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Abstract

The uncertainty in the output of a deterministic model, due to the uncertainty in the parameters of the model, is analyzed and compared to current procedures of using average values for the uncertain parameters. The present analysis considers an analytical rainfall-runoff flood frequency model where the infiltration parameter is considered as a stochastic variable. The same conceptual procedure can be used to analyze fixed but uncertain (unknown) parameters.

Introduction

The analysis of flood frequency using distribution theory has the basic assumption that the probability of a flood of a given magnitude is constant and does not change with time. Thus, basins which change physically with time, due to changes in the river itself, through channelization for example, or due to urbanization of the watershed, can not be analyzed effectively by the distribution theory procedures of flood frequency analysis.

This problem has been recognized and some procedures have been applied to estimate the frequency curves. The most successful methods are those that analyze the rainfall as a

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stochastic process and then estimate the flood discharge by modelling the physical process of overland flow of the excess rainfall. This has been done analytically by Eagleson (1972), and through simulation by Leclerc and Schaake (1973), Ott and Linsley (1972), and others. Such frequency analyses have often been criticized (Ibbitt, 1972) on the basis that the deterministic catchment model has parameters which are unknown with certainty and whose values seem to be determined through "intuition" and best guesses. Wood and Rodriguez (1974) analyzed the uncertainty in the parameters of the probability distributions of floods by considering the parameters as random variables and applying Bayesian statistics. The resulting probability distributions of floods reflected the uncertainty in their parameters. In an analogous manner, the uncertainty in the flood frequency curve, due to uncertainty in the rainfall-runoff simulation modelling, may be analyzed. The uncertain parameters, whether they are in the probability density functions of the rainfall model or in the deterministic runoff model, may be regarded as random variables. The procedures of Bayesian statistics can then be applied.

While this paper is aimed at simulation modelling, the vehicle for the analysis will be Eagleson's (1972) analytical derivation. Eagleson's derivation is used in the analysis, and the extension to computer simulation modelling is straightforward.

General Theory of Derived Flood Frequency Analysis

Flood frequency analysis aims at finding the probability that a flood will have a discharge less than or equal to some value q_m. This probability is defined as the cumulative density function (CDF) evaluated at q_m and written as $F(q_m)$.

Consider the case when all parameters are known with certainty. The modelling procedure for $F(q_m)$ can be considered as a simple urn problem. A random sample is drawn from an urn which yields the values of the elements of $\underline{\theta}$, a vector that describes the rainfall event. In this analysis, the vector $\underline{\theta}$ will contain two elements, the average intensity, $\bar{i}$, and the storm duration, t_r. With the values of rainfall intensity and storm duration, the overland flow modelling predicts (perfectly) the resulting peak discharge. This sampling for the rainfall values is done for every storm; thus, the stochastic process of the flood discharges is a function of the stochastic process of the rainfall events and the deterministic runoff modelling.

It has been shown by Eagleson (1972) that there exists in the $\bar{i}$ - t_r plane a line of constant peak discharges, q_m, such that all combinations of $\bar{i}$ and t_r to the southwest of this boundary produce discharges less than q_m. This is shown in Figure 1. The probability of observing particular values of $\bar{i}$, t_r is given by their joint probability density function, $f(\bar{i},t_r)$. Finding the cumulative density function for the peak discharge from a rainfall event is equivalent to finding

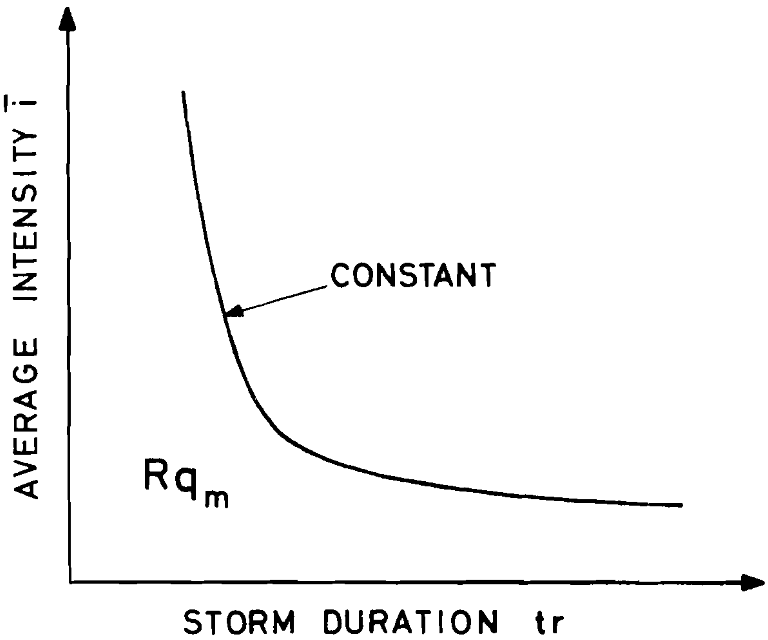

FIGURE 1. $\bar{i}$, t_r PLANE SHOWING PEAK DISCHARGE.

the cumulative density function for the rainfall parameters, $\bar{i}$ and t_r, that produce the peak discharge q_m. This is the problem of finding the volume under the joint density function of $\bar{i}$, t_r for the region Rq_m. This region has boundaries $\bar{i} = 0$, $t_r = 0$, and q_m = constant. The volume under $f(\bar{i},t_r)$, for this region is found by solving the integration

$$F(q_m) = \int_{Rq_m} f(\bar{i},t_r)\, d\bar{i}\, dt_r \quad . \tag{1}$$

The resulting volume is shown in Figure 2. The boundary q_m = constant is located by the modelling of the runoff, either by computer simulation or by analytical techniques. The shape and location of the boundary depend upon:

1) the shape of the rainfall event,
2) the modelling of the catchment response (overland flow) to the rainfall,
3) the values of the parameters in the catchment model.

Traditionally, the assessment of $F(q_m)$ has been to pick a storm pattern, choose a runoff model and set the parameters with the "best" available estimates. Such a procedure does not account for the uncertainty in the region Rq_m due to parameter uncertainty.

Now consider the case where the parameters are unknown and can be treated as random variables. Such uncertain parameters can be divided into two categories. The first category consists of those parameters that are fixed but unknown. A "true" value is thought to exist and, through

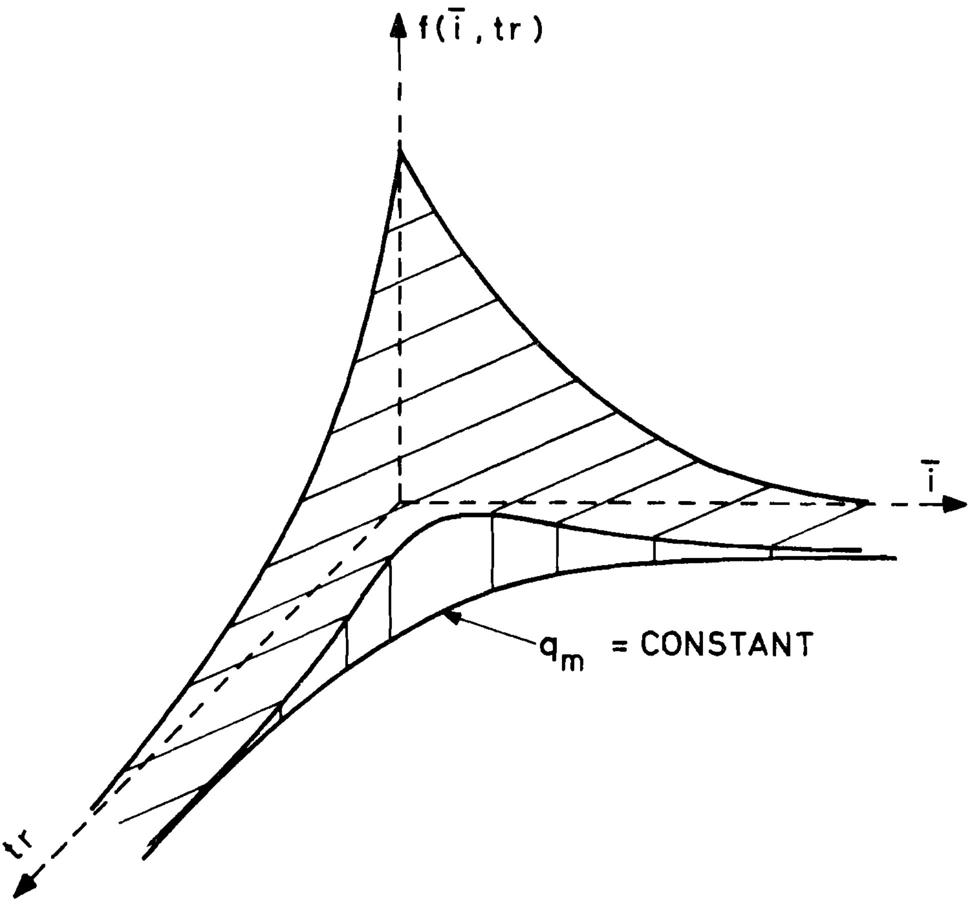

FIGURE 2. GRAPHICAL REPRESENTATION OF $F(q_m)$.

more data, better information may be obtained. Such variables would be the parameters of the runoff modelling, such as stream length or slope. The second category of uncertain parameters are those parameters that vary from rainfall event to rainfall event. Such a parameter would be infiltration. Let infiltration be modelled as a constant water loss, ϕ, over the rainfall event. Then the value of ϕ can be viewed as a stochastic process along with the rainfall event, and these two processes join together to generate peak discharges.

Again, handling these uncertain parameteres can be viewed as an urn sampling problem. The difference between the two types of uncertain parameters is important because it governs at what point "sampling" is done. Assume for the moment that the only uncertain parameters are those that vary from rainfall event to rainfall event and that the water loss ϕ is the only uncertain parameter. Then the sampling would be to choose from one urn a value of the rainfall intensity and storm duration set. From a second urn, a value for the water loss is obtained, which, combined with the runoff model and the rainfall values, produces the flood peak. The cumulative for the flood peak that accounts for the uncertainty in ϕ can be calculated by

$$\tilde{F}(q_m) = \int_{\phi} f(\phi) \left[\cdot \int_{Rq_m|\phi} f(\bar{i}, t_r)\, d\bar{i}\, dt_r \right] d\phi, \qquad (2)$$

where

$f(\phi)$ is the density function for the water loss and $Rq_m|\phi$ is the region in the $\bar{i} - t_r$ plane where the flood peak is less than or equal to q_m. This region is conditional upon ϕ.

The cumulative $\tilde{F}(q_m)$ will be called the Bayesian cumulative of q_m and is the expected value of the cumulative, taking parameter uncertainty into account.

When there exist parameters that are fixed but uncertain, Equation (2) is followed, but conditional upon the uncertain parameters. Then, at the end, the cumulative is weighed by the probability density function for the fixed but uncertain parameters. For example, assume that the rainfall pdf has two parameters, ξ and λ, which are unknown. Since it is assumed that the pdf is fixed but uncertain, the parameter uncertainty is introduced at the end. If the cumulative of Q_{max} is desired, where Q_{max} is the largest of n events and where the events are independent random occurrences, then $F_{Q_{max}}$ is found from

$$\tilde{F}_{Q_{max}} = \int_{\xi,\lambda} \tilde{F}^n(q_m|\xi,\lambda) \cdot f(\xi,\lambda)\, d\xi\, d\lambda \quad , \qquad (3)$$

where

$$\tilde{F}(q_m|\xi,\lambda) = \int_{\phi} f(\phi) \left[\int_{Rq_m|\phi} f(\bar{i},t|\xi,\lambda)\, d\bar{i}\, dt_r \right] d\phi .$$

$f(\xi,\lambda)$ is the probability density function for the fixed but uncertain rainfall parameters.

The analysis of the rainfall distribution in a Bayesian framework within the rainfall runoff analysis must be done at the end. The effect of parameter uncertainty is to introduce uncertainty as to the location of the boundary q_m = constant. The fixed but unknown parameters can be viewed as an uncertainty in the boundary due to a lack of information. The parameters that vary from event to event cause shifting in the boundary due to the interaction of stochastic processes.

There are also two density functions of interest that can be evaluated. The first is the marginal distribution of the exceedance probability at q_m. The exceedance probability, $G(q_m)$, is the probability of observing a flood greater than q_m. The marginal distribution of the exceedance probability, conditional upon the flood level q_m, will be written as $f[G(q_m)]$. The second marginal distribution of interest is the probability density function on the flood discharges, conditional upon an exceedance probability level; it will be written as $f[q|G(q)]$. The two density functions are displayed in Figure 3. These density functions are useful in performing sensitivity analysis on $G(q_m)$ and q_m due to the uncertainty in Rq_m.

They may play a larger role if, in a decision problem, the utility function for the decision set $\underline{A}$ depended upon the exceedance probability of the design discharge q_d. Under these conditions, the expected utility of a decision act, a_i, from the set $\underline{A}$, is given by

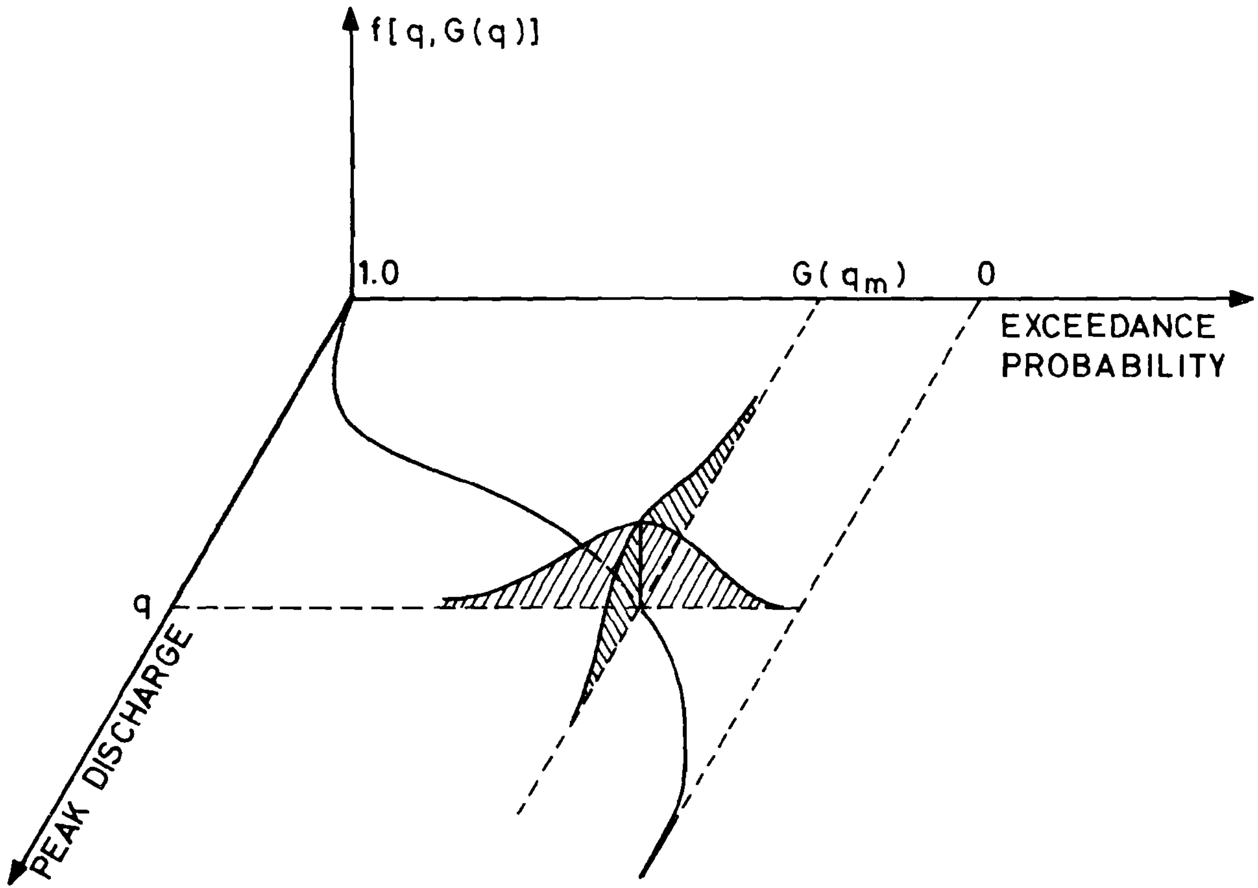

FIGURE 3. GRAPHICAL REPRESENTATION IN THE JOINT pdf FOR q AND G(q) IN THE q, G(q) PLANE.

$$E[u(a_i)] = \int u[a_i, G(q_d)] \cdot f[G(q_d)] \; dG(q_d) \quad . \quad (4)$$

The evaluation of (4) requires the density function $f[G(q_d)]$.

Derivation of the "Bayesian" Flood Frequency Curve

This section presents the analytical derivation of the marginal probability density functions for the exceedance probability, conditional upon a flood magnitude, $f[G(q_m)]$, and the marginal probability density function of the flood discharges, conditional upon the exceedance probability level, $f[q|G(q)]$. To fully focus upon the methodological aspects of the analysis and to permit analytical derivation of the required equations, the following assumptions will be employed:

1. All parameters will be known with certainty, except ϕ, the temporally and spatially averaged water loss rate of the rainfall event.
2. The rainfall event has a rectangular interior pattern.
3. Following Eagleson (1972), the joint probability density function for the average rainfall intensity $\bar{i}$ and storm duration t_r is of the form

$$f(\bar{i}, t_r) = \frac{\lambda\beta}{K} \exp\left[-\lambda\bar{i} - \frac{\beta}{K} t_r\right] \quad , \quad (5)$$

where

K is a factor to reduce point rainstorm depths to areal averages for events of common probability.

λ and β are parameters of the point rainfall density function.

All rainfall parameters are assumed known with certainty.

4. The response of the catchment to a rainfall event will follow Eagleson (1972). Eagleson analytically derived the peak discharge from a catchment by applying kinematic wave theory under the assumptions that the catchment can be modelled by an idealized flow plane and that the time of concentration of the stream is larger than the time of concentration for the catchment. Eagleson's catchment response will be used to define the boundary q_m = constant.

The extension to a simulation model is straightforward. The model will define lines of constant peak discharges in the $\bar{i} - t_r$ plane for given values of ϕ. The volume under the $f(\bar{i},t_r)$ surface, for the region Rq_m, can be found either by analytical procedures or by numerical procedures, depending upon the form $f(\bar{i},t_r)$ and the representation of the boundary of constant peak discharge.

Eagleson approximates the boundary q_m = constant by a function of the form

$$g(i) = B/i^m \quad , \tag{6}$$

taking m = 1/2 where

$$B = 2.97\left[\frac{A_r}{\alpha_c L_s}\right]^{1/2}\left[1 - \frac{655\ \alpha_s^{4/3}\ A_r^2}{\alpha_c L_s\ q_m}\right] \quad ,$$

A_r is area contributing to direct runoff

α_c and α_s are parameters of the catchment.

L_s is the stream length

$i = i_e - q_m/645\ A_r$, i_e being the average excess rainfall intensity.

For storm durations greater than the sum of the times of concentration for the catchment and the stream

$$q_m = 645\ A_r\ i_e \quad . \tag{7}$$

The analysis here, considers all rainfall events whereas Eagleson only considered events that produced direct runoff (excess rainfall events).

To find the cumulative for the peak discharge, $\tilde{F}(q_m)$, Equation (2) is applied. The inner integration is over the rainfall probability density function. The limits of integration cover the region Rq_m, which is a function of uncertain water loss parameters, ϕ. In fact, the region Rq_m in the $\bar{i} - t_r$ plane now becomes a volume in the $\bar{i} - t_r - \phi$ space, and the integration for $\tilde{F}(q_m)$ is done first for Rq_m, conditional upon ϕ. The integration over ϕ is then performed. Figure 4 shows the constant boundary in the $\bar{i} - t_r - \phi$ space and the volume, Rq_m, where the discharge is less than or equal to q_m.

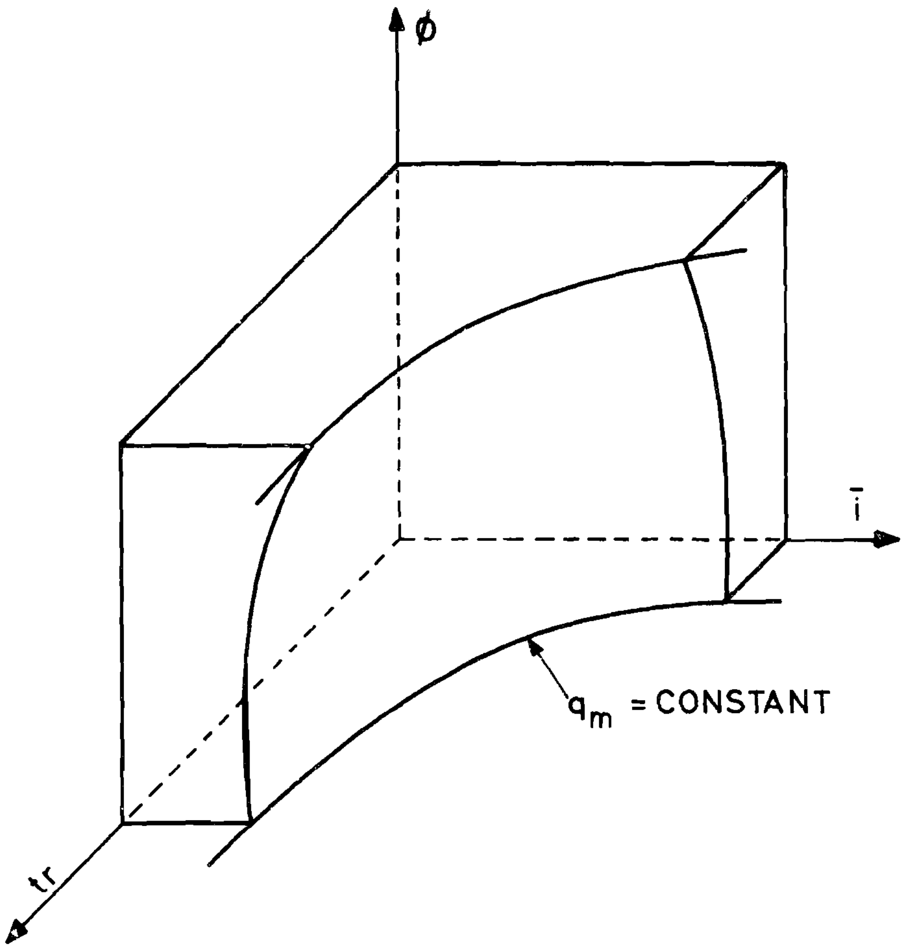

FIGURE 4. i, t_r, ϕ SPACE SHOWING PEAK DISCHARGE.

The integration of Equation (2), over the rainfall pdf, yields $\tilde{F}(q_m|\phi)$, and is evaluated by

$$\tilde{F}(q_m|\phi) = \int_{Rq_m|\phi} f(\bar{i},t_r)\, d\bar{i}\, dt_r \quad . \tag{8}$$

The region $Rq_m|\phi$ can be broken into two areas. The first has the boundaries

$$t_r = 0 \quad ,$$

$$t_r = \infty \quad ,$$

$$\bar{i} = 0 \quad ,$$

$$\bar{i} = \frac{q_m}{645\ A_r} + \phi \quad .$$

The solution to Equation (8) for these limits of integration will be represented by I_1. The solution for the following limits of integration will be represented by I_2. These limits are

$$\bar{i} = \frac{q_m}{645\ A_r} + \phi \quad ,$$

$$\bar{i} = \infty \quad ,$$

$$t_r = g(i_o) \quad ,$$

where $g(i_o)$ is a function of the form similar to Equation (6). The two areas of integration are shown in Figure 5 and are similar to the two regions Eagleson used to solve his function.

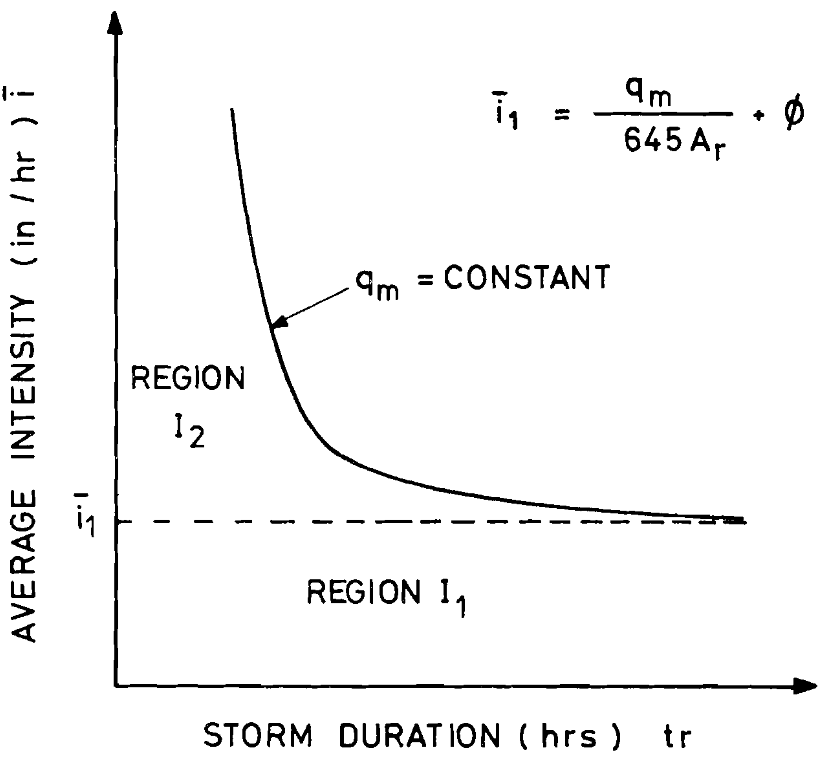

FIGURE 5. REGIONS OF INTEGRATION IN THE $\bar{i}, t_r$ PLANE.

$$I_1 = \int_0^\infty dt_r \int_0^{q_m/645\ A_r + \phi} \frac{\beta\lambda}{K} \exp\ (-\lambda t_r - \frac{\beta}{K}\ \bar{i})\ d\bar{i}$$

$$= 1 - \exp\ (-\beta q_m/645\ KA_r - \beta\phi/K)\ , \tag{9}$$

$$I_2 = \int_{q_m/645\ A_r + \phi}^\infty d\bar{i} \int_0^{g(\bar{i})} \frac{\beta\lambda}{K}\ (-\lambda t_r - \frac{\beta}{K}\ \bar{i})\ dt_r \ , \tag{10}$$

where $t_r = g(i_o)$. (11)

Letting

$$i_o = \bar{i} - (q_m/645\ A_r + \phi) \ , \tag{12}$$

Equation (10), becomes

$$I_2 = \int_0^\infty di_o \int_0^{g(i_o)} \frac{\beta\lambda}{K} \exp\ [-t_r - \frac{\beta}{K}$$

$$(i_o + q_m/645\ A_r + \phi)]\ dt_r \ , \tag{13}$$

$$I_2 = \exp\ (-\ \frac{\beta}{K}\ \frac{q_m}{645\ A_r} - \frac{\beta}{K}\ \phi)$$

$$\cdot \left[1 - \frac{\beta}{K} \int_0^\infty \exp\ [-\ \frac{\beta i_o}{K} - \lambda g(i_o)]\ di_o\right] \ . \tag{14}$$

When $g(i_o)$ is of the form of (6) then (14) integrates to

$$I_2 = \exp\ (-\ \frac{\beta q_m}{645\ KA_r} - \frac{\beta}{K}\ \phi) \cdot (1 - I_o)\ , \tag{15}$$

where

$$I_o = e^{-\sigma/m} \sigma^{-\sigma+1} \Gamma(\sigma) \quad ,$$

$$\sigma = \left[2.21 \frac{\beta\lambda^2 A_r}{K\alpha_c L_s} \left(1 - \frac{655\alpha_s^{4/3} A_r}{\alpha_c L_s^3 q_m^{1/3}}\right)^{1/3} \right]^{1/2} .$$

Thus

$$F(q_m) = 1 - I_o \cdot \exp\left(- \frac{\beta q_m}{645 A_r K} - \frac{\beta}{K} \phi\right) . \quad (16)$$

When considering the cumulative density function for q_m, conditional upon an excess rainfall event occurring, then (16) reduces to Eagleson's expression.

Often, decision makers are interested in the flood exceedance probability, $G(q_m) = 1 - F(q_m)$. Then, from (16) $G(q_m)$ is

$$G(q_m) = I_o \cdot \exp\left(- \frac{\beta q_m}{K\ 645\ A_r} - \frac{\beta}{K}\phi\right) . \quad (17)$$

Equation (12) provides a relationship between the exceedance probability for a given flood peak, q_m, and the water loss parameter, ϕ. If two random variables are functionally related, for example $y = g(x)$, and if the function is monotonic and continuous, then the following relationships hold

$$E\left[y^n\right] = \int_x g^n(x) \cdot f(x)\, dx \quad ,$$
$$f(y) = f(x) \cdot \left|\frac{dx}{dy}\right| \quad . \tag{18}$$

These relationships provide a procedure to obtain the marginal probability density function as well as the moments for the exceedance probability $G(q_m)$, given the peak discharge, and for the peak discharge, q, conditional upon the exceedance probability. These marginal density functions reflect the uncertainty in ϕ.

The form of these distributions depends upon the probability density function for ϕ, $f(\phi)$. Three forms will be examined. These are: $f(\phi)$ as a uniform pdf, a gamma-1 pdf, and an exponential. The latter is really a special case of the gamma-1.

Water Loss ϕ, uniformly distributed

Let $f(\phi)$ be represented by a uniform probability density function between ϕ_o and ϕ^o,

$$f(\phi) = \frac{1}{(\phi^o - \phi_o)} \quad , \qquad \phi_o \leq \phi \leq \phi^o \quad ,$$
$$= 0, \text{ otherwise} \quad , \tag{19}$$

and let $y = G(q_m)$. Then the Jacobian from (16) is

$$\left|\frac{dy}{d\phi}\right| = \frac{C\beta}{K}\, e^{-\phi\beta/K} \quad , \tag{20}$$

where

$$C = I_o \cdot \exp\left[\frac{-\beta q_m}{K\ 645\ A_r}\right] ,$$

$$f(y) = \frac{1}{(\phi^o - \phi_o)} \frac{K}{\beta y} ,$$

for (21)

$$C \cdot e^{-\phi^o \beta/K} \leq y \leq C \cdot e^{-\phi_o \beta/K}$$

$= 0$, otherwise.

The first two moments are

$$E[y] = \frac{CK}{\beta(\phi^o - \phi_o)} \left[e^{-\phi_o \beta/K}\right] , \tag{22}$$

$$E[y^2] = \frac{C^2 K}{2\beta(\phi^o - \phi_o)} \left[e^{-2\phi_o \beta/K} - e^{-2\phi^o \beta/K}\right] . \tag{23}$$

The decision maker is not only interested in the distribution of the exceedance probability at a particular flood discharge level, but, given an exceedance probability, he is also interested in the distribution of the flood discharges. This marginal probability density function can be found from Equations (16) and (18). Due to the complex nature of the discharge in (16), analytical derivation is only possible if the following assumption is valid: for a particular basin, I_o is constant over the range of flood discharges that are of interest. Table 1 shows that this assumption is a reasonable one; then the Jacobian, $|dq/d\phi|$,

Table 1. Values of I_o for Various Peak Discharges.

Discharge (cfs)	σ	I_o
100	.60199	.36384
1,000	.6190	.3496
5,000	.6249	.34465
10,000	.62661	.3432

(For catchment and rainfall parameters as given in Table 2.)

is from Equation (16),

$$\left|\frac{dq}{d\phi}\right| = 645\ A_r \quad . \tag{24}$$

The limits on q, for the derived distribution, may be obtained by rewriting Equation (16) as

$$\phi = \frac{K}{\beta}\ \ell n \left[\frac{I_o}{y}\right] - \frac{q_m}{645\ A_r} \quad . \tag{25}$$

For y, $[= G(q_m)]$, a constant and for no water loss ($\phi = 0$) q_m is a maximum and equal to

$$q_m = 645\ A_r \frac{K}{\beta}\ \ell n \left[\frac{I_o}{y}\right] \quad . \tag{26}$$

As the water loss increases, the discharge from the rainfall event must decrease until, at some value of ϕ, ϕ_m, there is no excess rainfall and no runoff. This value is:

$$\phi_m = \frac{K}{\beta} \ln\left[\frac{I_o}{Y}\right] \quad . \tag{27}$$

The probability that $q_m = 0$ is the probability that ϕ is greater than or equal to ϕ_m. The spike for $f_Q(q = 0)$ can be calculated by

$$f_Q(q = 0) = P(\phi \geq \phi_m) = \int_{\phi_m}^{\infty} f(\phi)\, d\phi \quad , \tag{28}$$

and the density function for q, $q > 0$, will be the derived density function from Equation (18) with limits

$$0 \leq q \leq 645\, A_r \frac{K}{\beta} \ln\left[\frac{I_o}{Y}\right] \quad . \tag{29}$$

With Equations (18), (19), and (24) the distribution f(q) is

$$f(q) = \frac{1}{(\phi^o - \phi_o) \cdot 645\, A_r} \quad , \tag{30}$$

and has limits

$$645\, A_r \left[\frac{K}{\beta} \ln\left[\frac{I_o}{Y}\right] - \phi^o\right] \leq q \leq 645\, A_r \left[\frac{K}{\beta} \ln\left[\frac{I_o}{Y}\right] - \phi_o\right] \tag{31}$$

if $\quad \phi^o < \frac{K}{\beta} \ln\left[\frac{I_o}{Y}\right]$.

If $\phi^o > \frac{K}{\beta} \ln\left(\frac{I_o}{Y}\right)$ and $\phi_o < \frac{K}{\beta} \ln\left(\frac{I_o}{Y}\right)$ then the limits are

$$0 \leq q \leq 645\ A_r\left[\frac{K}{\beta} \ln\left(\frac{I_o}{Y}\right) - \phi_o\right] \quad , \tag{32}$$

for $f(q|q > 0)$. The spike at $q = 0$ may be found from Equation (28) or from integrating Equation (30) between the limits

$$0 \leq q \leq 645\ A_r\left[\phi^o - \frac{K}{\beta} \ln\left(\frac{I_o}{Y}\right)\right] \quad . \tag{33}$$

The first two moments of $f(q)$ are

$$E[q] = 645\ A_r\left[A - \frac{(\phi^o + \phi_o)}{2}\right] \quad , \tag{34}$$

with the constraint of $E[q] \geq 0$ and where

$$A = \frac{K}{\beta} \ln\left(\frac{I_o}{Y}\right)$$

$$E[q^2] = (645\ A_r)^2 \cdot [A^2 - A(\phi^o + \phi_o) + \frac{1}{3}(\phi^o + \phi_o) - \frac{1}{3}\phi^o\phi_o] \quad , \tag{35}$$

again with $A = \frac{K}{\beta} \ln\left(\frac{I_o}{Y}\right)$.

Water Loss ϕ, Gamma-1 Distributed

Let ϕ be distributed with a probability density function of the form gamma-1, that is

$$f(\phi) = e^{-\alpha\phi} \phi^{r-1} \alpha^{r} \Gamma(r) \quad . \tag{36}$$

Using the same definitions for y and C as in the uniform pdf analysis and using the Jacobian as given in (20), then (18) gives

$$f(y) = A^{r} y^{A-1} C^{-A}[\ln(\frac{C}{y})]^{r-1}/\Gamma(r) \quad , \tag{37}$$

where

$$A = K\alpha/\beta \quad , \qquad 0 \leq y \leq 1 \quad .$$

The first two moments of y are

$$E[y] = C\left(\frac{\alpha}{\alpha + \frac{K}{\beta}}\right)^{r} \quad , \tag{38}$$

$$E[y^{2}] = C^{2} \cdot \left(\frac{\alpha}{\alpha + \frac{2K}{\beta}}\right)^{r} \quad . \tag{39}$$

For the distribution of q for a given exceedance level $G(q_m)$, again the approximation that $I_o \cong$ constant must be made.

The Jacobian from (16) is as given in Equation (24) and with Equations (18) and (36)

$$f(q|q > 0) = \frac{1}{645\ A_r} \cdot \exp\left[-\alpha(A-q/645\ A_r)\right] \cdot (A-q/645\ A_r)^{r-1} \cdot \alpha^{r}/\Gamma(r) \quad ,$$

$$0 < q < 645\ A_r \frac{K}{\beta} \ln\left(\frac{I_o}{y}\right) \quad , \tag{40}$$

where

$$A = \frac{K}{\beta} \ln\left[\frac{I_o}{y}\right] ,$$

f(q) has moments

$$E[q] = 645\ A_r(A - r/\alpha) \quad \text{, and}$$

$$E[q^2] = (645\ A_r)^2 \cdot \left[A^2 - 2Ar/\alpha + r(r + 1)/\alpha^2\right] \quad , (41)$$

where

$$A = \frac{K}{\beta} \ln\left[\frac{I_o}{y}\right] .$$

<u>Water Loss, ϕ, Exponentially Distributed</u>

Let ϕ be distributed exponentially. Then $f(\phi)$ is of the form,

$$f(\phi) = \alpha e^{-\alpha\phi} \quad , \tag{42}$$

which is a special case of the gamma-1 distribution when r = 1.

The marginal density function for the exceedance probability, with a peak discharge q_m and marginal density function for the discharge q at an exceedance level G(q), may be found by the application of Equations (16), (18) and (42). The marginals may also be found by taking the results from the gamma-1 analysis.

The results for the exceedance probability, $y = G(q_m)$, are

$$f(y) = Ay^{A-1} C^{-A} \quad , \qquad 0 < y < 1 \quad , \tag{43}$$

where

$$A = K\alpha/\beta$$

$$E[y] = C\left(\frac{\alpha}{\alpha + \beta/K}\right) \quad , \tag{44}$$

$$E[y^2] = C\left(\frac{\alpha}{\alpha + 2\beta/K}\right) \quad . \tag{45}$$

And for the discharge q, conditional upon q being greater than or equal to 0, the results are

$$f(q|q \geq 0) = \frac{\alpha}{645\ A_r} \exp\left(\frac{\alpha}{645\ A_r}\right) \cdot \left(\frac{I_o}{y}\right) \quad , \tag{46}$$

for

$$0 \leq q \leq 645\ A_r \frac{K}{\beta} \ln\left(\frac{I_o}{y}\right) \quad ,$$

$$E[q] = 645\ A_r[A - 1/\alpha] \quad , \tag{47}$$

$$E[q^2] = (645\ A_r)^2[A^2 - 2A/\alpha + 2/\alpha^2] \quad , \tag{48}$$

where

$$A = \frac{K}{\beta} \ln\left(\frac{I_o}{y}\right) \quad .$$

Recurrence Interval

The exceedance probability for the occurrence of flood events, $G(q_m)$, has been evaluated with the total series of independent rainfall events. Often hydrologists are interested in the exceedance probability of a flood peak as that peak relates to a partial duration series. When the number of flood events in this partial duration series equals N, the number of years of record, then the exceedance probability, for this particular partial duration series, can be found in the following manner (Eagleson, 1972).

Consider a record of N years which contains, on the average, θ rainfall events per year. There will be θN flood events, some of which will have a maximum discharge equal to 0 due to no excess rainfall. The r^{th} most severe event of the complete series will have an exceedance probability of

$$G(q_{mr}) = \frac{r}{\theta N + 1} \quad . \tag{49}$$

Now consider the annual exceedance series which is composed of the N largest flood events from the set of θN. The exceedance probability of q_{mr}, from the annual exceedance series, is

$$P[q_m > q_{mr}] = \frac{r}{N + 1} = \frac{1}{T_e} \quad , \tag{50}$$

where T_e is the recurrence interval measured in years. For $r \leq N$, (49) and (50) can be combined to give

$$\frac{1}{T_e} = \theta \cdot G(q_m) \quad , \tag{51}$$

assuming $N \gg 1$.

Equation (51) is used in the next section to compare the flood return periods obtained by the different modelling assumptions of the water loss parameter ϕ.

Example Application

The analytical results in this paper can be used to determine the effect of uncertainty in the water loss parameter, ϕ, upon the flood frequency curve. The expected frequency curve for a hypothetical catchment, with parameters as given in Table 2, will be determined for the three different probability modelling assumptions of ϕ. An indication of the variance in the process will be obtained by plotting the expected exceedance probability curve, $E[G(q_m)]$, with the expected exceedance probability curve plus and minus one standard deviation. These curves will be from the annual exceedance series, that is, a partial duration series of a length equal to the number of years of record. It should be visualized that there exists a surface in the $G(q_m) - q_m$ plane. This surface represents the joint probability density function. The three curves, $E[G(q_m)]$, $E[G(q_m)] + \sigma$, $E[G(q_m)] - \sigma$ represent three contours. For comparison, the frequency curve from the analysis which assumes ϕ is deterministic is also presented. In this analysis, the value of ϕ chosen is the mean value of $f(\phi)$.

Table 2. Catchment and Rainfall Parameters.

A_c = 100 sq. mi.

A_r = Ac/3 = 33.333 sq. mi.

$L_s = (3.\ A_c)^{\frac{1}{2}} = 17.32$ mi.

$\alpha_c = 10\ \text{sec}^{-1}$

$\alpha_s = .1\ \text{sec}^{-1}$

β = 30 hr/in.

$\lambda = .13\ \text{hr}^{-1}$

$K = .95\ (K = 1 - \exp[-1.1\lambda^{-\frac{1}{4}}] + \exp[-1.1\lambda^{-\frac{1}{4}} - .01\ A_r]$

(Eagleson, 1972)

θ = 109. events per year.

Figure 6 is for the case where the water loss is uniformly distributed with means $\bar{\phi}$ equal to .05 in/hr. Figures 7 and 8 are for the case where $f(\phi)$ is exponential with means of .03 in/hr and .05 in/hr respectively. Figures 9, 10, 11, and 12 are for $f(\phi)$ gamma-1 distributed with mean, $\bar{\phi}$, equal to .05 in/hr and coefficient of variation equal to .577, .477, .316, and .10 respectively.

The implications of the uncertainty in the frequency curve is evident from the curves. In decision problems, the expected exceedance probability, $E[G(q_m)]$ would be used. Take the case where $f(\phi)$ is exponential with a mean $\bar{\phi} = .05$. The error introduced by specifying that a peak discharge of 4500 cfs has a return period of 100 years, as predicted

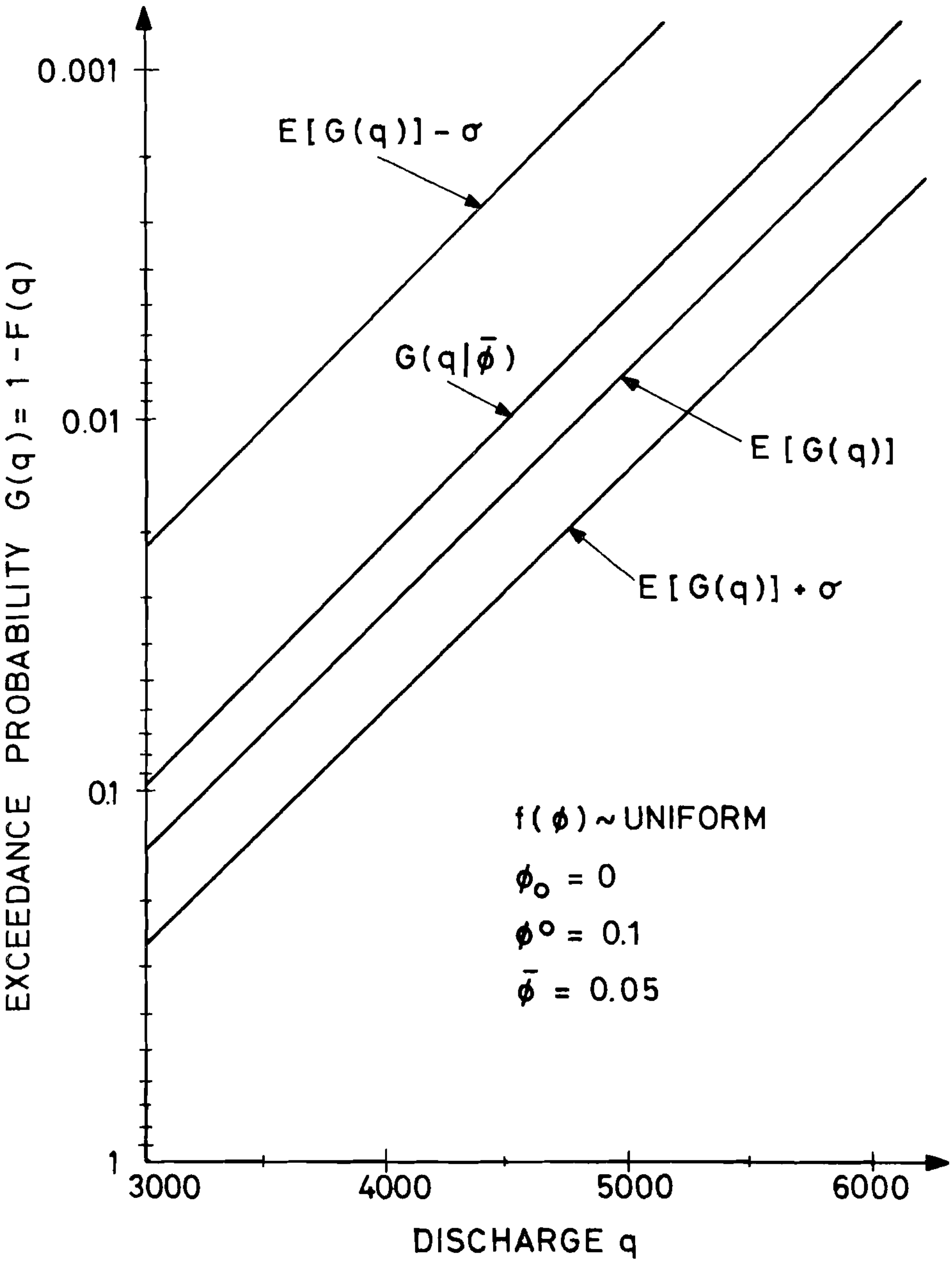

FIGURE 6. FREQUENCY CURVES FOR $f(\phi)$, UNIFORM WITH $\bar{\phi} = 0.05$ in/hr.

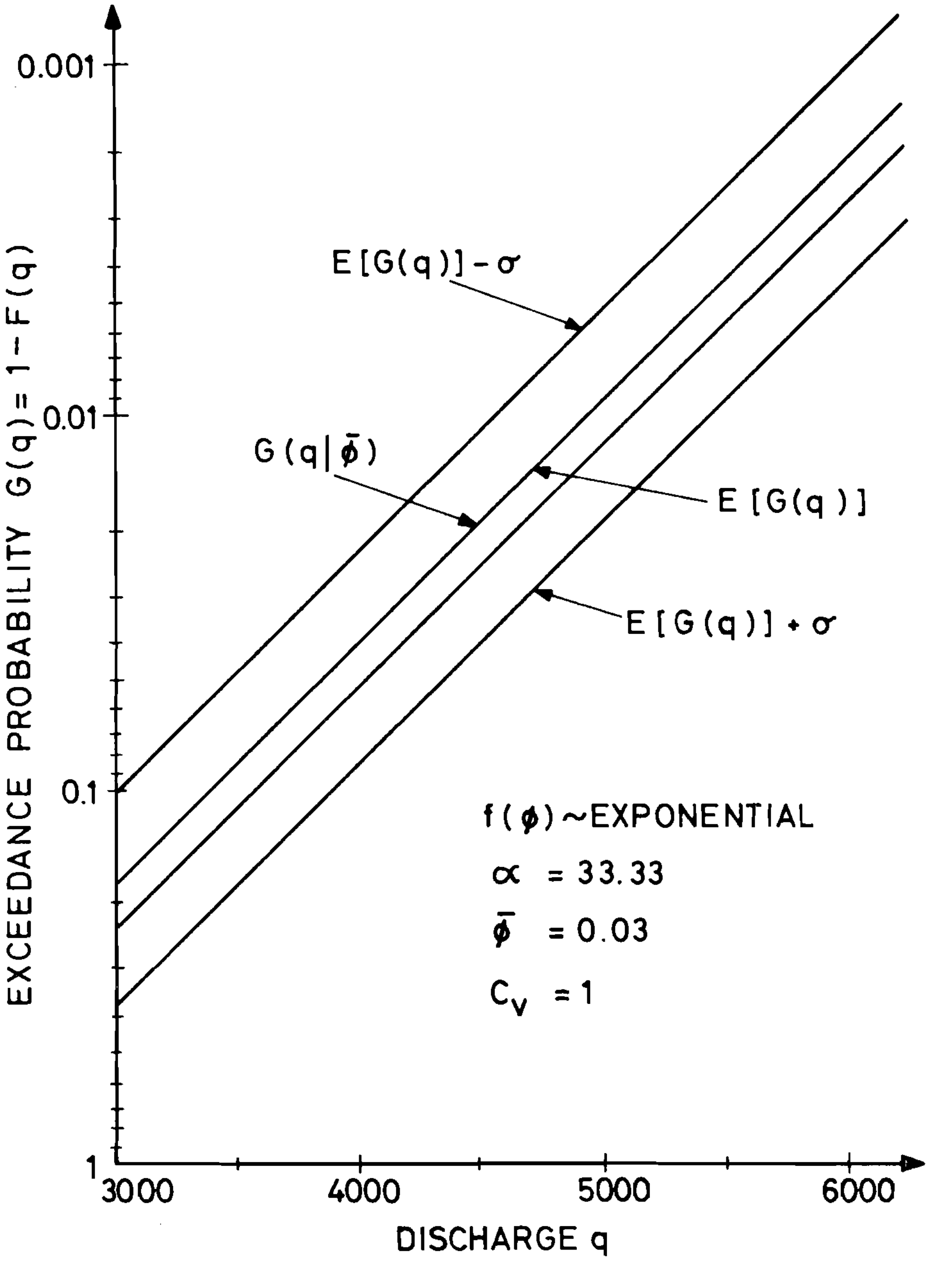

FIGURE 7. FREQUENCY CURVES FOR $f(\phi)$, EXPONENTIAL WITH $\bar{\phi} = 0.03$.

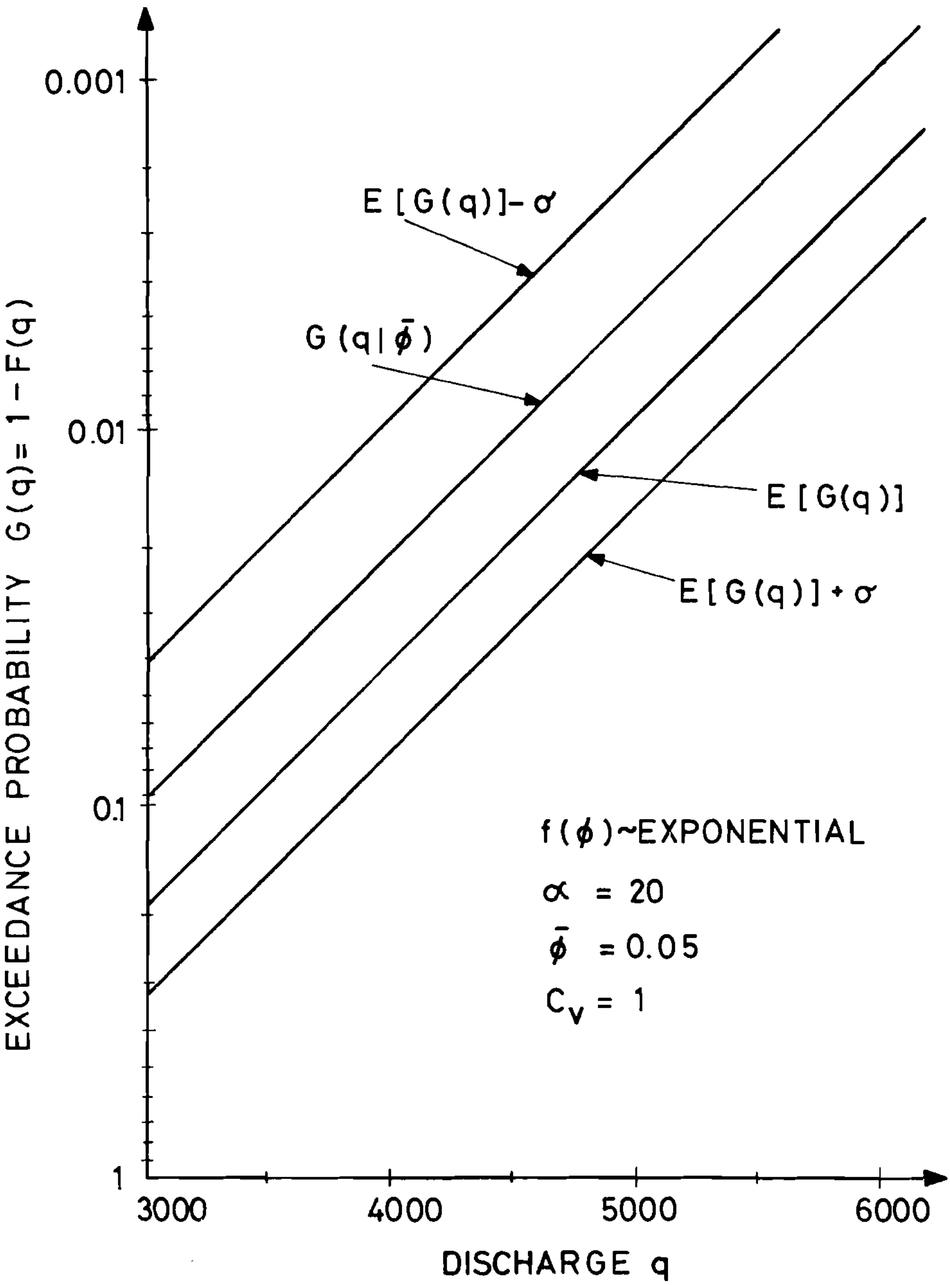

FIGURE 8. FREQUENCY CURVES FOR $f(\phi)$, EXPONENTIAL WITH $\bar{\phi} = 0.05$.

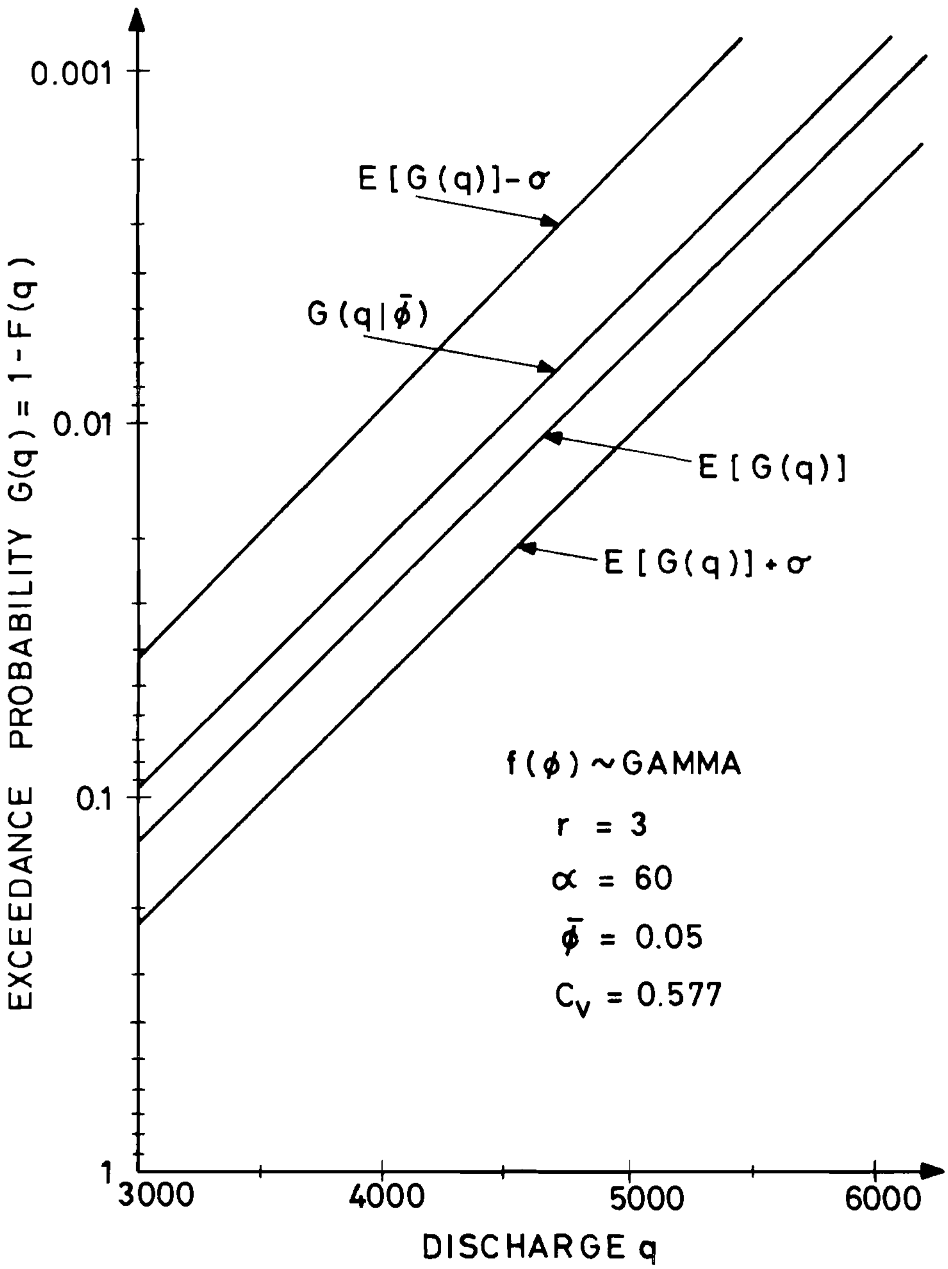

FIGURE 9. FREQUENCY CURVES FOR f(ϕ), GAMMA-1 WITH $\bar{\phi}$ = 0.05 AND C_V = 0.577.

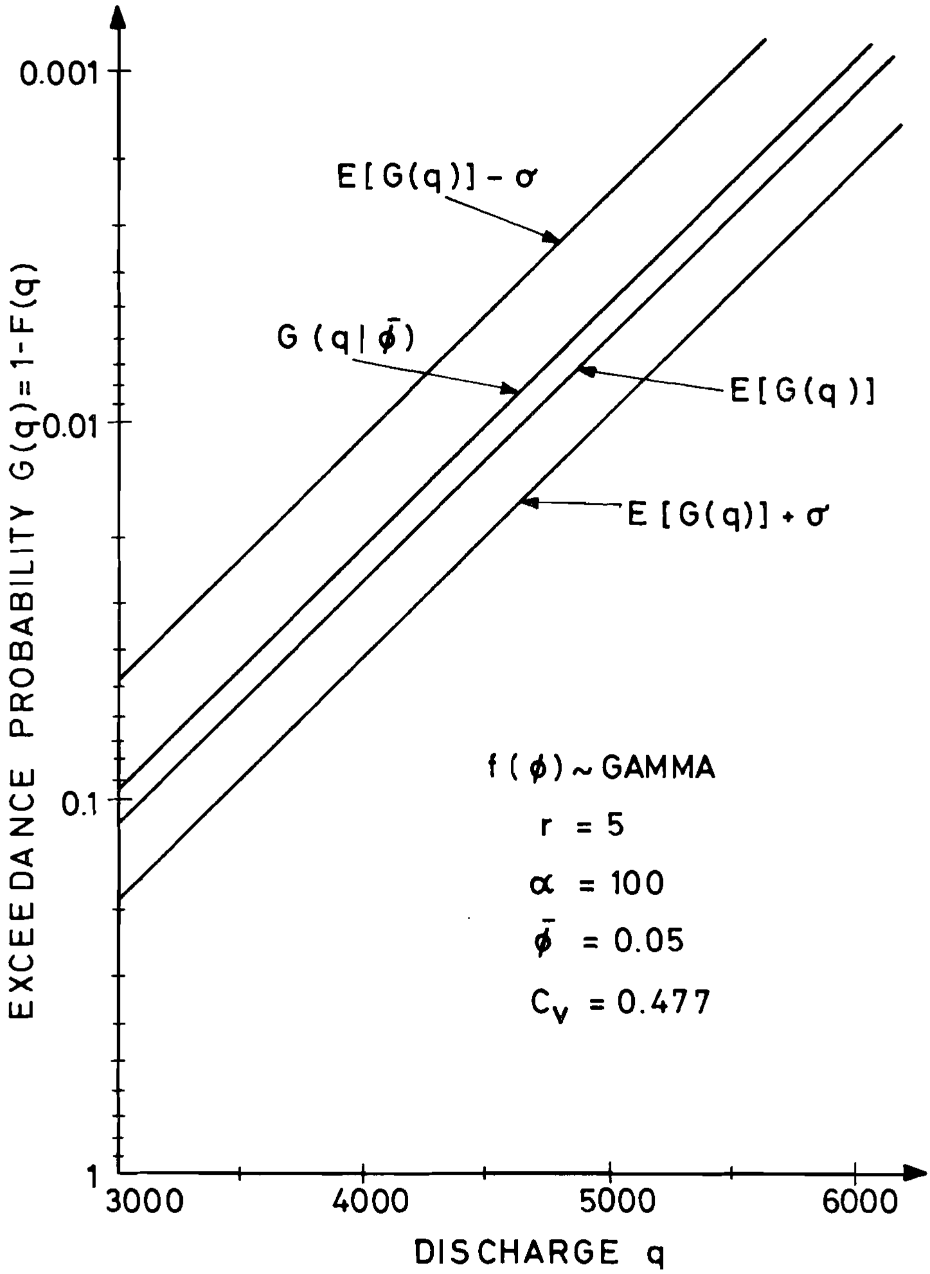

FIGURE 10. FREQUENCY CURVES FOR $f(\phi)$, GAMMA-1 WITH $\bar{\phi} = 0.05$ AND $C_V = 0.477$.

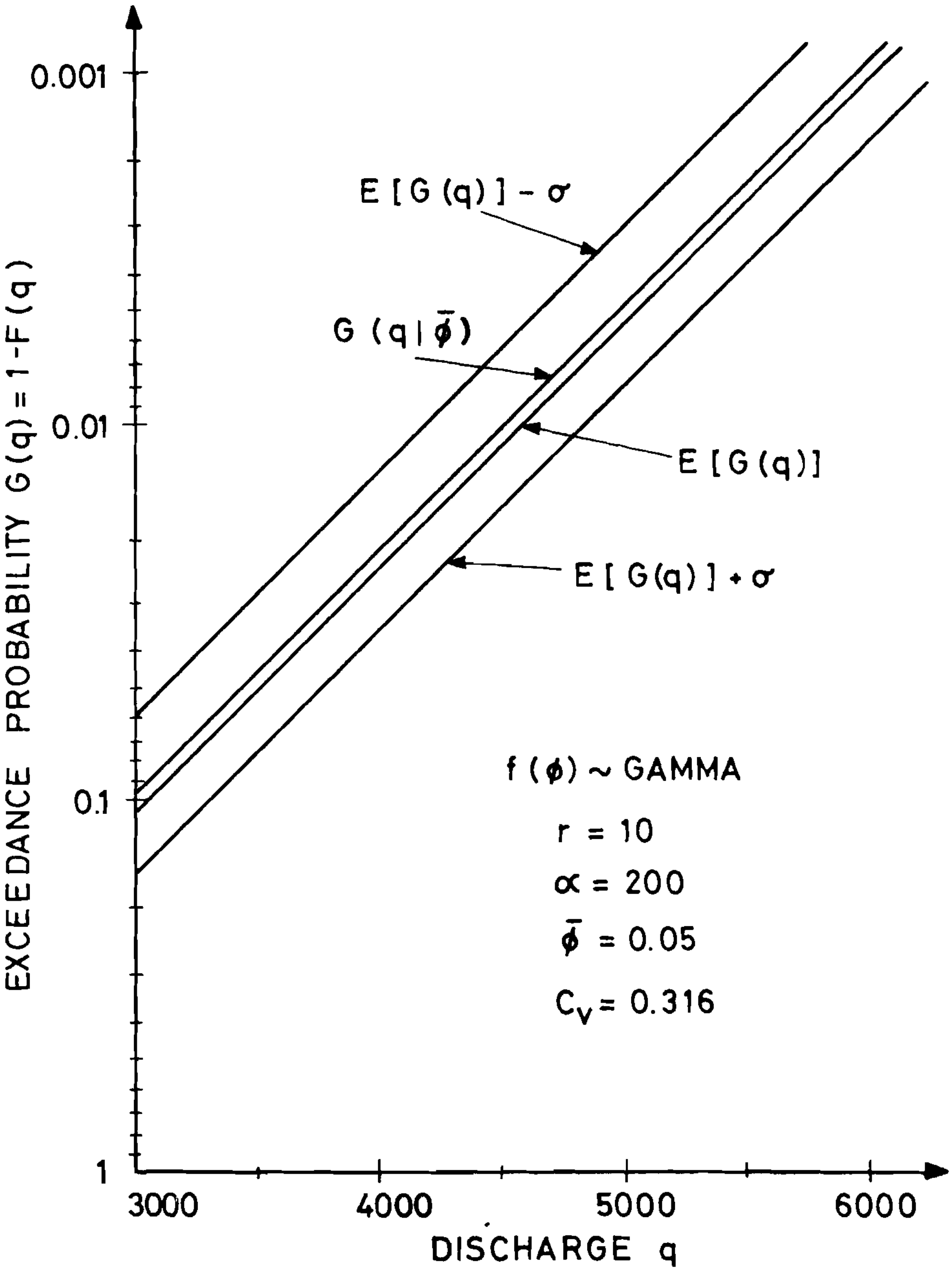

FIGURE 11. FREQUENCY CURVES FOR f(ϕ), GAMMA-1 WITH $\bar{\phi}$ = 0.05 AND C_V = 0.316.

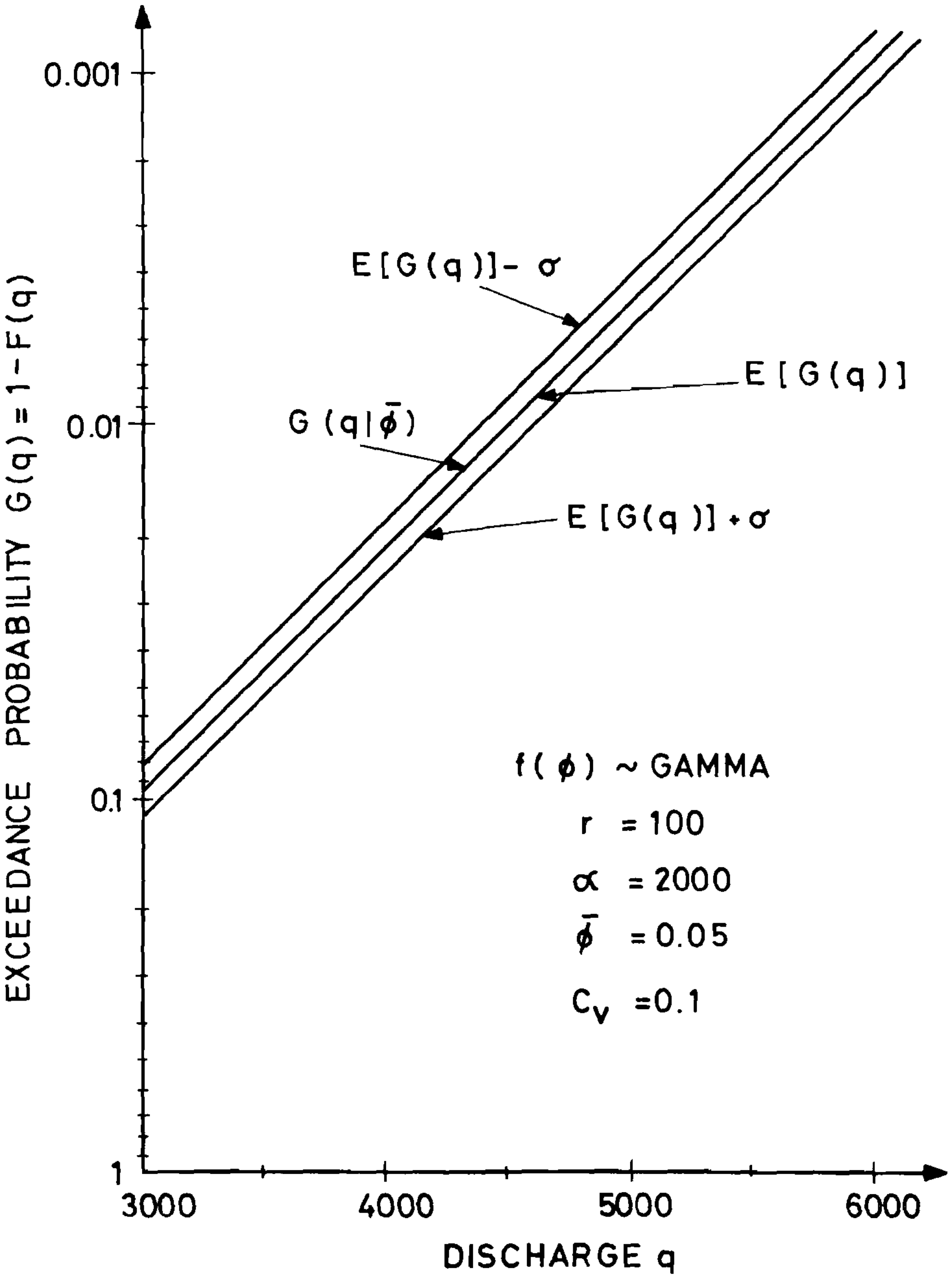

FIGURE 12. FREQUENCY CURVES FOR $f(\phi)$, GAMMA-1 WITH $\bar{\phi} = 0.05$ AND $C_V = 0.10$.

by the deterministic analysis, is substantial, since the stochastic analysis predicts that that peak discharge has a return period of 50 years. This error in accounting for parameter uncertainty may lead to serious design problems. When the information about ϕ is very good, which is represented by a tight distribution on ϕ (and shown in Figure 12), the difference between the two analyses is very small. Of course, this is expected.

This analysis only considered one uncertain parameter in the rainfall runoff modelling. The implications of considering many uncertain parameters are evident.

Conclusion

This paper analyzes the uncertainty in the output of a deterministic rainfall-runoff model due to the uncertainty in the models' parameters. Eagleson's derived flood frequency analysis is used to find the constant peak discharge boundary in the $\bar{i}$ - t_r plane, which in turn is used to define Rq_m, the region in which combinations of $\bar{i}$ and t_r yield discharges less than or equal to q_m. This boundary permitted the evaluation of the flood exceedance probability, $G(q_m)$ which is the probability that $q > q_m$. The uncertainty in the runoff model is represented by the water loss coefficient, ϕ, which results in uncertainty in the position of the constant peak discharge boundary for q_m and in the size and location of the region Rq_m. The expected flood exceedance probability, $E[G(q_m)]$, is found by

$$G(q_m) = E\left[G(q_m)\right] = 1 - \int_{\phi} f(\phi)\, d\phi \int_{Rq_m|\phi} f(\bar{i},t_r)\, d\bar{i}\, dt \quad , \tag{52}$$

which considers the uncertainty in ϕ.

Two probability density functions are obtained analytically. One is the peak discharge, conditional upon an exceedance probability level, and the other is the exceedance probability at a peak discharge level. This leads to the result that the use of a point estimate for the water loss ϕ underestimates the peak discharge for a given exceedance level, $G(q_m)$. Similarly, such a procedure underestimates the exceedance probability for a given peak discharge.

Continued research remains to be done on parameter uncertainty in rainfall runoff modelling. There are those parameters which vary from storm to storm--for example, the rainfall interior pattern--which are really stochastic processes and should be analyzed in such a framework. There are those parameters which are uncertain, due to statistical uncertainty. Their effect upon the region Rq_m has not been fully researched either. The area of parameter uncertainty in modelling the rainfall runoff process will provide many years of interesting work.

The extension of the theory presented here to other simulation models outside of hydrology--for example, water quality models--is straightforward. If simulation models are going to be applied for prediction, where the concern is an unknown future state of nature (an urbanized watershed, for example), then the probability distribution on the models' outputs should be estimated if the outputs are used to make meaningful decisions.

Furthermore, the analytical procedures presented here should be applied to the next step--the evaluation of the worth of data and their economic affects upon project designs. Uncertainty in various parameters (or types of parameters) have different affects upon the uncertainty of the model outputs which are used in decision making.

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