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Introduction

In Hansen and Koopmans [2], it was shown that the Scarf-Hansen fixed point algorithm may be applied to approximate an optimal invariant capital stock. They studied an economy with constant technology and non-reproducible resource availability, in which an initial capital stock is to be determined such that maximization of the discounted sum of future utility flows over an infinite horizon can be achieved by reconstituting that capital stock at the end of each period.

Despite the similarity in technology, the model differs from the growth-maximizing case studied by von Neumann [4]. Here the objective is one of maximizing the discounted utility

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of consumption. The optimal choices may be affected by the time preference parameter-- that is, by the utility discount factor α.

In this paper, we shall analyze the invariant capital stock problem from the viewpoint of the linear complementarity algorithm of Lemke [3] and of Cottle and Dantzig [1]. Our model is the same as that of Hansen and Koopmans except that we approximate their general concave utility function by one that is separable and piecewise linear. This assumption and approach yields a simpler constructive proof of existence of an optimal path than does the fixed point method. Our paper concludes with a small example which suggests that the algorithm may also be advantageous from the computational viewpoint because it exploits the linear structure of the system. In a finite number of pivot steps, this method obtains an exact solution.

Model Formulation

For convenience in the following proofs, the notation will differ slightly from that employed by Hansen and Koopmans (hereafter abbreviated H-K). Moreover, we will include a straightforward extension-- the non-stationary case in which there is an exogenously specified exponential rate of growth. The one-period growth factor β will be identical for all m resources and n activities.[1] With this modification, the problem is no longer one of computing an invariant capital stock, but rather one of computing invariant proportions under optimization.

For each time period, it is supposed that the following data are identical:

	dimensions
α = utility discount factor; $0 < \alpha < 1$	scalar
β = economy-wide growth factor; $0 < \alpha\beta < 1$	scalar
A = capital and current account input and output coefficient matrix	m x n
B = capital stock carryover coefficient matrix; $B \geq 0$	m x n
b = initial period's resource availabilities	m x 1
c = initial period's utility maximand coefficients	1 x n

Let x denote the nonnegative column vector of activity levels chosen for the initial time period (t = 0). Then, for all activities to increase by the identical one-period growth factor β, the activity levels during period t will be $x\beta^t$, and the resource availabilities will be $b\beta^t$. To satisfy the material balance relations for current and capital account inputs and outputs, the program must then meet the following conditions:

[capital and current account input requirements (+); or outputs produced (-) during period t]	$\leq$	[capital stocks carried over from period t-1]	+	[resources available during period t; constant growth rate from the initial period's resource availabilities]

$$\beta^t Ax \leq \beta^{t-1} Bx + \beta^t b$$

$$t = 1,2,.... \qquad (2.1.t)$$

Dividing through by the positive factor β^t and rearranging terms, we have the following <u>stationary</u> set of technology and resource constraints[2]

$$(A - \beta^{-1}B)x \leq b \tag{2.1}$$

$$x \geq 0 \ . \tag{2.2}$$

The activity levels x are to be chosen so as to maximize the discounted sum of future utility flows

$$\sum_{t=1}^{\infty} (\alpha^t c)(x\beta^t) = cx \sum_{t=1}^{\infty} (\alpha\beta)^t = cx\left[\frac{\alpha\beta}{1 - \alpha\beta}\right] . \tag{2.3}$$

Since $0 < \alpha\ \beta < 1$, the maximization of cx is equivalent to the maximization of (2.3) subject to constraints (2.1) and programming problem in terms of the primal unknowns x. The critical difference lies in the fact that the <u>dual</u> variables in successive time periods are related to each other by the utility discount factor α. Let the nonnegative row vector y denote the shadow prices for the initial time period (t = 0). Then, for relative prices to remain constant, the "present value" shadow prices for period t must be $\alpha^t y$. The dual feasibility conditions are then

$$\begin{bmatrix}\text{present value}\\ \text{of current}\\ \text{acount inputs}\\ \text{(+); or outputs}\\ \text{(-) produced}\\ \text{during period t}\end{bmatrix} \geq \begin{bmatrix}\text{present value}\\ \text{of capital}\\ \text{stocks carried}\\ \text{forward to}\\ \text{period t+1}\end{bmatrix} + \begin{bmatrix}\text{present value}\\ \text{of utility}\\ \text{received}\\ \text{during}\\ \text{period t}\end{bmatrix}$$

$$(\alpha^t y)A \geq (\alpha^{t+1} y)B + \alpha^t c \tag{2.4.t}$$

$$t = 1,2,\ldots.$$

Dividing (2.4.t) by the positive factor α^t and rearranging terms, we obtain the stationary set of dual feasibility conditions

$$y(A - \alpha B) \geq c \quad . \tag{2.4}$$

Before expressing the complementary slackness conditions for this stationary economy, it is convenient to define the matrices C and D

$$C = A - \beta^{-1} B \tag{2.5}$$

$$D = B(\beta^{-1} - \alpha) \geq 0 \ , \tag{2.6}$$

and to change the primal and dual inequality constraints (2.1) and (2.4) into equalities by introducing slack vectors $u \geq 0$ and $v \geq 0$ respectively. Thus (2.1) and (2.4) can be rewritten

$$v = b - Cx \geq 0 \tag{2.7}$$

$$u = -c + y(C + D) \geq 0 \tag{2.8}$$

where $D \geq 0$ follows from the assumptions $B \geq 0$ and $0 < \alpha\beta < 1$. N.B. If $D = 0$, this may be solved through conventional linear programming methods. In general, however, $D \neq 0$.

To find an invariant set of proportions for this economy, we seek nonnegative vectors u, v, x and y satisfying (2.7), (2.8) and the complementary slackness conditions

$$ux + yv = 0 \quad . \tag{2.9}$$

It is convenient to summarize conditions (2.7) - (2.9) in the Lemke Complementarity Tableau, Figure 1. To avoid awkwardness in the subsequent notation, transposition symbols have been omitted. In the next to rightmost column there is an artificial variable θ whose vector of coefficients is defined by

$$\begin{aligned} e &= (1,1\ldots,1)^T \quad \text{n-vector} \\ f &= (1,\ldots,1)^T \quad \text{m-vector} \; . \end{aligned} \tag{2.10}$$

The Lemke algorithm may always be initiated by assigning a sufficiently high value to θ, setting $x = 0$, $y = 0$ and solving for $u > 0$, $v > 0$. After a finite number of iterations, the algorithm must terminate in one of two ways: (1) a complementary solution with the artificial variable $\theta = 0$; or (2) an unbounded "ray" solution. In our principal theorem, it will be shown that -- given a certain "key" hypothesis regarding the existence of bounded optima for the primal and dual systems -- the Lemke algorithm cannot terminate in a ray, and that it therefore may be employed to find a solution corresponding to invariant proportions for this economy.

The set $\{(u,v,x,y,\theta) \geq 0 : u = y(C + D) + e\theta - c,$ $v = - Cx + f\theta + b\}$ forms an unbounded convex polyhedral set which we will refer to as the set of feasible solutions. A solution will be called almost-complementary if $ux + yv = 0$. If in addition $\theta = 0$, it will be called complementary. We seek a feasible complementary solution by iteratively moving from one extreme (basic) almost-complementary feasible solution to a neighboring one.

Complementary (slack) variables	Variables Primal	Dual	Artificial variable	
	x	y	θ	Constant column
u =	0	C + D	e	- c
v =	- C	0	f	b

Notes:

1) $D \geq 0$.

2) For a complementary solution, $\theta = 0$; $ux + yv = 0$; that is, for each i, $u_i x_i = 0$ and for each j, $y_j v_j = 0$. Along an almost-complementary path, $\theta > 0$; $ux + yv = 0$.

Figure 1. Complementarity Tableau.

Termination in a Ray

Before stating the key hypothesis and the complementarity construction theorem, we will list a series of propositions related to termination along a ray. Note that the Lemke algorithm may always be initiated along an extreme ray [$x = 0, y = 0, u = e\theta^r - c, v = f\theta^r + b, \theta = \theta^r$] by assigning a sufficiently high value to the artificial variable θ, (e.g. $\theta = +\infty$) and then letting $\theta \geq 0$ decrease until an extreme point solution is obtained. The algorithm will generate a path of almost-complementary solutions moving along edges of the polyhederal set from one extreme point to the next, stopping if $\theta = 0$ or if on a pivot step an unbounded edge is generated (an extreme ray). If the algorithm terminates in a ray, then let

$(x^*,y^*,u^*,v^*,\theta^*)$ denote the finite (extreme-point) end of the ray. It corresponds to an almost-complementary solution. Let $(x^h,y^h,u^h,v^h,\theta^h) \geq 0$ denote the homogeneous part of the ray solution, and let the scalar λ denote the value of the incoming variable that can be increased indefinitely to generate the ray. For the almost-complementary solution corresponding to the finite end of the ray, we have

$$u^* = -c + y^*(C + D) + e\theta^* \qquad (3.1)$$
$$v^* = b - Cx^* + f\theta^* \quad , \quad \theta^* > 0 \quad . \qquad (3.2)$$

The homogeneous part of the ray solution may be written

$$u^h = y^h(C + D) + e\theta^h \qquad (3.3)$$
$$v^h = -Cx^h + f\theta^h \quad . \qquad (3.4)$$

Points $(x^r,y^r,u^r,v^r,\theta^r)$ along the ray are then given parametrically in terms of $\lambda \geq 0$ by

$$\begin{aligned} \theta^r &= \theta^* + \lambda\theta^h \\ u^r &= u^* + \lambda u^h \\ v^r &= v^* + \lambda v^h \\ x^r &= x^* + \lambda x^h \\ y^r &= y^* + \lambda y^h \quad , \end{aligned} \qquad (3.5)$$

where θ^h, u^h, v^h, x^h, $y^h \geq 0$, but are not all equal to zero because the homogeneous solution to generate a ray must be non-trivial. It will now be shown that along the ray, either $cx^h > 0$ or $y^hb < 0$, but not both. This will be established through a series of six propositions.

Proposition 1. Along a ray,

$$u^*x^* = u^*x^h = u^hx^* = u^hx^h = 0 \quad , \tag{3.6}$$

and

$$y^*v^* = y^*v^h = y^hv^* = y^hv^h = 0 \quad . \tag{3.7}$$

Proof. Almost-complementarity implies that for all $\lambda \geq 0$

$$u^rx^r = (u^* + \lambda u^h)\,(x^* + \lambda x^h) = 0 \quad ,$$

and

$$y^rv^r = (y^* + \lambda y^h)\,(v^* + \lambda v^h) = 0 \quad .$$

Each of the above terms is nonnegative, and their sum is zero.

Proposition 2. Along a ray,

$$y^hDx^h = 0 \quad ,$$

and

$$\theta^h\,[ex^h + y^hf] = 0 \quad .$$

Proof. By proposition 1,

$$u^hx^h = [y^h(C + D) + e\theta^h\,]\,x^h = 0$$

$$y^hv^h = y^h \quad [-Cx^h + f\theta^h\,] \quad = 0 \quad .$$

Adding terms,

$$y^hDx^h + [ex^h + y^hf]\;\theta^h = 0 \quad .$$

Each of the above terms is nonnegative, and their sum is zero.

Proposition 3. Along a ray, $\theta^h = 0$.

Proof. From proposition 2, we already know that at least one of the two following statements holds:

a) $\theta^h = 0$

b) $x^h = 0$ and $y^h = 0$.

If both a) and b) hold, there would be a contradiction for, from (3.3) and (3.4), this would imply that $(u^h, v^h) = 0$. We would then have a trivial homogeneous solution which could not be used to generate a ray.

Now we shall show that if (b) is true and not (a), there is also a contradiction, for then (3.3) and (3.4) would imply that $(u^h, v^h) > 0$. In turn, proposition 1 would imply that $(x^*, y^*) = (x^h, y^h) = 0$. We now show that the final ray $(u^r, v^r, x^r, y^r, \theta^r)$ would then be identical with the initial ray, for these facts, together with (3.1) - (3.5), imply that the final extreme ray is of the form $[x^r = 0, y^r = 0, u^r = e\theta^r - c, v^r = f\theta^r + b, \theta = \theta^r]$ where $\theta^r = [\theta^* + \lambda\theta^h]$, $0 \leq \lambda \leq \infty$. But this is the same parametric format that defines the initial ray--a contradiction, for Lemke's algorithm cannot return to the initial ray along an almost-complementary path. Hence $\theta^h = 0$.

Proposition 4. Along a ray,

a) $y^h\ C\ x^h = 0$

b) $y^*\ C\ x^h = 0$

c) $y^h(C + D)\ x^h = 0$

d) $y^h(C + D)\ x^* = 0$.

Proof. Since $\theta^h = 0$, from (3.4),

$$\text{a)} \quad y^h C x^h = y^h [Cx^h - f\theta^h] = -y^h v^h = 0$$

$$\text{b)} \quad y^* C x^h = y^* [Cx^h - f\theta^h] = y^* v^h = 0 \quad .$$

Similarly, from (3.3),

$$\text{c)} \quad y^h(C + D)\, x^h = u^h x^h = 0$$

$$\text{d)} \quad y^h(C + D)\, x^* = u^h x^* = 0 \quad .$$

Proposition 5. Along a ray,

either $(v^h, x^h) = 0$ or $(u^h, y^h) = 0$, but not both.

Proof. In proposition 3, we have already noted that both statements cannot hold, for if $(x^h, y^h, \theta^h) = 0$, this would imply a trivial homogeneous solution and not a ray.

The next step will be an argument based upon a simplex tableau for the homogeneous system of (3.3) and (3.4):

unknowns	v	x	u	y	θ^h	constant column
	0	0	I - (C + D)		-e	= 0
	I	C	0	0	-f	= 0

From proposition 3, recall that $\theta^h = 0$. Hence if the incoming basic variable is a component of v or of x, the representation of its column can only have non-zero weights on basic columns corresponding to basic variables among the components of v or x (there is a zero weight on the θ^h column). Thus there are no non-zero weights among u and y, i.e. $(u^h, y^h) = 0$. Similarly, if the incoming variable is a component of the u or y vectors, its "representation" $(v^h, x^h) = 0$.

Proposition 6. Along a ray,

either a) $cx^h > 0$

or b) $y^h b < 0$, but not both.

Proof.

a) Suppose that $(u^h, y^h) = 0$, but that $(v^h, x^h) \neq 0$. Then (b) cannot hold. To prove that $cx^h > 0$, note that (3.1) and (3.6) imply.

$$u^* x^h = 0 = [-c + y^* (C + D) + e\,\theta^*]\, x^h \quad .$$

By proposition 4, $y^* C x^h = 0$. Because $D \geq 0$,

$$cx^h = y^* D x^h + e x^h \theta^* > 0 \quad ,$$

where strict inequality must hold because equality $\Rightarrow e x^h = 0 \Rightarrow x^h = 0$ and [by proposition 4 and (3.4)] $\Rightarrow v^h = 0$, a trivial homogeneous solution--a contradiction.

b) Similarly, suppose that $(v^h, x^h) = 0$, but that $(u^h, y^h) \neq 0$. Now (a) cannot hold. To prove (b):

$$y^h v^* = 0 = y^h\, [b - Cx^* + f\theta^*]$$

By proposition 4, $y^h C x^* = -y^h D x^*$. Hence

$$y^h b = -y^h D x^* - y^h f \theta^* < 0 \quad .$$

where again strict inequality must hold by an argument similar to that given in (a) above.

Key Hypothesis and Complementarity Construction Theorem

To ensure that the Lemke algorithm will not terminate along a ray, we shall make the following plausible key hypothesis:[3] The set of linear programming solutions to the two following problems is each non-empty and bounded:

$$\text{(P)} \quad \text{maximize} \quad cx \quad \text{subject to} \quad Cx \leq b, \quad x \geq 0$$

$$\text{(D)} \quad \text{minimize} \quad yb \quad \text{subject to} \quad y(C+D) \geq c, \quad y \geq 0 \; .$$

Call these optimal solutions, respectively, $\hat{x}$ and $\hat{y}$.

Complementarity Construction Theorem. If the key hypothesis holds, the Lemke algorithm cannot terminate in a ray. The algorithm will therefore construct a complementary solution satisfying (2.7)-(2.9).

Proof. Assume on the contrary termination in a ray, then according to proposition 6, either

$$cx^h > 0 \quad \text{or} \quad y^h b < 0 \, , \quad \text{but not both.}$$

In the first case, we may obtain an unbounded solution to P by setting $x = \hat{x} + \lambda x^h$, where again $\lambda > 0$.

In the second case, we may obtain an unbounded solution to D by setting $y = \hat{y} + \lambda y^h$, where again $\lambda > 0$.

In either case, we contradict the key hypothesis of bounded linear programming solutions. Hence the algorithm cannot terminate in a ray.

Numerical Results

In order to apply the Lemke algorithm to the numerical example studied by Hansen and Koopmans, it was necessary to modify the problem formulation. The log of their maximand turned out to be a sum of concave functions which are replaced by piecewise linear approximations. Their one-period utility function is:

$$u(y) = \prod_{i=1}^{3} (y_i)^{0.2} \quad , \tag{5.1}$$

where (in their notation) y_i denotes the quantity consumed of item $i = 1,2,3$. Taking logarithms -- and recalling that the logarithmic function is monotone increasing -- we maximize log u(y) in place of the maximand (5.1) and write it as a sum of separable concave functions

$$\log u(y) = \sum_{i=1}^{3} (0.2) \log y_i \quad . \tag{5.2}$$

Next, suppose we have sufficient prior information about the problem so that it is known that an optimal value of y_i will lie between some lower limit $\bar{y}_{i1}$ and an upper limit $\bar{y}_{iJ}$. Moreover, let there be J grid points $\bar{y}_{ij}$ such that

$$\bar{y}_{i1} \leq \bar{y}_{i2} \cdots \leq \bar{y}_{ij} \cdots \leq \bar{y}_{iJ} \quad .$$

For each grid point, we introduce a nonnegative unknown x_{ij} to denote the interpolation weight placed upon the j th level of demand for item i. That is, the unknown y_i is replaced by the linear function

$$y_i = \sum_{j=1}^{J} \bar{y}_{ij} \, x_{ij} \quad , \qquad i = 1,2,3. \tag{5.3}$$

The problem is formulated so that the interpolation weights will add up to unity:

$$\sum_{j=1}^{J} x_{ij} = 1 \quad , \qquad i = 1,2,3. \tag{5.4}$$

Finally, the original utility function (5.1) is replaced by the following piecewise linear approximation

$$\log u(y) \approx \sum_{i=1}^{3} .2 \sum_{j=1}^{J} (\log \bar{y}_{ij}) \, x_{ij} \quad . \tag{5.5}$$

Since the logarithm is a strictly concave function, it is guaranteed that in an optimal solution, there will be a positive intensity assigned to no more than two of the unknown x_{ij} for each item i. Moreover, the optimal grid points will be adjacent to each other. For an application of this technique to development planning, see e.g. Westphal [5, p.61].

The numerical efficiency of this interpolation weight technique will depend upon the goodness of the initial choice of grid points. If hundreds of grid points are specified, there will be hundreds of unknowns x_{ij} for each item i, and this will lead to hundreds of rows and columns in the complementarity matrix of Figure 1. For purposes of this numerical experiment, we selected only four grid points for each of the three consumption goods. For example, with $\alpha = .7$, H-K calculated the stationary value of $y_1 = .215$. Making use of this prior information, our grid values were chosen as follows:

$$\bar{y}_{11} = .18 \; ; \; \bar{y}_{12} = .20 \; ; \; \bar{y}_{13} = .22 \; ; \; \bar{y}_{14} = .24 \quad .$$

With this approximation, the complementarity tableau contained 32 rows and 32 columns, excluding the artificial and constant columns shown in Figure 1. Charles Engles applied the Lemke algorithm for three different values of α. Since both the primal and dual solutions were in close agreement with those reported by Hansen and Koopmans, Table 1 contains only the numerical values of the one-period maximand. The linear complementarity method required 2.5 - 3.0 seconds on an IBM 360/67 -- excluding the time required to compile the program. Hansen and Koopmans reported that to obtain a terminal primitive set, the computing time was 14 minutes on an IBM 1130, and that this would be equivalent to about one minute on an IBM 360/50. In itself, this experiment is inconclusive, for we made use of the H-K results in our selection of utility function grid points.[4] Nonetheless, the results are sufficiently promising so that further work seems warranted in comparing the fixed-point and the linear complementarity algorithms on this class of models.

Table 1. Comparison of Numerical Results

	One-period utility discount factor,	0.7	0.8	0.9
One-period utility attained	Obtained by Hansen and Koopmans, applying the fixed-point algorithm	.48855	.52216	.55935
	Obtained by Charles Engles, applying the linear complementarity algorithm	.48904	.52209	.55939

Footnotes

[1] For labor or for renewable natural resources such as forests, it might be appropriate to postulate a growing availability -- or perhaps a constant future level. In this case, $\beta \geq 1$. For non-renewable natural resources such as petroleum, the earth contains only a finite stock. If such resources are essential, the economy could decline exponentially, and we would then have $\beta < 1$.

[2] This one-period problem corresponds to constraints (3.1A)-(3.1D) in H-K. Similarly, the dual conditions (2.4) will correspond to (3.2b) and (3.3).

[3] The key hypothesis is analogous to those underlying H-K Theorem 1 and Lemma 1.

[4] To avoid use of prior information, we could have solved one complementarity problem with a coarse grid, a second with a fine grid, a third with a still finer grid, etc.

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