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An interdisciplinary approach to classical and quantum random walk: an activity with secondary school students

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Abstract

The thesis is situated within the “I SEE project” (Inclusive STEM Education to Enhance the capacity to aspire and imagine future careers), a triennial ERASMUS+ project, started in 2016 and ended in September 2019, coordinated by the Department of Physics and Astronomy of the University of Bologna (https://iseeproject.eu/).
The main aim of the project is to design of teaching modules on advanced interdisciplinary STEM topics, such as climate change, quantum computers or artificial intelligence, for secondary school students. The approach developed toward these STEM topics reflects the need of updating Science Education and preparing students to address the socio-scientific challenges that the world compels us to face.

The work of this thesis concerns the design and the development of an activity on classical and quantum random walk that aims to: i. carry out the intrinsic interdisciplinarity of the topic, ii. highlight the differences between the logic at the basis of the classical and quantum computers. It is designed in the light of a follow-up of the I SEE project, the IDENTITIES project (https://identitiesproject.eu/). IDENTITIES (Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenges) is an ERASMUS+ project, started in September 2019 and coordinated by the Department of Physics and Astronomy of the University of Bologna. This project, more focused on interdisciplinary themes, provides a lens to explore the random walk as a context in which we can investigate, on one hand, the peculiarities of the disciplines involved and, on the other, their intertwining.

A future-oriented part, my main contribution to the activity, is carried out in order to foster students to explore the wide range of application and implication that random walk hides and to show them the variety of STEM carriers that can be involved.

Sommario

La tesi si colloca all'interno del "progetto I SEE" (Inclusive STEM Education to Enhance the capacity to aspire and imagine future careers), un progetto triennale ERASMUS +, avviato nel 2016 e terminato a settembre 2019, coordinato dal Dipartimento di Fisica e Astronomia dell'Università di Bologna (https://iseeproject.eu/).
L'obiettivo principale di I SEE è la progettazione di moduli didattici su argomenti STEM avanzati, come il cambiamento climatico, i computer quantistici o l'intelligenza artificiale, per studenti delle scuole secondarie. L'approccio sviluppato per affrontare questi temi riflette la necessità di aggiornare la didattica delle scienze con lo scopo di preparare gli studenti ad affrontare le sfide socio-scientifiche che il mondo ci pone davanti.

Il lavoro di questa tesi riguarda la progettazione e lo sviluppo di un'attività sul random walk classico e quantistico che mira a: i. esplorare l'intrinseca interdisciplinarità del tema, ii. mettere in luce le differenze tra la logica alla base dei computer classici e quantistici.
Come follow-up del progetto I SEE è nato il progetto IDENTITIES (Integrate Disciplines to Elaborate Novel Teaching approach to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenge). Si tratta di un progetto ERASMUS +, avviato a settembre 2019 e coordinato dal Dipartimento di Fisica e Astronomia dell'Università di Bologna. Questo progetto, più incentrato su temi interdisciplinari, ha fornito una lente per esplorare il random walk come un contesto in cui indagare, da un lato, le peculiarità delle discipline coinvolte e, dall'altro, il loro intreccio.

Il mio principale contributo riguarda lo sviluppo di un'attività orientata al futuro con lo scopo di fare esplorare agli studenti l'ampia gamma di applicazioni e implicazioni del random walk e per mostrare loro la varietà di carriere STEM che possono essere coinvolte.

Introduction

The present thesis is situated within the I SEE project (Inclusive STEM Education to Enhance the capacity to aspire and imagine future careers), an ERASMUS+ project coordinated by the University of Bologna that started in 2016 (http://www.iseeproject.eu/) and finished in September 2019. The aim of the I SEE project is the development of teaching-learning modules on STEM topics such as climate change, carbon sequestration, artificial intelligence, and quantum technologies. I contributed to the design of the second implementation of a teaching module on Quantum computers for secondary school students. In particular, I participated in the realization of an activity on classical and quantum random walk that is located in a follow-up of the I SEE project: the IDENTITIES project. This project is much more focused on the theme of interdisciplinarity and it provide us a lens on which analyze and reconstruct the topic for secondary school students.

The module we designed has been implemented within the PLS (Piano Lauree Scientifiche) project, organized by the department of Physics and Astronomy of University of Bologna in January-February 2020.

The thesis is divided in three chapters and the conclusions.
The first chapter is focused on the I SEE project. After a brief introduction in which I problematize the importance of address contemporary STEM challenges in Science Education, the I SEE project and its teaching goals are introduced. In the last part of the chapter an overall description of the developed module on the quantum computer is presented.

The second chapter is dedicated to the random walk. Once the probabilistic tools necessary to understand the topic are resumed, the classical and quantum random walk are deepened from a mathematical and physical point of view.

The third chapter, original part of the thesis, concerns the educational reconstruction of an activity on the random walk. In the first part of the chapter a description of the aims of the activity and of what we actually did with students are presented. The last part is dedicated to the activity, held my me, regarding an exploration of the broad spectrum of application of classical and quantum random walk, from Physics to art.

In the conclusions some students reaction to the interdisciplinary approach are discussed.

Chapter 1: I SEE project and the teaching module on quantum computing

1.1 Science and STEM education

"If the role of education is to prepare learners for their future, how can education prepare learners for an uncertain future?" (Branchetti, Cutler, Laherto, Levrini, Palmgren, Tasquier & Wilson, 2018)

This is an important question in the contemporary society, a question that arises spontaneously due to the uncertainty of modern time. In this society, science and technology play a crucial role enhancing the need to develop skills and competences to navigate the complexity and the uncertainty we are experiencing, and to broaden the future's perceptions of the young generation. On the contrary, in the era of technical and social acceleration (Rosa H., 2010), science and technology are contributing, especially in the young, to increasing a sense of anxiety and fear. The constant changes and the acceleration of the evolution of our society, assisted by an economic crisis, caused an impairment in young people's educational and professional possibilities. This is also showed by the European union trough the Eurobarometer survey, that pointed out the exclusion from economic and social life suffered by young people. Furthermore, the survey highlighted their difficulties to project themselves into the future (Eurobarometer,2014).

Science education is becoming more and more aware of the need to prepare young people to deal with an uncertain future and to develop a pedagogy that can understand the anxiety that young people feel. There is the need to update the science education. The path that science education is taking is to foster the interest toward STEM subjects as a context to explore the relation between science disciplines and the societal challenges.

The STEM subjects are, as the acronym would suggest, the disciplines related to Science, Technology, Engineering and Mathematics. This acronym was created in the late 1990's in the USA by the National Science Foundation. The contemporary challenges that world requires to face mark the needs of combining these disciplines to create more integrated and interdisciplinary approach to learning where "rigorous academic concepts are coupled with real world applications and students use STEM in contexts that make connections between school, community, work, and the wider world" (Committee on STEM Education, 2018).

STEM education is not the simple combination, but it is a multidisciplinary, interdisciplinary and transdisciplinary approach that search for an integration between these disciplines. There are different definitions of STEM education, one of the most significative to recall its multidisciplinary character is the one provided by Kennedy and Odell: "an approach to teaching that is larger than its constituent parts" (National High School Alliance, nd, in Kennedy & Odell, 2014).

- An important feature of STEM education is the boundary crossing between the different disciplines in order to have a better comprehension of this. As identified by Vasquez, there are four increasing levels of integration:
- Disciplinary: concepts and skills are learned separately in each discipline.
- Multidisciplinary: Concepts and skills are learned separately in each discipline but within a common theme.
- Interdisciplinary: Closely linked concepts and skills are learned from two or more disciplines with the aim of deepening knowledge and skills.
- Transdisciplinary: Knowledge and skills learned from two or more disciplines are applied to real-world problems and projects, thus helping to shape the learning experience. (Vasquez et al 2013)

In Vasquez's idea, the transdisciplinary level is the most important because the task of STEM education is to help solving real-world problems and to provide the skills to understand our uncertain time. STEM education has also the duty to prepare young people to the jobs of the future, the STEM carriers, as referred by a report from the Committee on STEM Education National Science and Technology Council (USA) of May 2013: *"The jobs of the future are STEM jobs: The demand for professionals in STEM fields is projected to outpace the supply of trained workers and professionals. Additionally, STEM competencies are increasingly required for workers both within and outside specific STEM occupations*." (Holdren, Marrett, & Suresh, 2013).

We can now understand the importance of including STEM education in science education.
In the next section, I present the I SEE project and how it contributed to the development of a STEM approach to improve the young people's capacities to deal with their future.

1.2 I SEE project

The I SEE project (https://iseeproject.eu/), an acronym that stands for Innovative STEM Education to Enhance the capacity to aspire and imagine future careers, is a European project, coordinated by the University of Bologna and finished in September 2019. It involved eight different partners:

Three secondary schools: the Liceo A. Einstein in Rimini (Italy), the Helsinki Normal Lyceum (Finland), and the Hamrahlid College of Reykjavik (Iceland).
Two universities departments: Department of Physics of the University of Bologna (UNIBO) and the Department of Physics of the University of Helsinki (UH)
Three associations: The Icelandic Environment Association (IEA), the Association for Science Education (ASE, UK) and he Fondazione Golinelli (FG).

The project tried to solve the issues embedded by science education, pointed out in the first paragraph, in order to provide the students with the abilities to understand the future and aspire to STEM careers. The principal aim of the project is to realize teaching module on STEM topic in order to foster students' identities as capable persons and responsible citizens in a fast changing world (Branchetti et al., 2018).

A central aspect is the development of "*future-scaffolding skills*" that will help the students to understand and face up to the future. For *future-scaffolding skills* we "*refer to the capability of organising knowledge in the present, imagining futures and moving dynamically and consciously, back and forth, globally-locally between different times and dimensions*" (Levrini, Tasquier, Branchetti, & Barelli, 2019). Some of these skills are strategic thinking and planning, risk taking, thinking beyond the realm of possibilities, managing uncertainty, creative thinking, modelling, and argumentation.

The teaching modules developed within the I SEE project concern four cross-cutting topics: climate change, carbon sequestration, artificial intelligence, and quantum computing. All these topics share some characteristics: they are future-oriented, STEM topics, and they are relevant from a personal and social point of view. The present thesis is focused on the quantum computing module.

Although the topics are different, all the I SEE modules have the same structure. The three main phases of the typical structure are (Fig.1):

Encountering the focal issue;

Developing of conceptual ad epistemological knowledge about the topic;
Synthesizing the ideas and putting them into practice with future oriented activities.

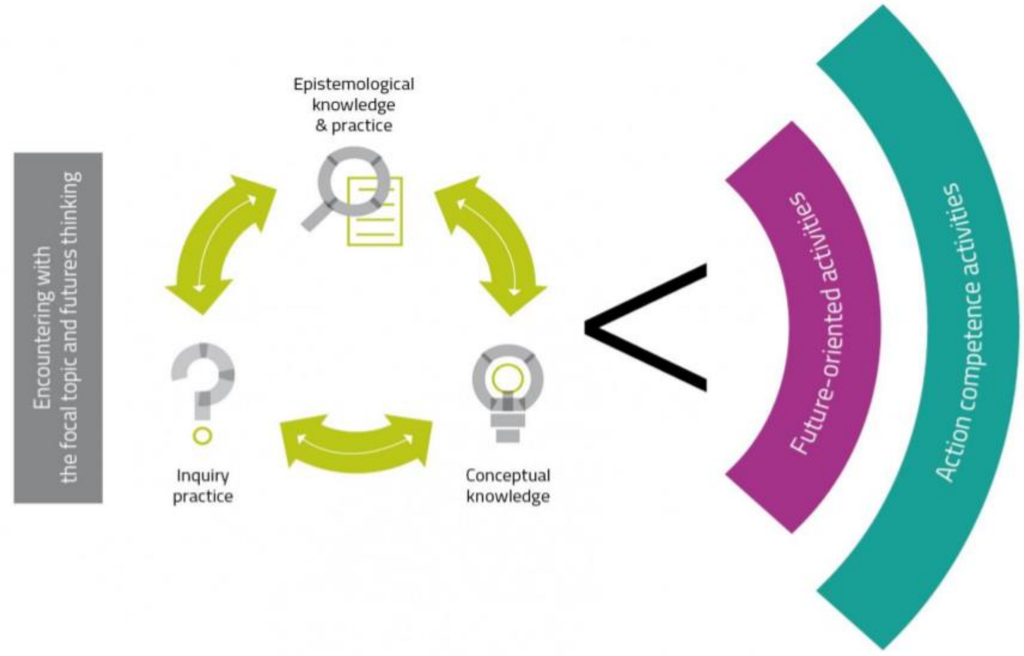

Figure 1.1: structure of an I SEE project module

Every module begins with the students encountering the focal issues, that is necessary to develop a preliminary idea of the main topic. In this preliminary phase, we stress on the relevance of the topic and on the multidimensional aspect of the module. Initially all the instruments needed to understand and face up the topic, such as the conceptual and epistemological scientific knowledge required and the specific language, are presented to the students. Then the problematic aspects of the topic and an analysis of the stakeholders involved in the theme are introduced.

The second phase can be divided in three sub-phases:
Conceptual knowledge (CK)
Epistemological knowledge and practice (EKP)
Inquiry practice (IP).
Conceptual knowledge refers to the disciplinary content knowledge, in this phase the topic is introduced from the conceptual point of view. Then the aim moves on the epistemic practice such as, modelling, arguing, and explaining, and this is achieved with a series of group activities where the students are guided also to grasp the shift in the epistemological paradigm. Other activities are organised to use and acquire inquiry skills such as formulating hypotheses,

designing inquiry, and moving from models to experiments and vice versa. Studies in the field have shown that including these three features in the science contents may foster a deeper and meaningful learning (Tasquier, Levrini, & Dillon, 2016).
The last phase is focused on future-oriented practices. These future-oriented activities can be divided in four groups:

Activities to flesh out the future-oriented structure of scientific discourse, language, and concepts.
Activities inspired by future studies or by the working life and social matters.
Exposure activities to enlarge the imagination about possible STEM careers.
Action competence activities.

This is the most creative part of the I SEE module, where the students have to deal with a real problem. This is also a moment in which, knowledge and practices acquired along the whole sequence begin to transform into skills in action. This final part particularly contributes to the creation in the students of the *future scaffolding skills.*

1.3 The quantum computing module

The module on quantum technologies is an output of the Erasmus+ project I SEE, described in the previous section. However, in the last year it was elaborated to match new aims foreseen by another project that is a sort of follow up of I SEE: the IDENTITIES project (https://identitiesproject.eu/). IDENTITIES project (*Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenges*) is still coordinated by the University of Bologna and started just when I SEE finished, in September 2019. The overarching goals of the new project are i) to design novel teaching approaches on interdisciplinarity in science and mathematics to innovate pre-service teacher education for contemporary challenges and ii) to explore emergent advanced STEM themes (i.e. quantum technologies, artificial intelligence, nano-technologies) and curricular interdisciplinary topics (i.e. cryptography, parabola, etc.) as contexts to explore inter-multi-trans-disciplinary forms of knowledge organization and to design classroom activities and new models of co-teaching. With the lenses of IDENTITIES the module on quantum computing was re-thought in order to explore the topic as a STEM topic, focusing on the links and interweaving between physics, mathematics, and computer science.

The teaching module was implemented twice within the PLS (Piano Lauree Scientifiche) laboratories organized by the University of Bologna. The first pilot study was realized in February/March 2019 and involved 25 secondary school students (17/18 years old). I participate to the second round that took place in January/February 2020 and involved 22 secondary school students. The course was structured as an I SEE module with a combination of lectures, interactive/laboratory activities, and group activities. Some activities, including the activity of random walk, was a novelty of the second edition, added to match the goals of IDENTITIES.

The main purposes of the course, in principle with the characteristics of the I SEE project modules, were:

To introduce the basic quantum physics knowledge necessary to understand the differences between a classical and a quantum computer and to grasp the potentiality of these new technologies
To consider the scientific, cultural, and social implications of these new quantum technologies (quantum computers, quantum simulators and quantum internet)
To grasp the multidimensionality of the theme and to detect the possibilities of STEM carriers
To develop critical thought that can guide in this fast changing world and in the upcoming future.

The module consisted in six meetings of three hours each. Each meeting included activities to introduce students to the conceptual and epistemological dimensions of the topic, and a part dedicated to future-oriented activities. In table 1 the timetable of course is reported.

Table 1: timetable of the Quantum computing course

Day	Concept&epistemological -oriented activities	Future-oriented activities
1	History of classical computers Introduction and history of quantum technologies	Future-oriented activity “quantum computing &...”
2	The basics concepts for quantum computer (state, superposition principle, Qubit, state evolution and measurement)	Delivery of students’ output on “quantum computing &...”
3	Introduction to multi-qubit systems and entanglement Cryptography	Futures and action competence activity: the Eve city

4	Quantum teleportation	Futures and action competence activity: the Eve city
5	Classical and Quantum random walk	Futures and action competence activity: the Eve city
6	Delivery of students' outputs on futures and action competence activity (cancelled due Covid epidemic)	

The lessons were held by:
Prof. Olivia Levrini, a researcher in Physics Education, coordinator of the course;
Prof. Elisa Ercolessi, a theoretical physicist with expertise in quantum computing;
Paola Fantini, a retired secondary school teacher with professional expertise in classical architectures and algorithms;
Sara Satanassi, a PhD student in Physics;
Me, Dario Casali, Bachelor student in Physics.

1.3.1 The lectures

The first day started with an introduction of the course presented by Professor Olivia Levrini. After a brief presentation of the characteristic of an I SEE project module on quantum computing, she asked some questions about the future to the students. The students' answers showed and confirmed the uncertainty and the apprehension that the young people feel for the future (justifying one of the main reasons for the realization of the course).

Then the meeting proceeded with a lecture held by Paola Fantini, with an introduction of classical computing and classical computer. The aims of the lecture were:
to introduce what classical computer architecture is
to introduce the binary logic that characterises classical computer,
to retrace the evolution of classical computer (from hardware to software).
The leading thread of the lecture, and of the entire course, was the presentation of the parallelism between experiment and computation. To reach this goal, the three phases of computer processing (input information – processing information – output information) have been used to re-read the phases of an experiment: state preparation – state manipulation/evolution – measurement (Satanassi, 2019; Satanassi, S., Ercolessi, E., Levrini, O., under review). The lecture was very engaging because Paola Fantini joined an historical

exposition on the evolution of the classical computers and her personal history (as an Italian early adopter of the computer technology).
More specifically, after a brief introduction of Von Neumann architecture, the concept of Bit and the logic gates were presented through a story extracted from an article published in 1988 by Dewdney called "an ancient rope-and-pulley computer is unearthed in the jungle of Apraphul" (Dewdney, 1988). The story tells of an invented island where mechanical devices were found and such devices reproduced logic gates realized with systems of ropes and pulleys. Then the computational representation of logic gates and their functioning were shown, introducing students to the concept of circuit. Finally a brief evolution of quantum computers has been presented, passing through the four generations of computers (from information processing through vacuum tubes, to the modern microchips).

The second part of the lecture was held by Sara Satanassi, who introduced quantum technologies as a current scientific challenge with strong implications on the society. Following a similar path to Fantini's part, after an introduction of the history and evolution of the quantum computers, she took a glance on the principal quantum concepts (superposition principle and Qubit) in order to open up students' vision on the differences between the classic and quantum computer. Finally she presented some fields of application of quantum technologies and the impact of these technologies on different dimensions such as the dimension of research, society, politics, ethics, environment, and education.

The final part of the meeting was dedicated to a group activity: the aim was to introduce students to the Quantum Manifesto, an official document that highlights the importance of such technologies in the European scenario, and guide them to explore in group the principal scientific and societal fields of interest of the quantum technologies (Scientific and technologic research, politic, society and communication).

The second day started with the delivery of the group activity, the students in groups presented their explorations on the different implication and on the possible applications of the quantum technologies to the whole class.

Then, a deeper lecture on the basic concepts of quantum physics and quantum computing were presented to students. The lecture was held by Professor Elisa Ercolessi. Through the approach of the comparison between experiment and computation, already seen in previous lectures, and through the Stern and Gerlach experiment, she introduced the concepts of quantum state, evolution of the state, superposition principle and measurement. Finally, she passed to the computational representation of the same concept introducing the concept of Qubit, quantum logic gates and respective truth table.

The third day Professor Elisa Ercolessi introduced two-Qubit systems and the concept of entanglement. The lecture continued with the presentation, after a brief historical excursus, of quantum cryptography, in particular of the BB84 protocol, through the simulation of QUVIS by the University of St. Andrews (https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/cryptography-bb84/Quantum_Cryptography.html).

The second half of the lesson was held by Olivia Levrini, that introduced students to the future-oriented activities. Firstly she presented to the students the concept of the "future-cone" (fig.2).

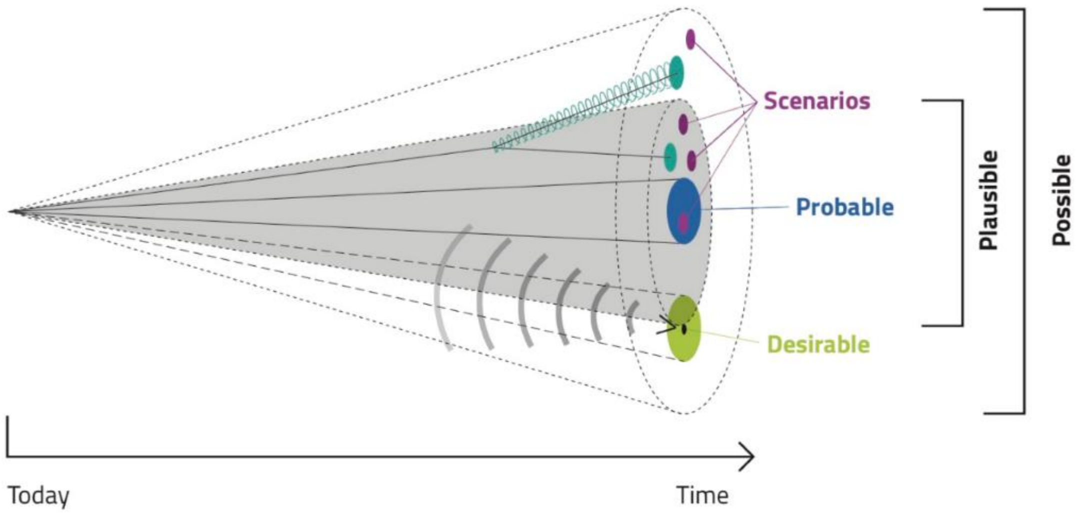

Figure 1.2: I SEE interpretation of the Future-cone by Voros.

The future-cone is an important representation of the vision of the "futures". It is a fundamental step to communicate to students, that the future should not be thought as singular but as plural. There are different possible futures, some are plausible, some probable and some are preferable. Each future is imaginable by depicting scenarios, which are "stories about distinct futures" (Miller, 2006, p. 98).

Then a short story about an imaginary city called Eve was presented to students. They, divided in groups, had to put themselves in the shoes of the major of the city and decide whether or not to invest in a big quantum artificial intelligence to re-invent the Eve city. The students had consider the possible implications of their choice, to find the stakeholders, the needs and the interests of the citizens and their relationships. At the end they had to decide whether to adopt or not the technology and to imagine the possible impact of their choice.

The fourth day was mainly focused on quantum teleportation. The activity was held by Sara Satanassi, taking in account the complexity of the topic, she used the approach of the comparison between experiment and circuit and articulated the discourse on four different levels: the narrative, the logical, the technical- mechanistic, and mathematical one (Satanassi, S., Fantini, P., Spada, R., Levrini, O., under review). Initially she contextualised the problem to be solved (how teleport the state of the photon from one side to the other of the Danube) through an introductive story (narrative level). Then she passed to the description the quantum teleportation experiment (technical-mechanistic level), realized by the group of Zeilinger in 2004 (Ursin, Jennewein, Aspelmeyer, Kaltenbaek, Lindenthal, Walther, P., & Zeilinger, 2006). She mainly focused on the crucial moments that make teleportation possible without going into the details of the experimental tools. Finally she presented the circuital representation (logical level) of the teleportation protocol, recalling the logic gates introduced in the previous meeting and presenting the calculation using the bra-ket formalism (mathematical level).

The meeting continued with the pursuance of the EVE city's group activity where three possible scenarios were presented to the students and they had to decide the one they prefer. The students with the same choices formed the group and presented to the rest of the group the motivations for their choice. The activity fostered students to think about an actual problem of the society and try to address it within the chosen scenario, paying attention to the hypothetical crucial steps or contingencies that could lead to the problem solution.

The fifth day was centered on the classical and quantum random walk, with the lens on interdisciplinarity between mathematics, physics, and computer science. This is the main focus on my thesis and topic of the third chapter. I contributed to the design and the development of the activities carried out in this meeting.

The sixth day should be dedicated to the delivery of the future-oriented activity. Unfortunately, because of the health crisis caused by the covid-19, we had to cancel it.

In the next chapter the classical and quantum random walk are treated from a mathematical and physical perspectives.

Chapter 2: A probabilistic study of the classical and quantum random walk

With the term *random walk,* we refer to a probabilistic theory based on a discrete parameter stochastic process $X(t)$, with random variable X describing the position as a function of time t. In order to be able to talk about random walk we have to be familiar with certain concepts of the probabilistic theory like random variables and probabilistic distributions. In particular we are going to analyse the normal (or gaussian) distribution and the binomial distribution, even studying links between the two.
After the introduction of the random walk from a mathematical point of view, we will analyse the differences between the classical and the quantum random walk and we will explore the application of the binomial distribution to the random walk. The Galton board will be presented as the physical counterpart both for the classical and for the quantum case. This example is not only an important physic application, but it can also bring to a better comprehension of the random walk's concept.

The following brief review is mainly excerpted from the book *"Introduction to error analysis, the study of uncertainties in physical measurements*" by Taylor J. (1997).

2.1 A glance at the probability theory

2.1.1 The concept of probability and of probability density function

A stochastic variable (or random variable) is a quantity X, defined by a set of possible values and by a probability distribution over these values. This set of possible values can assume discrete or continue values.

The set can be discrete, like the outcome of a coin toss (head or tail) or the number of electrons in the conduction band of a semiconductor. The set can be continue, like a component of the velocity of a Brownian particle.

In the discrete case, the variables have an associate probability p_i, with the following properties:

$$p_i \geq 0$$

$$\sum p_i = 1 \quad (1.1)$$

The probability density function, $P(x)$, is defined as:

$$P_X(x) = \sum p_i \delta(x - x_i) \qquad (1.2)$$

For continue variables, the probability distribution function, $P(x)$, complies the following the properties:

$$P(x) \geq 0$$

$$\int_{-\infty}^{+\infty} P(x)dx = 1 \quad (1.3)$$

where 1.3 is the normalization property.

$P(x)dx$ represents the probability that X attains a value within x and $x + dx$, so the probability density function, that is the probability in an interval $[a < x < b]$, is defined as:

$$Prob(a < x < b) = \int_a^b P(x)dx \qquad (1.4)$$

Through the probability distribution function, we can also define the cumulative distribution function, that is the probability that X will take a value less than or equal to x:

$$F(x) = \int_{-\infty}^{x} P(x')dx' \qquad (1.5)$$

We have to introduce other few concepts that will be useful for the following explanation of the distributions, the moments.

The moments provide information on the amplitude and shape of the probability density$P_X(x)$. The nth moment of X is defined as:

$$\langle x^n \rangle = \int_{-\infty}^{+\infty} x^n P_X(x)dx. \qquad (1.6)$$

this general definition assumes specific meanings for certain value of the n. The average is defined as:

$$\langle x \rangle = \int_{-\infty}^{+\infty} x P_X(x)dx. \qquad (1.7)$$

Using the definition of average we can define another useful value, the variance:

$$Var(X) = \langle x^2 \rangle - \langle x \rangle^2 \qquad (1.8)$$

With the variance we can also obtain the standard deviation as the square root of the variance:

$$\sigma = \sqrt{Var(X)} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \qquad (1.9)$$

The last concept we need is the characteristic function that is the Fourier transformation of the probability function. It is an alternative route to analytical studies compared to the probability or the cumulative density functions. The characteristic function $f_X(k)$ is defined as:

$$f_X(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{+\infty} P(x) e^{ikx}\, dx = \sum_{n=0}^{\infty} \frac{(ik)^n \langle x^n \rangle}{n!} \qquad (1.10)$$

If the characteristic function is known, we can obtain the probability function as:

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_X(k)\, e^{-ikx} dk \qquad (1.11)$$

2.1.2 Binomial distribution

In order to explain the binomial distribution we can start with the example of an N number of independent coin tosses. As we know the result of a coin toss can be head or tail. In this example we have a number n_h of heads and a number n_t of tails that must follow the equation $N = n_h + n_t$.

The probability of obtaining a number n_h of heads is given by the binomial distribution:

$$P(n_h) = \frac{N!}{n_h! - (N - n_h!)} p^{n_h} q^{N-n_h} \qquad (1.12)$$

where p is the probability to obtain head and $q = 1 - p$ is the probability to obtain tail.

We can check that the normalization property is verified:

$$\sum_{n_h=0}^{N} P(n_h) = \sum_{n_h=0}^{N} \frac{N!}{n_h! - (N - n_h!)} p^{n_h} q^{N-n_h} = (p + q)^N = 1^N = 1 \qquad (1.13)$$

We can define the average number of heads (or tails) on a number N of attempts:

$$<n_h> = \sum_{n_h=0}^{N} n_h \frac{N!}{n_h! - (N - n_h!)} p^{n_h} q^{N-n_h} = pN \qquad (1.14)$$

The standard deviation is:

$$\sigma_{n_h} = \sqrt{Np(1-p)} = \sqrt{Npq} \qquad (1.15)$$

2.1.3 Gaussian distribution

In the limit $of\ N \to \infty$ (N number of tries) the binomial distribution can be approximated to the gaussian (or normal) distribution.

The general form of the Gaussian distribution is:

$$P(x) = Ce^{-\frac{1}{2(Ax^2 - Bx)}} \qquad (-\infty < x < \infty) \qquad (1.16)$$

where A, B and C are positive constants: A determines the width, B determines the position of the peak and C is the normalization constant. These parameters can be expressed using the mean, μ, and the standard deviation, σ, and they become:

$$A = \frac{1}{\sigma^2} \qquad\qquad B = -A\mu \qquad C = \left(\frac{A}{2\pi}\right)^{1/2} e^{-B^2/2A} \quad (1.17)$$

Now the Gaussian distribution can be written as:

$$P(x) = \sqrt{\frac{1}{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (1.18)$$

2.2 Random walk: an application of binomial distribution

After briefly reviewing the mathematical tools, let us now address some applications and examples of these distributions. Our major interest is for the random walk. It can be seen as an application of the binomial distribution. The simplest form of random walk is the classical bidimensional random walk. Let us consider a particle that can move in a unidimensional space (figure 2.1), making single steps on the left or on the right.

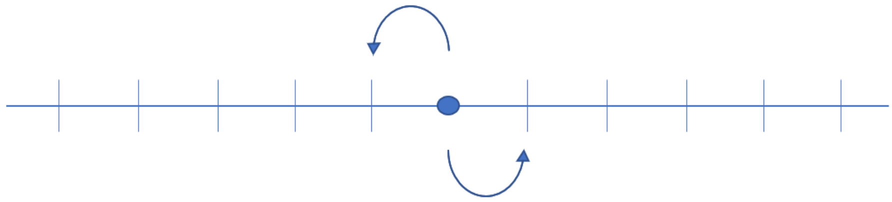

figure 2.1: unidimensional random walk

We want to know the probability of being in a position n $(-\infty < n < \infty)$ after r steps. To each step corresponds a stochastic variable X_j (j = 1,2, …,r), taking the value Δ, $-\Delta$ with the same probability of $\frac{1}{2}$. The position after r steps is:

$$Y_N = \sum_{j=0}^{r} X_j \qquad (1.19)$$

The corresponding characteristic function is:

$$f_{Y_N}(k) = (\cos(k\Delta))^N \qquad (1.20)$$

We can express this function with a Taylor series:

$$f_{Y_N}(k) = (\cos(k\Delta))^N = (1 - \frac{k^2\Delta^2}{2!} + \cdots)^N \approx 1 - \frac{Nk^2\Delta^2}{2!} + \cdots \qquad (1.21)$$

We can easily find that the mean is $\langle Y_N \rangle = 0$, and the standard deviation is $\sigma_{Y_N} = \Delta\sqrt{N}$

It is possible to find the characteristic function considering the distance Δ and the infinitesimal time τ between two consecutive steps:

$$f_Y(k,(N+1)\tau) - f_Y(k,N\tau) = (\cos(k\Delta) - 1)\, f_Y(k,N\tau) = (-\frac{k^2\Delta^2}{2!} + \cdots) f_Y(k,N\tau)$$
(1.22)

We can define the random walk diffusion coefficient $D = \left(\frac{\Delta^2}{2\tau}\right)$ and within the limit for D finite, that is when N → ∞, τ → 0, $\Delta \to 0$ and labelling $N\tau = t$ we can notice that:

$$\lim_{N\to\infty}\lim_{\tau\to 0}\frac{f_Y(k,(N+1)\tau) - f_Y(k,N\tau)}{\tau} = \frac{\partial f_Y(k,t)}{\partial t} \qquad (1.23)$$

And

$$\lim_{N\to\infty}\lim_{\tau\to 0}\lim_{\Delta\to 0}(-\frac{k^2\Delta^2}{2\tau} + \cdots) f_Y(k,N\tau) = -Dk^2 f_Y(k,t) \qquad (1.24)$$

That brings to:

$$\frac{\partial f_Y(k,t)}{\partial t} = -Dk^2 f_Y(k,t) \qquad (1.25)$$

When $f_Y(k,0) = 1$ the eq. 1.24 has the following solution:

$$f_Y(k,t) = e^{-Dk^2 t} \qquad (1.26)$$

The probability distribution has the form:

$$P(y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky} e^{-Dk^2 t}\, dk = \sqrt{\frac{1}{4\pi Dt}}\, e^{-\frac{y^2}{4Dt}} \qquad (1.27)$$

The probability density function of a random walker is a gaussian and its standard deviation is $\sigma = \sqrt{2Dt}$

2.2.1 Galton board

The Galton board is a device invented by Francis Galton to provide an empirical proof of the central limit theorem and normal distribution. The device is structured as a vertical board with interleaved rows of pegs. A ball, thrown into the apparatus, moves under gravitation and it bounces off the pegs while falling. When the ball hit a peg, it randomly moves right or left with the same probability of 1/2.

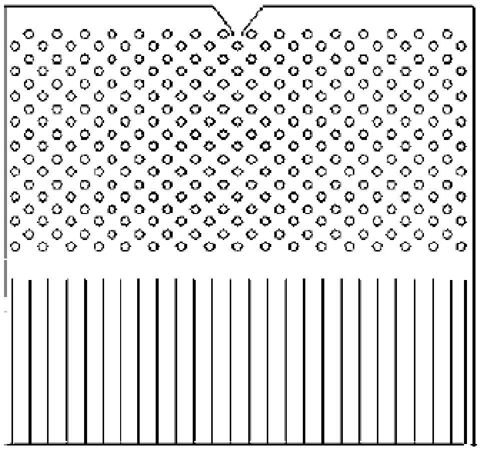

Figure 2.2: Galton board structure

Calling X_k the horizontal deviation caused by a peg in row k, it can only take the values:

$$X_k = \begin{cases} 1 & \mathit{when\ the\ ball\ goes\ right} \\ -1 & \mathit{when\ the\ ball\ goes\ left} \end{cases}$$

The probability function takes the values:

$$P_{X_k}(n) \begin{cases} \frac{1}{2} \ \ if\ n = 1 \\ \frac{1}{2} \ \ if\ n = -1 \end{cases}$$

The horizontal distance from the centre X is given by $X = \sum_{k=1}^{N} X_k$

The horizontal distance X is a discrete binomial variable as a sum of Bernoulli independent variables. We can easily find the probability function:

$$P_X = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \frac{1}{2}^n \quad with\ p = \frac{1}{2} \quad (1.28)$$

We can see an example of the probability distribution for a classic Galton board in table 1.

Table 1: Probability of being at position X after N steps for a classic Galton board

	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$		$\frac{1}{2}$				
2				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3			$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$		
4		$\frac{1}{16}$		$\frac{1}{4}$		$\frac{3}{8}$		$\frac{1}{4}$		$\frac{1}{16}$	
5	$\frac{1}{32}$		$\frac{5}{32}$		$\frac{5}{16}$		$\frac{5}{16}$		$\frac{5}{32}$		$\frac{1}{32}$

2.3 Quantum random walk

In this paragraph we introduce the quantum random walk in the discrete case.

The main reference we used for the presentation of the topic in this section is a paper titled "*Quantum random walks: an introductory overview*" by Kempe J. (Kempe, 2003).

Let us consider a particle in a unidimensional space, whose position is described by a wave-packet $|\psi_{x_0}\rangle$ localised around the position x_0. The transition of the particle after a step of length l can be expressed by the unitary operator:

$$U_l = e^{-\frac{iPl}{\hbar}} \quad \text{such that} \quad U_l|\psi_{x_0}\rangle = |\psi_{x_0-l}\rangle. \qquad (1.29)$$

Let us also assume a particle with a spin of $-\frac{1}{2}$. It is useful to represent the z component of the spin operator with S_z and to denote its eigenstates by $|\uparrow\rangle$ and $|\downarrow\rangle$, so that $S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$ and $S_z|\downarrow\rangle = \frac{\hbar}{2}|\downarrow\rangle$.

We can put $\hbar = 1$ (like in the natural system) to simplify the notation, so the particle is described by the tensor:

$$|\Psi\rangle = \alpha^{\uparrow}\, |\uparrow\rangle \otimes |\psi^{\uparrow}\rangle + \alpha^{\downarrow}\, |\downarrow\rangle \otimes |\psi^{\downarrow}\rangle \qquad (1.30)$$

The first part of (1.30) is the component of the wave-function of the particle in the spin-$|\uparrow\rangle$ space and the second one in the spin-$|\downarrow\rangle$ space. Furthermore normalization requires that $|\alpha^{\uparrow}|^2 + |\alpha^{\downarrow}|^2 = 1$.

The time development corresponding to a step of length l of particle with the spin $-\frac{1}{2}$ can now be described by the unitary operator:

$$U = e^{-2iS_z \otimes Pl} \qquad (1.31)$$

This operator induces a conditional translation on the particle depending on its internal spin-degree of freedom. If the spin of the particle is initially in the state $|\uparrow\rangle$, so that its wave-function is of the form $|\uparrow\rangle \otimes |\psi_{x_0}\rangle$, an application of the operator U transforms the wave-function into $|\uparrow\rangle \otimes |\psi_{x_0-l}\rangle$ and the particle will be shifted by a step l to left. The opposite case is when the spin of the particle is initially in the state $|\downarrow\rangle$, so that its wave-function is of the form $|\downarrow\rangle \otimes |\psi_{x_0}\rangle$, an application of the operator U transforms the wave-function to $|\downarrow\rangle \otimes |\psi_{x_0+l}\rangle$ and the particle will be shifted by l to right.

An important behavior occurs when the spin of the particle, localized in x_0, is not in an eigenstate of S_z, but it is in a superposition state:

$$|\Psi_{\boldsymbol{in}}\rangle = (\alpha^{\uparrow} |\uparrow\rangle + \alpha^{\downarrow} |\downarrow\rangle) \otimes |\psi_{x_0}\rangle \qquad (1.32)$$

The application of the operator U brings to:

$$U|\Psi_{\boldsymbol{in}}\rangle = \alpha^{\uparrow}|\uparrow\rangle \otimes |\psi_{x_0-l}\rangle + \alpha^{\downarrow}|\downarrow\rangle \otimes |\psi_{x_0+l}\rangle \qquad (1.33)$$

Now if we want to measure the spin in the S_z basis, the particle will be in the state $|\uparrow\rangle \otimes |\psi_{x_0-l}\rangle$, localized around $x_0 + l$ with the probability $p^{\uparrow} = |\alpha^{\uparrow}|^2$ or in the state $\alpha^{\downarrow}|\downarrow\rangle \otimes |\psi_{x_0+l}\rangle$, localized around $x_0 - l$ with the probability $p^{\downarrow} = |\alpha^{\downarrow}|^2$.

This last procedure coincides to a random walk on a line, where after a step the particle is on average displaced by $l(p^{\uparrow} - p^{\downarrow})$.

If we repeat the process T times, we will find that the particle is on average displaced by:

$$\langle x \rangle = Tl(p^{\uparrow} - p^{\downarrow}) = Tl(|\alpha^{\uparrow}|^2 - |\alpha^{\downarrow}|^2) \qquad (1.34)$$

And the variance of its distribution will be:

$$\sigma^2 = Tl|\alpha^{\uparrow}|^2|\alpha^{\downarrow}|^2 = Tlp^{\uparrow}p^{\downarrow} \qquad (1.35)$$

2.3.1 The discrete case

Let H_p be the Hilbert space spanned by the position of the particle. We describe the random walk in one dimension, so on a line or on a circle. In the first case the space is spanned by the base $\{|i\rangle : i \in \boldsymbol{Z}\}$, in the second case (a circle of perimeter of lenght N) we have $H_p = \{|i\rangle : i = 0 \dots N-1\}$ with $|i\rangle$ corresponding to a particle localized in position i.

The total space is constituted by the Hilbert space H_p and a "coin-space" (the spin space) H_c spanned by the base $\{|\uparrow\rangle, |\downarrow\rangle\}$, so the total space is given by $H = H_c \otimes H_p$.

The conditional translation is expressed by the shift operator defined as:

$$S = |\uparrow\rangle\langle\uparrow| \otimes \sum_i |i+1\rangle\langle i| + |\downarrow\rangle\langle\downarrow| \otimes \sum_i |i-1\rangle\langle i| \quad (1.36)$$

where $i \in \boldsymbol{Z}$ in the case of the line or $0 \leq i \leq N-1$ in the case of the circle. S transforms the basis state:

$$|\uparrow\rangle \otimes |i\rangle \xrightarrow{S} |\uparrow\rangle \otimes |i+1\rangle \qquad (1.37)$$

and

$$|\downarrow\rangle \otimes |i\rangle \xrightarrow{S} |\downarrow\rangle \otimes |i-1\rangle. \qquad (1.38)$$

The first step of the random walk is a rotation in the coin-space H_c. Let us take the initial state of the random walk in the $|0\rangle$ state while the coin is in the state $|\uparrow\rangle$, $|\Phi_{in}\rangle = |\uparrow\rangle \otimes |0\rangle$. After an iteration (a rotation in the coin space followed by the application of the S operator) we want to have a shift on the right ($|1\rangle$) with a probability of $\frac{1}{2}$ and a shift on the left ($|-1\rangle$) with the same probability. A commonly balanced unitary coin is the Hadamard coin H:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (1.39)$$

Using the H coin, it esay to see that it is balanced:

$$|\uparrow\rangle \otimes |0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |0\rangle \xrightarrow{S} \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |1\rangle + |\downarrow\rangle \otimes |-1\rangle) \qquad (1.40)$$

Measuring the coin state in the standard basis gives $\{|\uparrow\rangle \otimes |1\rangle, |\downarrow\rangle \otimes |-1\rangle\}$ with probability $\frac{1}{2}$. After this measurement there is no correlation between the position left. If we continue the process, with a measurement at each iteration, we obtain a classic random walk represented by a classic Galton board.

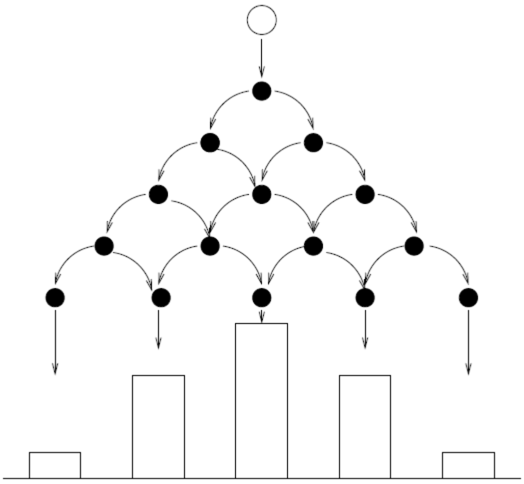

Figure 2.3: Galton's board

In the quantum random walk, we do not measure the coin register during intermediate iteration, to keep the quantum correlations between different positions and let them interfere in the following steps.

The quantum random walk of T steps is defined by the transformation U^T, where U is given by:

$$U = S \cdot (H \otimes I) \qquad (1.41)$$

In order to show the differences of the quantum walk from its classical counterpart let us evolve the walk (without intermediate measurements), starting from the initial state

$$|\Phi_{in}\rangle = |\downarrow\rangle \otimes |0\rangle \qquad (1.42)$$

$$|\Phi_{in}\rangle \overset{U}{\rightarrow} \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |1\rangle - |\downarrow\rangle \otimes |-1\rangle) \overset{U}{\rightarrow} \frac{1}{2}(|\uparrow\rangle \otimes |2\rangle - (|\uparrow\rangle - |\downarrow\rangle) \otimes |0\rangle + |\downarrow\rangle \otimes |-2\rangle) \overset{U}{\rightarrow} \frac{1}{2\sqrt{2}}(|\uparrow\rangle \otimes |3\rangle + |\downarrow\rangle \otimes |1\rangle + |\uparrow\rangle \otimes |-1\rangle - 2|\downarrow\rangle \otimes |-1\rangle - |\downarrow\rangle \otimes |-3\rangle)$$

(1.43)

In table 2, the probability distribution for this case is reported.

Table 2: Probability of being at position X after N steps for a quantum Galton board with the initial state: $|\Phi_{in}\rangle = |\downarrow\rangle \otimes |0\rangle$.

	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$		$\frac{1}{2}$				
2				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3			$\frac{1}{8}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		
4		$\frac{1}{16}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{1}{16}$	
5	$\frac{1}{32}$		$\frac{17}{32}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{5}{32}$		$\frac{1}{32}$

We can see a very different behavior of the quantum random walk: it is no more approaching a gaussian distribution. An important thing to notice is the asymmetry of the distribution, that is shifted to the left. This is explained by the characteristics of the Hadamard coin that treats differently the two directions $|\uparrow\rangle$ and $|\downarrow\rangle$, it only multiplies the phase by -1 in the case $|\downarrow\rangle$. This induces more cancellations for paths going right-wards (destructive interference) while particles that move to the left interfere constructively.

It is also possible to obtain a symmetric distribution, it is necessary to start the walk with a superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$, $|\Phi_{Syn}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \otimes |0\rangle$. Table 3 shows an example of a symmetric distribution.

Table 3: Probability of being at position X after N steps for a quantum Galton board with the symmetric initial state: $|\Phi_{Syn}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \otimes |0\rangle$

	-4	-3	-2	-1	0	1	2	3	4
0					1				
1				$\frac{1}{2}$		$\frac{1}{2}$			
2			$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$		
3		$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$	
4	$\frac{1}{16}$		$\frac{3}{8}$		$\frac{1}{8}$		$\frac{3}{8}$		$\frac{1}{16}$

The next chapter is dedicated to the design of the activity and to the exploration of some applications and implications about classical and quantum random walk, my principal contribution.

Chapter 3: Planning and implementation of an educational activity

In this chapter the development of the fifth day of the module, centered on classical and quantum random walk, is presented. The aims of the activity were to:

a. regain the basic concept of quantum physics, introduced in the previous meeting, necessary and sufficient to comprehend the new logic at the basis of quantum technologies;
b. explore the classical and quantum random walk with an interdisciplinary lens (mathematic, physics, computer science);
c. highlight the differences between classical and quantum logic;
d. reflect on the nature of probability in the two cases.

The activity lasted about two hours and was structured as follows:

- Introduction of classical random walk from an interdisciplinary perspective;
- Introduction of quantum random walk and comparison with the classical case;
- Presentation of some applications of the random walk.

The structure of the random walk activity comprehends all the educational aspects of the I SEE module (Section 1.X). In fact, the activity includes, first of all, an introductory overview of the topic, followed by the treatment of the conceptual elements of the random walk through a teamwork activity aimed to solve a problem. Then, a brief interactive lecture is held and, finally, the presentation of visualization tools such as simulations and examples complete the slot. The activity was thought not only to introduce students to the conceptual aspects of random walk and to appropriate formalism but also to communicate the epistemological aspects behind the topic, with a special focus on the concept of random and on the differences between classical and quantum logic. In the end we wanted to give a vision on the possible applications of the random walk to enlarge the imagination about the impact on the random walk on different fields, even beyond the physical and mathematical ones, and about the plurality of possible STEM careers involved.

3.1 The teaching activity

The first part of the activity was dedicated to the classical random walk. In order to introduce students to the topic and to its mathematization we started with a team-work activity in which they had to solve the well-known problem of the “Drunk sailor’s” (figure 3.1).

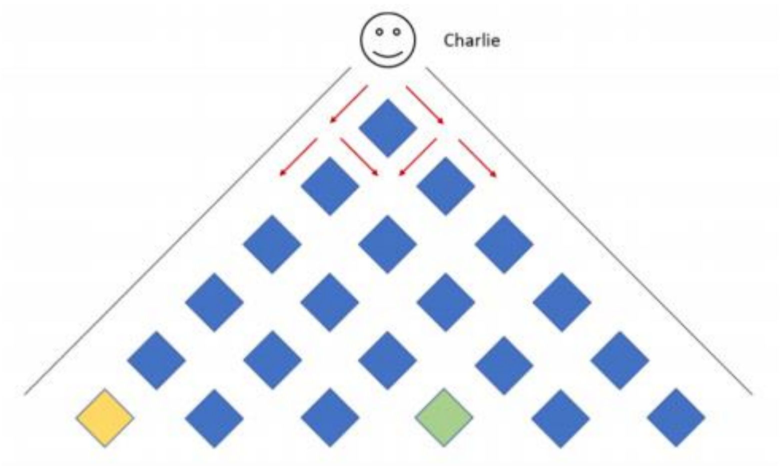

Figure 3.1 “Drunk sailor’s” random walk

The task was:

> “A mathematician, an experimental physicist and a computer scientist are in a cafe debating on the best way to solve the following problem:
>
> Charlie, after a long evening of vices and extravagances out of town, returns, a bit staggering, to Eve city. As soon as he crosses the door of the city, a problem arises: Charlie does not remember where he lives and what is the way to go back. So he starts walking across the streets, proceeding randomly -but without ever turning back! -, hoping to find the right way again.
>
> - What is the probability that Charlie will reach his house (green square, figure 3.1) by moving randomly?
> - What about the probability that Charlie randomly reaches his friend Bob's house (yellow square, figure 3.1)? Is it the same?
>
> The problem made the experts arguing for hours. The issues that most of all aroused discussion concerned: what does it mean to model "walking randomly" for a physicist, a mathematician, and a computer scientist? What is the best method to obtain information on the final probability (measure, calculate and compute)? Among the experts, there were who was looking for rules or models to formalize the problem, who was looking for phenomenological examples/contexts in which design measures, who was looking for input and output data/variables and process strategies (algorithms) to get the computer to solve the problem in the fastest/most effective/and elegant way.”

The groups were asked to reflect on the methods and tools that characterized the disciplines and to try to answer to the following questions:

- What tools and what forms of reasoning would you use to model the problem and support your position in the discussion?
- How would you reformulate the problem from your perspective (mathematics, physics, and computer science) in order to use "your" conceptual, formal, and methodological tools?
- How would you solve it?

The students were divided in three kinds of groups: a group that played the role of mathematicians, that of experimental physicists and that of computer scientists. They students were asked to tell what type of group they preferred and the groups were formed according to students' preferences. The mathematicians' group solved the problem through the Tartaglia triangle and the permutations. The physicists' group thought that it was possible to model the problem for example with the Galton board. The computer scientists' group proposed to solve the problem with the realization of a computer code that, step by step, calculates the probability with the formula.
This introductory problem was designed both to build the interdisciplinary lens that characterize the whole meeting, and to test students' inquiry skills. In particular, we wanted to investigate their ability of making hypothesis and solve problems with different approaches.
Furthermore, through the "Drunk sailor's" problem we shed light on a profound epistemological aspect: the concept of "random".

The second part of the activity was dedicated to take stock of the classical random walk and to introduce the quantum random walk with an interdisciplinary lens.
After resuming the probabilistic aspect of the classical random walk (figure 3.2), the students were introduced to the concept of probability distribution (mathematical point of view).

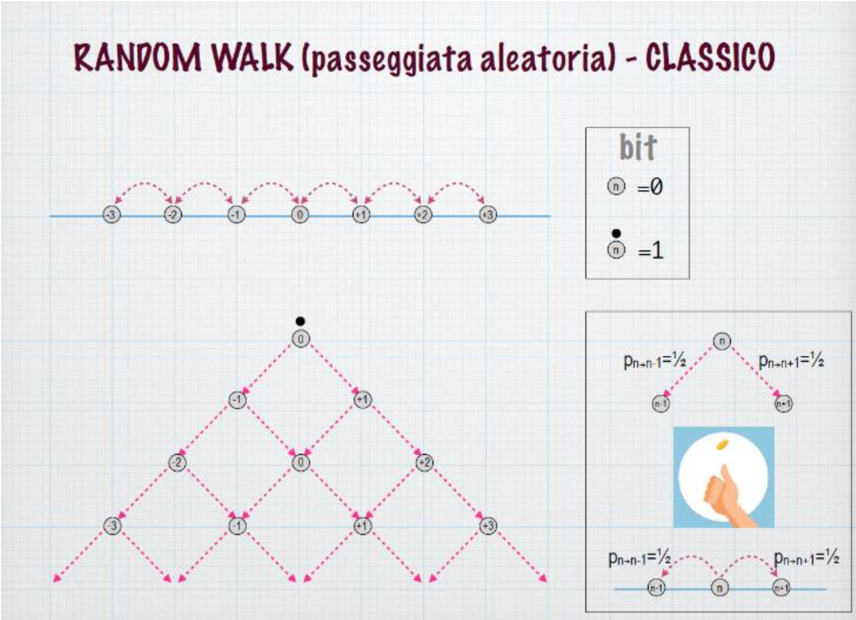

Figure 3.2 Classical random walk summary slide

The drunk's path to come back home is stressed to be similar to the path of a sober person who decides, at each intersection, whether to go right or left by flipping a coin. At each toss the person is equally likely to move to the right or left and the probability distribution obtained is a binomial distribution (figure 3.3).

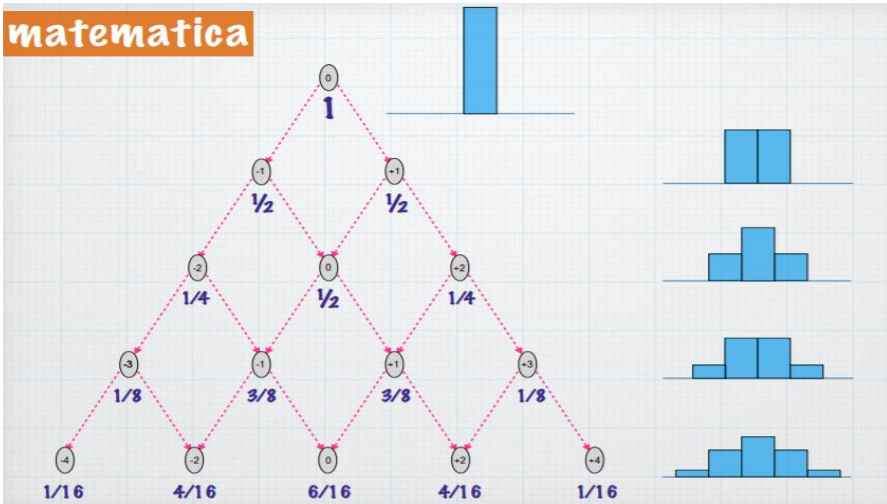

Figure 3.3: binomial distribution of the discrete random walk

From a physical point of view, the Galton machine was presented through a video (https://www.youtube.com/watch?v=4HpvBZnHOVI) in order to not only show how a random walk can physically be realized but also to test, against an experiment, the "mathematical hypothesis" about the probability distribution.

Finally, from a computational point of view, two methods were discussed. The first was the one presented from the group of students who played the role of computer scientists: we know the equation and we implement it on a computer to calculate the probability. The second possible method was introduced in the discussion and consists in the simulation of the coin with n algorithm that generates random numbers (Figure 3.4).

```
# Python code for 1-D random walk.
import random
import numpy as np
import matplotlib.pyplot as plt

# Probability to move up or down
prob = [0.05, 0.95]

# statically defining the starting position
start = 2
positions = [start]

# creating the random points
rr = np.random.random(1000)
downp = rr < prob[0]
upp = rr > prob[1]


for idownp, iupp in zip(downp, upp):
    down = idownp and positions[-1] > 1
    up = iupp and positions[-1] < 4
    positions.append(positions[-1] - down + up)

# plotting down the graph of the random walk in 1D
plt.plot(positions)
plt.show()
```

Figure 3.4: algorithm for generating random numbers in python.

The presentation of these two methods was important to highlight the different ways of processing between a simulator and a computer, important point of the previous meetings. The quantum random walk was now introduced fostering students to think about the initial problem in the case in which the drunker was “quantum”.

The basic concepts of quantum physics necessary to understand the topic had been introduced in the previous lectures, and now they were resumed. Particular importance was given to the concepts of state, state processing and superposition principle (figure 3.5).

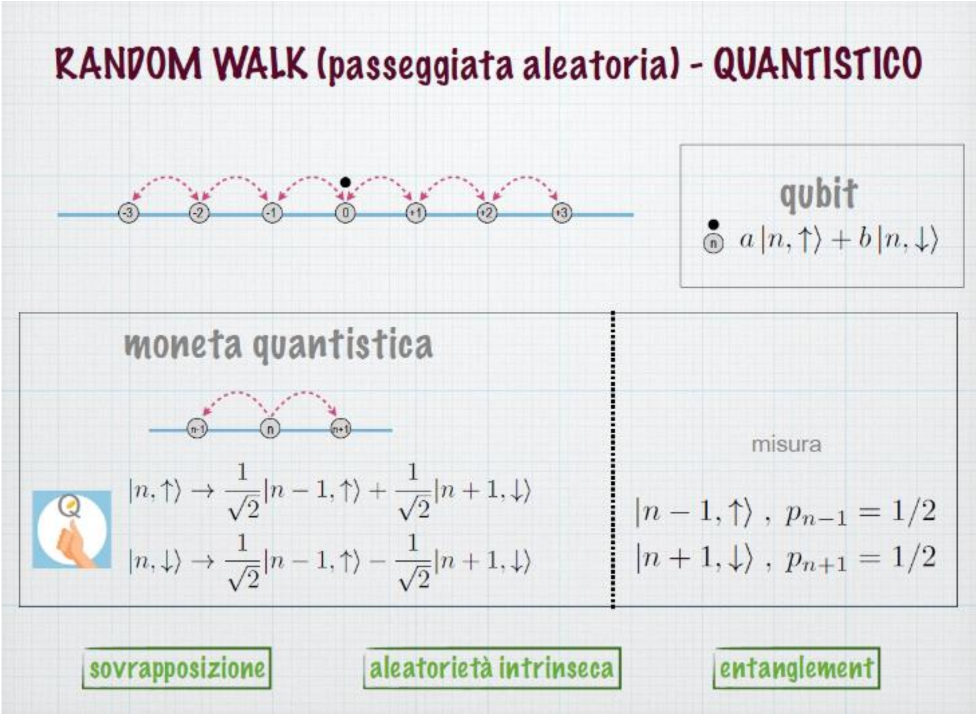

Figure 3.5: Introduction to quantum random walk

In order to build an analogy with the classical case, the flip coin and the shift rule were redefined in the quantum case. So, the classical coin was replaced by the Hadamard logic gate that transforms $|0\rangle$ and $|1\rangle$ as

$$|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

while the shift operator remained the same: based on the outcome of the “toss” the “quantum drunkard” moves to the right or to the left.

Once all the “ingredients” were explained, the quantum random walk was presented from a mathematical point of view with the Dirac formalism (figure 3.6).

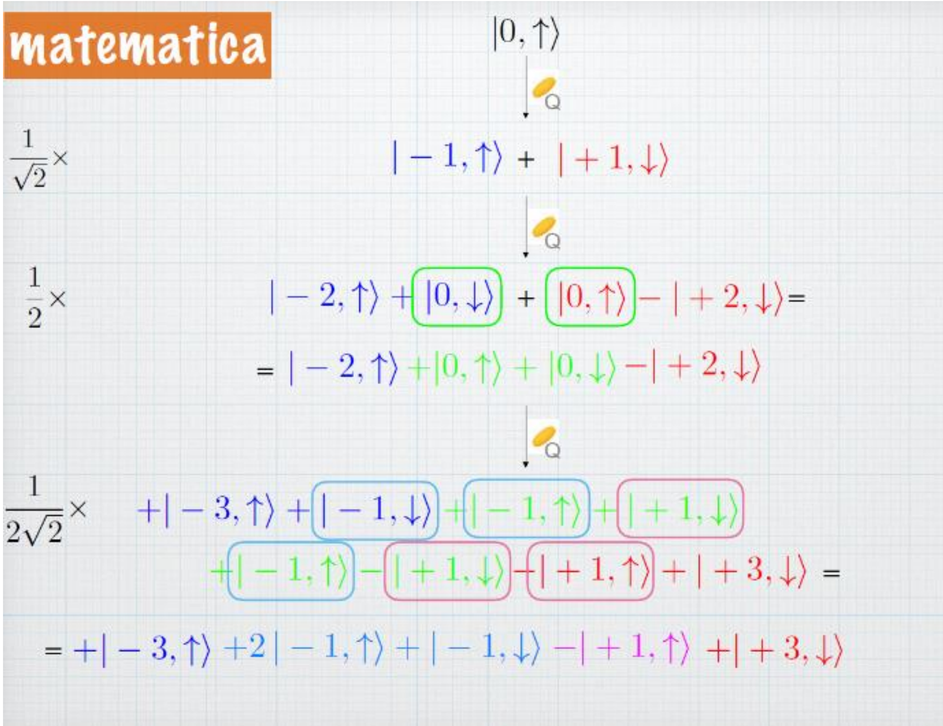

Figure 3.6: mathematical passages to solve the “quantum drunk problem”

The various mathematical passages to solve the “quantum drunk problem” were reconstructed with help of the students that participated actively. Through the calculation we showed how, starting from a particular state ($|0,\uparrow\rangle$), the shape of the probability distribution was different from the classical case: the distribution was no longer centred and changed shape according to the initial state (Figure 3.7).

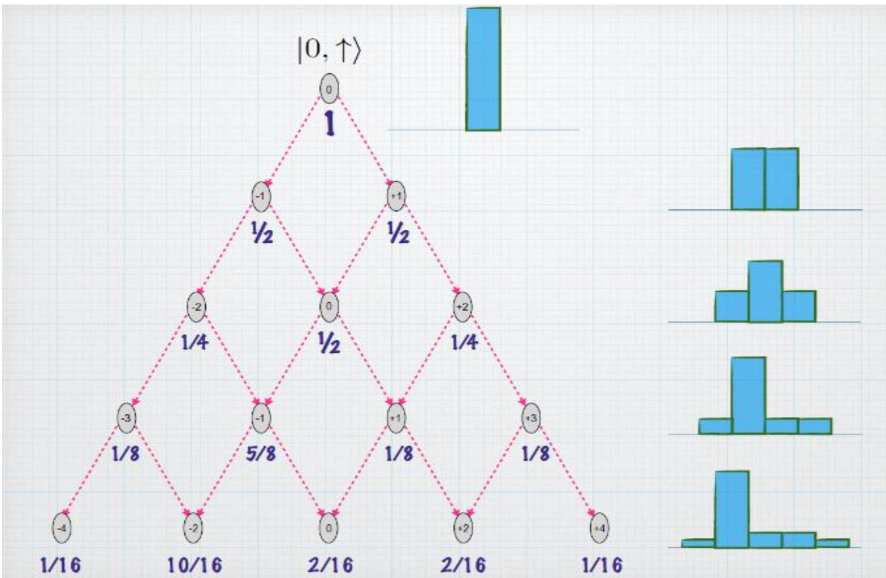

Figure 3.7: probability distribution in the quantum case

Then the case of a symmetric shape of the probability distribution were discussed (Figure 3.8).

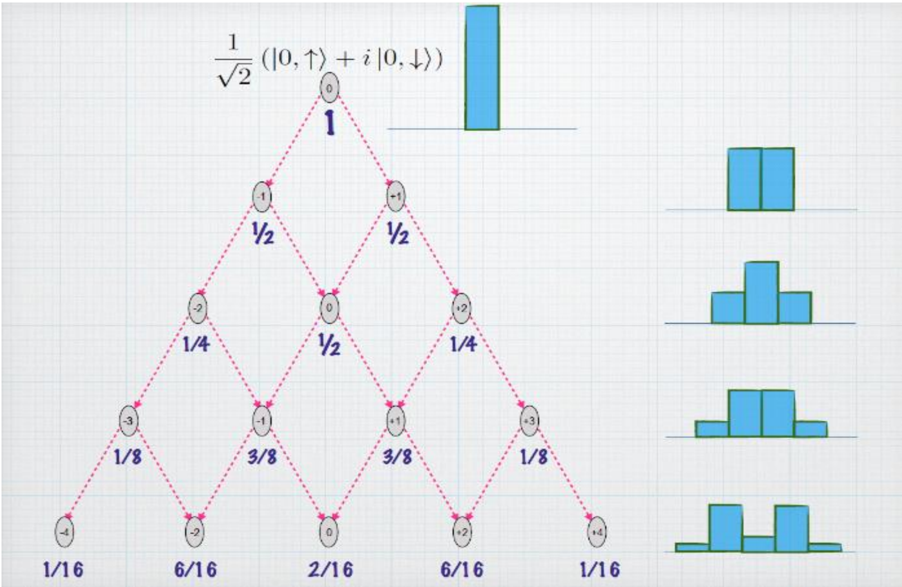

Figure 3.8: symmetric shape of probability distribution in the quantum case

From a physical point of view we introduced the example of the boson sampling (Figure 3.9). Our aim was not to present the very complicated phenomenon, but to use it as a visual tool.

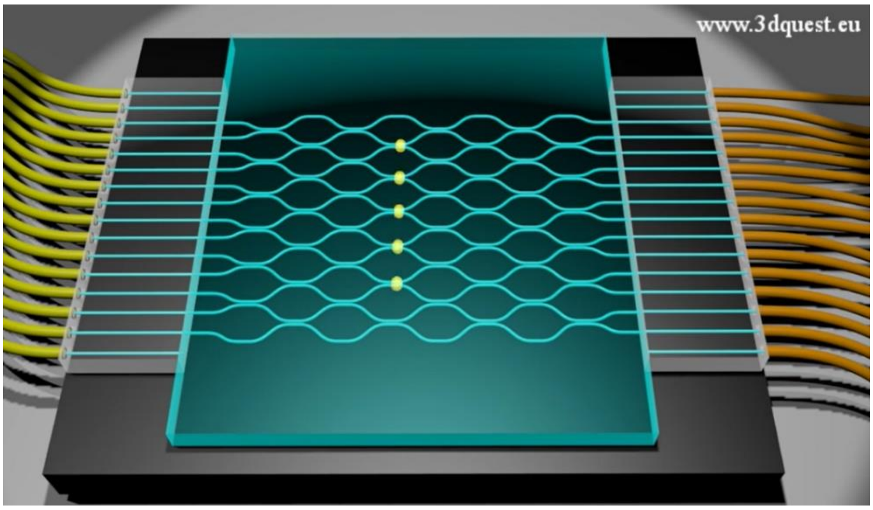

Figure 3.9: boson sampling (https://www.youtube.com/watch?v=jiodj5b8Z1E)

From a computational point of view, the code to implement the quantum random walk was presented (Figure 3.10). Through this perspective, we shed light on the nature of the probability in the classical and quantum case. As regards the nature of probability in the case of classical computation, based on classical and therefore deterministic physics, it is not possible to generate completely random numbers (in fact, we talk about pseudo-random number generator), for which randomness is not intrinsic. In the quantum case, on the other hand, the probability lies in the properties of the quantum object, that is of ontological nature.

```python
# Numero di siti
L = 1000
# Crea stato iniziale
# Il vettore di stato è una matrice Lx2 (gdl spaziali x gdl spin)
Psi = np.zeros( (L,2), dtype=complex )
Psi[np.int(L/2),0] = 1.0  / math.sqrt(2)
Psi[np.int(L/2),1] = 1.0j / math.sqrt(2)
InitState = Psi.copy() # salva lo stato iniziale che non si sa mai

# Porta di Hadamard
H = np.array( [[1,1], [1,-1]], dtype=complex ) * math.sqrt(0.5)
# Operatori di shift
Splus  = np.zeros( (L,L), dtype=complex )
Sminus = np.zeros( (L,L), dtype=complex )
for k in range(L-1):
    Splus[k,k+1]  = 1
    Sminus[k+1,k] = 1

# Esegue il walk di 1 step
def Walk(State):
    # Coin flip
    for k in range(L):
        State[k,:] = H @ State[k,:]
    # Shift
    State[:,0] = Splus  @ State[:,0]
    State[:,1] = Sminus @ State[:,1]
    return State

# Calcola l'ampiezza di probabilità
def Prob(State):
    return np.sqrt( abs(State[:,0])**2 + abs(State[:,1])**2)

```

Figure 3.10: Quantum random walk

Finally a simulation, built with a PhD student (Sunny Pradhan), was presented, and discussed to show the differences between the classical and quantum random walk (Figure 3.11).

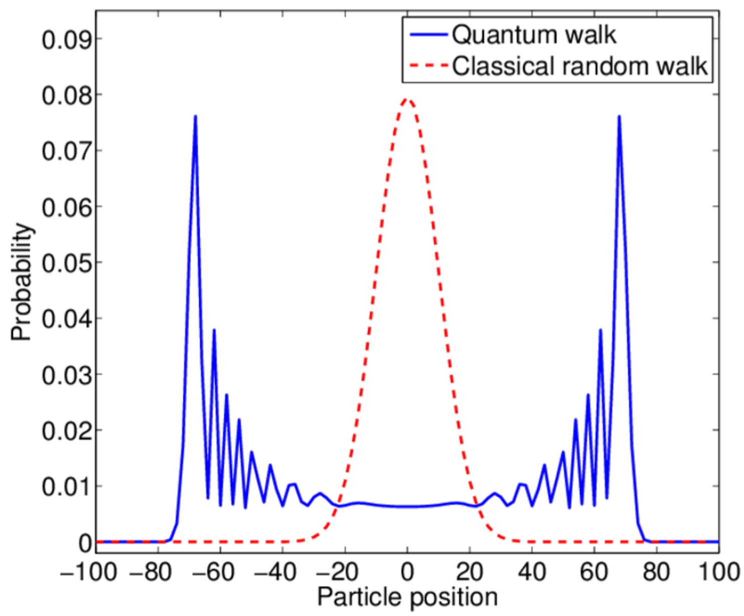

Figure 3.11: comparison between probability distribution in classical and quantum random walk

In particular, we insisted on the following aspects:

- The quantum random walk “samples” the “position space” of the particle faster than the classical case;
- The particle in the quantum case can assume many more positions than in the classical one.

The activity proved to be very reach and allowed us to:

a. regain the basic concept of quantum physics and of computation introduced in the previous meeting;
b. present the classical and quantum random walk with an interdisciplinary approach showing how the peculiarities of the different involved disciplined interplay and interact to build a more comprehensive picture of random walk;
c. show where the difference between classical and quantum computation lies;
d. reflect on the nature of probability in the two cases.

3.2 The importance of the simulations in the Physics Education

When we were planning the fifth meeting, we realized that the theme of the activity, the classical and quantum random walk, could be an arduous topic for the students and that we need more than a single point of view. We thought that the simple mathematical explanation of the topic could not be enough to communicate the topic and we decided to introduce some simulations to improve not only the understanding but also the visualization.

The use of computer simulations for science education is not new in the literature. In fact, computer simulations make complex systems accessible for students of varying ages, abilities and learning levels. The computer, instead of the student, can assume responsibility of processing the underlying mathematics in order to let the student begin exploring a complex system by first focusing on conceptual understanding (Rieber, L. P., Tzeng, S. C., & Tribble, K., 2004). The strength of the computer simulation is the interactivity, a simulation in fact is not a static reproduction and the students can interact with the program. Nevertheless, it is incorrect to believe that more interactive a simulation is and more the students learn. It is important not to abuse of this interactivity which can otherwise have negative effects on the learning. In the case of simulation, this can occur when there are too many possible choices, because of an interface made too heavy by unnecessary details or because the interactivity simply suffocates the students’ capabilities of reflection (Landriscina, 2009).

The controversial interactivity is not the only advantage of the simulation, that offers also practical advantages such as freedom from time and space, safety, and cheapness. Physics is an experimental science where observation, measuring and theoretical speculations are processes that cannot be separated from the physical knowledge construction, even in the classroom and the simulations brings the experiments in the classroom (Concari, Giorgi, Cámara, & Giacosa, 2006). The absence of restriction in time and space allows several consecutive attempts without fear of making mistakes and the possibility of testing alternative hypothesis. These possibilities give the opportunity to develop several cognitive processes:

- integrate the information coming from different sources
- connect new knowledge to the one already known
- recover analogies capable of favoring one's understanding
- produce explanations
- coordinate representation and different perspectives
- to create inferences
- to abandon concepts that are no longer useful

All this can facilitate the construction of new mental schemes or the replacement of the already existing with new ones.

The simulation we realized for the activity aims to show the probability distribution in the classical and quantum case in order to draw the differences in terms of the basic logic. We introduced the interactivity through the possibility of inserting several factors in input. One of them is the choice between the classical coin and the "Hadamard coin" (or other transformation that create a superposition state). Other input variables are: the initial state and the number of "balls".

3.3 Applications of the random walk

3.3.1 The activity on the application in class

The final part of the random walk activity, held by me, was a future-oriented activity, a presentation on the possible applications of the random walk in various fields from art to finance. Table 3.1 shows the applications that I presented.

Table 3.1 Fields of interest and corresponding topics of the different applications

Field	Topic
Art	"Quantum Cloud" sculpture
Digital image processing	Image segmentation

Analysis algorithm	PageRank
Finance	High speed trading
	Random walk theory
Videogame programming	Artificial intelligence

I started with an introduction where I explained the aim of the presentation in which random walk was presented not only as simple mathematical algorithm, but also as a powerful tool in various fields, that go far from mathematics and physics. For this purpose, I showed the image of the sculpture “Quantum cloud” (figure 3.12) a work of art designed through the random walk.

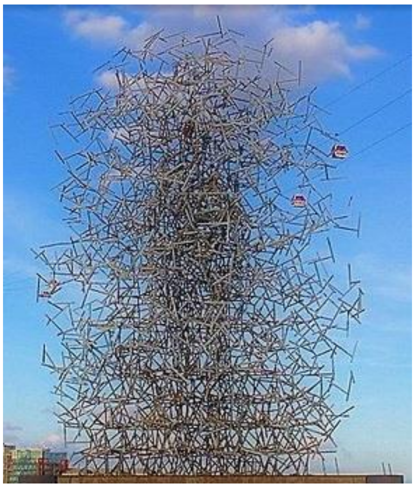

Figure 3.12 the sculpture “Quantum cloud”

I continued the presentation with a quick overview of digital image processing with the random walker method applied on the image segmentation (figure 3.13).

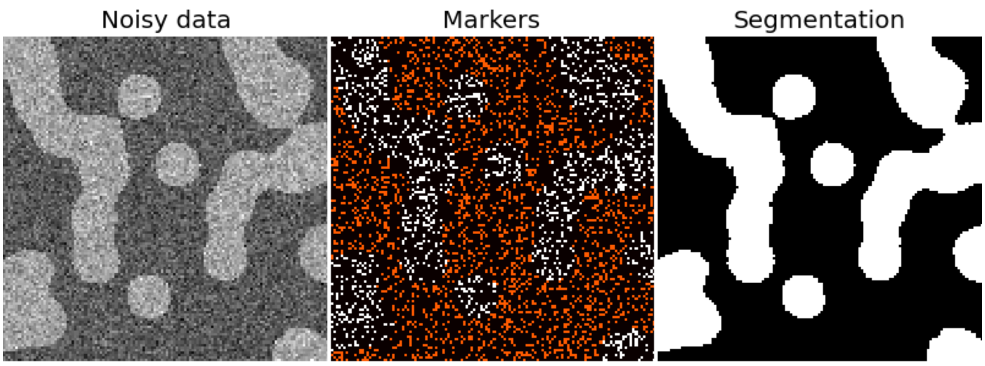

Figure 3.13 example of image segmentation through random walker method

As for the field of the network analysis, I briefly showed the structure of the PageRank algorithm, used by Google to give a weight to the different sites on the web. This algorithm was developed starting from the classic random walk. I continued with a part on the finance application of the random walk with the examples of high-speed trading and with the “random walk theory”. The first is a practice where you act on the market with a large number of transactions at very high speed, with the aim of building maps of the markets. The second one is a theory of the evolution of the markets based on the mathematical concept of random walk, according to which there is no correlation between future and past market trends.

The last example that I decided to show was the application in the evolution of artificial intelligence (AI) in videogames. I explained that one of the first examples of AI in the videogame history is represented by the four ghosts in Pac-man where the orange one is programmed through the random walk (figure 3.14).

Figure 3.14 Pac-man screenshot

3.3.2 The importance of the applications

We see in the first chapter the structure of an I SEE module, and one of the three main phases of every module concerns the future oriented activities. As we said there are four kinds of future oriented activities in the I SEE modules: activities to flesh out the future-oriented structure of scientific discourse, activities inspired by future studies, exposure activities and action competence activities. The presentation of the applications belongs to the future oriented activities as an exposure activity with the aim of enlarging the imagination about possible STEM careers. In particular, activities of this type are based on the idea that a student, in order to be able to choose among alternative futures, has to be exposed to the sense of them. The exposure activities make STEM careers more attractive because, it is conjectured in I SEE, they will not only help students directly experience the acquisition of authentic professional competences but they will also support students to cope rationally, emotionally, creatively, and responsively with their future (Branchetti et al., 2018). This kind of activities shows the possible fields of applications of a scientific topic, is an important instrument of the future oriented activities, because we can demonstrate how close the STEM themes are to reality and helpful for facing it.

Conclusions

The present thesis is situated within the I SEE project, an Erasmus + project with the overarching goal of designing and developing teaching modules on STEM (Science, Technology, Engineering and Mathematic) topics. In particular, I participated to the design and the implementation of a module of quantum computers.

The I SEE project has developed an approach to deal with a big challenge that the Science Education is facing: the inclusion of STEM in teaching because STEM subjects can be a context to explore the relation between science disciplines and societal challenges. The developed approach keeps together the conceptual and epistemological dimensions and the future dimension of a STEM topic. The interaction between these three dimensions is supposed to improve students' capacities to navigate the complexity and the uncertainty of the present and enlarge their visions of the future, supporting possible ways of acting in the present with one's eye on the horizon. As follow-up of the project I SEE, the project IDENTITIES is much more focused on the theme of interdisciplinarity and because of this new activities have been designed for the 2020 edition of the module of quantum technologies.

In particularly, it was new and informed also by the project IDENTITES the activity to which I directly contributed. It concerns the classical and quantum random walk and, as well as a future-oriented activity. It is an activity strongly characterized by the aim to show the interdisciplinarity between physics, mathematics, and computer science. More specifically, the educational reconstruction of the topic allowed us to touch the most important aspects of the entire module: i. the intrinsic interdisciplinary nature of quantum technologies and ii. the difference between the logic at the basis of classical and quantum computers. In fact showing the random walk from a mathematical, physical, and computational perspective, the student could explore the peculiarities of the disciplines involved and how they intertwine to build a more comprehensive picture of the topic.
The designed interdisciplinary approach seemed to be effective and to have impacted students' understanding of the topic.
A student commented:

> *"This type of approach helped me to have a comparison of the various disciplines on the same topic and therefore to complete it in all its aspects. Mathematics is based on the elaboration of numbers and on certain procedures, the physics on measurements and experiments, computer science on logical procedures"*

The quotation stresses the peculiarities of the disciplines he perceived and the role that a multi-perspective vision can provide to grasp many aspects of the topic.

Another students said:

> “[...] *the three disciplines complement each other as mathematics provides formulas that are applied and exploited by physics to formulate laws which in turn are transported to a computational dimension where they are easily verified*”.

This student tried to find a thread, a sort of *concecutio* that regulates the roles and the purposes of the three disciplines.

Finally, the future-oriented dimension, typical of I SEE project, prompted me to pay particular attention on the application and implication of the random walk in various fields from physics to art. This activity fosters the students to deal with the wide range of STEM carriers that these technologies can involve.

I think that one of the main difficulties that Science Education has to overcome is to make explicit to the students the importance of the STEM themes and to expand the vision on the future STEM careers. This is one of the goals that I hope to continue to contribute for the future as well.

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