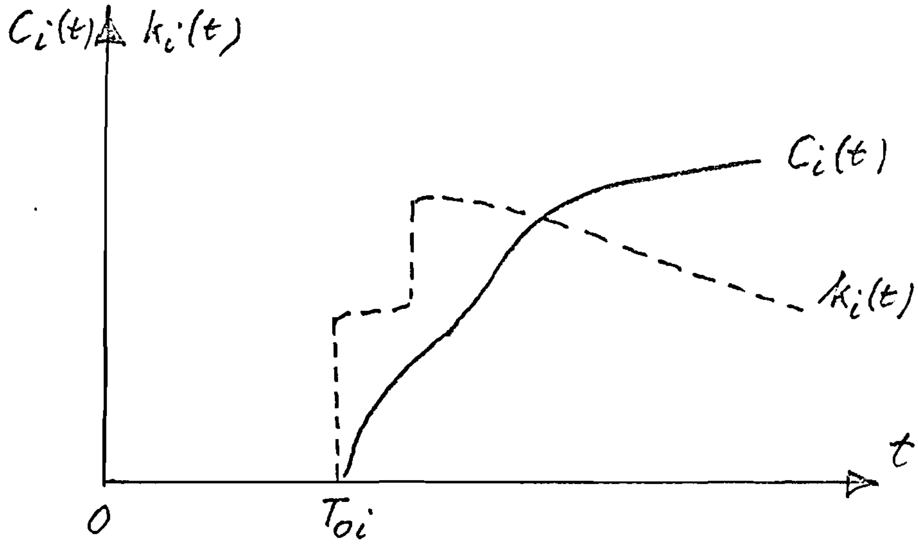
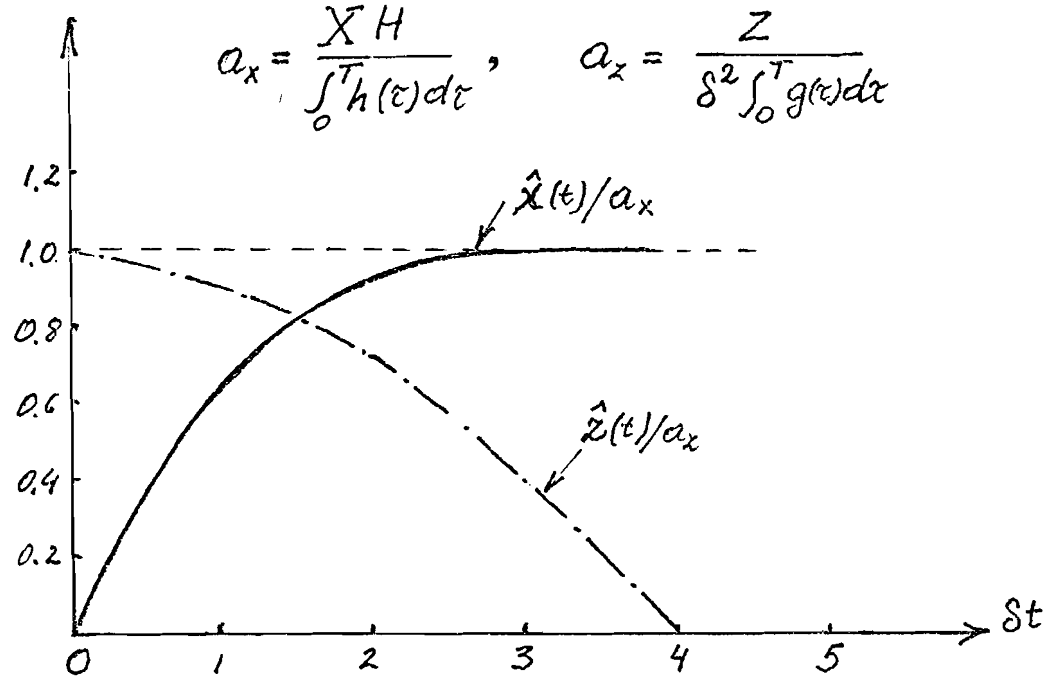


Fig. 3



$a_x = \frac{XH}{\int_0^T h(\tau)d\tau}$, $a_z = \frac{Z}{\delta^2 \int_0^T g(\tau)d\tau}$
1.2
1.0
0.8
0.6
0.4
0.2
$\hat{x}(t)/a_x$
$\hat{z}(t)/a_z$
0
1
2
3
4
5
δt

Fig. 4.