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I. Introduction

In portfolio analysis, the basic setting is that of an individual or a group of individuals making inferences and decisions in the face of uncertainty about future security prices and related variables. Formal models for decision making under uncertainty require inputs such as probability distributions to reflect a decision maker's uncertainty about future events and utility functions to reflect a decision maker's preferences among possible consequences [30]. Moreover, when a series of interrelated decisions is to be made over time, the decision maker should 1) revise his probability distributions as new information is obtained and 2) take into account the effect of the current decision on future decisions. In terms of formal models of the decision-making process, probability revision can be accomplished by using Bayes' theorem and the interrelationships among the decisions can be taken into consideration by using dynamic programming to determine optimal decisions. Since portfolio selection and revision involves a series of interrelated decisions made over time, formal portfolio models should, insofar as possible,

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incorporate these features. A search of the extensive literature concerning portfolio models indicates, however, that such models have ignored one or both of these features.

Since Markowitz [18] developed his original model of portfolio selection, a considerable amount of work has been conducted in the area of mathematical portfolio analysis, and much of this work is summarized by Sharpe [31] and Smith [33]. Although the emphasis in portfolio analysis has been primarily on single-period models and portfolio selection, multiperiod models and portfolio revision are investigated by Tobin [35], Smith [32], Mossin [21], Pogue [22], Chen, Jen, and Zionts [3], and Hakansson [13,14]. In addition, general multiperiod models of consumption-investment decisions are developed by Hakansson [10,11,12], Merton [19], Samuelson [29], Fama [6], and Meyer [20]. However, it is generally assumed that the probability distributions of interest are completely specified and that they are unaffected by new information, implying that the portfolio revision models do not involve probability revision over time. Bayesian models have received virtually no attention in the portfolio literature. Mao and Särndal [17] present a simple, discrete, single-period Bayesian model in which the returns from securities are related to the level of general business activity and information is obtained concerning business conditions. Kalymon [16] develops a model that is similar to the inferential model presented in Winkler [37] and discussed in Section II of this paper, but his paper is primarily concerned with measuring "risk" in terms of the variance of returns;

the implications of changes in the relevant distributions over time with respect to multiperiod portfolio models are not investigated.

The purpose of this paper is to present general models for portfolio selection and revision that utilize Bayesian inferential procedures to formally update probability distributions as new information is obtained. Both single-period (myopic) models and multiperiod models are considered. In Section II a Bayesian inferential model is discussed, and in Section III the portfolio selection and revision models are presented. Section IV illustrates the models with examples involving linear and quadratic utility, and a brief summary and discussion is presented in Section V.

II. A Bayesian Model for Forecasting Future Security Prices

Suppose that a particular security is under consideration, and let $\tilde{x}_i$ represent the price of that security at time i, where i = 0 corresponds to the current time. The objective in developing a model to forecast a future price such as $\tilde{x}_t$ is to include restrictions that simplify the analysis without greatly limiting the realism of the model. As a starting point, a very simple model will be presented. The model deals with price differences, $\tilde{d}_i = \tilde{x}_i - \tilde{x}_{i-1}$, and assumes that the price differences are independent and identically distributed and that the distribution of $\tilde{d}_i$ belongs to a certain family of distributions which may be indexed by the parameter (or vector of parameters) $\tilde{\theta}$. Given a prior distribution, $f(\theta)$, the marginal distribution at time 0 of $\tilde{x}_t$, which is called a predictive distribution in Bayesian terminology, can be found.

For example, suppose that $\tilde{d}_i$ is normally distributed with unknown mean $\tilde{\mu}$ and known variance σ^2 and that the prior distribution of $\tilde{\mu}$ is a normal distribution with mean m_0 and variance σ^2/n_0. Then at time 0, the marginal distribution of $\tilde{d}_i$ is a normal distribution with mean m_0 and variance $(n_0 + 1)\sigma^2/n_0$, and the predictive distribution $f(x_t|x_0)$ is a normal distribution with mean $x_0 + tm_0$ and variance $(n_0 + t)\sigma^2/n_0$. Note that the particular choice of distributions greatly simplifies matters. Since $\tilde{d}_i$ is normally distributed for each i, the sum of price differences $\sum_{j=1}^{t} \tilde{d}_j = \tilde{x}_t - x_0$ is normally distributed. Given that $\tilde{\mu}$ is also normally distributed, the derivation of $f(x_t|x_0)$ is quite simple.

Bayes' theorem is used to revise the distributions of $\tilde{\theta}$ and of $\tilde{x}_t$ as new information in the form of observed prices becomes available. To simplify matters, it is assumed that the only relevant information available in the i^{th} time period (the period from time i-1 to time i) is $\tilde{x}_i$. For the example utilizing normality assumptions, the distribution of $\tilde{\mu}$ at time $i(i = 1,\ldots,t - 1)$, $f(\mu|x_0,\ldots,x_i)$, is normal with mean $m_i = (n_0m_0 + x_i - x_0)/(n_0 + i)$ and variance $\sigma^2/n_i = \sigma^2/(n_0 + i)$. The predictive distribution of $\tilde{x}_t$ at time i is normal with mean $x_i + (t - i)m_i$ and variance $(n_i + t - i)\sigma^2/n_i$.

Perhaps the most important aspect of the implementation of a model of this nature is the determination of the necessary inputs, which include the length of the time intervals, the definition of price, the measure of price shifts, the statistical model for the data-generating process, and the prior distribution. As in any modelling situation, the inputs must be chosen to provide a suitable balance between realism and manageability.

From a decision theoretic standpoint, the average, high, and low prices of a security during a period may be of greater interest than the closing price at the end of the period. The model in this paper can be formulated in terms of closing prices, average prices, high prices, low prices, or possibly yet other definitions of price. The definition of price may affect other details of the model (e.g. the variance of an average price might be expected to be smaller than that of a closing price), so it is necessary to carefully specify which definition is to be used (see [36]).

In the literature concerning probability distributions relating to security prices, the variable of interest is frequently the difference in the natural logarithms of prices. Replacing $\tilde{d}_i$ with $\tilde{\Delta}_i = \log \tilde{x}_i - \log \tilde{x}_{i-1}$ would be a convenient modification of the model if the process that generates differences in log prices can be represented by a reasonably tractable family of distributions. For instance, the normal family of distributions, which is relatively easy to work with, may provide a closer fit to differences in log prices than to straight differences in prices. Furthermore, a difference in log prices is the logarithm of $1 + \tilde{r}_i$, where $\tilde{r}_i = (\tilde{x}_i - \tilde{x}_{i-1})/\tilde{x}_{i-1}$. If the time periods are short enough that values of $\tilde{r}_i$ far from zero are very unlikely, then $\tilde{\Delta}_i$ is approximately equal to $\tilde{r}_i$, which is a convenient variable to consider in portfolio problems.

The model is flexible in terms of the choice of a statistical model to represent the data-generating process as well as in terms of the choice of variables. The example assumed a normal data-generating process, but empirical evidence (e.g. see [4]) suggests that the distribution of price changes of securities is non-Gaussian and can be represented most generally in terms of the family of stable distributions (which includes the normal distribution as a special case). Unfortunately, the family of stable distributions is more difficult to work with than the normal distribution [5,7,8]. Of course, statistical models other than the normal and stable models might also be considered [23,25,26]. An important question in the choice of a family of distributions for $\tilde{d}_i$ is the sensitivity of the inferences and decisions produced by the model to variations in the distribution of $\tilde{d}_i$. If such inferences and decisions tend to be somewhat insensitive to moderate deviations from normality, then the normal family might be a useful approximation to the distribution of $\tilde{d}_i$.

The model is also flexible in terms of the choice of a prior distribution. For the sake of tractability in the application of Bayes' theorem, it is convenient if this distribution is conjugate with respect to the family of distributions chosen to represent the data-generating process (see [28]). Otherwise, it may be necessary to use numerical methods to revise the distributions of interest. In the example presented earlier in this section, the normal distribution for $\tilde{\mu}$ is a conjugate distribution. If the conjugate family is considered too restrictive, it can be broadened considerably without much loss in tractability by allowing mixtures of conjugate distributions. For instance, if the conjugate family is the family of normal distributions, only symmetric, unimodal conjugate prior distributions are available; mixtures of normal distributions, on the other hand, include asymmetric and multimodal distributions. In a study by Bartos [2], distributions for future security prices assessed subjectively by security analysts frequently were multimodal, suggesting that mixtures of conjugate distributions may provide good representations of subjective prior opinions. Of course, even within a family of conjugate distributions or mixtures of conjugate distributions, the problem of choosing a specific distribution remains. Various aspects of the assessment of probability distributions for future security prices are discussed by Bartos [2], Fried [9], Stäel von Holstein [34], and Winkler [36].

The general model presented in this section is reasonably flexible, and various extensions make it even more flexible. For example, it can be extended to the situation in which several securities are of interest and variables other than simply the security prices are considered. Such variables might include economic indicators, variables related to particular industries, variables related to individual securities (e.g. earnings per share), or even forecasts of future values of certain variables. For details concerning such extensions, see [37].

III. Portfolio Selection and Revision

The model described in Section II is of some interest in a purely inferential sense, but that aspect is overshadowed by the potential interest in the model as a basis for making decisions. A portfolio selection and revision procedure utilizing a Bayesian model of security price movements has the desirable feature of updating the probability distributions of interest as new information is obtained. In this section both a single-period portfolio model and a multiperiod model are presented.

Assume that a decision maker (e.g. a portfolio manager) has wealth W_0 (which may be in the form of cash or in the form of an existing portfolio of securities) at time 0 and that he wants to determine an optimal portfolio to hold during the first time period. If W_0 consists of cash, this is a portfolio selection problem; if W_0 consists of a portfolio, it is a portfolio revision problem. In either case, of course, the decision making problem for subsequent periods will be a portfolio revision problem.

It is assumed that the portfolio will be chosen from M risky securities (securities with uncertain rates of return) and one risk-free security (a security with a positive rate of return that is known but may vary from period to period). The risk-free security is labelled security 0, and the risky securities are securities 1 through M. W_i represents the decision maker's wealth at time i (i = 0,1,...), and a_i^k denotes the total amount invested in security k (k = 0,...,M) at the end of period i - 1 (i.e. at time i) before the portfolio is revised at time i. Thus,

$$W_i = \sum_{k=0}^{M} a_i^k \quad ,$$

and the portfolio before revision at time i can be represented by the 1 x (M + 1) vector $\underset{\sim}{a}_i = (a_i^0, a_i^1, \ldots, a_i^M)$. Furthermore, p_i^k and q_i^k represent the amount of security k that is purchased and sold, respectively, at time i. After revision, then, the total amount invested in security k at time i is $a_i^k + p_i^k - q_i^k$. The rate of return on security k during period i + 1 is denoted by r_{i+1}^k, so the amount invested in security k at time i + 1 before revision is

$$(1 + r_{i+1}^k)(a_i^k + p_i^k - q_i^k) \quad ,$$

and the total wealth at time i + 1 is simply

$$W_{i+1} = \sum_{k=0}^{M} (1 + r_{i+1}^k)(a_i^k + p_i^k - q_i^k) \quad .$$

The decision variables at time i are the vectors $\underset{\sim}{p}_i = (p_i^0, p_i^1, \ldots, p_i^M)$ and $\underset{\sim}{q}_i = (q_i^0, q_i^1, \ldots, q_i^M)$, and the uncertainty facing the decision maker involves future rates of return, $\tilde{\underset{\sim}{r}}_j = (r_j^0, \tilde{r}_j^1, \ldots, \tilde{r}_j^M)$, for j = i + 1, i + 2, ... (The uncertainty only involves the last M elements of $\tilde{r}_j$, since r_j^0, the return on the risk-free security during period j, is known.) Inferential models such as the model presented in Section II can be used to update the probability distribution of $\tilde{r}_j$. The details of such models are not required for the purposes of this section, but the examples in Section IV will illustrate the use of a specific Bayesian inferential model in portfolio selection and revision.

A. A Single-Period Model

The distinguishing feature of a single-period portfolio model, as opposed to a multiperiod model, is that the decision maker behaves myopically in the sense that he never looks more than one period into the future. At time i, he chooses a portfolio to maximize $E_i U(\tilde{W}_{i+1})$, the expected utility of his wealth at time i + 1, where the subscript on the expectation operator indicates that expectations are taken with respect to the decision maker's joint probability distribution at time i.

First, consider the case in which there are no transactions costs. Then at time i the decision maker wants to choose p_i and q_i to

$$\text{Max} \quad E_i U\left[\sum_{k=0}^{M} (1 + \tilde{r}_{i+1}^k)(a_i^k + p_i^k - q_i^k)\right] \quad ,$$

subject to the following constraints:

$$\sum_{k=0}^{M} p_i^k = \sum_{k=0}^{M} q_i^k \quad ,$$

$$0 \leqq q_i^k \leqq a_i^k \quad , \qquad k = 0,\ldots,M \quad ,$$

$$p_i^k \geqq 0 \quad , \qquad k = 0,\ldots,M \quad ,$$

and

$$p_i^k q_i^k = 0 \quad , \qquad k = 0,\ldots,M \quad .$$

The first constraint states that the total amount of securities purchased must equal the total amount sold, the next 2M + 2 constraints require that all amounts purchased and sold be nonnegative and that the amount sold of any security cannot exceed the amount currently invested in that security (i.e. short sales are not allowed), and the final M + 1 constraints are included to preclude the possibility of simultaneously purchasing and selling positive amounts of the same security. Because there are no transactions costs, simultaneously purchasing 20 shares and selling 10 shares of a security is equivalent to purchasing 10 shares and selling none. If the final M + 1 constraints were not included, the decision making problem as stated above would have an infinite number of solutions corresponding to a single optimal portfolio; precluding simultaneous purchasing and selling results in a one-to-one correspondence between a choice of $(\underset{\sim}{p}_i,\underset{\sim}{q}_i)$ and the resulting portfolio, $\underset{\sim}{a}_i + \underset{\sim}{p}_i - \underset{\sim}{q}_i$.

As stated above, the decision making problem is one of portfolio revision. If the decision maker's initial wealth W_0

is in the form of cash, then $a_0^k = q_0^k = 0$ for all m, and the problem is one of portfolio selection

$$\text{Max} \quad E_0 U\Big[\sum_{k=0}^{M} (1 + \tilde{r}_1^k)\, p_0^k\Big] \quad ,$$

subject to

$$\sum_{k=0}^{M} p_0^k = W_0$$

and

$$p_0^k \geqq 0 \quad , \qquad k = 0, \ldots M \quad .$$

Next, suppose that there are transactions costs, represented by the positive, increasing functions C_p^k and C_q^k where $C_p^k(z)$ is the transactions cost associated with purchasing an amount z of security k and $C_q^k(z)$ is the transactions cost associated with selling an amount z of security k. At time i, the decision maker wants to choose $\underset{\sim}{p}_i$ and $\underset{\sim}{q}_i$ to

$$\text{Max} \quad E_i U\Big[\sum_{k=0}^{M} (1 + \tilde{r}_{i+1}^k)(a_i^k + p_i^k - q_i^k)\Big] \quad ,$$

subject to the constraint set

$$G_i = \{\underset{\sim}{p}_i, \underset{\sim}{q}_i \mid \sum_{k=0}^{M} [p_i^k + C_p^k(p_i^k)] = \sum_{k=0}^{M} [q_i^k - C_q^k(q_i^k)] \quad ,$$

$$0 \leqq q_i^k \leqq a_i^k \quad , \qquad k = 0, \ldots, M \quad ,$$

and $\qquad p_i^k \geqq 0 \quad , \qquad k = 0, \ldots, M\} \quad .$

The first constraint reflects the fact that transactions costs reduce the total amount of securities that can be purchased as a result of selling other securities. In selling q_i^k of security k, the decision maker only receives $q_i^k - C_q^k(q_i^k)$, and in order to purchase p_i^k of security k, he must spend $p_i^k + C_p^k(p_i^k)$. Note that if $C_p^k \equiv C_q^k \equiv 0$, the first constraint is identical to the first constraint in the zero-transactions-cost case. Also, if $C_p^k(p_i^k) = cp_i^k$ and $C_q^k(q_i^k) = cq_i^k$ (i.e. if there is a constant per-unit transactions cost of c for both purchasing and selling), the first constraint can be written in the form

$$(1 + c) \sum_{k=0}^{M} p_i^k = (1 - c) \sum_{k=0}^{M} q_i^k \quad ,$$

in which case the total amount of securities purchased can only be $(1 - c)/(1 + c)$ times as great as the total amount of securities sold. Obviously, since $c > 0$, $(1 - c)/(1 + c) < 1$. Also, unless $c < 1$, the transactions costs would be confiscatory.

The constraints included in the first model in this section to prevent simultaneous purchasing and selling of the same security ($p_i^k q_i^k = 0$, $k = 0,\ldots,M$) are not needed when transactions costs are always positive. If $p_i^k q_i^k > 0$, reducing both p_i^k and q_i^k by $z = \min\{p_i^k, q_i^k\}$ yields the same amount of security k in the portfolio but changes the transactions costs associated with security k from $C_p^k(p_i^k) + C_q^k(q_i^k)$ to $C_p^k(p_i^k - z) + C_q^k(q_i^k - z)$. This change is a reduction because C_p^k and C_q^k are increasing functions. The amount thus saved could always be invested in the risk-free security to yield a certain return of $r_{i+1}^0 > 0$, thereby increasing

$E_i U(\tilde{W}_{i+1})$, assuming of course that U is monotone increasing. Therefore, the optimal solution to the portfolio revision problem in the case of positive transactions costs will never involve simultaneous purchasing and selling of the same security.

If the decision maker's initial wealth W_0 is in the form of cash, the portfolio selection problem with positive transactions costs is to

$$\text{Max} \quad E_0 U[\sum_{k=0}^{M} (1 + \tilde{r}_1^k) p_0^k] \quad ,$$

subject to

$$G_0^* = \{\underset{\sim}{p}_0 | \sum_{k=0}^{M} [p_o^k + c_p^k(p_0^k)] = W_0$$

and

$$p_0^k \geqq 0, \qquad k = 0,\ldots,M\} \quad .$$

To avoid the possibility of holding cash, it is assumed that the expected return from at least one security is large enough to assure that the decision maker will be fully invested. This can be guaranteed, for instance, by requiring that $zr_i^0 > c_p^0(z) + c_q^0(z)$ for all i and z, implying that holding the risk-free security is always better than holding cash.

Although the single-period portfolio models presented in this section are myopic by definition, they do provide for portfolio revision on the basis of new information. This information includes the past returns on securities and any other information that is included in the inferential model used to update probability distributions for future returns.

B. A Multiperiod Model

Single-period models ignore the dynamic nature of the portfolio selection and portfolio revision problems. Mossin [21, p.215] states, "In a multiperiod theory the development through time of total wealth becomes crucial and must be taken into account." The most general multiperiod model involves an infinite horizon, but the model presented in this section assumes a finite horizon of t periods (t = 1 corresponds to the single-period model). That is, at time 0, the decision maker wants to maximize the expected utility of $\tilde{W}_t$, the wealth at the end of the finite horizon, taking into consideration the uncertainties involving future returns and the possibility of revising the portfolio at times 1,2,...,t - 1. This requires a dynamic programming formulation whereby the optimal solution is determined through backward induction, starting with the decision at time t - 1 and working backward to the decision at time 0.

At time t - 1, there is only one period remaining until time t, so the single-period model is applicable. Assuming positive transactions costs, the decision maker should choose $\underset{\sim}{p}_{t-1}$ and $\underset{\sim}{q}_{t-1}$ to

$$\text{Max} \quad E_{t-1}U\left[\sum_{k=0}^{M} (1 + \tilde{r}_t^k)(a_{t-1}^k + p_{t-1}^k - q_{t-1}^k)\right] \quad ,$$

subject to the constraint set G_{t-1}. The solution of this problem for any given $\underset{\sim}{a}_{t-1}$ yields the optimal portfolio revision at time t - 1.

Before time t - 1, of course, $\underset{\sim}{a}_{t-1}$ is not known, but previous decisions must be related to the decision at time t - 1. Define $U^*_{t-1}(\underset{\sim}{a}_{t-1})$ to be the expected utility corresponding to the optimal solution to the portfolio revision problem at time t - 1, given $\underset{\sim}{a}_{t-1}$:

$$U^*_{t-1}(\underset{\sim}{a}_{t-1}) = \max_{\underset{\sim}{p}_{t-1},\underset{\sim}{q}_{t-1}} E_{t-1}U[\sum_{k=0}^{M} (1 + \tilde{r}^k_t) \cdot (a^k_{t-1} + p^k_{t-1} - q^k_{t-1})] \quad ,$$

where the maximization is subject to the constraint set G_{t-1}, of course. The decision maker's objective at time t - 2, then, should be to choose $\underset{\sim}{p}_{t-2}$ and $\underset{\sim}{q}_{t-2}$ to maximize $E_{t-2}[U^*_{t-1}(\underset{\sim}{a}_{t-1})]$. But

$$a^k_{t-1} = (1 + \tilde{r}^k_{t-1})(a^k_{t-2} + p^k_{t-2} - q^k_{t-2}) \quad ,$$

so the portfolio revision problem at time t - 2 can be written

$$\max_{\underset{\sim}{p}_{t-2},\underset{\sim}{q}_{t-2}} E_{t-2}\left\{\max_{\underset{\sim}{p}_{t-1},\underset{\sim}{q}_{t-1}} E_{t-1}U[\sum_{k=0}^{M} (1 + \tilde{r}^k_t)\{(1 + \tilde{r}^k_{t-1}) \cdot (a^k_{t-2} + p^k_{t-2} - q^k_{t-2}) + p^k_{t-1} - q^k_{t-1}\}]\right\} \quad ,$$

subject to the constraint set $\{G_{t-2},G_{t-1}\}$. If the maximum value of this objective function at time t - 2 is denoted by $U^*_{t-2}(\underset{\sim}{a}_{t-2})$, the decision at time t - 3 is to choose $\underset{\sim}{p}_{t-3}$ and $\underset{\sim}{q}_{t-3}$ to maximize $E_{t-3}[U^*_{t-2}(\underset{\sim}{a}_{t-2})]$.

The process of backward induction continues until the initial portfolio revision problem is reached. At time 0, the decision maker should choose $\underset{\sim}{p}_0$ and $\underset{\sim}{q}_0$ to maximize

$E_0[U_1^*(\underset{\sim}{a}_1)]$. That is, he should

$$\max_{\underset{\sim}{p}_0,\underset{\sim}{q}_0} E_0\left\{ \max_{\underset{\sim}{p}_1,\underset{\sim}{q}_1} E_1 \{ \max_{\underset{\sim}{p}_2,\underset{\sim}{q}_2} E_2 \quad \ldots \{ \max_{\underset{\sim}{p}_{t-1},\underset{\sim}{q}_{t-1}} E_{t-1}U(W_t)\}\ldots\}\right\}$$

where the maximization at time i is subject to the constraint set G_i. (If the decision maker's initial wealth W_0 is in the form of cash rather than an initial portfolio $\underset{\sim}{a}_0$, the constraint set at time 0 is G_0^*.)

The multiperiod portfolio model, like the single-period model, provides for portfolio revision on the basis of new information. In addition, it takes into consideration the potential effects of a portfolio revision decision on future portfolio revision decisions.

The determination of the necessary inputs for the portfolio selection and revision models presented in this section is an important and often very difficult aspect of the implementation of such models. The determination of inputs for the inferential model was discussed in Section II. Of course, the decision making problem may dictate which uncertain quantities are of interest and may therefore affect some details of the inferential model. On the other hand, it is possible that certain assumptions concerning the inferential model may lead the decision maker to reformulate the decision making model in slightly different terms.

In addition to the inferential inputs, the portfolio models require the assessment of a utility function, the choice of functions to represent transactions costs, and the choice of a time horizon in the multiperiod model. The utility function, which represents the decision maker's relative preferences among various amounts of money, is perhaps the most crucial of these inputs. In theory, at least, one can elicit a decision maker's utility function by asking questions concerning his preferences between two gambles and by adjusting the pairs of gambles to determine various points on his utility function [24]. This process is not quite as simple as it sounds, however, and more work is needed regarding the assessment of utility functions. Moreover, it is convenient from the standpoint of tractability if the utility function can be approximated closely by a simple mathematical function. (This is analogous to the desirability of choosing a prior distribution that is a member of the conjugate family of distributions.) Some of the simple functions that have been used to represent utility functions in various applications are linear, quadratic, exponential, and logarithmic functions, and the relative merits of these and other functions have been widely debated. With respect to portfolio analysis, most studies have used the mean-variance approach, which, in the absence of distributional assumptions, implies a quadratic utility function. However, some recent articles have seriously questioned the applicability of quadratic utility (e.g. [15]). A final point with respect to utility theory is that if the portfolio selection and revision problem involves a group or a corporation rather than a single individual, the difficulties encountered in determining an appropriate utility function to be used in the model are intensified (e.g. [1]).

The choice of functions C_p^k and C_q^k to represent transactions costs technically should be determined by the institutions controlling the purchasing and selling of securities. These functions are subject to modification from time to time, however, and they are not always convenient to use from the standpoint of tractability. Matters are greatly simplified if it can be assumed that $C_p^k \equiv C_p$ and $C_q^k \equiv C_q$ for all k and that $C_p \equiv C_q \equiv C$. Under these assumptions, it is only necessary to determine a single function C. Under these assumptions, it is only necessary to determine a single function C. Some possibilities are a stepwise linear function (e.g. [22]), a linear function with a fixed charge [$C(z$ $b + cz$], and a linear function without a fixed charge [$C(z) = cz$

In some problems the selection of a time horizon t may be simple (e.g. if a portfolio must be liquidated at a certain date in the future), but in most cases it is by no means obvious. The sensitivity of portfolio decisions to the choice of t is of considerable importance. It may be that the optimal portfolio at time 0 varies little as the number of periods until the horizon increases beyond some finite t. The choice of t may involve a trade-off between computational ease and the accuracy of the initial portfolio allocation.

IV. Examples

The purpose of this section is to illustrate how, given a particular inferential model and a particular portfolio selection and revision model, an optimal portfolio can be selected. The examples to be presented are purposely quite simple, involving one risk-free security, one risky security ($M = 1$), and a time horizon of two periods ($t = 2$) in the multiperiod case. A constant per-unit transactions cost of $c \geq 0$ is assumed, with $c = 0$ corresponding to the case of no transactions costs.

The inferential model assumes a stationary normal data-generating process for the log price changes, $\tilde{\Delta}_i^1 = \log \tilde{x}_i^1 - \log \tilde{x}_{i-1}^1$, of the risky security. The data-generating process has known variance σ^2 and unknown mean $\tilde{\mu}$. (This differs from the example in Section II in that $\tilde{\Delta}_i^1$ instead of $\tilde{d}_i^1$ is assumed to be generated by a normal process.) The decision maker's prior distribution for $\tilde{\mu}$ at time 0 is a normal distribution with mean m_0 and variance $n_0^{-1}\sigma^2$. At time i, then, the distribution of $\tilde{\mu}$ is a normal distribution with mean $m_i = (n_0 m_0 + \sum_{j=1}^{i} \Delta_j^1)/(n_0 + i)$ and variance $(n_0 + i)^{-1}\sigma^2$, and the predictive distribution of $\tilde{\Delta}_{i+1}^1$ is a normal distribution with mean m_i and variance $(n_0 + i + 1)\sigma^2/(n_0 + i)$. Furthermore, it is assumed that the time periods are short enough (implying the potential values of $\tilde{\Delta}_i^1$ are small enough) that $\tilde{\Delta}_i^1$ provides a very close approximation to r_i^1 -- in this section, $\tilde{\Delta}_i^1$ and r_i^1 are considered to be interchangeable. The risk-free security, of course, has fixed return r_i^0 in period i.

A. Linear Utility

The easiest situation to deal with in terms of utility is the situation in which the decision maker's utility function is linear with respect to money. U can then be taken as U(W) = W, so the decision maker's objective is to maximize expected terminal wealth. Linear utility and the optimal portfolios generated under the assumption of linear utility seem unrealistic, but the use of linear utility serves to demonstrate in a simple manner the differences among the different portfolio models presented in Section III.

First, consider the single-period model. At time i, the decision maker should choose $\underset{\sim}{p}_i$ and $\underset{\sim}{q}_i$ to

$$\text{Max } E_i\left[(1 + r_{i+1}^0)(a_i^0 + p_i^0 - q_i^0) + (1 + \tilde{r}_{i+1}^1)(a_i^1 + p_i^1 - q_i^1)\right]$$

subject to

$$G_i = \{\underset{\sim}{p}_i, \underset{\sim}{q}_i \mid (1 + c)(p_i^0 + p_i^1) = (1 - c)(q_i^0 + q_i^1)\} \quad ,$$

$$0 \leqq q_i^k \leqq a_i^k \quad , \quad k = 0,1 \quad ; \quad \text{and } p_i^k \geqq 0 \quad ,$$

$$k = 0,1\} \quad .$$

If c = 0, the constraints $p_i^k q_i^k = 0$, k = 0,1, must be added.

Taking expectations, the objective function can be expressed

$$\text{Max} \quad [(1 + r_{i+1}^0)(a_i^0 + p_i^0 - q_i^0) + (1 + m_i)(a_i^1 + p_i^1 - q_i^1)] \quad .$$

This is a simple linear programming problem, and the solution is

$$(p_i^0, p_i^1) = \begin{cases} (\beta^{-1} a_i^1, 0) & \text{if } 1 + r_{i+1}^0 \geqq \beta(1 + m_i) \quad , \\ (0,0) & \text{if } \beta^{-1}(1 + m_i) < 1 + r_{i+1}^0 \\ & \quad < \beta(1 + m_i) \quad , \\ (0, \beta^{-1} a_i^0) & \text{if } 1 + r_{i+1}^0 \leqq \beta^{-1}(1 + m_i) \quad , \end{cases}$$

where $\beta = (1 + c)/(1 - c)$ is the amount of one security that must be sold to buy one unit (i.e. $1) of the other security. Since $M = 1, q_i^0 = \beta p_i^1$ and $q_i^1 = \beta p_i^0$, solutions in this section, therefore, are just given in terms of $\underset{\sim}{p}_i$.

In the case of zero transactions costs ($c = 0$), $\beta = 1$, and the optimal strategy at time i is

$$(p_i^0, p_i^1) = \begin{cases} (a_i^1, 0) & \text{if } r_{i+1}^0 \geqq m_i \quad , \\ (0, a_i^0) & \text{if } r_{i+1}^0 < m_i \quad . \end{cases}$$

At each time period, then, the decision maker invests everything in the security with the higher expected return for the next period. In fact, this result generalizes to the case of more than two securities. If $c > 0$, it is possible that the optimal portfolio will not contain the security with the higher expected return for the next period. The effect of nonzero transactions costs is to make it less likely that a portfolio will actually be changed at any particular time. (Note that for any value of c, though, if the portfolio is changed, it will be changed to a portfolio containing only one security.) For instance, if the optimal portfolio at time 0 consists solely of the risk-free security (i.e. if $a_0^1 + p_0^1 - q_0^1 = 0$), then the probability of including any of the risky security in the optimal portfolio at time 1 (implying that the decision maker will switch entirely to the risky security) is

$$P[1 + r_2^0 \leqq \beta^{-1}(1 + \tilde{m}_1)] = P(\tilde{m}_1 \geqq \beta r_2^0 + \beta - 1) \quad .$$

Without transactions costs, the probability of switching to the risky security at time 1 is just $P(\tilde{m}_1 \geqq r_2^0)$. Since $\beta > 1$ for nonzero transactions costs, $\beta r_2^0 + \beta - 1 > r_2^0$, so the decision maker is less likely to switch when $c > 0$.

If the initial wealth W_0 is in the form of cash, $a_0^0 = a_0^1 = q_0^0 = q_0^1 = 0$, and the optimal $\underset{\smile}{p}_0$ is given by

$$(p_0^0, p_0^1) = \begin{cases} ([1 + c]^{-1}W_0, 0) & \text{if } r_1^0 \geqq m_0 \quad , \\ (0, [1 + c]^{-1}W_0) & \text{if } r_1^0 < m_0 \quad . \end{cases}$$

In this situation, the optimal portfolio will always consist of just one security, ignoring the borderline situation in which <u>any</u> portfolio is optimal (at time 0, this situation occurs if $r_1^0 = m_0$). If W_0 consists of an initial portfolio a_0 with $a_0^0 > 0$ and $a_0^1 > 0$ (i.e. an initial "diversified" portfolio), the decision maker will retain a diversified portfolio as long as the difference between r^0_{i+1} and m_i is small enough that $(p^0_i, p^1_i) = (0,0)$, even if U is linear.

In the multiperiod portfolio model with t = 2, the decision at time 1 is identical to that of the single-period model

$$(p^0_1, p^1_1) = \begin{cases} (\beta^{-1}a^1_1, 0) & \text{if } 1 + r^0_2 \geq \beta(1 + m_1) \ , \\ (0,0) & \text{if } \beta^{-1}(1 + m_1) < 1 + r^0_2 \\ & \qquad < \beta(1 + m_1) \ , \\ (0, \beta^{-1}a^0_1 & \text{if } 1 + r^0_2 \leq \beta^{-1}(1 + m_1) \ . \end{cases}$$

At time 0, the decision maker should choose $\underset{\sim}{p}_0$ and $\underset{\sim}{q}_0$ to

$$\underset{\underset{\sim}{p}_0, \underset{\sim}{q}_0}{\text{Max}} \ E_0\Big[\underset{\underset{\sim}{p}_1, \underset{\sim}{q}_1}{\max} \ E_1\{(1 + r^0_2)[(1 + r^0_1)(a^0_0 + p^0_0 - q^0_0) + p^0_1 - q^0_1] + (1 + \tilde{r}^1_2)[(1 + \tilde{r}^1_1) \cdot (a^1_0 + p^1_0 - q^1_0) + p^1_1 - q^1_1]\}\Big]$$

subject to the constraint set $\{G_0, G_1\}$. The objective function simplifies to

$$\underset{p_0, q_0}{\text{Max}} \ E_0[(1 + r^0_2(1 + r^0_1)(a^0_0 + p^0_0 - q^0_0) + (1 + m_1)(1 + \tilde{r}^1_1)(a^1_0 + p^1_0 - q^1_0) + \underset{\underset{\sim}{p}_1, \underset{\sim}{q}_1}{\max} \{(1 + r^0_2)(p^0_1 - q^0_1) + (1 + m_1)(p^1_1 - q^1_1)\}] \ ,$$

or

$$\max_{\underset{\sim}{p}_0, \underset{\sim}{q}_0} \quad (1 + r_2^0)(1 + r_1^0)(a_0^0 + p_0^0 - q_0^0) + \{(1 + m_0)^2 + n_0^{-1}\sigma^2\} (a_0^1 + p_0^1 - q_0^1)$$

$$+ \int_A (1 + r_1^1)[\beta^{-1}(1 + r_2^0) - (1 + m_1)](a_0^1 + p_0^1 - q_0^1) f(m_1) \, dm_1$$

$$+ \int_B (1 + r_1^0)[\beta^{-1}(1 + m_1) - (1 + r_2^0)](a_0^0 + p_0^0 - q_0^0) f(m_1) \, dm_1 \quad ,$$

where

$$A = \{m_1 | 1 + r_2^0 \geqq \beta(1 + m_1)\}$$

and

$$B \quad \{m_1 | 1 + r_2^0 \leqq \beta^{-1}(1 + m_1)\} \quad .$$

(The region $\beta^{-1}(1 + m_1) < 1 + r_2^0 < \beta(1 + m_1)$ is omitted because in that region, $p_1^0 = p_1^1 = q_1^0 = q_1^1 = 0$, and thus the relevant partial expectation is zero.) At time 0, the distribution of $\tilde{m}_1$ is a normal distribution with mean m_0 and variance $n_0^{-1}(n_0 + 1)^{-1}\sigma^2$. The objective funciton, then, reduces to

$$\text{Max} \quad K_0(a_0^0 + p_0^0 - q_0^0) + K_1(a_0^1 + p_0^1 - q_0^1) \quad ,$$

where

$$K_0 = (1 + r_2^0)(1 + r_1^0) + E_{0(B)}[(1 + r_1^0)\{\beta^{-1}(1 + \tilde{m}_1) - (1 + r_2^0)\}] \quad ,$$

$$K_1 = (1+m_0)^2 + n_0^{-1}\sigma^2 + E_{0(A)}[(1+\tilde{r}_1^1)\{\beta^{-1}(1+r_2^0) - (1+\tilde{m}_1)\}] ,$$

and $E_{0(A)}$ and $E_{0(B)}$ denote partial expectations, taken at time 0, over the sets A and B, respectively. The partial expectations reflect the effect of the anticipated second-period decision on the first-period decision. The solution to this linear programming problem is

$$(p_0^0, p_0^1) = \begin{cases} (\beta^{-1}a_0^1, 0) & \text{if } K_0 \geqq \beta K_1 , \\ (0,0) & \text{if } \beta^{-1}K_1 < K_0 < \beta K_1 , \\ (0, \beta^{-1}a_0^0) & \text{if } K_0 \leqq \beta^{-1}K_1 . \end{cases}$$

This solution is of the same general form as the solution at time 0 for the single-period model, with K_0 replacing $1 + r_1^0$ and K_1 replacing $1 + m_0$. If the decision maker's initial wealth is in the form of cash, the solution is

$$(p_0^0, p_0^1) = \begin{cases} ([1 + c]^{-1}W_0, 0 & \text{if } K_0 \geqq K_1 , \\ (0, [1 + c]^{-1}W_0) & \text{if } K_0 < K_1 . \end{cases}$$

B. Quadratic Utility

Unless the initial wealth is in the form of a diversified portfolio and transactions costs are high enough relative to differences in expected returns to make it optimal to retain the initial diversified portfolio, the decison maker with linear utility will always invest all of his wealth in a single security. Since real-world investors do not usually behave in this fashion, linear utility is a very questionable assumption. Traditionally, mathematical portfolio analysis has focused on the mean-variance approach, which (in the absence of distributional assumptions) implies some sort of quadratic utility function. A quadratic utility function of the form

$$U(W) = W - bW^2 \text{ for } W < 1/2b \quad ,$$

where $b > 0$, is assumed in this subsection. This utility function implies that the decision maker is risk-averse and that he becomes more risk-averse as W increases.

In the single-period model, the decision maker's objective at time i should be to choose $\underset{\sim}{p}_i$ and $\underset{\sim}{q}_i$ to

$$\begin{aligned} \text{Max} \quad E_i\{&(1 + r^0_{i+1})(a^0_i + p^0_i - q^0_i) + (1 + \tilde{r}^1_{i+1})(a^1_i + p^1_i - q^1_i) \\ &- b[(1 + r^0_{i+1})(a^0_i + p^0_i - q^0_i) \\ &+ (1 + \tilde{r}^1_{i+1})(a^1_i + p^1_i - q^1_i)]^2\} \quad , \end{aligned}$$

subject to G_i. This can be simplified to

$$\text{Max} \ [J_1(p^0_i)^2 + J_2(p^1_i)^2 + J_3 p^0_i + J_4 p^1_i] \quad ,$$

subject to

$$0 \leqq p^0_i \leqq \beta^{-1} a^1_i \text{ and } 0 \leqq p^1_i \leqq \beta^{-1} a^0_i \quad ,$$

where

$$J_1 = -b(1 + r^0_{i+1})^2 - b\beta^2\{(1 + m_i)^2 + (n_0 + i + 1)(n_0 + i)^{-1}\sigma^2\}$$

$$+ 2b\beta(1 + r^0_{i+1})(1 + m_i) \quad ,$$

$$J_2 = -b\beta^2(1 + r^0_{i+1})^2 - b\{(1 + m_i)^2 + (n_0 + i + 1)(n_0 + i)^{-1}\sigma^2\}$$

$$+ 2b\beta(1 + r^0_{i+1})(1 + m_i) \quad ,$$

$$J_3 = (1 + r^0_{i+1}) - \beta(1 + m_i) - 2ba^0_i(1 + r^0_{i+1})^2$$

$$+ 2b\beta a^1_i\{(1 + m_i)^2 + (n_0 + i + 1)(n_0 + i)^{-1}\sigma^2\}$$

$$+ 2b(\beta a^0_i - a^1_i)(1 + 2^0_{i+1})(1 + m^1_i) \quad ,$$

and

$$J_4 = -\beta(1 + r^0_{i+1}) + (1 + m_i) + 2b\beta a^0_i(1 + r^0_{i+1})^2$$

$$-2ba^1_i\{(1 + m_i)^2 + (n_0 + i + 1)(n_0 + i)^{-1}\sigma^2\}$$

$$+ 2b(\beta a^1_i - a^0_i)(1 + r^0_{i+1})(1 + m_i) \quad .$$

Note that the cross-product terms involving $p^0_i p^1_i$ in the original objective function drop out, since $p_i p_i = \beta^{-1} p^0_i q^0_i = 0$. The solution to this quadratic programming problem is

$$(p^0_i, p^1_i) = \begin{cases} (\beta^{-1}a^1_i, 0) & \text{if } -J_3/2J_1 \geqq \beta^{-1}a^1_i \ , \\ (-J_3/2J_1, 0) & \text{if } 0 < -J_3/2J_1 < \beta^{-1}a^1_i \ , \\ (0,0) & \text{if } -J_3/2J_1 \leqq 0 \text{ and } -J_4/2J_2 \leqq 0 \ , \\ (0, -J_4/2J_2) & \text{if } 0 < -J_4/2J_2 < \beta^{-1}a^0_i \ , \\ (0, \beta^{-1}a^0_i) & \text{if } -J_4/2J_2 \geqq \beta^{-1}a^0_i \ . \end{cases}$$

If $c = 0$, $J_1 = J_2$ and $J_3 = -J_4$, and the solution takes on the form

$$(p_i^0, p_i^1) = \begin{cases} (a_i^1, 0) & \text{if } -J_3/2J_1 \geqq a_i^1 \ , \\ (-J_3/2J_1, 0) & \text{if } 0 < -J_3/2J_1 < a_i^1 \ , \\ (0, J_3/2J_1) & \text{if } 0 < J_3/2J_1 < a_i^0 \ , \\ (0, a_i^0) & \text{if } J_3/2J_1 \geqq a_i^0 \ . \end{cases}$$

If the initial wealth W_0 is in the form of cash, then the decision maker should choose $\underset{\sim}{p}_0$ and $\underset{\sim}{q}_0$ at time 0 to

$$\text{Max} \quad E_0\{(1 + r_1^0)p_0^0 + (1 + \tilde{r}_1^1)p_0^1 - b[(1 + r_1^0)p_0^0 + (1 + \tilde{r}_1^1)p_0^1]^2\} \ ,$$

subject to G_0^*. The solution is

$$(p_0^0, p_0^1) = \begin{cases} ([1 + c]^{-1}W_0, 0) & \text{if } L \leqq 0 \ , \\ ([1 + c]^{-1}W_0 - L, L) & \text{if } 0 < L < (1 + c)^{-1}W_0 \\ (0, [1 + c]^{-1}W_0) & \text{if } L \geqq (1 + c)^{-1}W_0 \ , \end{cases}$$

where

$$L = L_1/2bL_2 \ ,$$

$$L_1 = (1 + m_0) - (1 + r_1^0) + 2b(1 + c)^{-1}W_0\{(1 + m_0^1)^2 - (1 + m_0^1)(1 + r_1^0) + n_0 + 1)\sigma^2/n_0\} \ ,$$

and

$$L_2 = [(1 + m_0^1) - (1 + r_1^0)]^2 + (n_0 + 1)\sigma^2/n_0 \ .$$

In the multiperiod model with t = 2, the decision at time 1 is identical to that of the single-period model. At time 0, the decision maker should choose $\underset{\sim}{p}_0$ and $\underset{\sim}{q}_0$ to

$$\underset{\underset{\sim}{p}_0,\underset{\sim}{q}_0}{\text{Max}} \; E_0\Big[\underset{\underset{\sim}{p}_1,\underset{\sim}{q}_1}{\max} \; E_1(\tilde{W}_2 - b\tilde{W}_2^2)\Big] \quad ,$$

subject to $\{G_0, G_1\}$, where

$$\tilde{W}_2 = (1 + r_2^0)[(1 + r_1^0)(a_0^0 + p_0^0 - q_0^0) + p_1^0 - q_1]$$
$$+ (1 + \tilde{r}_2^1)[(1 + \tilde{r}_1^1)(a_0^1 + p_0^1 - q_0^1) + p_1^1 - q_1^1] .$$

This is a problem of the form

$$\underset{\underset{\sim}{p}_0,\underset{\sim}{q}_0}{\text{Max}} \; J_1^*(p_0^0)^2 + J_2^*(p_0^1)^2 + J_3^* p_0^0 + J_4^* p_0^1$$

subject to

$$0 \leqq p_0^0 \leqq \beta^{-1} a_0^1 \text{ and } 0 \leqq p_0^1 \leqq \beta^{-1} a_0^0 \quad .$$

The algebraic expressions for J_1^*, J_2^*, J_3^*, and J_4^* are quite long and are functions of the decision variables through partial expectations over sets such as

$$\{m_1 | 0 < -J_3/2J_1 < \beta^{-1} a_i^1\}$$

(see the solution to the portfolio problem at time 1 under quadratic utility). In order to conserve space, these expressions are not presented here. In general, this maximization problem must be solved numerically rather than analytically.

C. Numerical Examples

To illustrate the two-security, two-period models presented in this section, suppose that a decision maker has initial wealth $W_0 = 1$, consisting of equal amounts of the risk-free security and the risky security (i.e. $a_0^0 = a_0^1 = .5$). Moreover, the risk-free security has a fixed return of .02 in each period ($r_1^0 = r_2^0 = .02$), and the decision maker's uncertainty about the risky security can be summarized by $m_0 = .035$ (the expected return), $\sigma^2 = .001$, and $n_0 = 2$. The constant per-unit transactions cost is $c = .006$.

Assuming linear utility, the optimal decision at time 0 under both the single-period model and the multiperiod model is $q_0^0 = a_0^0 = .5$. That is, the decision maker should sell all of the risk-free security, and the resulting portfolio is $\underset{\sim}{a}_0 + \underset{\sim}{p}_0 - \underset{\sim}{q}_0 = (0, .994)$. At time 1, the decision maker should make no change if $m_1 \geqq .0078$, but he should switch entirely to the risk-free security otherwise. The transactions costs are high enough that the switch should be made only if $\tilde{r}_1^1 \leqq -.047$, and the probability of this event, as seen by the decision maker at time 0, is only .017. Incidentally, if W_0 consists of cash, the optimal portfolio is $\underset{\sim}{p}_0 = (0, .994)$.

The single-period and multiperiod models do not, in general, yield identical solutions. For instance, if m_0 were .03 instead of .035, the solution to the multiperiod model would be unchanged but the solution to the single-period model would be $\underset{\sim}{p}_0 = \underset{\sim}{q}_0 = (0,0)$. The transactions costs are more crucial when the time horizon is only one period; even though $E_0(\tilde{r}_1^1) - r_1^0 = .01$ and the decision maker is not risk-averse, the single-period model does not result in any selling of the risk-free security.

Assuming quadratic utility with $b = .37$ (implying that the utility function is defined for $W \leq 2.7$), the optimal decision at time 0 under the single-period model is $p_0^1 = .08$, yielding $\underset{\sim}{a}_0 = \underset{\sim}{p}_0 - \underset{\sim}{q}_0 = (.419, .580)$. Under the multiperiod model, $p_0^1 = .458$, yielding $\underset{\sim}{a}_0 + \underset{\sim}{p}_0 - \underset{\sim}{q}_0 = (.037, .958)$. Thus, in the multiperiod model, more of the holdings of the risk-free security are transferred to the risky security than in the single-period model, in spite of the small amount of prior information. If more prior information were available (i.e. if n_0 were larger), the variances of the predictive distributions for $\tilde{r}_1^1$ and $\tilde{r}_2^1$ would be smaller, thereby making the risky security even more attractive to a risk-avoider with a quadratic utility function.

The examples indicate that the single-period and multi-period models may lead to quite different portfolios, although the differences (as well as the sensitivity of the objective functions to such differences) obviously depend on the exact nature of the situation and the assumptions that are made about the situation.

V. Summary and Discussion

In this paper we have presented models for portfolio selection and revision that utilize Bayesian inferential procedures to formally update probability distributions of uncertain quantitites that are relevant to the decision making problem. In these models, the decison maker selects an initial portfolio and earns some return on this portfolio, and at the same time he is learning more about the process that "generates" security price changes as well as changes in other variables of interest. This additional information about the process is useful when the decision maker contemplates revision of the portfolio. Even in a single-period model, this "learning effect" occurs. Furthermore, in a multiperiod model, the decision maker chooses a portfolio with an eye toward its ramifications for future portfolio revision decisions.

Although the specific models presented in this paper are admittedly quite simple and ignore many considerations that may be important in real world portfolio selection and revision, the general approach, as summarized in the preceding paragraph, seems to be a reasonable description of the actual behavior of individuals who make portfolio selection and revision decisions. Such individuals gather a considerable amount of information over time, both from the "tape" and from other sources, both in terms of "hard data" and in terms of what might be called "soft data" (e.g. verbal information that must be interpreted by the decision maker). As such information is gathered, the decision maker's opinions about the potential returns of various securities are modified, and such modifications may lead to revision of the portfolio. Moreover, by analogy with the multiperiod model, it is not unreasonable to suggest that a successful portfolio manager, like a successful chess player, is always thinking ahead and contemplating the effects of current decisions on future "moves." The point of this discussion is not to claim that individuals responsible for portfolio decisions actually use models like those presented in this paper; sophisticated, realistic models of this nature are simply not available. However, the general approach of these models is intuitively appealing and may be a good approximation to the procedures used in practice by portfolio managers.

Given that the general approach is appealing, the next question concerns the realism of the specific models presented in this paper. Obviously these models are but a first step, and, as noted previously, many important factors are omitted from consideration. The determination of inputs for the models (e.g. the set of securities and other variables, the statistical model for the data-generating process, the prior distribution, the utility function) is a crucial consideration that has already been discussed in Sections II and III. In addition, further extensions of these models need to be investigated if a realistic model is desired. Such extensions might include tax effects, short sales, borrowing and lending, costs associated with updating probabilities and determining an optimal portfolio revision strategy, the effect of positive or negative increments in the available wealth due to extraneous factors (i.e. income to and consumption from the portfolio over time), legal and/or policy restrictions, time preferences (i.e. preferences among different "wealth paths" that may lead to the same terminal wealth), and nonstationarity in the data-generating process. This list is intended to be illustrative, not exhaustive, but hopefully it includes most of the important factors. Some of these factors might be incorporated into the model without too much difficulty (e.g. the inclusion of short sales appears to require only a modification of the constraints)--others might be more troublesome in the sense that they may complicate the model. In any event, such extensions, which would make the portfolio selection and revision models more realistic, are fertile grounds for further research.

The actual implementation of the models presented in this paper, whether as research tools to investigate the general nature of optimal portfolio revision strategies in various types of situations or as operational procedures to assist decision makers in selecting and revising portfolios, depends not only on the realism of the models but also on the ease with which the optimal solutions can be determined. Of course, this is quite dependent upon the exact choice of inputs; as in just about any mathematical modelling situation, there is a trade-off between realism and tractability. For example, normal distributions are generally easier to work with than other families of distributions that may provide better approximations to reality; the use of certain simple mathematical functions (e.g. linear, quadratic, exponential, or logarithmic functions) to represent the decision maker's utility for money is convenient, but such functions often may be poor approximations to a decision maker's utility function. Of course, the question of tractability relates to the use to which the model is to be put and to the desired form of the solution. Obtaining general analytical solutions like those presented in Section IV for linear and quadratic utility requires that the inputs be in reasonably simple form. On the other hand, if the primary concern is determining numerical solutions for specific cases, much more flexibility is possible in the choice of inputs because numerical methods can be used in solving the problem. The use of numerical methods implies, for example, that a decision maker's utility function can be approximated as closely as desired even though it may bear no resemblance to any of the simple mathematical functions commonly used to represent utility functions. With respect to implementation, the question of tractability, both in analytical terms and in numerical terms, is quite important, and some work regarding tractability under various conditions is currently being conducted.

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