A Game Theory Approach to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control for A Class of Stochastic Time-Varying Systems with Randomly Occurring Nonlinearities

Lifeng Ma[a], Zidong Wang[b], Yuming Bo[a], Zhi Guo[a]

[a] *School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China.*

[b] *Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom.*

Abstract

This paper is concerned with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for a class of stochastic time-varying systems with nonlinearities. The nonlinearities are described by statistical means and could cover several kinds of well-studied nonlinearities as special cases. The occurrence of the addressed nonlinearities is governed by two sequences of Bernoulli distributed white sequences with known probabilities. Such nonlinearities are named as randomly occurring nonlinearities (RONs) as they appear in a probabilistic way. The purpose of the problem under investigation is to design a controller such that the closed-loop system achieves the expected $\mathcal{H}_2$ performance requirements with a guaranteed $\mathcal{H}_\infty$ disturbance attenuation level. A sufficient condition is given for the existence of the desired controller by means of the solvability of certain coupled matrix equations. By resorting to the game theory approach, an algorithm is developed to obtain the controller gain at each sampling instant. A numerical example is presented to show the effectiveness and applicability of the proposed method.

Keywords: Discrete time-varying systems, Nonlinear stochastic systems, Randomly occurring nonlinearities, Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control, Game theory.

In control engineering practice, it is always desirable to design controllers that are able to satisfy multiple performance objectives, such as the $\mathcal{H}_2$

Email address: Zidong.Wang@brunel.ac.uk (Zidong Wang)

performance index evaluating the transient system behaviors [18] and the $\mathcal{H}_\infty$ performance index scaling the system robustness against external disturbances as well as parameter uncertainties [6, 16, 17]. In particular, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control scheme serves as a typical multiobjective controller design technique by which the closed-loop system could simultaneously achieve the pre-specified $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices. Therefore, in the past few decades, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has attracted considerable research interests and, accordingly, a number of theoretical and practical results have been reported in the literature, see e.g. [7, 9, 11, 19].

So far, several approaches have been developed to tackle the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. For linear deterministic systems, the algebraic approach, time-domain Nash game approach, convex optimization approach and linear matrix inequality (LMI) approach have been applied in [1, 5, 8, 12], respectively, to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems. Parallel to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problems have also gained quite a lot of research interests, see e.g. [2, 15] where the minimization approach and Nash game approach have been employed. As far as the nonlinear *deterministic* systems are concerned, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been dealt with in [10] with the solution characterized in terms of the cross-coupled Hamilton-Jacobi-Issacs (HJI) partial differential equations. Due to the difficulty in solving HJI equations, a T-S fuzzy model has been proposed in [3] to approximate the nonlinear system with hope to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback control problem.

It is now well recognized that the stochastic systems have been playing more and more important roles in control theory and applications. Accordingly, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control and filtering problems for stochastic systems have recently stirred much research interest and some initial results have been available in the literature, see e.g. [4, 20, 21, 23]. Specifically, in [20], a unified LMI framework has been established for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem for a class of uncertain stochastic systems. By means of game theory, the mixed stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been solved with state and disturbance dependent noises for continuous-time case in [4] and discrete-time case in [23], respectively. When it comes to the nonlinear stochastic system, the reported results are relatively fewer due mainly to the complicated dynamics of the nonlinearities. Moreover, most of the existing literature has focused on the *time-invariant* nonlinear stochastic systems despite the fact that real-world systems are usually time-varying, that is, all the system parameters are explicitly dependant on time. In fact, for a truly time-varying nonlinear stochastic system, we would be more interested in the transient behaviors over a *finite time-interval*, e.g., the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance over a finite horizon. Unfortunately, for *time-varying* stochastic *nonlinear* systems, the $\mathcal{H}_2/\mathcal{H}_\infty$ control as well as filtering problems have not been adequately investigated yet.

Another issue that should be brought to our attention is the so-called randomly occurring nonlinearities (RONs) [13]. In many real-world engineering practices, due to random abrupt changes of the environmental circumstances, the system may be corrupted by nonlinear disturbances in a *probabilistic* way. As opposed to the traditional deterministic disturbances, such nonlinear disturbances could be either existent or non-existent at a specific time and the existence is governed by certain Bernoulli distributed white sequences with known probabilities. For example, in the maneuvering target tracking problem, it is quite common for the targets to maneuver in a nonlinear way occasionally in order to get rid of the potential track. In this case, the RONs can be ideally applied to describe such a maneuvering behavior. Another case that can be perfectly interpreted by RONs is the networked control systems (NCSs). In NCSs, due to the limited bandwidth, the network inevitably suffers from the poor stability and variational reliability that give rise to probabilistic nonlinear disturbances. Such phenomena of RONS have already been verified by many experiments. Unfortunately, up to now, the phenomena of RONs have not gained adequate research attention especially in the context of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem.

Summarizing the discussion made by far, in this paper, we aim to employ the game theory method to deal with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for a special type of nonlinear stochastic systems where the nonlinearities are described by statistical means and could cover several nonlinearities frequently appearing in engineering practices. Such nonlinearities appear in a probabilistic way and are therefore named as randomly occurring nonlinearities. The purpose of the problem under investigation is to design a controller such that the closed-loop system achieves the pre-specified $\mathcal{H}_\infty$ disturbance attenuation level and, at the same time, the energy of system output is minimized when the worst-case disturbance happens. The main contribution of this paper lies in: (i) a unified framework is established to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem when the system is both *time-varying* and *nonlinear*; (ii) the controller design technique developed in this paper is presented in terms of the solvability of certain Riccati-like difference equations that have a much lower computation complexity than the commonly used LMI technique, and is therefore more convenient and effective for practical design; and (iii) the phenomena of randomly occurring nonlinearity are considered in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. As a conclusion, the control problem considered in this paper is significant in both the theoretical and practical senses.

The rest of the paper is organized as follows. In Section 1, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for a class of discrete time-varying nonlinear stochastic systems is formulated. A sufficient condition for satisfying the pre-specified $\mathcal{H}_\infty$ performance index is given 2. Section 3 proposes the state feedback design algorithm. In Section 4, a numerical example is presented to show the effectiveness of the proposed algorithm. Section 5 gives the conclusion.

Notation The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The notation $X \geq Y$ (respectively $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively positive definite). $\mathbb{E}\{x\}$ stands for the expectation of stochastic variable x and $\mathbb{E}\{x|y\}$ for the expectation of x conditional on y. The superscript "T" denotes the transpose. $\|a\|^2$ represents $a^{\mathrm{T}}a$ where a is a vector, while $\|a\|_A^2$ means $a^{\mathrm{T}}Aa$. $\mathrm{tr}(A)$ means the trace of matrix A. $\mathrm{diag}\{F_1, F_2, \ldots, F_m\}$ denotes a block diagonal matrix whose diagonal blocks are given by F_1, F_2, $\ldots$, F_m.

1. Problem Formulation

Consider the following discrete-time nonlinear stochastic system defined on $k \in [0, N-1]$:

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k + D_k \omega_k + \alpha_k f_k(x_k, u_k, k) \\ y_k = C_k x_k + \beta_k g_k(x_k, u_k, k), \quad x_0 = x_0. \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^r$ is the system output, $\omega_k \in \mathbb{R}^p$ is the external disturbance belonging to $l_2([0, N-1], \mathbb{R}^p)$. A_k, B_k, C_k and D_k are time-varying matrices with appropriate dimensions.

For notation simplicity, let $h_k \triangleq \begin{bmatrix} f_k^{\mathrm{T}}(x_k, u_k, k) & g_k^{\mathrm{T}}(x_k, u_k, k) \end{bmatrix}^{\mathrm{T}}$. For all x_k and u_k, h_k is assumed to satisfy:

$$\mathbb{E}\{h_k | x_k\} = 0, \tag{2}$$

$$\mathbb{E}\left\{h_k h_j^{\mathrm{T}} | x_k\right\} = 0, \qquad k \neq j \tag{3}$$

$$\mathbb{E}\left\{h_k h_k^{\mathrm{T}}|x_k\right\} = \sum_{i=1}^{q}\left[\begin{array}{cc} \varPi_{ik}^{11} & \varPi_{ik}^{12} \\ (\varPi_{ik}^{12})^{\mathrm{T}} & \varPi_{ik}^{22} \end{array}\right]\mathbb{E}\left\{\left(x_k^{\mathrm{T}}\varGamma_{ik}x_k + u_k^{\mathrm{T}}\varXi_{ik}u_k\right)\right\}, \quad (4)$$

where $\varPi_{ik}^{11} \geq 0$, $\varPi_{ik}^{22} \geq 0$, $\varGamma_{ik} \geq 0$, $\varXi_{ik} \geq 0$ and $\varPi_{ik}^{12}$ $(i = 1, 2, \ldots, q)$ are known matrices with compatible dimensions.

Remark 1. *As discussed in [22], the nonlinear descriptions* (2)-(4) *can characterize several well-studied nonlinear stochastic systems, such as systems with state-dependent multiplicative noises, nonlinear systems with random sequences whose powers depend on sector-bound nonlinear function of the state, nonlinear systems with a random sequence whose power depends on the sign of a nonlinear function of the state, to name just a few. It is worth pointing out, in [23], a mixed multiobjective controller has been designed for a type of discrete-time systems with state-dependent noise, which is a special case of the stochastic systems we study in this paper. Moreover, in this paper, we also consider the phenomena of RONs which will be illustrated in detail in the following section. Such a model is applied to describe the way in which the time-varying working conditions affect the system structure. Compare with [26], the model we studied in this paper is much closer to the engineering practice not only in the type of nonlinearities but also in the way how these nonlinearities affect the system.*

The random variables $\alpha_k \in \mathbb{R}$ and $\beta_k \in \mathbb{R}$, which account for the phenomena of RONs, are assumed to be independent from h_k and take values of 0 and 1 with

$$\mathrm{Prob}\{\alpha_k = 1\} = \mu, \qquad \mathrm{Prob}\{\alpha_k = 0\} = 1 - \mu, \quad (5)$$

$$\mathrm{Prob}\{\beta_k = 1\} = \nu, \qquad \mathrm{Prob}\{\beta_k = 0\} = 1 - \nu. \quad (6)$$

Hence it can be easily obtained that

$$\mathbb{E}\{\alpha_k\} = \mathbb{E}\{\alpha_k^2\} = \mu. \quad (7)$$

and

$$\mathbb{E}\{\beta_k\} = \mathbb{E}\{\beta_k^2\} = \nu. \quad (8)$$

Remark 2. *As discussed in [13], the Bernoulli distributed white sequences α_k and β_k are employed to describe the randomly occurring stochastic nonlinearities. Such a phenomenon often occurs with the changes of the environmental circumstances, for example, random failures and repairs of the* *components, changes in the interconnections of subsystems, sudden environment changes, modification of their types and/or intensity. Obviously, if* $\alpha_k = 0$ *(*$\beta_k = 0$*), there is no nonlinearity in the state (output) equation; if* $\alpha_k = 1$ *(*$\beta_k = 1$*), there is a nonlinear disturbance with first and second moments specified by (2)-(4).*

For system (1), given a $\gamma > 0$, we consider the following performance index:

$$J_1 \triangleq \mathbb{E}\left\{\sum_{k=0}^{N-1}\left(\|y_k\|^2_{R_k} - \gamma^2\|\omega_k\|^2_{W_k}\right)\right\} - \gamma^2\|x_0\|^2_{S_k}, \tag{9}$$

where $R_k > 0$, $W_k > 0$ and $S_k > 0$ are weight matrices.

In this paper, our aim is to determine a state feedback control law $u_k = K_k x_k$ such that

($\mathcal{R}1$) For a pre-specified $\mathcal{H}_\infty$ disturbance attenuation level $\gamma > 0$, $J_1 < 0$ holds.

($\mathcal{R}2$) When there exists a worst-case disturbance ω_k^*, the output energy defined by $J_2 \triangleq \mathbb{E}\{\sum_{k=0}^{N-1}\|y_k\|^2\}$ is minimized.

2. $\mathcal{H}_\infty$ Performance

In this section, we first analyze the $\mathcal{H}_\infty$ performance for the time-varying stochastic system with randomly occurring nonlinearities. Then, a sufficient condition is given for satisfying the pre-specified $\mathcal{H}_\infty$ requirement. To this end, setting $u_k = 0$, we obtain the corresponding unforced system for system (1) as follows:

$$\begin{cases} x_{k+1} = A_k x_k + D_k\omega_k + \alpha_k f_k(x_k, k) \\ \quad y_k = C_k x_k + \beta_k g_k(x_k, k), \quad x_0 = x_0. \end{cases} \tag{10}$$

We now consider the following recursion:

$$\begin{aligned} Q_k = {} & A_k^{\mathrm{T}} Q_{k+1} A_k + \sum_{i=1}^{q} \mathit{\Gamma}_{ik}\Big(\mathrm{tr}[\mu Q_{k+1}\mathit{\Pi}_{ik}^{11}] + \mathrm{tr}[\nu R_k \mathit{\Pi}_{ik}^{22}]\Big) \\ & + C_k^{\mathrm{T}} R_k C_k + \left(D_k^{\mathrm{T}} Q_{k+1} A_k\right)^{\mathrm{T}} \mathit{\Theta}_k^{-1}\left(D_k^{\mathrm{T}} Q_{k+1} A_k\right), \end{aligned} \tag{11}$$

with $Q_N = 0$ and

$$\mathit{\Theta}_k \triangleq \gamma^2 W_k - D_k^{\mathrm{T}} Q_{k+1} D_k. \tag{12}$$

Lemma 1. *Consider the system* (10) *with given positive scalar* $\gamma > 0$ *and weight matrices* W_k, R_k *and* S_k. *A sufficient condition for* J_1 *in* (9) *to be negative for all nonzero* $(\{\omega_k\}, x_0)$ *is that there exists a solution* Q_k *to* (11) *such that* $\Theta_k > 0$ *and* $Q_0 < \gamma^2 S_k$.

Proof 1. *By defining*

$$\tilde{J}_k \triangleq x_{k+1}^{\mathrm{T}} Q_{k+1} x_{k+1} - x_k^{\mathrm{T}} Q_k x_k \tag{13}$$

we have

$$\begin{aligned}\mathbb{E}\{\tilde{J}_k\} =& \mathbb{E}\{(A_k x_k + D_k\omega_k + \alpha_k f_k(x_k,k))^{\mathrm{T}} Q_{k+1} \\ & \times (A_k x_k + D_k\omega_k + \alpha_k f_k(x_k,k))\} - x_k^{\mathrm{T}} Q_k x_k \\ =& x_k^{\mathrm{T}} \left(A_k^{\mathrm{T}} Q_{k+1} A_k - Q_k + \sum_{i=1}^{q} \mathit{\Gamma}_{ik} \mathrm{tr}[\mu Q_{k+1} \mathit{\Pi}_{ik}^{11}] \right) x_k \\ & + 2x_k^{\mathrm{T}} A_k^{\mathrm{T}} Q_{k+1} D_k \omega_k + \omega_k^{\mathrm{T}} D_k^{\mathrm{T}} Q_{k+1} D_k \omega_k.\end{aligned} \tag{14}$$

Adding the following zero term

$$\|y_k\|_{R_k}^2 - \gamma^2\|\omega_k\|_{W_k}^2 - (\|y_k\|_{R_k}^2 - \gamma^2\|\omega_k\|_{W_k}^2) \tag{15}$$

to the right side of (14) *and then taking mathematical expectation results in*

$$\begin{aligned}&\mathbb{E}\{\tilde{J}_k\} \\ =& x_k^{\mathrm{T}} \left(A_k^{\mathrm{T}} Q_{k+1} A_k - Q_k + C_k^{\mathrm{T}} R_k C_k + \sum_{i=1}^{q} \mathit{\Gamma}_{ik} \Big(\mathrm{tr}[\mu Q_{k+1} \mathit{\Pi}_{ik}^{11}] + \mathrm{tr}[\nu R_k \mathit{\Pi}_{ik}^{22}] \Big) \right) x_k \\ & + 2x_k^{\mathrm{T}} A_k^{\mathrm{T}} Q_{k+1} D_k \omega_k + \omega_k^{\mathrm{T}} \left(-\gamma^2 W_k + D_k^{\mathrm{T}} Q_{k+1} D_k \right) \omega_k \\ & - \mathbb{E}\left\{ \|y_k\|_{R_k}^2 - \gamma^2\|\omega_k\|_{W_k}^2 \right\}.\end{aligned} \tag{16}$$

By applying completing squares method, we have

$$\mathbb{E}\{\tilde{J}_k\} = -(\omega_k - \omega_k^*)^{\mathrm{T}} \Theta_k (\omega_k - \omega_k^*) - \mathbb{E}\left\{ \|y_k\|_{R_k}^2 - \gamma^2\|\omega_k\|_{W_k}^2 \right\}, \tag{17}$$

where

$$\omega_k^* = \Theta_k^{-1} D_k^{\mathrm{T}} Q_{k+1} A_k x_k. \tag{18}$$

Then, it follows that

$$\begin{aligned}\mathbb{E}\{\sum_{k=0}^{N-1}\tilde{J}_k\} =&\mathbb{E}\{x_N^{\mathrm{T}}Q_Nx_N\} - x_0^{\mathrm{T}}Q_0x_0\\ =&-\sum_{k=0}^{N-1}\Big((\omega_k-\omega_k^*)^{\mathrm{T}}\Theta_k(\omega_k-\omega_k^*)\\ &+\mathbb{E}\left\{\|y_k\|_{R_k}^2-\gamma^2\|\omega_k\|_{W_k}^2\right\}\Big).\end{aligned} \tag{19}$$

Since $\Theta_k > 0$, $Q_N = 0$ and $Q_0 - \gamma^2 S_k < 0$, we obtain

$$\begin{aligned}J_1 =&\mathbb{E}\left\{\sum_{k=0}^{N-1}\left(\|y_k\|_{R_k}^2-\gamma^2\|\omega_k\|_{W_k}^2\right)\right\}-\gamma^2\|x_0\|_{S_k}^2\\ =&x_0^{\mathrm{T}}Q_0x_0-\sum_{k=0}^{N-1}(\omega_k-\omega_k^*)^{\mathrm{T}}\Theta_k(\omega_k-\omega_k^*)-\gamma^2\|x_0\|_{S_k}^2<0,\end{aligned} \tag{20}$$

which ends the proof.

3. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Design

In this section, we shall design a state feedback controller for system (1) such that the design objectives ($\mathcal{R}1$) and ($\mathcal{R}2$) are satisfied simultaneously. It turns out that the solvability of the addressed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be determined by the solvability of certain coupled backwards Riccati-type matrix equations. A computational algorithm is proposed in the sequel to solve such set of matrix equations.

3.1. State-Feedback Controller Design

Implementing $u_k = K_kx_k$ to system (1), we consider the following closed-loop stochastic nonlinear system over the finite horizon $[0,\ \ N-1]$:

$$\begin{cases} x_{k+1} = (A_k + B_kK_k)x_k + D_k\omega_k + \alpha_kf_k(x_k, K_kx_k, k)\\ \quad y_k = C_kx_k + \beta_kg_k(x_k, K_kx_k, k), \quad x_0 = x_0.\end{cases} \tag{21}$$

The following theorem presents a sufficient condition for the closed-loop system (21) to guarantee both the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performances at the same time.

Theorem 1. *For a pre-specified $\mathcal{H}_\infty$ performance index $\gamma > 0$ and properly chosen weight matrices W_k, R_k and S_k, there exists a state feedback controller $u_k = K_k x_k$ for system* (21) *such that the requirements* ($\mathcal{R}1$) *and* ($\mathcal{R}2$) *are achieved simultaneously if the following coupled matrix equations have solutions Q_k and P_k* ($0 < k \leqslant N-1$)*:*

$$\begin{cases} Q_k = (A_k + B_k K_k)^{\mathrm{T}} Q_{k+1} (A_k + B_k K_k) + C_k^{\mathrm{T}} R_k C_k \\ \qquad + \sum_{i=1}^{q} \left(\varGamma_{ik} + K_k^{\mathrm{T}} \varXi_{ik} K_k \right) \left(\mathrm{tr}[\mu Q_{k+1} \varPi_{ik}^{11}] + \mathrm{tr}[\nu R_k \varPi_{ik}^{22}] \right) \\ \qquad + \left(D_k^{\mathrm{T}} Q_{k+1} (A_k + B_k K_k) \right)^{\mathrm{T}} \varTheta_k^{-1} \left(D_k^{\mathrm{T}} Q_{k+1} (A_k + B_k K_k) \right), \\ \varTheta_k = \gamma^2 W_k - D_k^{\mathrm{T}} Q_{k+1} D_k > 0, \\ Q_N = 0, \quad Q_0 < \gamma^2 S_k, \end{cases} \tag{22}$$

$$\begin{cases} P_k = (A_k + D_k T_k)^{\mathrm{T}} P_{k+1} (A_k + D_k T_k) \\ \qquad + \sum_{i=1}^{q} \varGamma_{ik} \Big(\mathrm{tr}[\mu P_{k+1} \varPi_{ik}^{11}] + \mathrm{tr}[\nu \varPi_{ik}^{22}] \Big) \\ \qquad + C_k^{\mathrm{T}} C_k - (A_k + D_k T_k)^{\mathrm{T}} P_{k+1} B_k \varOmega_k^{-1} B_k^{\mathrm{T}} P_{k+1} (A_k + D_k T_k) \\ \varOmega_k = \sum_{i=1}^{q} \varXi_{ik} \Big(\mathrm{tr}[\mu P_{k+1} \varPi_{ik}^{11} + \mathrm{tr}[\nu \varPi_{ik}^{22}] \Big) + B_k^{\mathrm{T}} P_{k+1} B_k > 0, \\ P_N = 0, \end{cases} \tag{23}$$

where the matrix T_k and the feedback gain K_k can be recursively computed by:

$$\begin{cases} T_k = \varTheta_k^{-1} \left(D_k^{\mathrm{T}} Q_{k+1} (A_k + B_k K_k) \right), \\ K_k = -\varOmega_k^{-1} B_k^{\mathrm{T}} P_{k+1} (A_k + D_k T_k). \end{cases} \tag{24}$$

Proof 2. *According to Lemma 1, the closed-loop system* (21) *achieves $J_1 < 0$ if there exists a solution Q_k to equation* (22) *such that $\varTheta_k > 0$. If Q_k exists, the worst-case disturbance can be expressed by $\omega_k^* = T_k x_k$. Moreover, $J_{1\max} = x_0^{\mathrm{T}} Q_0 x_0$ when $\omega_k = \omega_k^*$.*

In the following stage, under the situation of worst-case disturbance, we shall try to find out the parametric expression of K_k which is able to minimize the output energy J_2. To this end, when the worst-case disturbance happens, *the original system (1) can be formulated by:*

$$\begin{cases} x_{k+1} = (A_k + D_k T_k)x_k + B_k u_k + \alpha_k f_k(x_k, u_k, k) \\ y_k = C_k x_k + \beta_k g_k(x_k, u_k, k), \quad x_0 = x_0. \end{cases} \tag{25}$$

Similar to the derivation of Lemma 1, we first define

$$\tilde{J}_{2k} \triangleq x_{k+1}^{\mathrm{T}} P_{k+1} x_{k+1} - x_k^{\mathrm{T}} P_k x_k. \tag{26}$$

Denoting $\bar{A}_k = A_k + D_k T_k$*, we have*

$$\begin{aligned} \mathbb{E}\{\tilde{J}_{2k}\} =& \mathbb{E}\{(\bar{A}_k x_k + B_k u_k + \alpha_k f_k(x_k, u_k, k))^{\mathrm{T}} P_{k+1} \\ & \times (\bar{A}_k x_k + B_k u_k + \alpha_k f(x_k, u_k, k))\} - x_k^{\mathrm{T}} P_k x_k \\ =& x_k^{\mathrm{T}} \left(\bar{A}_k^{\mathrm{T}} P_{k+1} \bar{A}_k - P_k + \sum_{i=1}^{q} \mathit{\Gamma}_{ik} \mathrm{tr}[\mu P_{k+1} \mathit{\Pi}_{ik}^{11}] \right) x_k \\ & + u_k^{\mathrm{T}} \left(B_k^{\mathrm{T}} P_{k+1} B_k + \sum_{i=1}^{q} \mathit{\Xi}_{ik} \mathrm{tr}[\mu P_{k+1} \mathit{\Pi}_{ik}^{11}] \right) u_k \\ & + 2 x_k^{\mathrm{T}} \bar{A}_k^{\mathrm{T}} P_{k+1} B_k u_k. \end{aligned} \tag{27}$$

Then, it follows that

$$\begin{aligned} & \mathbb{E}\{\tilde{J}_{2k} + \|y_k\|^2 - \|y_k\|^2\} \\ =& x_k^{\mathrm{T}} \left(\bar{A}_k^{\mathrm{T}} P_{k+1} \bar{A}_k - P_k + \sum_{i=1}^{q} \mathit{\Gamma}_{ik} \mathrm{tr}[\mu P_{k+1} \mathit{\Pi}_{ik}^{11}] + C_k^{\mathrm{T}} C_k + \sum_{i=1}^{q} \mathit{\Gamma}_{ik} \mathrm{tr}[\nu \mathit{\Pi}_{ik}^{22}] \right) x_k \\ & + u_k^{\mathrm{T}} \left(B_k^{\mathrm{T}} P_{k+1} B_k + \sum_{i=1}^{q} \mathit{\Xi}_{ik} \mathrm{tr}[\mu P_{k+1} \mathit{\Pi}_{ik}^{11}] + \sum_{i=1}^{q} \mathit{\Xi}_{ik} \mathrm{tr}[\nu \mathit{\Pi}_{ik}^{22}] \right) u_k \\ & + 2 x_k^{\mathrm{T}} \bar{A}_k^{\mathrm{T}} P_{k+1} B_k u_k - \mathbb{E}\{\|y_k\|^2\}. \end{aligned} \tag{28}$$

Completing the squares of u_k*, it is obtained that*

$$\mathbb{E}\{\tilde{J}_{2k} + \|y_k\|^2 - \|y_k\|^2\} = \mathbb{E}\{(u_k - u_k^*)^{\mathrm{T}} \mathit{\Omega}_k (u_k - u_k^*) - \|y_k\|^2\}, \tag{29}$$

where $\mathit{\Omega}_k$ *is defined in* (23) *and* $u_k^* = K_k x_k$ *with* K_k *defined in* (24)*. Furthermore, we have*

$$\begin{aligned} J_2 &= \mathbb{E}\{\sum_{k=0}^{N-1} \|y_k\|^2\} \\ &= \mathbb{E}\left\{ \sum_{k=0}^{N-1} (u_k - u_k^*)^{\mathrm{T}} \mathit{\Omega}_k (u_k - u_k^*) \right\} + x_0^{\mathrm{T}} P_0 x_0 - x_N^{\mathrm{T}} P_N x_N. \end{aligned} \tag{30}$$

Since $P_N = 0$, it is easy to see that, when $u_k = u_k^*$, the following relationship

$$J_{2\min} = x_0^{\mathrm{T}} P_0 x_0 \tag{31}$$

is true, which means that

$$J_1(u_k^*, \omega_k^*) \geqslant J_1(u_k^*, \omega_k), \quad J_2(u_k^*, \omega_k^*) \leqslant J_2(u_k, \omega_k^*). \tag{32}$$

We can now conclude that (u_k^*, ω_k^*) is a Nash equilibria we intend to find. The proof is complete.

Notice that the solvability of the addressed controller design problem is presented as the feasibility of four coupled matrix-valued equations. In the following subsection, we propose a numerical recursive algorithm to obtain the value of the required controller gain K_k.

3.2. Computational Algorithm

The following algorithm shows how to solve the addressed problem with known parameters and terminal values Q_N and P_N.

Algorithm of Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem

- Step 1. Select the proper initial values. Set $k = N-1$, then $Q_{k+1} = Q_N$ and $P_{k+1} = P_N$ are known.
- Step 2. Calculate Θ_k, Ω_k with known Q_{k+1} and P_{k+1} via the second equation of (22) and (23), respectively.
- Step 3. With the obtained Θ_k and Ω_k, solve the following coupled matrix equations:

$$\begin{cases} T_k = \Theta_k^{-1} \left(D_k^{\mathrm{T}} Q_{k+1}(A_k + B_k K_k) \right) \\ K_k = -\Omega_k^{-1} B_k^{\mathrm{T}} P_{k+1}(A_k + D_k T_k) \end{cases} \tag{33}$$

  to obtain the values of T_k and K_k.

- Step 4. Solve the first equation of (22) and (23) to get Q_k and P_k respectively.
- Step 5. If $k = 0$, EXIT. Else, set $k = k - 1$ and go to Step 2.

Remark 3. *By the recursive algorithm developed above, we can obtain the controller gain K_k step by step at every sampling instant. However, in Step 2, one needs to check whether the obtained Θ_k is positive definite or not. If not, then the addressed problem is not solvable. Moreover, in Step 5, when the recursion is finished, we should also check if the initial condition $Q_0 < \gamma^2 S_k$ is satisfied with the obtained Q_0. If not, the addressed problem is not solvable as well, and then we need to change the pre-specified parameters' values and repeat the iterative algorithm again.*

4. Numerical Example

In this section, a numerical example is presented to demonstrate the effectiveness of the method proposed in this paper.

First, assume the stochastic nonlinearities $f_k(x_k, u_k, k)$ and $g_k(x_k, u_k, k)$ have the following bilinear form:

$$
\begin{aligned}
f_k &= \sum_{i=1}^{2} \begin{bmatrix} a_{ik} \\ b_{ik} \end{bmatrix} (l_{ik}x_{1k}\zeta_{1k} + m_{ik}x_{2k}\zeta_{2k} + n_{ik}u_k\zeta_{3k}), \\
g_k &= \sum_{i=1}^{2} \varrho_{ik}(l_{ik}x_{1k}\zeta_{1k} + m_{ik}x_{2k}\zeta_{2k} + n_{ik}u_k\zeta_{3k}),
\end{aligned}
$$

where x_{1k} and x_{2k} stand for the first and second component of the system state respectively, and ζ_{jk} $(j = 1, 2, 3)$ represents three mutually independent Gaussian white noise sequences. a_{ik}, b_{ik}, ϱ_{ik}, l_{ik}, m_{ik}, n_{ik} are time-varying coefficients.

Let the exogenous disturbance input be $\omega(k) = \exp(-k/35) \times n(k)$ where $n(k)$ is uniformly distributed over $[-0.5, 0.5]$. The probability is assumed to be $\delta = 0.9$.

Now, consider the following discrete time-varying nonlinear stochastic system with time-varying parameters as follows:

For $k = 0$;

$$
\begin{aligned}
&A_0 = \begin{bmatrix} 0.5 & -0.1 \\ -0.3 & 0.07 \end{bmatrix}, B_0 = \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix}, D_0 = \begin{bmatrix} 0.2 \\ -0.8 \end{bmatrix}, H_0 = 0.1, \\
&S_0 = I, C_0 = \begin{bmatrix} 0.5 & -0.6 \end{bmatrix}, R_0 = I, W_0 = I, \mu = 0.16, \nu = 0.25, \\
&a_{10} = 0.6324, a_{20} = 0, b_{10} = 0, b_{20} = 0.7746, \varrho_{10} = 0.6, \varrho_{20} = 0.8, \\
&l_{10} = l_{20} = 0.4472, m_{10} = m_{20} = 0.7071, n_{10} = 0.7, n_{20} = 0.5477,
\end{aligned}
$$

hence, we can easily obtain that

$$\varGamma_{10} = \varGamma_{20} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \varPi_{10}^{11} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}, \varPi_{20}^{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0.6 \end{bmatrix},$$
$$\varPi_{10}^{22} = 0.36, \varPi_{20}^{22} = 0.64, \varXi_{10} = 0.49, \varXi_{20} = 0.3,$$

For $k = 1$,

$$A_1 = \begin{bmatrix} 0.4 & -0.08 \\ -0.1 & 0.05 \end{bmatrix}, B_1 = \begin{bmatrix} 0.3 \\ -0.6 \end{bmatrix}, D_1 = \begin{bmatrix} 0.1 \\ -0.5 \end{bmatrix}, H_1 = 0.1,$$
$$S_1 = I, C_1 = \begin{bmatrix} 0.3 & -0.1 \end{bmatrix}, R_1 = I, W_1 = I, \mu = 0.16, \nu = 0.25,$$
$$a_{11} = 0.8367, a_{21} = 0, b_{11} = 0, b_{21} = 0.4472, \varrho_{11} = 0.3162, \varrho_{21} = 0.5477,$$
$$l_{11} = l_{21} = 0.3162, m_{11} = m_{21} = 0.5477, n_{11} = n_{21} = 0.3162,$$

hence, we can easily obtain that

$$\varGamma_{11} = \varGamma_{21} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \varPi_{11}^{11} = \begin{bmatrix} 0.7 & 0 \\ 0 & 0 \end{bmatrix}, \varPi_{21}^{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$\varPi_{11}^{22} = 0.1, \varPi_{21}^{22} = 0.3, \varXi_{11} = 0.1, \varXi_{21} = 0.1.$$

For $k = 2$,

$$A_2 = \begin{bmatrix} 0.2 & -0.05 \\ -0.1 & 0.08 \end{bmatrix}, B_2 = \begin{bmatrix} 0.03 \\ -0.5 \end{bmatrix}, D_2 = \begin{bmatrix} 0.05 \\ -0.8 \end{bmatrix}, H_2 = 0.1, S_2 = I,$$
$$C_2 = \begin{bmatrix} 0.1 & -0.2 \end{bmatrix}, R_2 = I, W_2 = I, \mu = 0.16, \nu = 0.25,$$
$$a_{12} = 0.8944, a_{22} = 0, b_{12} = 0, b_{22} = 0.7071, \varrho_{12} = 0.8944, \varrho_{22} = 0.7071,$$
$$l_{12} = l_{22} = 0.4472, m_{12} = m_{22} = 0.3162, n_{12} = n_{22} = 0.6324,$$

hence, we can easily obtain that

$$\varGamma_{12} = \varGamma_{22} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \varPi_{11}^{11} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0 \end{bmatrix}, \varPi_{21}^{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$\varPi_{11}^{22} = 0.8, \varPi_{21}^{22} = 0.5, \varXi_{12} = 0.4, \varXi_{22} = 0.4.$$

For $k = 3$,

$$A_3 = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}, D_3 = \begin{bmatrix} 0.03 \\ -0.5 \end{bmatrix}, H_3 = 0.1, S_3 = I,$$
$$C_3 = \begin{bmatrix} 0.2 & -0.5 \end{bmatrix}, R_3 = I, W_3 = I, \mu = 0.16, \nu = 0.25,$$
$$a_{13} = 0.7071, a_{23} = 0, b_{13} = 0, b_{23} = 0.7071, \varrho_{13} = 0.6324, \varrho_{23} = 0.7746,$$
$$l_{13} = l_{23} = 0.3162, m_{13} = m_{23} = 0.3162, n_{13} = 0.4472, n_{23} = 0.5477,$$

hence, we can easily obtain that

$$\varGamma_{13} = \varGamma_{23} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \varPi_{13}^{11} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \varPi_{23}^{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$\varPi_{13}^{22} = 0.4, \varPi_{23}^{22} = 0.6, \varXi_{13} = 0.2, \varXi_{23} = 0.3.$$

Choosing $\gamma = 1$, we apply the *Algorithm of Mixed* $\mathcal{H}_2/\mathcal{H}_\infty$ *Control Problem* from the instant $k = 3$ to $k = 0$ with $Q_4 = 0$ and $P_4 = 0$:

- Set $k = 3$.
- Then, we obtain
$$\varTheta_3 = 1, \quad \varOmega_3 = 0.05.$$
- With obtained $\varTheta_3$ and $\varOmega_3$, we have
$$T_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$
- With known $\varTheta_3$, $\varOmega_3$, T_3 and K_3, we have
$$Q_3 = \begin{bmatrix} 0.0650 & -0.1000 \\ -0.1000 & 0.2750 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0.0650 & -0.1000 \\ -0.1000 & 0.2750 \end{bmatrix}.$$

By similar recursions, it can be obtained that

$$\varTheta_2 = 0.8158, \ \ \varOmega_2 = 0.2139,$$
$$T_2 = \begin{bmatrix} -0.0161 & 0.0091 \end{bmatrix}, \ K_2 = \begin{bmatrix} -0.4518 & 0.2559 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 0.11222 & -0.0373 \\ -0.0373 & 0.0852 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0.0840 & -0.0203 \\ -0.0213 & 0.0756 \end{bmatrix},$$
$$\varTheta_1 = 0.9738, \ \ \varOmega_1 = 0.0758,$$
$$T_1 = \begin{bmatrix} 0.0068 & -0.0075 \end{bmatrix}, \ K_1 = \begin{bmatrix} -0.2709 & 0.0769 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.1144 & -0.0327 \\ -0.0327 & 0.0452 \end{bmatrix}, \ P_1 = \begin{bmatrix} 0.0683 & -0.0186 \\ -0.0198 & 0.0402 \end{bmatrix},$$
$$\varTheta_0 = 0.9560, \ \ \varOmega_0 = 0.1509,$$
$$T_0 = \begin{bmatrix} 0.0336 & -0.0073 \end{bmatrix}, \ K_0 = \begin{bmatrix} -0.1461 & 0.0291 \end{bmatrix},$$
$$Q_0 = \begin{bmatrix} 0.3401 & -0.3078 \\ -0.3078 & 0.4925 \end{bmatrix}, \ P_0 = \begin{bmatrix} 0.2870 & -0.2964 \\ -0.2967 & 0.4883 \end{bmatrix}.$$

In order to show the effectiveness of the proposed controller design algorithm, choosing the initial value of the system state $x_0 = [0.1 \;\; 0.1]^{\mathrm{T}}$, first we define the system noise attenuation index as

$$J_3 = \frac{\sum_{k=0}^{N-1} \|y_k\|^2_{R_k}}{\sum_{k=0}^{N-1} \|\omega_k\|^2_{W_k} + \|x_0\|^2_{S_k}},$$

then, we will show in the following table that the different performances with or without applying the control scheme.

TABLE I

	J_3	$\bar{J}_3$	J_2	$\bar{J}_2$
$k = 0$	0.0004	0.0004	0.0001	0.0001
$k = 1$	0.0060	0.0062	0.0067	0.0069
$k = 2$	0.0156	0.0179	0.0181	0.0206
$k = 3$	0.0165	0.0254	0.0241	0.0372

In the table listed above, J_2 and J_3 represent the performances after implementing the proposed control strategy, while $\bar{J}_2$ and $\bar{J}_3$ stand for that without control. It is clearly shown in the table that both the disturbance attenuation level and the output energy have been largely improved by applying the proposed control algorithm. Therefore, the proposed algorithm is capable of obtaining the required controller gain K_k in an effectively recursive way.

5. Conclusion

In this paper, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design problem has been dealt with for a class of nonlinear stochastic systems with randomly occurring nonlinearities that are characterized by two Bernoulli distributed white sequences with known probabilities. The stochastic nonlinearities addressed cover several well-studied nonlinearities in the literature. For the multiobjective controller design problem, the sufficient condition of the solvability of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been established by means of the solvability of four coupled matrix-valued equations. Then, a recursive algorithm has been developed to obtain the value of feedback controller step by step at every sampling instant. A numerical example has been given to show the effectiveness of the proposed method. It should be noted that based on the result of this paper, we can apply the game theory approach to much more complex system with more general nonlinearities. The result will appear in the near future.

References

[1] D. Bernstein and W. Haddad. LQG control with an $\mathcal{H}_\infty$ performance bound: A riccati equation approach. *IEEE Transactions on Automatic Control*, Vol. 34, No. 3, pp. 293–305, 1989.

[2] D. Bernstein and W. Haddad. Steady-state Kalman filtering with an $\mathcal{H}_\infty$ error bound. *Systems & Control Letters*, Vol. 12, Iss. 1, pp. 9–16, 1989.

[3] B. Chen, C. Tseng and H. Uang. $\mathcal{H}_2/\mathcal{H}_\infty$ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach. *IEEE Transactions on Fuzzy Systems*, Vol. 49, No. 1, pp. 45–57, 2008.

[4] B. Chen and W. Zhang. Stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control with state-dependent noise, *IEEE Transactions on Automatic Control*, Vol. 34, No. 3, pp. 293–305, 2004.

[5] X. Chen and K. Zhou. Multiobjective $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design. *SIMA Journal of Control optimization*, Vol. 40, Iss. 2, pp. 628–660, 2001.

[6] H. Gao, J. Lam and C. Wang. Induced l_2 and generalized $\mathcal{H}_\infty$ filtering for systems with repeated scalar nonlinearities, *IEEE Transactions on Signal Processing*, Vol. 53, No. 11, pp. 4215–4226, 2005.

[7] H. Gao, J. Lam and C. Wang. New approach to mixed H_2/H_∞ filtering for polytopic discrete-time systems, *IEEE Transactions on Signal Processing*, Vol. 53, No. 8, pp. 3183–3192, 2005.

[8] P. Khargonekar and M. Rotea. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: a convex optimization approach. *IEEE Transactions on Automatic Control*, Vol. 36, No. 7, pp. 824–837, 1994.

[9] M. Khosrowjerdi, R. Nikoukhah and N. Safari-Shad, Fault detection in a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ seetting. *Proceedings of IEEE Conference on Decision and Control*, pp. 1461–1466, 2003.

[10] W. Lin. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of nonlinear systems, *Proceedings of IEEE Conference on Decision and Control*, New Orleans, LA, December, pp. 333–338, 1995.

[11] H. Rotstein, M. Sznaier and M. Idan. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering theory and an aerospace application. *International Journal of Robust and Nonlinear Control*, Vol. 6, Iss. 4, pp. 347–366, 1996.

[12] C. Scherer, P. Gahient and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, Vol. 42, pp. 896–911, 1997.

[13] B. Shen, Z. Wang, H. Shu and G. Wei. Robust $\mathcal{H}_\infty$ finite-horizon filtering with randomly occurred nonlinearities and quantization effects, *Automatica*, Vol. 46, pp. 1743–1751, 2010.

[14] X. Shen and L. Deng. Game theory approach to discrete $\mathcal{H}_\infty$ filter design, *IEEE Transcations on Signal Processing*, Vol. 45, No. 4, pp. 1092–1095, 1997.

[15] Y. Theodor and U. Shaked. A dynamic game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ estimation. *International Journal of Robust and Nonlinear Control*, Vol. 6, Iss. 4, pp. 331–345, 1996.

[16] Z. Wang, F. Yang, D. W. C. Ho and X. Liu. Robust H_∞ filtering for stochastic time-delay systems with missing measurements, *IEEE Transactions on Signal Processing*, Vol. 54, No. 7, pp. 2579–2587, 2006.

[17] Z. Wang, Y. Liu and X. Liu. H_∞ filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities, *Automatica*, Vol. 44, pp. 1268–1277, 2008.

[18] L. Xie, L. Lu, D. Zhang and H. Zhang. Improved robust H_2 and H_∞ filtering for uncertain discrete-time systems, *Automatica*, Vol. 40, No. 5, pp. 873–880, 2004.

[19] C. Yang and Y. Sun. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback design for microsatelite attitude control. *Control Engineering Practice*, Vol. 10, pp. 951–970, 2002.

[20] F. Yang, Z. Wang, Y. Hung and H. Shu. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for uncertain systems with reginal pole assignment. *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 41, No. 2, pp. 438–448, 2005.

[21] F. Yang, Z. Wang and D. W. C. Ho. Robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for a class of nonlinear stochastic systems. *IEE Proceedings on Control Theory Applications*, Vol. 153, Iss. 2, pp. 175–184, 2006.

[22] E. Yaz. On the optimal state estimation of a class of discrete-time nonlinear systems, *IEEE Transactions on Automatic Control*, Vol. 34, No. 9, pp. 1127–1129, 1987.

[23] W. Zhang, Y. Huang and H. Zhang. Stochastic $\mathcal{H}_\infty/\mathcal{H}_2$ control for discrete-time systems with state and disturbance dependent noise, *Automatica*, Vol. 43, pp. 513–521, 2007.