Relative Gromov-Witten theory and Vertex Operators

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Abstract

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In this thesis, we report on two projects applying representation theoretic techniques to solve enumerative and geometric problems, which were carried out by the author during his pursuit of Ph.D. at Columbia.

We first study the relative Gromov-Witten theory on $T^*\mathbf{P}^1 \times \mathbf{P}^1$ and show that certain equivariant limits give relative invariants on $\mathbf{P}^1 \times \mathbf{P}^1$. By formulating the quantum multiplications on $\mathrm{Hilb}(T^*\mathbf{P}^1)$ computed by Davesh Maulik and Alexei Oblomkov as vertex operators and computing the product expansion, we demonstrate how to get the insertion operator computed by Yaim Cooper and Rahul Pandharipande in the equivariant limits.

Brenti proves a non-recursive formula for the Kazhdan-Lusztig polynomials of Coxeter groups by combinatorial methods. In the case of the Weyl group of a split group over a finite field, a geometric interpretation is given by Sophie Morel via weight truncation of perverse sheaves. With suitable modifications of Morel's proof, we generalize the geometric interpretation to the case of finite and affine partial flag varieties. We demonstrate the result with essentially new examples using $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$.

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Chapter 1

Introduction

The unifying theme of this thesis is the application of representation theoretic and harmonic analytic techniques to enumerative, geometric, and arithmetic problems. In the following sections, we introduce the motivation, notation, and state the main statements dealt with in the following two projects.

1. Investigate the interaction among equivariant relative Gromov-Witten theory on $\mathbf{P}^1 \times \mathbf{P}^1$, $T^*\mathbf{P}^1 \times \mathbf{P}^1$, classical Severi degree problem on $\mathbf{P}^1 \times \mathbf{P}^1$ and quantum cohomology on $\mathrm{Hilb}(T^*\mathbf{P}^1)$[Wan19];

2. Find a generalization of a nonrecursive formula for Kazhdan-Lusztig polynomials in the case of finite and affine partial flag varieties. [WZ19].

1.1 Relative Gromov-Witten theory and vertex operators

Gromov-Witten theory is a modern approach to address questions in enumerative geometry that were asked more than a century ago by Chasles, Zeuthen,Schubert, and the Italian school, of which one typical example is the Severi degree problem:

How many algebraic curves in $\mathbf{P}^2$ of geometric genus g and degree d pass through $3d + g - 1$ general points?

For example, when $g = 0$ and $d = 1$, the answer is 1—there is a unique line passing through 2 generic points. The Severi problem for rational curves in all degrees can be solved by considering intersection numbers on the moduli space of stable morphisms from rational curves to $\mathbf{P}^2$, the associated quantum cohomology ring structure yields the following striking result for N_d, the number of degree d rational plane curves passing through $3d - 1$ general points:

$$N_d = \sum_{d_1+d_2=d; d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

The first few terms $N_1 = 1$, $N_2 = 1$, $N_3 = 12$, $N_4 = 620...$, already vastly generalize our understanding of enumerative geometry of the projective plane. Moreover, Witten's conjecture [Wit91] was first solved by Kontsevich [Kon92]. In 2001, a new proof was given in Okounkov-Pandharipande [OP09], which involves Hurwitz numbers and matrix integral techniques. Generalizations of Witten's 1990 conjecture [Wit91] were proposed in [EHX97] and special cases were proven by Givental for toric Fano manifolds [Giv01] and by Okounkov-Pandharipande [OP06] for algebraic curves.

In Okounkov-Pandharipande's approach [OP09], the representation theory of symmetric groups is an essential ingredient. To enumerate the Hurwitz number of algebraic curves, it's enlightening to view them as the matrix coefficients of an operator M_H. The punchline is that, M_H is diagonalized by Schur functions, thus the generating function has a very simple form in the basis of Schur functions. Moreover, in this formalism, the tensor products and induced representations on the representation theoretic side yield a natural understanding of the cut-and-join structures of Hurwitz numbers. Cooper and Pandharipande give a Fock space formalism of the Severi degree problem on $\mathbf{P}^1 \times \mathbf{P}^1$. By degenerating $\mathbf{P}^1 \times \mathbf{P}^1$, the absolute Gromov-Witten invariants can be encoded in the matrix coefficients of the point-insertion operator $M_S(u,Q)$ in the relative Gromov-Witten theory, and eigenvalues of special cases are computed. More precisely, let the variables Q_1 and Q_2 correspond to the curve classes of the fibers of the first and second projections to $\mathbf{P}^1$ respectively. The generating function for possibly disconnected Severi degrees $N^{\bullet}_{g,(d_1,d_2)}$ is defined to be

$$Z^{\mathbf{P}^1\times\mathbf{P}^1} = 1 + \sum_{g\in\mathbf{Z}} u^{g-1} \sum_{(d_1,d_2)\neq(0,0)} N^{\bullet}_{g,(d_1,d_2)} \frac{t^{2d_1+2d_2+g-1}}{(2d_1+2d_2+g-1)!} Q_1^{d_1} Q_2^{d_2}.$$

The vector $v = \sum_{d_1\geq 0} |(1)^{d_1}, \emptyset\rangle = \sum_{d_1\geq 0} \alpha_{-1}^{d_1}(\mathrm{pt})$ in $\mathcal{F}[\mathbf{P}^1]$, the Fock space on $\mathrm{H}^{\bullet}(\mathbf{P}^1)$. See section 4.2.3 for more details of the notations. Cooper and Pandharipande prove the following theorems.

Theorem 1.1.1 ([CP17a])**.** *The partition function for Severi degrees of* $\mathbf{P}^1 \times \mathbf{P}^1$ *is*

$$Z^{\mathbf{P}^1\times\mathbf{P}^1} = e^{\frac{tQ_2}{u}} \langle v| Q_1^{|\cdot|} M_S(u,Q_2) |v\rangle$$

Proposition 1.1.2 ([CP17a])**.** *The eigenvalues of* $M_S(0,Q)$ *on the space of energy* s *are*

$$\{(|\mu| - |\nu|)\sqrt{Q}\}_{|\mu|+|\nu|=s}.$$

On the other hand, the Hilbert/Gromov-Witten correspondence was initiated by Okounkov-Pandharipande [OP10] and solved for $\mathbf{C}^2$ [OP10] and $\mathcal{A}_n$-resolutions [MO09, Mau09]. In our $\mathcal{A}_1 = T^*\mathbf{P}^1$ case, heuristically, by viewing a curve in $T^*\mathbf{P}^1 \times \mathbf{P}^1$ as a $\mathbf{P}^1$-family of points in $T^*\mathbf{P}^1$— roughly a $\mathbf{P}^1$ in $\mathrm{Hilb}^{\bullet}(T^*\mathbf{P}^1)$, the equivariant quantum cohomology of $\mathrm{Hilb}^{\bullet}(T^*\mathbf{P}^1)$ is essentially the same as the equivariant Gromov-Witten theory of $T^*\mathbf{P}^1 \times \mathbf{P}^1$. The Fock space $\mathrm{QH}^{\bullet}_T(\mathrm{Hilb}(T^*\mathbf{P}^1))$ is isomorphic to the space of boundary conditions in the relative Gromov-Witten theory. Denote the two divisor on $\mathrm{Hilb}(T^*\mathbf{P}^1)$ as (2) and $(1,\omega)$, for more details, see section 3.2.1. The quantum multiplication operator $M_{(1,\omega)}$ [MO09] on $\mathrm{Hilb}(T^*\mathbf{P}^1)$ corresponds to a divisor insertion operator in the Gromov-Witten theory of $T^*\mathbf{P}^1 \times \mathbf{P}^1$. By taking the limit, it reduces to the point-insertion operator $M_S(u, Q)$ in [CP17a]. Via this link, we first match the relative invariants.

Proposition 1.1.3. *All relative Gromov-Witten invariants of $\mathbf{P}^1 \times \mathbf{P}^1$ appear as the coefficients of $(t_1+t_2)^{2d+g-1}$-terms of the equivariant relative Gromov-Witten invariants of $T^*\mathbf{P}^1 \times \mathbf{P}^1$.*

Based on the invariant level comparison and the following proposition,

Proposition 1.1.4. *The divisor $(1, \omega)$ is the difference of the first Chern classes of the tautological bundle $\mathcal{V}_1$ and $\mathcal{V}_0$ and the cup product acts diagonally in the fixed point basis. More precisely,*

$$(1, \omega) = c_1(\mathcal{V}_1) - c_1(\mathcal{V}_0)$$

$$(1, \omega) \cup |\lambda, \mu\rangle = (t_1|\lambda| + t_2|\mu|)|\lambda, \mu\rangle.$$

We first match the classical part $M^{\mathrm{cl}}_{(1,\omega)}$ of a quantum multiplication operator on $\mathrm{Hilb}(T^*\mathbf{P}^1)$ [MO09] with the type-A part of $M_S(u, Q)$ in [CP17a]:

Proposition 1.1.5. $\mathrm{val}_{t_1+t_2} M^{\mathrm{cl}}_{(1,\omega)} = M^A_S(u, Q)$.

Then by using Frenkel-Kac construction [FK81] of the basic representation of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$, the purely quantum part of the quantum multiplications on $\mathrm{Hilb}(T^*\mathbf{P}^1)$ computed by Maulik-Oblomkov can be realized as vertex operators. Computing the operator product expansion (OPE) and taking equivariant limits, we match the purely quantum part of the quantum multiplication operator in $\mathrm{Hilb}(T^*\mathbf{P}^1)$ with the type-B part of $M_S(u, Q)$ in [CP17a]:

Proposition 1.1.6. *The qs-coefficient of Ω_+ in [MO09] is the operator corresponding* *to the type-B curve counting in [CP17b]. More precisely,*

$$[qs]\Omega_+ = \sum_{k\neq 0} : f(k)e(-k) := \sum_{|\mu|=|\nu|>0} \alpha_{-\mu}\alpha_\nu.$$

They're vertex operators of $\widehat{\mathfrak{sl}}_2$.

All combined, we recover $M_S(u,Q)$ in the limit of $M_{1,\omega}$:

Theorem 1.1.7 ([Wan19])**.** *$M_S(u,Q)$ is an equivariant limit of $M_{(1,\omega)}$:*

$$M_S^A(u,Q) = \mathrm{val}_{t_1+t_2} M_{1,\omega}^{\mathrm{cl}}$$
$$M_S^B(u,Q) = [qs]\Omega_+.$$

After utilizing the Beilinson spectral sequence on Hirzebruch surfaces to give a monad description of $\mathrm{Hilb}(T^*\mathbf{P}^1)$ as quiver varieties and match the tautological bundles, we also demonstrate how to diagonalize the quantum multiplication operator and compute its eigenvalues via Bethe equations following Aganagic-Okounkov [AO17], Nekrasov-Shatashvili [NS09, NS10].

Theorem 1.1.8. *The eigenvalues of $M_{(1,\omega)}$ are given by $\sum_i (x_i - y_i)$, where x_i, y_i are the roots of the the Bethe equations*

$$q = \frac{a+x_i}{a+x_i+\hbar}\prod_j \frac{x_i-y_j-\hbar}{y_j-x_i-\hbar}\prod_{j\neq i}\frac{x_j-x_i-\hbar}{x_i-x_j-\hbar}$$
$$s = \frac{a+y_i}{a+y_i+\hbar}\prod_j \frac{y_i-x_j-\hbar}{x_j-y_i-\hbar}\prod_{j\neq i}\frac{y_j-y_i-\hbar}{y_i-y_j-\hbar}.$$

This gives a full computation of the Severi degree problem for $\mathbf{P}^1\times\mathbf{P}^1$ and for $C\times\mathbf{P}^1$ by degeneration.

The strategy is depicted in the following diagram

$$\begin{tikzcd}
\mathrm{GW}(T^*\mathbf{P}^1 \times \mathbf{P}^1) \arrow[rr, dash, "\text{Hilb/GW}"] \arrow[d, dash, "\text{Localization}"'] & & \mathrm{QH}^\bullet_T(\mathrm{Hilb}(T^*\mathbf{P}^1)) \arrow[d, dash, "\text{[AO17]}"] \\
\mathrm{GW}(\mathbf{P}^1 \times \mathbf{P}^1) \arrow[dr, dash, "\text{Degeneration}"'] & & \text{Quiver of } \mathrm{Hilb}(T^*\mathbf{P}^1) \arrow[dl, dash] \\
& \text{Severi problem on } \mathbf{P}^1 \times \mathbf{P}^1 &
\end{tikzcd}$$

1.2 Nonrecursive formulas for KL polynomials

Kazhdan-Lusztig polynomials and R-polynomials were defined in [KL79, KL80] as the matrix coefficients of an involution in the standard basis of the Hecke algebra $\mathcal{H}_W$ of a Coxeter group W. Brenti proves a non-recursive formula for the Kazhdan-Lusztig polynomials of Coxeter groups by combinatorial methods [Bre98]. In the case of the Weyl group of a split group over a finite field, a geometric interpretation of the formula on G/B is given by Sophie Morel [Mor11] via the weight truncation theory of perverse sheaves developed by her. With suitable modifications of Morel's proof, in a joint work with Yihang Zhu [WZ19], we generalize the geometric interpretation to the case of finite and affine partial flag varieties G/P. In the parabolic case corresponding to a subgroup $W_J \hookrightarrow W$, let W^J be the set of minimal length representatives of W/W_J. For any two elements $\tau \leq \sigma$ in W^J, certain generalized Kazhdan-Lusztig polynomials $P^J_{\tau,\sigma}$ and $R^J_{\tau,\sigma}$ can de defined [Deo87]. Let $\tau_{\leq d}$ denote the truncation operator in $\mathbf{Q}[t^{\pm\frac{1}{2}}]$, that is $\tau_{\leq d}(\sum_{i\in\mathbf{Z}} a_i t^{\frac{i}{2}}) = \sum_{i\leq d} a_i t^{\frac{i}{2}}$. In this setting, we prove an analogue of [Mor11]:

Theorem 1.2.1 ([WZ19])**.** *For any τ, $\sigma \in W^J$, the generalized Kazhdan-Lusztig poly-*

nomial can be computed from the generalized R-polynomials:

$$P^J_{\tau,\sigma} = \tau_{\ell(\sigma)-\ell(\tau)-1} \sum_{\tau=v_1<\cdots<v_r<\sigma} (-1)^r (T_1 \circ \cdots \circ T_{r-1} \circ T_r)\mathbf{1},$$

where $\mathbf{1}$ *is the constant polynomial* 1 *and*

$$T_r(f) = \tau_{\ell(\sigma)-\ell(v_{r+1})}(R^J_{v_{r+1},\sigma} \cdot f)$$

On the one hand, the Kazhdan-Lusztig polynomial is the Poincaré series of the perverse sheaf $\mathrm{IC}_{\overline{X}_\sigma}$—the intermediate extension of the constant $\mathbf{Q}_\ell$-sheaf on the open smooth locus X_σ [KL80]. On the other hand, Morel's result [Mor08] enables one to construct $\mathrm{IC}_{\overline{X}_\sigma}$ via weight truncation instead of the usual cohomological truncation. The advantage is that weight truncation is compatible with the Frobenius action, and this makes it possible to compute the Poincaré series step by step. Two geometric inputs are important for the actual computation. One is a refinement of the Bruhat decomposition of a flag variety [Deo85, Deo87]. Deodhar's original construction was combinatorial, and we can give a proof via a generalized Bott-Samelson resolution of $\overline{X}_\sigma$ and Bialynicki-Birula decomposition, similar to the proof for finite G/B case in [Mor11]. The other is that the embedding $X_\tau \hookrightarrow \overline{X}_w$ has a very nice open neighborhood of the form $X_\tau \times (\overline{X}_\sigma \cap X^\tau)$. These geometric facts together with the Grothendieck-Lefschetz trace formula reduce the computation required in [Mor11] to a point counting problem on $X_\sigma \cap X^\tau$. Due to a result by Kazhdan-Lusztig [KL80], the point counting on $X_\sigma \cap X^\tau$ just gives the corresponding R-polynomial. Via this link, when we compute the Poincaré series of the intersection cohomology complex by counting points via Frobenius action, we bridge the generalized R-polynomials with the generalized Kazhdan-Lusztig polynomials with the help of Morel's result [Mor08].

1.3 Structure of this thesis

This thesis is organized as follows. In Chapter 2 we recall the key concepts and constructions for some infinite dimensional Lie algebras and their highest weight representations. They are important for the description of the quantum multiplication operators and the geometry of the generalized partial flag varieties. Then in Chapter 3 we describe the geometry and equivariant quantum cohomology of Hilbert schemes of $T^*\mathbf{P}^1$ via weight spaces in the basic representation of $\widehat{\mathfrak{gl}}_2$. In Chapter 4 we introduce the moduli space of relative Gromov-Witten theory, the corresponding virtual fundamental classes and localization theorem. In Chapter 5, we utilize the equivariant localization techniques and explore the relations between certain vertex operators, we demonstrate how to get the invariants on $\mathbf{P}^1 \times \mathbf{P}^1$ from the invariants of $T^*\mathbf{P}^1 \times \mathbf{P}^1$. Then we match the operators in [CP17b] and [MO09]. In Chapter 6 we recall the basic concepts and constructions of perverse sheaves that are important for our understanding of the intersection cohomology of Schubert varieties in generalized partial flag varieties, and we also review Morel's weight truncation theory with examples. In Chapter 7, we explore local geometry of generalized partial flag varieties and then state and prove non-recursive formulas for finite and affine G/P following [Mor11].

Chapter 2

Infinite dimensional Lie algebras and representations

A basic strategy to get infinite dimensional Lie algebras is to take a finite dimensional Lie algebra $\mathfrak{g}$, consider the corresponding loop algebra $\mathfrak{g}[t, t^{-1}]$, and then consider central extensions or add certain derivation when necessary. If we start from the abelian Lie algebra $\mathbf{C}$, we get the Heisenberg algebra, if we take a semisimple Lie algebra $\mathfrak{g}$, we get Kac-Moody algebra $\widehat{\mathfrak{g}}$. We'll briefly review the general construction and demonstrate with the Heisenberg algebra $\mathcal{H}$, and the Kac-Moody Lie algebras $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$. The Heisenberg algebra [KR87] and its representations will be important in our description of different cohomology theories of Hilbert Schemes in section 3.2 and the space of relative conditions of relative Gromov-Witten theory in section 4.2.3. The structure of semisimple and Kac-Moody Lie algebras will be used in the description of the geometry of the partial flag varieties in section 7.1, the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$ are key for representation theoretic presentation of the quantum cohomology ring of Hilbert schemes and quantum multiplication by divisors in section 3.2.

2.1 The Heisenberg algebra

Starting from the abelian Lie algebra $\mathbf{C}$, Heisenberg algebra $\mathcal{H}$ is the one dimensional central extension of the loop algebra $\mathbf{C}[t, t^{-1}]$. In other words, the Heisenberg algebra has a basis $\{\alpha_n, n \in \mathbf{Z}; c\}$, and the commutation relations

$$[c, \alpha_n] = 0$$

$$[\alpha_k, \alpha_l] = k\delta_{k+l}c.$$

Note that α_0 is also a central element, it's called the zero mode. Some authors just ignore α_0 in the definition of a Heisenberg algebra.

The Fock space $\mathcal{F} = \mathbf{C}[x_1, x_2, \dots]$ is the polymonial ring with infinite many variables. Given $\mu, \hbar \in \mathbf{C}$, the Fock space representation of $\mathcal{H}$ on $\mathcal{F}(k > 0)$:

$$\alpha_k = \epsilon_k \frac{\partial}{\partial x_k}$$

$$\alpha_{-k} = \hbar \epsilon_k^{-1} k x_k$$

$$\alpha_0 = \mu I$$

$$c = \hbar I$$

In the thesis, all incarnations of this representation have $\hbar = 1$, $\epsilon_k = 1$ and $\mu = 1$.

Moreover, we can consider the Heisenberg (super)algebra associated to a super space V [EG00, Nak99], that is a decomposition $V = V_+ \oplus V_-$ of V into even subspace and odd subspace, with a non-degenerate bilinear form, for homogeneous element v and w, satisfying

$$\langle v, w\rangle = (-1)^{\deg v \deg w} \langle w, v\rangle.$$

Let $W = V \otimes \mathbf{C}[t, t^{-1}]$, we equip W with a bilinear form

$$\langle v \otimes t^k, w \otimes t^\ell \rangle = k\delta_{k+\ell}\langle v, w\rangle.$$

By abuse of notation, the Heisenberg superalgebra $\mathcal{H}$ is defined to be the free Lie superalgebra divided by the two-sided ideal generated by $[v \otimes t^k, w \otimes t^\ell] = \langle v \otimes t^k, w \otimes t^\ell\rangle \mathbf{1}$. This is slightly different from the definition in [Nak99] since we include all the zero modes. The Heisenberg algebra has a representation on the Fock space

$$\mathcal{F}[V] \cong \bigotimes_{m=0}^{\infty} \mathrm{Sym}^\bullet(V_+ \otimes t^m) \otimes \wedge^\bullet(V_- \otimes t^m).$$

The cohomology space $\mathrm{H}^\bullet(X)$ with the pairing $\int_X \gamma_1 \cdot \gamma_2$ is our prototype of such V. The Heisenberg superalgebra and corresponding Fock space will appear as space of boundary conditions in relative Gromov-Witten theory on $\mathbf{P}^1 \times \mathbf{P}^1$ and $T^*\mathbf{P}^1 \times \mathbf{P}^1$, as well as the equivariant cohomology ring of $\mathbf{Hilb}^\bullet(T^*\mathbf{P}^1)$, the Hilbert scheme of points on $T^*\mathbf{P}^1$. We also note that, in all our applications $V = V_+$, the Fock space only contains the symmetric part, this agrees with the construction of the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$.

Define the degree of the monomial $x_1^{j_1} \dots x_k^{j_k}$ as $\sum_k kj_k$, we also identify it with a partition μ given by j_k-rows with k boxes. Let $\mathcal{F}_k$ be the subspace of $\mathcal{F}$ spanned by the monomials of degree k. Then $\mathcal{F} = \bigoplus_{k \geq 0} \mathcal{F}_k$ and $\dim \mathcal{F}_k = p(k)$, the number of partitions of size k. Thus the dimension generating function is the same as the generating function of partitions

$$\sum_{k \geq 0} (\dim \mathcal{F}_k) q^k = \sum_{k \geq 0} \dim p(k) q^k = \prod_{k \geq 1} \frac{1}{1 - q^k}.$$

Similarly, the Fock space on a super space $V = V_+ \oplus V_-$ has generating function

$\prod_{k\geq 1}\frac{(1+q^k)^{\dim V_-}}{(1-q^k)^{\dim V_+}}$. In practice of this thesis $V = V_+ = \mathrm{H}^\bullet(X)$, then the generating function is just $\big(\prod_{k\geq 1}\frac{1}{1-q^k}\big)^{\dim \mathrm{H}^\bullet(X)}$, which can be viewed as the generating function of the space of all possible combinations of $n = \dim \mathrm{H}^\bullet(X)$ partitions. Directly related to the degree is the so-called energy operator

$$E = \sum_{k\geq 0}\alpha_{-k}\alpha_k = \sum_k kx_k\frac{\partial}{\partial x_k}.$$

The monomial $x_1^{j_1}\dots x_k^{j_k}$ is an eigenspace of E with eigenvalue exactly its degree $\sum_k kj_k$, that is the size of the partition. $\mathcal{H}$ admits an antilinear anti-involution ω, such that $\omega(\alpha_n) = \alpha_{-n}$, $\omega(c) = c$. By [KR87], the corresponding Fock space $\mathcal{F}$ carries a unique Hermitian form which is contravariant with respect to ω and such that $\langle \mathbf{1},\mathbf{1}\rangle = 1$ for the vacuum vector $v_\emptyset = \mathbf{1}$. The monomials $x_1^{j_1}\dots x_k^{j_k} = \alpha_{-1}^{j_1}\dots\alpha_{-n}^{j_n}$ for an orthogonal basis have norms given by

$$\langle\alpha_{-1}^{j_1}\dots\alpha_{-n}^{j_n}|\alpha_{-1}^{j_1}\dots\alpha_{-n}^{j_n}\rangle = \prod_{k=1}^{n}(j_k)!(k)^{j_k} =: \mathfrak{z}(\mu).$$

In later sections, this will be used to describe the intersection paring on Hilbert schemes as well as package the Severi degrees on $\mathbf{P}^1\times\mathbf{P}^1$ after a little modification

2.2 Kac-Moody Algebras and highest weight representations

Kac-Moody Lie algebras are generalizations of finite-dimensional simple Lie algebras. They include finite-dimensional simple Lie algebras as special cases but most Kac-Moody Lie algebras are infinite dimensional. Kac-Moody Lie algebras share many concepts and results similar to the finite dimensional counterparts. For example, the root system, Weyl group, weight lattice, the integrable highest weight representation theory and the Weyl character formulas. These concepts are important both for our Gromov-Witten theory project and the Kazhdan-Lusztig polynomials for finite and infinite G/P project. Thus in this section, we first review the notion of Kac-Moody Lie algebras and their highest weight representations.

Let A be a (generalized) Cartan matrix. This is a $\ell \times \ell$ matrix $A = (a_{ij})$ with $a_{ii} = 2$, $\forall i \in I = \{1, 2, \ldots, \ell\}$, $a_{ij} \leq 0$ for distince $i, j \in I$, such that $a_{ij} = 0$ if and only if a_{ji}=0. In later discussions, we always assume that the generalized Cartan matrix is indecomposable and symmetrizable. Indecomposable means that it cannot be rearranged into two diagonal blocks a change of basis of the vector space; and symmetrizable means that DA is symmetric for some invertible diagonal matrix D. $\mathfrak{g}$ admits an invariant symmetric bilinear form if the corresponding Cartan matrix is diagonalizable. For in this case, we can define the Casmir operator and the corresponding representation theory behaves good. The transpose A^t of A is also a generalized Cartan matrix, the corresponding Kac-Moody Lie algebra $\mathfrak{g}_{A^t}$ is the dual of $\mathfrak{g}_A$ in the sense that the roots and coroots are interchanged.

Definition 2.2.1 (Kac-Moody Lie algebra $\mathfrak{g}_A$)**.** For any generalized Cartan matrix A and for any field K of characteristic 0, $\mathfrak{g}_A$ denote the Lie algebra generated over K by the 3ℓ elements $h_1, \ldots, h_\ell$, $e_1, \ldots, e_\ell$, and $f_1, \ldots, f_\ell$ subject to the following relations

$$
\begin{aligned}
&[h_i, h_j] = 0, \ [f_i, e_j] = \delta_{ij} h_i \\
&[h_i, e_j] = a_{ij} e_j, \ [h_i, f_j] = -a_{ij} f_j, \ \forall i, j \in I = \{1, \ldots, \ell\} \\
&(\mathrm{ad} e_i)^{-a_{ij}+1} e_j = 0, \ (\mathrm{ad} f_i)^{-a_{ij}+1} f_j = 0. \text{ for distinct } i, j. \text{ (Serr relations)}.
\end{aligned}
$$

We call $\mathfrak{g}_A$ the Kac-Moody algebra over K associated to A.

Let $\Gamma = \bigoplus_{i \in I} \mathbf{Z} \alpha_i$ be a free abelian group generated by $\{\alpha_1, \ldots, \alpha_\ell\}$. $\mathfrak{g}_A$ has a Γ-graded Lie algebra structure. Let $\deg(e_i) = \alpha_i$, $\deg(h_i) = 0$, $\deg(f_i) = -\alpha_i$. For any $\alpha \in \Gamma$, $\mathfrak{g}_\alpha$ denotes the subspace of $\mathfrak{g}_A$ consisting of all elements of degree α. Then we have $\mathfrak{h} := \mathfrak{g}_0 = Kh_1 \oplus \cdots \oplus Kh_\ell$. A nonzero element $\alpha \in \Gamma$ is called a root if $\mathfrak{g}_\alpha \neq 0$. Let Δ denote the set of all roots. Then Δ is contained in $\Gamma_+ \cup \Gamma_-$, where $\Gamma_+ := \{\alpha = \sum_{i \in I} k_i \alpha_i | k_i \geq 0\}$ and $\Gamma_- = -\Gamma_+$. Set $\Delta_+ = \Gamma_+ \cap \Delta$, $\Delta_- = -\Delta_+$. They're called the positive root system and negative root system of $\mathfrak{g}_A$ respectively. For any $\alpha = \sum_{i \in I} k_i a_i$, $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{-\alpha}$)is the subspace of $\mathfrak{g}_A$ generated by elements of the form

$$[e_{i_1}[e_{i_2} \ldots [e_{i_{r-1}}, e_{i_r}] \ldots]]$$
$$\text{resp. } [f_{i_1}[f_{i_2} \ldots [f_{i_{r-1}}, f_{i_r}] \ldots]],$$

where e_i (resp. f_i) appears exactly $|k_i|$ times. Specially, $\mathfrak{g}_\alpha = Ke_i$ and $\mathfrak{g}_{-\alpha} = Kf_i$. $\{\alpha_i, \ldots, \alpha_\ell\}$ is called the set of simple roots. The vector space $\mathfrak{h}$ contains vectors $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$, which is called the set of simple coroors, such that $\langle \alpha_i^\vee, \alpha_j \rangle = \alpha_j(\alpha_i^\vee) = a_{ij}$.

Affine Lie algebras (e.g. $\widehat{\mathfrak{sl}}_2$), as special cases of Kac-Moody algebras, can also be described as central extensions of loop algebras associated to finite dimensional semisimple Lie algebras. Start from a finite dimensional simple Lie algebra $\mathfrak{g}$, then one may make a central extension of the loop algebra $\mathfrak{g}[t, t^{-1}]$, see the appendix .

$$0 \to \mathbf{C} \cdot c \to \mathfrak{g}' \to \mathfrak{g}[t, t^{-1}] \to 0$$

To get the affine Lie algebra $\widehat{\mathfrak{g}}$, we add another basis element d, which acts on $\mathfrak{g}'$ by derivation. If $\mathfrak{h}$ is the Cartan algebra of $\mathfrak{g}$, then the Cartan algebra of $\mathfrak{g}'$ is $\mathfrak{h} \oplus \mathbf{C} \cdot c$, that of $\widehat{\mathfrak{g}}$ is $\mathfrak{h} \oplus \mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$. For an untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ [KR87]. The root system $\widehat{\Delta} = \Delta \oplus \mathbf{Z}\delta$, where Δ is the root system of $\mathfrak{g}$. Real roots consist of $\alpha + n\delta$ with $\alpha \in \Delta$, and $n \in \mathbf{Z}$. A root $\alpha + n\delta$ is positive if either $n = 0$ and $\alpha \in \Delta_+$ or $n > 0$. The imaginary roots consist of $n\delta$ with $n \neq 0$. For a real root α, let r_α be the reflection in the hyperplane orthogonal to α. The Weyl group is defined to be the Coxeter group

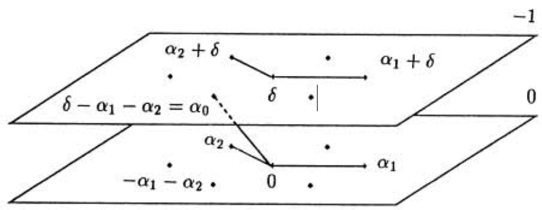


Figure 1: $\widehat{\mathfrak{sl}}_3$ root system

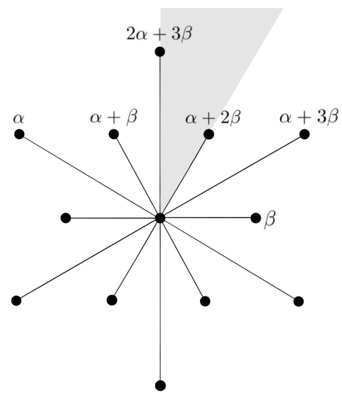


Figure 2: G_2 root/weight system

generated by the $\ell+1$ reflections associated to the simple roots α_i ($i=0,\dots,\ell$).

Example 2.2.2 (Root system of affine $\widehat{\mathfrak{sl}}_3$)**.** The root system of $\widehat{\mathfrak{sl}}_3$ is generated by $\{\alpha_0,\alpha_1,\alpha_2\}$, the imaginary root $\delta=\alpha_1+\alpha_1+\alpha_2$. See Figure 1 taken from [KMPS90] [1] .

Definition 2.2.3 (Weight lattice and dominant weights)**.** The subset of $\mathfrak{h}^*$ characterized by $\lambda(\alpha_i^\vee)\in\mathbf{Z}$ for all the coroots $\alpha_i^\vee$ is called the weight lattice P. A weight $\lambda\in P$ is called dominant if

$$\langle\lambda,\alpha_i^\vee\rangle=\lambda(\alpha_i^\vee)\geq 0$$

for all simple coroots $\alpha_i^\vee$. Let P_+ be the set of dominant weights.

Example 2.2.4 (Dominant weights of G_2)**.** The root system of G_2 is generated by the long root α and short root β. The weight lattice of G_2 is the same as its root lattice,

[1] This use is covered by fair use, in accordance with Section 107 of US Copyright Law.

the dominant weights are spanned by $\{2\alpha+3\beta, \alpha+3\beta\}$. See figure 2, credit to Samuel Mundy.

The structure theory of these (infinite dimensional) Lie algebras has deep interaction of the geometry of homogeneous spaces, such as flag varieties G/B and partial flag varieties G/P. For example, the Picard group of G/B can be described by the cokernel of the Cartan matrix [Pop74]. The Bruhat decomposition of a finite dimensional flag variety (resp. partial flag varieties) can be described by the Bruhat order in the Weyl group (resp. cosets of the Weyl group) in chapter 7. The Bruhat decomposition for the affine Grassmannian $\mathbf{Gr}_G$ is parametrized by the dominant weights P_+ in section 7.1.

Moerover, a Kac-Moody Lie algebra has a triangular decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-,$$

where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are locally nilpotent Lie algebras by our description above and the Serre relations. If V is a $\mathfrak{g}$-module, we have a decomposition of V into weight subspaces

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda.$$

In many cases V_λ is finite-dimensional vector space, and $h \in \mathfrak{h}$ acts by $hv = \lambda(h)v$ for $h \in \mathfrak{h}$, $v \in V_\lambda$. λ is called a weight if and only if $V_\lambda \neq 0$. V_λ is called the weight space of weight λ and its dimension is the multiplicity of V_λ. Specially, the root space decomposition of $\mathfrak{g}_A$ we discussed above is the same as the weight space decomposition of $\mathfrak{g}_A$ viewed as a $\mathfrak{g}$-module under the adjoint representation.

Let $V = \bigoplus\limits_{\lambda\in\mathfrak{h}^*} V_\lambda$ be a $\mathfrak{g}$-module with a weight space decomposition, $v \in V$ is called a highest weight vector if it's annihilated by $\mathfrak{n}_+$. If the space of highest weight vectors is one-dimensional, and if V is generated by a highest weight vector v, then $Kv = V_\lambda$

for a weight λ, λ is called the highest weight of V, and V is called a highest weight representation. Let α be a real root, then the two one-dimensional spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ generate a Lie algebra isomorphic to $\mathfrak{sl}_2$.

Definition 2.2.5 (Integrable $\mathfrak{g}$-module)**.** The $\mathfrak{g}$-module V is called **integrable** if for each real root α the representation of $\mathfrak{sl}_2$ obtained this way integrates to a representation of the Lie group SL_2.

Remark 2.2.6. Since this group contains an element that stabilizes $\mathfrak{h}$ and induces the corresponding simple reflection on the weight lattice, integrability implies that the weight multiplicities are invariant under the action of the Weyl group.

For andy $\lambda \in \mathfrak{h}^*$, then there is a universal highest weight module $M(\lambda)$ such that any highest weight module with highest weight λ is a quotient of $M(\lambda)$. $M(\lambda)$ is called the Verma module of weight λ. Specially, $M(\lambda)$ has a unique irreducible quotient denoted $L(\lambda)$.

As special cases of heighest weight representations of the form $L(\lambda)$, we shall describe in the next section the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$. They'll be indispensable in the relative Gromov-Witten theory project.

2.3 Examples: Basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$

First, we've recalled in the previous section the construction of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated to a complex semisimple Lie algebra $\mathfrak{g}$ and its highest weight representations. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, Δ be the root system in $\mathfrak{h}^*$ and Q be the lattice generated by the root system. Let $\langle , \rangle$ be an invariant bilinear pairing on $\mathfrak{g}$ such that $\langle \alpha, \alpha \rangle = 2$ for a long root α. Unwinding the definition, the affine Lie algebra $\widehat{\mathfrak{g}}$

characterized by the following conditions

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$
$$[x(k) \oplus a_1 c \oplus b_1 d, y(l) \oplus a_2 c \oplus b_2 d] = [x, y](k + l) + k\delta_{k+l}\langle x, y\rangle + b_1 l y(l) - b_2 k x(k).$$

The basic representation ρ is an irreducible representation V of $\mathfrak{g}$ and there exists a vector $v \in V$ such that $\rho(\mathfrak{g}[t])v = 0$ and $\rho(c)v = v$. The basic representation plays a central role in our description of the quantum cohomology ring of $\mathrm{Hilb}(T^*\mathbf{P}^1)$. Here we recall the Frenkel-Kac construction of the basic representation. It's based on the notion of a Heisenberg system $(\widetilde{\mathfrak{s}}, Q)$, where $\widetilde{\mathfrak{s}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbf{C}c$ can be viewed as a generalization of the oscillator algebra. Let $\mathfrak{s}_- = t^{-1}\mathfrak{h}[t^{-1}]$ be the negative part of the Heisenberg algebra. We define the Fock space $\mathcal{F}$ to be the symmetric algebra $\mathrm{Sym}^\bullet(\mathfrak{s}_-)$ of $\mathfrak{s}_-$. Then the basis representation as a vector space is given by

$$L = \mathrm{Sym}^\bullet(\mathfrak{s}_-) \otimes \mathbf{C}[Q],$$

where $\mathbf{C}[Q]$ is the group algebra of the root lattice Q. The actions can be described as follows, for any $k \neq 0$, $h(k)$ acts trivially on $\mathbf{C}[Q]$ and as a multiplication by $h(k)$ if $k \leq 0$ and as a derivation of $\mathrm{Sym}^\bullet(\mathfrak{s}_-)$ if $k > 0$, namely $\rho(h(k))(h'(l)) = k\delta_{k+l}\langle h, h'\rangle$. On the other hand $h(0)$ acts trivially on $\mathrm{Sym}^\bullet((\mathfrak{s}_-))$ and as a derivation on $\mathbf{C}[Q]$, namely $\rho(h)(e^\alpha) = \alpha(h)e^\alpha$, we denote it by ∂_h. The action of the off-diagonal element E_α are given by the so-called vertex operators associated with the root $\alpha \in Q$. To be more precise, $e(z) = \sum_{k\in\mathbf{Z}} e(k)z^{-k-1}$ is given by

$$X(\alpha, z) := \exp(\sum_{k\geq 1} \frac{z^k}{k}\alpha(-k)) \exp(t_\alpha + (\ln z)\partial_\alpha) \exp(-\sum_{k\geq 1} \frac{z^{-k}}{k}\alpha(k))$$

where t_α means Q acts on $\mathbf{C}[Q]$ by $t_\alpha(e^\beta) = \epsilon(\beta,\alpha)e^{\beta+\alpha}$, ϵ is a 2-cocycle of the group Q with values in $\{\pm 1\}$ such that $\epsilon(\alpha,\beta)\epsilon(-\alpha,\beta) = e^{i\pi\langle\alpha,\beta\rangle}$ and $\epsilon(\alpha,-\alpha) = \epsilon(\alpha,0) = 1$. Note that the middle term in the vertex algebra can also be written as $z^{\frac{1}{2}\langle\alpha,\alpha\rangle}e^\alpha \exp(\ln z)\partial_\alpha$ and $\exp(\ln z)\partial_\alpha = \sum_{k\in\mathbf{Z}} z^k P_k$ where $P_k(e^\beta) = \delta_{k,\langle\beta,\alpha\rangle}e^\beta$ [FK81, page 47]. Finally, choose dual bases u_i, u^i of $\mathfrak{h}$, we define the action of d to be $-D_0$, where $D_0 = \sum_{i=1}^{\dim\mathfrak{g}}(\frac{1}{2}u_i(0)u^i(0) + \sum_{k\geq 1} u_i(-n)u_i(n)$.

Now we specialize the Frenkel-Kac construction to the $\widehat{\mathfrak{sl}}_2$ case to describe the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{gl}}_2$. The loop algebra $\mathfrak{sl}_2[t,t^{-1}]$ is generated by

$$\alpha(k) = t^k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e(k) = t^k \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f(k) = t^k \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The affine Lie algebra $\widehat{\mathfrak{sl}}_2$ (type $A_1^{(1)}$) is defined in terms of a two-step central extension (see the appendix for more explanation on the central extension) of the loop algebra $\mathfrak{sl}_2[t,t^{-1}]$. Namely $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2[t,t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$ where c is a central element and d is a derivation. The commutators are given by

$$\begin{aligned} [x(k), y(l)] &= [x,y](k+l) + k\delta_{k,-l}\mathrm{tr}(xy)c \\ [d, x(k)] &= kx(k) \\ [d, c] &= 0. \end{aligned}$$

The Cartan subalgebra and its dual are given by $\widehat{\mathfrak{h}} = \mathbf{C}\alpha \oplus \mathbf{C}c \oplus \mathbf{C}d$ and $\widehat{\mathfrak{h}}^* = \mathbf{C}\alpha^* \oplus \mathbf{C}\Lambda \oplus \mathbf{C}d$. The invariant symmetric non-degenerate bilinear form $\langle,\rangle$ on $\widehat{\mathfrak{sl}}_2$ is given by

$\langle , \rangle$	$x(k)$	c	d
$y(l)$	$\delta_{k+l}\mathrm{tr}(xy)$	0	0
c	0	0	1
d	0	1	0

Thus sets of dual bases are given by

$$\{\alpha(k), e(k), f(k), c, d\} \text{ and}$$
$$\{\alpha(k)^* = \frac{1}{2}\alpha(-k), e(k)^* = f(-k), f(k)^* = e(-k), \Lambda = c^* = d, \delta = d^* = c\}.$$

The Frenkel-Kac construction above gives us the basic representation L_Λ of highest weight Λ. The negative part of the Heisenberg algebra is $\mathfrak{s}_- = \bigoplus_{k\geq 1} \alpha(-k)$. The Fock space $\mathcal{F} = \mathrm{Sym}^\bullet(\mathfrak{s}_-)$ thus can be identified with the space of all partitions or the ring of infinitely many variables $\mathbf{C}[x_1, x_2, \dots]$. The root system is one-dimensional thus we have $\mathbf{C}[Q] \cong \mathbf{Z}$. Then the basic representation is $L_\Lambda = \mathcal{F} \otimes \mathbf{C}[Q] = \mathrm{Sym}^\bullet(\mathfrak{s}_-) \otimes \mathbf{C}[Q]$. In our case, $Q = \mathbf{Z}\alpha$. Let $q = e^\alpha \in \mathbf{C}[Q]$, we have

$$L_\Lambda = \mathbf{C}[x_1, x_2, \dots; q^{\pm 1}]$$

The operators are given as follows

$\alpha(-k)$	α	$\alpha(k)$	c	d
nx_n	$2q\frac{\partial}{\partial q}$	$2\frac{\partial}{\partial x_n}$	1	$-(q\frac{\partial}{\partial q})^2 - \sum_{n\geq 1} nx_n \frac{\partial}{\partial x_n}.$

For $\widehat{\mathfrak{gl}}_2$, we have $\widehat{\mathfrak{gl}}_2 = \widehat{\mathfrak{sl}}_2 \oplus \bigoplus_{k\in\mathbf{Z}} h(k)$, where $h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. That is, it's the direct sum of $\widehat{\mathfrak{sl}}_2$ and the Heisenberg algebra generated by $\{h(k)\}_{k\in\mathbf{Z}}$. The Cartan subalgebra and its dual is just the ones of $\widehat{\mathfrak{sl}}_2$ with h or $h^* = \frac{1}{2}h$ added. Then the basic representation of the same highest weight is

$$V_\Lambda = \mathcal{F} \otimes L_\Lambda = \mathbf{C}[y_1, y_2, \ldots; x_1, x_2, \ldots; q^{\pm 1}],$$

where we have identified the Fock space of $\{h(k)\}_{k \in \mathbf{Z}}$ with $\mathbf{C}[y_1, y_2, \ldots]$. Viewed as an element in $\widehat{\mathfrak{gl}}_2$, d acts as

$$d \mapsto -\sum_{n \geq 1} n y_n \frac{\partial}{\partial y_n} - \sum_{n \geq 1} n x_n \frac{\partial}{\partial x_n} - (q \frac{\partial}{\partial q})^2.$$

In 3.2.7, we'll use these descriptions to embed $\mathrm{H}^\bullet(\mathrm{Hilb}^\bullet(T^*\mathbf{P}^1))$ into the basic representation of $\widehat{\mathfrak{gl}}_2$. In section5.3, we'll use them to match the purely quantum part of the the quantum multiplication operator in [MO09] with that of in [CP17b].

Chapter 3

Hilb/GW correspondence

3.1 Preliminaries on quantum cohomology

Let $\beta \in \mathrm{H}_2(X, \mathbf{Z})$ be en effective curve class. Given (equivariant) cohomology classes $\gamma_1, \ldots, \gamma_n$, the n-pointed genus g Gromov-Witten invariant is defined by

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{k=1}^{n} \mathrm{ev}_k^*(\gamma_k),$$

where $\mathrm{ev}_k : \overline{M}_{g,n}(X, \beta) \to X$ is the evaluation map associated to the k-th marked point and the integration is against the T-equivariant virtual fundamental class of the moduli space of genus 0 stable morphism along the pushforward map

$$\mathrm{H}_T^*(\overline{M}_{g,n}(X, \beta)) \to \mathrm{H}_T^*(\mathrm{pt}).$$

The quantum multiplication and quantum cohomology is defined via the three-point genus 0 Gromov-Witten invariants, to be more precise

$$\langle \gamma_1, \gamma_2 \star \gamma_3 \rangle_X = \langle \gamma_1, \gamma_2 \cup \gamma_3 \rangle + \sum_{\beta \in \mathrm{H}_2^{\mathrm{eff}}(X, \mathbf{Z})} \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{0,3,\beta} q^\beta$$

We'll focus on the case where γ_2 is a divisor. Namely $\gamma_2 \in \mathrm{H}^2_T(X)$. Note that in this case we have the divisor equation in this situation

$$\langle \gamma_1, D, \gamma_3 \rangle^X_{0,3,\beta} = \langle D, \beta \rangle \langle \gamma_1, \gamma_3 \rangle^X_{0,3,\beta}$$

For the Hilbert schemes of interest, the moduli spaces of genus 0 stable morphisms are not proper, however the T-fixed locus is necessarily proper. This is simply because the crepant resolution $\mathrm{Hilb}(\mathcal{A}_n) \to \mathbf{C}^2/\mathbf{Z}_{n+1}$ is a proper map to an affine variety. Under certain torus action, the base variety is contracted to the cone point 0, thus the T-fixed locus is a closed subvariety of $\pi^{-1}(0)$, which is proper. In this case, the integral above can be defined by the pushforward of its equivariant residue

$$\mathrm{H}^*_T(\overline{M}_{0,n}(X,\beta))_{loc} \to \mathrm{H}^*_T(\mathrm{pt})_{loc}.$$

We denote $\mathbf{Q}[[q^\beta]]$ the algebra of formal power series in q^β, where $\beta \in \mathrm{H}_2^{\mathrm{eff}}(X, \mathbf{Z})$ are effective curve classes. The equivariant quantum cohomology ring $\mathrm{QH}^\bullet_T(X) = \mathrm{H}^\bullet_T(X) \otimes_{\mathbf{Q}} \mathbf{Q}[[q^\beta]]$ is a deformation of the ordinary equivariant cohomology ring $\mathrm{H}^\bullet_T(X)$, equipped with a ring structure via the quantum multiplication we defined above.

3.2 Quantum cohomology of Hilbert schemes

3.2.1 Geometry of Hilbert schemes

The Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points on X parametrizes zero-dimensional subschemes of X of length n. When X is a quasiprojective surface, $\mathrm{Hilb}^n(X)$ is a smooth irreducible quasiprojective algebraic variety of dimension $2n$. The classical cohomology [Nak99, Gro96] of Hilbert schemes of points on surfaces can be described well by representation theory of Heisenberg algebras. We'll focus on the $\mathcal{A}_n$ surfaces, which can be realized as the minimal resolution of the quotient $\mathcal{A}_n \to \mathbf{C}^2/\mathbf{Z}_{n+1}$, where the cyclic group $\mathbf{Z}_{n+1}$ acts on $\mathbf{C}^2$ by $(z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2)$, ζ is the $n+1$-th root of unity. When X is a $\mathcal{A}_n$ surface, the quantum cohomology and quantum multiplications can also be described nicely by representation theory of $\widehat{\mathfrak{gl}}_{n+1}$ [MO09]. As a crepant resolution of the quotient, the exceptional divisors on $\mathcal{A}_n$ consists of a chain of rational curves $E_1, \ldots, E_n$. The negative of the intersection matrix is just the Cartan matrix of type A_n. Thus $\langle E_k, E_k \rangle = -2$, $\langle E_k, E_\ell \rangle = 1$ if $|k-\ell| = 1$. Moreover $\mathrm{H}^2(\mathcal{A}_n, \mathbf{C}) = \bigoplus_k \mathbf{C} \cdot E_k$. We denote the dual basis by $\{\omega_1, \ldots, \omega_n\}$. That is $\langle \omega_k, E_\ell \rangle = \delta_{k\ell}$. We can also identify $\mathrm{H}_2(\mathcal{A}, \mathbf{Z})$ with the type A_n root lattice by sending E_k to the k-th simple root of $\mathfrak{gl}_{n+1}$, the one take value $a_{ii} - a_{jj}$ on the diagonal matrix $\mathrm{diag}(a_{11}, \ldots, a_{nn})$. Consequently, the positive roots correspond to $\alpha_{k\ell} = E_k + \cdots + E_\ell$, we'll use this terminology in the description of the Gromov-Witten invariants on $\mathcal{A}_n \times \mathbf{P}^1$, see section 3.3.

The diagonal action of $T = (\mathbf{C}^\times)^2$ on $\mathbf{C}^2$ commutes with the $\mathbf{Z}/\mathbf{Z}_{n+1}$ action, thus it lifts to torus actions on $\mathcal{A}_n$ as well as $\mathrm{Hilb}(\mathcal{A}_n)$. Under this action $\mathcal{A}_n$ has $n+1$ fixed points $p_0, \ldots, p_n$ with tangent weights at the fixed point p_k given by

$$
\begin{aligned}
w_i^L &= (n+2-k)t_1 + (1-k)t_2 \\
w_i^R &= (-n+k-1)t_1 + kt_2.
\end{aligned}
$$

The T-fixed curves are the n exceptional rational curves, the noncompact fibre at p_0 and p_n.

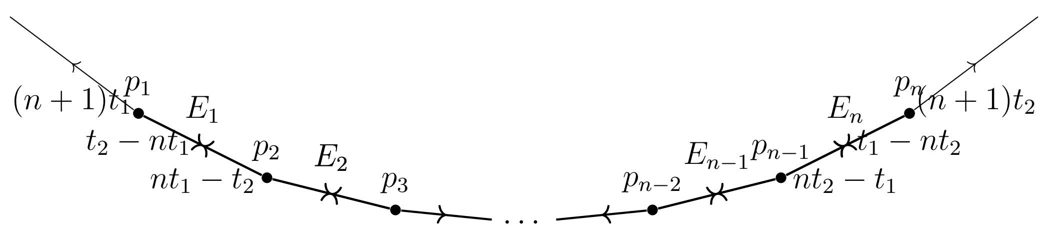


Figure 3: Torus weight of $\mathcal{A}_n$

Since our project is mainly about the $\mathcal{A}_1 = T^*\mathbf{P}^1$ case and for clarity. We specialize the discussion above to this simplest case. The geometry for $T^*\mathbf{P}^1$ is very clear. The exceptional locus E of $T^*\mathbf{P}^1$ is just a copy of $\mathbf{P}^1$ with self-intersection number -2. E is a generator of $\mathrm{H}^2(T^*\mathbf{P}^1, \mathbf{C})$, together with the identity element $\mathbf{1}$, they generate the whole cohomology ring of $\mathbf{H}(T^*\mathbf{P}^1, \mathbf{C})$. Denote ω the dual basis of E, that is $\langle \omega, E\rangle = 1$, actually $\omega = -\frac{1}{2}E$.

The T-action acts both on $T^*\mathbf{P}^1$ and $\mathrm{Hilb}(T^*\mathbf{P}^1, \mathbf{C})$. With this torus action, we have two fixed points $p_0 = 0$ and $p_1 = \infty$. The tangent weights are given in the following figure .

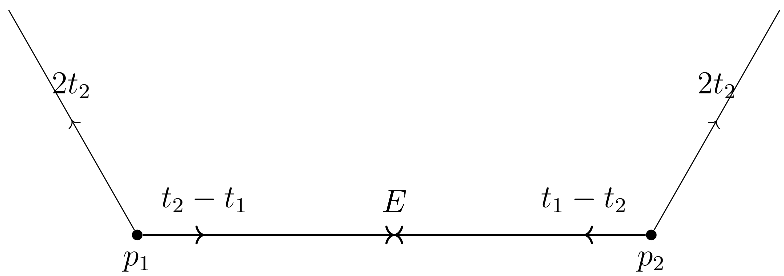


Figure 4: Torus weight of $T^*\mathbf{P}^1$

Note that we have a simple identification of $\mathrm{H}_2(T^*\mathbf{P}^1, \mathbf{Z})$ with the rank one root lattice of type A_1.

3.2.2 Monad and quiver description

Let $\pi : \Sigma_n = \mathbf{P}(\mathcal{E}^\vee) = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \to \mathbf{P}^1$ be the n-th Hirzebruch surfaces. Let $H = c_1(\mathcal{O}_{\Sigma_n}(1))$, E be the class of $\mathbf{P}(\mathcal{O}(-n))$, its self-intersection equals to $-n$, F be a fiber class of π. We have $E = H - nF$. The Chow ring of Σ_n is given by $A(\Sigma_n) = A[\mathbf{P}^1][H]/(H^2 - nFH) = \mathbf{Z}[F, H]/(F^2, H^2 - nFH)$, naturally one has $\mathrm{Pic}(\Sigma_n) = \mathbf{Z}H \oplus \mathbf{Z}F$. From now on, we denote $\mathcal{E}(p, q) = \mathcal{E} \otimes \mathcal{O}_{\Sigma_n}(pH + qF)$ for any sheaf of $\mathcal{O}_{\Sigma_n}$-modules. Recall that the relative Euler sequence computes the relative canonical sheaf

$$0 \to \mathcal{O}_{\Sigma_n}(-1, 0) \to \pi^*(\mathcal{E}^\vee) \to T_{\Sigma_n/\mathbf{P}^1}(-1, 0) \to 0.$$

$$\Omega_{\Sigma_n/\mathbf{P}^1} = \mathcal{O}_{\Sigma_n}(-2,0) \otimes \pi^*(\det \mathcal{E}) = \mathcal{O}_{\Sigma_n}(-2,n)$$
$$T_{\Sigma_n/\mathbf{P}^1} = \mathcal{O}_{\Sigma_n}(2,-n)$$
$$\omega_{\Sigma_n} = \mathcal{O}_{\Sigma_n}(-2,n-2).$$

For later computations, we need the following lemmata.

Lemma 3.2.1.

$$\mathrm{H}^0(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \textit{ if and only if } \begin{cases} p \geq 0 \\ np+q \geq 0 \end{cases}$$
$$\mathrm{H}^1(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \textit{ if and only if } \begin{cases} p \geq 0 \\ q \leq -2 \end{cases} \textit{ or } \begin{cases} p \leq -2 \\ q \geq n \end{cases}$$
$$\mathrm{H}^2(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \textit{ if and only if } \begin{cases} p \geq -2 \\ np+q \leq -(n+2) \end{cases}.$$

Proof. See [BBR15, Lemma 3.1] ☐

The classical Beilinson spectral sequence is a way to describe torsion-free sheaves on $\mathbf{P}^2$ as the cohomology of certain three-term complexes—the so-called monad. With the isomorphism between the Hilbert scheme of points on $\mathbf{C}^2$ and the moduli space of rank 1 torsion-free sheaves on $\mathbf{P}^2$ which are trivial over infinity, the Hilbert scheme can be realized as the quiver variety with one vertex and one loop, see [Nak99, Theorem 2.1]. The essential part of the construction is a resolution of the diagonal in $\mathbf{P}^2 \times \mathbf{P}^2$. For Hirzebruch surfaces, the diagonal can also be resolved(for references to the details of the construction , we refer to [Buc87] or [AB09]). We first briefly recall the Beilinson-type spectral sequence on Σ_n, following the approach in [Nak99] we give a monad description of $\mathrm{Hilb}(T^*\mathbf{P}^1)$ as a Nakajima quiver variety of type $A_1^{(1)}$, and we show that the two tautological bundles corresponding to the two vertices are exactly $\mathcal{O}_X(1)^{[n]}$

and $\mathcal{O}_X^{[n]}$ [Buc87]. We denote $p_i : \Sigma_n \times \Sigma_n$ be the projections to the two factors, $p : X \times X \to \mathbf{P}^1 \times \mathbf{P}^1$ be the product of the ruling π. Let $\Delta_{\mathbf{P}^1} \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the diagonal divisor on $\mathbf{P}^1 \times \mathbf{P}^1$ and Δ be the diagonal divisor on $\Sigma_n \times \Sigma_n$, $L = \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(\Delta_{\mathbf{P}^1})$. Consider the line bundle

$$\mathcal{F} = p_1^*(T_{\Sigma_n/\mathbf{P}^1}(-1,0)) \otimes p_2^*(\mathcal{O}_{\Sigma_n}(1,0)) = \mathcal{O}_{\Sigma_n}(1,-n) \boxtimes \mathcal{O}_{\Sigma_n}(1,0).$$

A rank 2 locally free sheaf $\mathcal{G}$ is defined by an extension

$$0 \to \mathcal{F} \to \mathcal{G} \to p^*(L) \to 0.$$

Note that $p^*(L) = \mathcal{O}_{\Sigma_n}(0,1) \boxtimes \mathcal{O}_{\Sigma_n}(0,1)$. Buchdahl proved that[Buc87] the diagonal Δ can be realized as the zero locus of a global section s of $\mathcal{G}$, then the Koszul complex of s gives us the resolution of the diagonal $C^\bullet \twoheadrightarrow \mathcal{O}_\Delta$:

$$0 \to \wedge^2 \mathcal{G}^\vee \to \mathcal{G}^\vee \xrightarrow{s} \mathcal{O}_{\Sigma_n \times \Sigma_n} \to \mathcal{O}_\Delta \to 0.$$

Then the Beilinson spectral sequences comes from different ways of computing the image of $p_1^*\mathcal{E}$ under the composite functor $p_{2*}(\mathcal{O}_\Delta \otimes -)$. On the one hand, we trivially have $p_{2*}(\mathcal{O}_\Delta \otimes p_1^*\mathcal{E}) = \mathcal{E}$. If we take the cohomology of $C^\bullet$ first in the double complex for the hyper direct image $\mathbf{R}^\bullet p_{2*}(C^\bullet \otimes p_1^*\mathcal{E})$, the trivial identity tells us exactly that the corresponding spectral sequence degenerate at the E_2 page, namely

$$E_2^{p,q} = R^q p_{2*}(\mathrm{H}^p(C^\bullet \otimes p_1^*\mathcal{E})) = \begin{cases} \mathcal{E} & (p,q) = (0,0) \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, we can also take the direct image first, then we get the so-called Beilinson spectral sequence for Σ_n.

Theorem 3.2.2 ([Buc87]). *For any torsion free sheaf $\mathcal{E}$ on Σ_n, there exists a spectral sequence, depending on $\mathcal{E}$:*

$$E_1^{p,q} = R^q p_{2*}(\wedge^{-p}\mathcal{G}^\vee \otimes p_1^*\mathcal{E}) \Rightarrow \begin{cases} \mathcal{E} & p+q=0 \\ 0 & \textit{otherwise} \end{cases}.$$

Remark 3.2.3. It's originally stated for locally free sheaves, the same argument works for torsion free sheaves with very little modification. Also notice that we can use either $p_{1*}(p_2^*\mathcal{E} \otimes \mathcal{O}_\Delta)$ or $p_{2*}(p_1^*\mathcal{E} \otimes \mathcal{O}_\Delta)$. In general the two spectral sequences are different. We chose the one above simply because we can get better vanishing control on the E_1-page.

Take the exterior powers of the dual of the defining sequence of $\mathcal{G}$ and tensor with $p_1^*\mathcal{E}$, Since $\mathcal{E}$ is torsion-free, $- \otimes p_1^*\mathcal{E}$ is an exact functor. We have

$$0 \to \wedge^{-p-1}\mathcal{F}^\vee \otimes p^*(L^\vee) \otimes p_1^*\mathcal{E} \to \wedge^{-p}\mathcal{G}^\vee \otimes p_1^*\mathcal{E} \to \wedge^{-p}\mathcal{F}^\vee \otimes p_1^*\mathcal{E} \to 0.$$

If $p=0$, the first sheaf in the sequence above vanishes, $\wedge^{-p}\mathcal{G}^\vee \otimes p_1^*\mathcal{E} = p_1^*\mathcal{E}$, if $p=-2$, the last sheaf in the sequence above vanishes, $\wedge^{-2}\mathcal{G}^\vee \otimes p_1^*\mathcal{E} = \mathcal{F}^\vee \otimes p^*(L^\vee) \otimes p_1^*\mathcal{E} = \mathcal{E}(-1, n-1) \boxtimes \mathcal{O}_{\Sigma_n}(-1,-1)$. Take the associated long exact sequence one has

$$E_1^{0,q} = \mathrm{H}^q(\mathcal{E}) \otimes \mathcal{O}_{\Sigma_n}(0,0) \tag{3.1}$$

$$\cdots \to \mathrm{H}^q(\mathcal{E}(0,-1)) \otimes \mathcal{O}_{\Sigma_n}(0,-1) \to E_1^{-1,q} \to \mathrm{H}^q(\mathcal{E}(-1,n)) \otimes \mathcal{O}_{\Sigma_n}(-1,0) \to \dots \tag{3.2}$$

$$E_1^{-2,q} = \mathrm{H}^q(\mathcal{E}(-1,n-1)) \otimes \mathcal{O}_{\Sigma_n}(-1,-1). \tag{3.3}$$

Lemma 3.2.4. *Let $\mathcal{E}$ be a torsion-free sheaf on Σ_n, trivial at infinity. We have*

$$\mathrm{H}^0(\mathcal{E}(p,q)) = 0 \textit{ for } np+q \le -1$$

$$\mathrm{H}^2(\mathcal{E}(p,q)) = 0 \textit{ for } np+q \ge -(n+1).$$

Proposition 3.2.5. *For any torsion free sheaf $\mathcal{E}$ on Σ_n can be realized as a monad:*

$$0 \to \mathrm{H}^1(\mathcal{E}(-2, n-1)) \otimes \mathcal{O}_{\Sigma_n}(0, -1) \to E_1^{-1,1} \otimes \mathcal{O}_{\Sigma_n}(1, 0) \to \mathrm{H}^1(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n}(1, 0) \to 0.$$

where the $E_1^{-1,1} \otimes \mathcal{O}_{\Sigma_n}(1, 0)$-term can be computed from

$$0 \to \mathrm{H}^1(\mathcal{E}(-1, -1)) \otimes \mathcal{O}_{\Sigma}(1, -1) \to E_1^{-1,1} \otimes \mathcal{O}_{\Sigma_n}(1, 0) \to \mathrm{H}^1(\mathcal{E}(-2, n)) \otimes \mathcal{O}_{\Sigma_n}(0, 0) \to 0.$$

Moreover, the second sequence splits.

Proof. The trick here is that $\mathcal{E}$ can't be realized as a monad just from the spectral sequence. However, for $\mathcal{E}(-1, 0)$, the Beilinson spectral sequence becomes

$\mathrm{H}^2(\mathcal{E}(-2, n-1)) \otimes \mathcal{O}(-1, -1)$	$E_1^{-1,2}$	$\mathrm{H}^2(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n}$
$\mathrm{H}^1(\mathcal{E}(-2, n-1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1)$	$E_1^{-1,1}$	$\mathrm{H}^1(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n}$
$\mathrm{H}^0(\mathcal{E}(-2, n-1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1)$	$E_1^{-1,0}$	$\mathrm{H}^0(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n}$

By Lemma 3.2.4, all the four corner terms vanish. Then Theorem 3.2.2 forces $E_1^{-1,2}$ and $E_1^{-1,0}$ to be zeros. Thus only the $q = 1$ terms in the spectral sequence survive and the spectral sequence degenerates at the E_1-page. This proves that $\mathcal{E}(-1, 0)$ is the cohomology of

$$0 \to \mathrm{H}^1(\mathcal{E}(-2, n-1)) \otimes \mathcal{O}_{\Sigma_n}(-1, -1) \to E_1^{-1,1} \to \mathrm{H}^1(\mathcal{E}(-1, 0)) \otimes \mathcal{O}_{\Sigma_n} \to 0.$$

Tensoring it with $\mathcal{O}_{\Sigma_n}(1, 0)$ gives the first statement. To compute $E_1^{-1,1} \otimes \mathcal{O}_{\Sigma_n}(1, 0)$, Lemma 3.2.4 shows that $\mathrm{H}^0(\mathcal{E}(-1, -1)) = 0$ and $\mathrm{H}^2(\mathcal{E}(-1, -1)) = 0$ for any Σ_n. Sequence 3.2 degenerates to the second statement in the proposition. It splits because $\mathrm{Ext}^1(\mathcal{O}_{\Sigma_n}(1, -1), \mathcal{O}_{\Sigma_n}) = \mathrm{H}^1(\mathcal{O}_{\Sigma_n}(-1, 1)) = 0$, the last equality comes from Lemma 3.2.1. $\square$

With the information above the the method similar to that in [Nak99, Page 18], we can realize $\mathrm{Hilb}(T^*\mathbf{P}^1)$ as a $\widehat{A}_1$-quiver variety. We briefly recall this description here. For a quiver Q and vertex set V and edge set E. Given a GIT stability parameter $\theta \in \mathbf{R}^{|V|}$, and two V-graded vector spaces $V = \bigoplus_{k\in V} V_k$ and $W = \bigoplus_{k\in V} W_k$, the space of framed representation of the quiver is defined to be

$$\mathrm{Rep}_Q^{\mathrm{framed}} := \bigoplus_{k\to\ell\in E} \mathrm{Hom}(V_k, V_\ell) \oplus \bigoplus_{k\in V} \mathrm{Hom}(W_k, V_k).$$

W is called the framing. $T^*\mathrm{Rep}_Q^{\mathrm{framed}}$ has a standard symplectic form. The action of the structure group $G_V := \prod_{k\in V} GL(V_k)$ is an Hamiltonian action, the corresponding moment map is denoted by μ. The Nakajima quiver variety is defined to be a GIT quotient with respect to the stability condition θ

$$\mathcal{M}_\theta := \mu^{-1}(0)//_\theta G_V.$$

It has irreducible components $\mathcal{M}_\theta(v, w)$ parametrized by the dimension vectors $v = (\dim V_1, \dots, \dim V_{|V|})$ and $w = (\dim W_1, \dots, \dim W_{|V|})$. Specially, let $v_0 = (1, \dots, 1)$ and $w_0 = (1, 0, \dots, 0)$.

Theorem 3.2.6 ([Nak99, Kuz07])**.** *For any integer $m \geq 0$ and stability condition θ such that $\sum_{k=1}^{n+1} \theta_k > 0$, there is an isomorphism*

$$\mathcal{M}_\theta(mv_0, w_0) \cong \mathrm{Hilb}^m(\mathcal{A}_n).$$

Moreover, when θ is generic, $\mathcal{M}_\theta$ is always a smooth algebraic variety and is a fine moduli space of quiver representations. It has a universal family $\mathscr{V} = \bigoplus_{k\in V} \mathscr{V}_k$. $\mathcal{M}_\theta(v, w)$ has $|V|$ tautological bundles $\{\mathcal{V}_0, \dots, \mathcal{V}_n\}$. For $\mathcal{M}_\theta(nv_0, w_0) \cong \mathrm{Hilb}^n(\mathcal{A}_n)$, we can describe it as follows, if $Z \subset \mathcal{A}_{n+1}$ is 0-dimensional subscheme of length n. The fibre of $\mathcal{V}_k$ restricted at Z is given by

$$\mathcal{V}_k|_Z = \mathrm{H}^0(\mathcal{A}_{n+1}, \mathscr{V}_k \otimes \mathcal{O}_Z).$$

3.2.3 Fock space formalism

To start, we know that $\mathrm{H}^\bullet_T(T^*\mathbf{P}^1, \mathbf{Q}) = \mathrm{H}^\bullet(T^*\mathbf{P}^1, \mathbf{Q}) \otimes \mathbf{Q}[t_1, t_2]$. Consider the Heisenberg algebra $\mathcal{H}$ generated over the field $\mathbf{Q}(t_1, t_2)$ by a central element c, $\{\alpha_k(\gamma)\}$ for $\gamma \in \mathrm{H}^\bullet_T(T^*\mathbf{P}^1)$ and $k \in \mathbf{Z} \setminus \{0\}$. The Lie algebra structure of $\mathcal{H}$ is given by

$$[\alpha_k(\gamma_1), \alpha_\ell(\gamma_2)] = -k\delta_{k+l}\langle \gamma_1, \gamma_2 \rangle c$$

$$[c, \alpha_k(\gamma)] = 0.$$

Note that the pairing $\langle \gamma_1, \gamma_2 \rangle$ is commonly computed by equivariant localization formula. Actually, the only nonvanishing Lie bracket is essentailly $[\alpha_k(\omega), \alpha_{-k}(E)] = -kc$. As noted in [MO09], we can pick a basis of cohomology for which the denomenator of any intersection pairing are never divisible by $\hbar = (t_1 + t_2)$, that is $\mathcal{H}$ can be defined over $R = \mathbf{Q}[t_1, t_2]_{(t_1+t_2)}$, the ring of rational functions with nonnegative valuation at $t_1 + t_2$.

The Fock space $\mathcal{F}$ is freely generated over $\mathbf{Q}[t_1, t_2]$ by the commutation relations of $\alpha_{-k}(\gamma)$ on the vacuum vector $v_\emptyset$. $\mathcal{F}$ has a natural grading induced by defining $\deg(v_\emptyset) = 0$ and $\deg(\alpha_{-k}(\gamma)) = k$ which is compatible with the number of points grading of $\mathrm{Hilb}^m(T^*\mathbf{P}^1)$. Base change to $\mathbf{Q}(t_1, t_2)$, the intersection pairing on gives a nondegenerate paring on $\mathcal{F} \otimes \mathbf{Q}(t_1, t_2)$, namely $\langle v_\emptyset, v_\emptyset \rangle = 1$ and specifying the adjoint $\alpha_k(\gamma)^* = (-1)^k \alpha_{-k}(\gamma)$.

A natural basis of $\mathcal{F}$ can be described by cohomology-weighted partitions $\overrightarrow{\mu} = \{(\mu_i, \gamma_i)\}_{i=1}^{\ell(\mu)}$, where $\{\mu_i\}_{i=1}^{\ell(\mu)}$ is a partition and $\gamma_i = \mathbf{1}$ or E. A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu,\nu\rangle = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)}\prod_{i=1}^{\ell(\mu)}\alpha_{-k}(\mathbf{1})\prod_{j=1}^{\ell(\nu)}\alpha_{-\nu_j}(E)v_\emptyset$$

where

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)|\prod_{i=1}^{\ell(\mu)}\mu_i.$$

We also denote this basis element by $\overrightarrow{\mu}$ if it's clear in the context. The cohomological degree of $\overrightarrow{\mu}$ is $2(|\mu|-\ell(\mu))+\sum\deg(\gamma_i)$. The intersection pairing in this basis is given by

$$\langle\overrightarrow{\mu},\overrightarrow{\nu}\rangle = \frac{\delta_{\overrightarrow{\mu}\,\overrightarrow{\nu}}}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)}.$$

We denote the boundary divisor on $\mathrm{Hilb}^n(T^*\mathbf{P}^1)$ by D, in the Nakajima basis, see section 3.2.4 formalism, we have $D = -\{(2,1),(1,1)^{n-2}\}$, $\mathrm{H}^2(\mathrm{Hilb}^n(T^*\mathbf{P}^1))$ is 2-dimensional, the other generator is given by $(1,\omega) = \{(1,\omega),(1,1)^{n-1}\}$. Note that D could be thought as a tangency condition operator and ω could be thought as an insertion at a fibre condition.

For our Hilbert scheme $\mathrm{Hilb}(T^*\mathbf{P}^1)$, We also define the generating function

$$\langle\gamma_1,\dots,\gamma_n\rangle^{\mathrm{Hilb}} = \sum_\beta\langle\gamma_1,\dots,\gamma_n\rangle^{\mathrm{Hilb}}_{0,n,\beta}q^{D\cdot\beta}s^{(1,\omega)\cdot\beta}.$$

Similarly, we define

$$\langle\gamma_1,\gamma_2\star\gamma_3\rangle = \langle\gamma_1,\gamma_2,\gamma_3\rangle^{\mathrm{Hilb}}$$

The quantum multiplication gives the equivariant quantum cohomolgy

$$\mathrm{QH}_T(\mathrm{Hilb}(T^*\mathbf{P}^1)) = \mathrm{H}_T(\mathrm{Hilb}(T^*\mathbf{P}^1))\otimes\mathbf{Q}(t_1,t_2)((q))[s]$$

a structure of supercommutative associative algebra. It's a deformation of the ordinary equivariant cohomology ring in two variables q and s. We denote the quantum multiplication operator with respect to D and $(1, \omega)$ by M_D and $M_{(1,\omega)}$ respectively.

3.2.4 Quantum multiplication and the basic representation of $\widehat{\mathfrak{gl}}_2$

In [MO09], the authors compute the quantum multiplication by divisors on $\mathrm{Hilb}(\mathcal{A}_n)$ in terms of $\widehat{\mathfrak{gl}}_{n+1}$-representations. We specialize the result in [MO09] to the case of $\mathcal{A}_1 = T^*\mathbf{P}^1$ and the $\widehat{\mathfrak{gl}}_2$ basic representation, see section 2.3. Consider the operator

$$\Omega_+ = \sum_{k\in\mathbf{Z}} : f(k)e(-k) : \log(1-(-q)^k s).$$

The normal ordering is given by

$$: f(k)e(-k) := \begin{cases} f(k)e(-k), \; k \le 0 \\ e(-k)f(k), \; k > 0. \end{cases}.$$

We also consider another operator

$$\Omega_0 = -\sum_{k\ge 1} (2t_1t_2\alpha_{-k}(1)\alpha_k(1) + \alpha_{-k}(E)\alpha_k(\omega)) \log(\frac{1-(-q)^k}{1-(-q)}).$$

The Ω_0 is defined in term of the Nakajima operators, so it naturally acts on the equivariant cohomology ring of $\mathrm{Hilb}(T^*\mathbf{P}^1)$. For Ω_+, it only acts on the whole basic representation a prior. However each summand $: f(k)e(-k) :$ commutes with the Cartan $\widehat{\mathfrak{h}}$, hence Ω_+ preserves each weight subspace. We shall identify $\mathrm{H}^\bullet_T(\mathrm{Hilb}(T^*\mathbf{P}^1))$ with a collection of weight subspaces in the basic representation. First we can embed the Heisenberg algebra into $\widehat{\mathfrak{gl}}_2 \otimes \mathbf{Q}(t_1, t_2)$ via

$$\alpha_{-k}(\mathbf{1}) \mapsto \mathrm{Id}(-k),\ \alpha_{-k}(\mathbf{1}) \mapsto \frac{\mathrm{Id}(-k)}{4t_1t_2}$$
$$\alpha_{-k}(E) \mapsto \begin{pmatrix} t^k & 0 \\ 0 & -t^k \end{pmatrix},\ c \mapsto 1.$$

Then we can identify the equivariant quantum cohomology of $T^*\mathbf{P}^1$ with certain weight spaces of the basic representation of $\widehat{\mathfrak{gl}}_2$.

Proposition 3.2.7. *The following map gives a $\mathcal{H}$-module isomorphism:*

$$\mathrm{H}_T(\mathrm{Hilb}^\bullet(T^*\mathbf{P}^1)) \to \bigoplus_{m=0}^{\infty} V_\Lambda[\Lambda - m\delta]$$
$$\alpha_{-\mu}(1)v_\emptyset \mapsto y_\mu$$
$$\alpha_{-\mu}(E)v_\emptyset \mapsto x_\mu.$$

Proof. α acts as $2q\frac{\partial}{\partial q}$ on L_Λ and acts trivially on $\mathcal{F}$. h also acts trivially, that is acts by 0. Thus $V[\Lambda - m\delta] = \{v \in V_\Lambda | h(v) = (\Lambda(h) - m\delta(h))(v)\}$. $\widehat{\mathfrak{h}}$ acts diagonally on monomials, for a given $y_\mu x_\nu q^k$, their eigenvalues are given by

α	h	c	d
$2k$	0	1	$k^2+\lvert\mu\rvert+\lvert\nu\rvert$

That means weight spaces in the basic representations are those spanned by mononials, more precisely,$\mathbf{C}y_\mu x_\nu q^k \cong V_\Lambda[\Lambda - m\delta + 2k\alpha^*] = V_\Lambda[\Lambda - m\delta + \frac{1}{2}k\alpha]$. We conclude that

$$\mathrm{H}_T(\mathrm{Hilb}^\bullet(T^*\mathbf{P}^1) \cong q^0C[y_1, \dots; x_1, \dots] = \bigoplus_{m=0}^{\infty} V_\Lambda[\Lambda - m\delta].$$

□

Proposition 3.2.8 ([MO09])**.** *We have the equality for the quantum multiplication of* *divisor classes*

$$M_D = M_D^{\mathrm{cl}} + (t_1+t_2) q \frac{d}{dq}\Omega(q,s)$$
$$M_{(1,\omega)} = M_{(1,\omega)}^{\mathrm{cl}} + (t_1+t_2) s \frac{d}{ds}\Omega_+(q,s).$$

3.3 Absolute Gromov-Witten theory of $\mathcal{A}_n \times \mathbf{P}^1$

Let $X = \mathcal{A}_n \times \mathbf{P}^1$ and $(\beta, m) \in \mathrm{H}_2(X, \mathbf{Z})$ be en effective curve class, where $\beta \in \mathrm{H}_2(\mathcal{A}_n)$. Given partitions $\mu_1, \ldots, \mu_n$, and n distinct points $z_1, \ldots, z_k$ on $\mathbf{P}^1$. The moduli space

$$\overline{M}_g^{\bullet}(X, (\beta, m), (\mu_1, \ldots, \mu_n))$$

parametrized **possibly disconnected** relative stable maps from a genus g source curve to X, with image given by the cohomology class (β, m) and the ramification profile given by partition μ_k on $\mathcal{A}_n \times z_k$. Note that the $\bullet$ notation means that we don't allow collapsed connected components in the domain. The virtual fundamental class has dimension

$$-K_{\mathcal{A}_n} \cdot \beta + 2m + \sum_k (\ell(\mu_k) - m) = \sum_k \ell(\mu_k) + (2-n)m.$$

Consider the cohomology-weighted partitions $\{\overrightarrow{\mu_k}\}_{k=1}^{\ell}$ of m, we as defined in 3.2.4, for each part μ_k^i, we can associate to the corresponding ramification point the evaluation map

$$\mathrm{ev}_k^s : \overline{M}_g(X, (\beta, m), (\mu_1, \ldots, \mu_k)) \to X \times z_k.$$

The relative Gromov-Witten invariants in this case is defined to be

$$\langle \overrightarrow{\mu}_1, \ldots, \overrightarrow{\mu}_n \rangle_{g,\beta}^X = \frac{1}{\prod \mathrm{Aut}(\mu_k)} \int_{[\overline{M}_g^{\bullet}(X)]}^{\mathrm{vir}} \prod_{k=1}^{n} \prod_{s=1}^{\ell(\mu_k)} \mathrm{ev}^*(\gamma_k^s).$$

Note that we suppress the m since it has been encoded in the partitions, namely $m = |\mu_k|$. The automorphism prefactors accounts for the fact that our relative conditions are unordered however to define the moduli space, we need ordered partitions.

We also define a generating function of relative invariants [Mau09]

$$Z'(X)_{\vec{\mu}_1,\ldots,\vec{\mu}_n} = \sum_{g,\beta} \langle \vec{\mu}_1, \ldots, \vec{\mu}_n \rangle^X_{g,\beta} u^{2g-2} s^{\beta\cdot\omega}.$$

Note once again that this generating function is defined by **possibly disconnected** domain curves. If we fix a connected domain component, Z' is just a product of the connected Gromov-Witten invariants. Thus it suffices to study the moduli space $\overline{M}^\circ(X,(\beta,m),\mu,\rho,\nu)$ for connected domain curves. We can split the generating function into"classic part"$(\beta = 0)$ and "quantum part" $(\beta \neq 0)$:

$$Z^\circ_{\beta=0}(X)_{\vec{\mu},\vec{\rho},\vec{\nu}} + Z^\circ_{\beta\neq 0}(X)_{\vec{\mu},\vec{\rho},\vec{\nu}}$$

.

Proposition 3.3.1 ([Mau09])**.** *If μ, ν are partitions of $m > 0$ and the cohomology classes labelling μ,ν are divisors, then we have*

$$u^{\ell(\mu)+\ell(\nu)-1} Z^\circ_{\beta\neq 0}(\mathcal{A}_n \times \mathbf{P}^1)_{\vec{\mu},(2),\vec{\nu}} = \frac{d}{du}\Theta^\circ(\vec{\mu},\vec{\nu})$$
$$u^{\ell(\mu)+\ell(\nu)} Z^\circ_{\beta\neq 0}(\mathcal{A}_n \times \mathbf{P}^1)_{\vec{\mu},(1,\omega_k),\vec{\nu}} = s_k \frac{d}{ds_k}\Theta^\circ(\vec{\mu},\vec{\nu})$$

Otherwise, these invariants vanish.

The $\Theta^\circ(\overrightarrow{\mu}, \overrightarrow{\nu})$ above is given by

$$\Theta^\circ(\overrightarrow{\mu}, \overrightarrow{\nu}) = \frac{t_1 + t_2}{|\mathrm{Aut}(\overrightarrow{\mu})||\mathrm{Aut}(\overrightarrow{\nu})|} \times \sum_{1 \le i < j \le n} \sum_{d=1}^{\infty} (du)^{\ell(\mu)+\ell(\nu)-2} \times$$
$$\frac{\prod_{k=1}^{\ell(\mu)} (\alpha_{i,j} \cdot \gamma_k) S(d\mu_k u) \prod_{k=1}^{\ell(\nu)} (\alpha_{i,j} \cdot \eta_k) S(d\nu_k u)}{dS(du)^2} (s_i \ldots s_{j-1})^d,$$

where $S(u) = \dfrac{\sin(\frac{u}{2})}{\frac{u}{2}}$ and $S(du)^2$ in the denominator means $(S(du))^2$.

3.4 Hilb/GW correspondence

Theorem 3.4.1 ([MO09]). *Under the variable substitution $q = e^{-iu}$ we have*

$$(-1)^m \langle \overrightarrow{\mu}, \overrightarrow{\nu}, \overrightarrow{\rho} \rangle^{\mathrm{Hilb}} = (-iu)^{-m+\ell(\mu)+\ell(\nu)+\ell(\rho)} Z'(\mathcal{A}_n \times \mathbf{P}^1)_{\overrightarrow{\mu}, \overrightarrow{\nu}, \overrightarrow{\rho}}$$

for $\overrightarrow{\nu} = D$ and $\overrightarrow{\nu} = (1, \omega)$.

Chapter 4

Relative Gromov-Witten theory

4.1 Preliminaries on relative Gromov-Witten theory

In this part, we give a brief review of the relative Gromov-Witten theory following [LLZ07]. In chapter 5, we'll compare the virtual fundamental classes of $T^*\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^1 \times \mathbf{P}^1$, virtual localization gives the change of variables that is needed to get the relative invariants of $\mathbf{P}^1 \times \mathbf{P}^1$ from those of $T^*\mathbf{P}^1 \times \mathbf{P}^1$.

4.1.1 The moduli space

Given a smooth projective variety X, and $D_1, \ldots, D_k$, smooth divisors on X, the relative Gromov-Witten invariants essentially count the number of stable morphisms from curves to X with certain intersection conditions on the divisors and we allow degeneration of X along D_i, a subtle point is that the degeneration is a 'rubber', in the sense that morphisms into the degeneration are viewed equal up to a $\mathbf{C}^\times$-action. We demonstrate the definition in the case of one smooth divisor $D \hookrightarrow X$, multiple divisors only use more indices. We first introduce some notations.

- $\Delta(D) = \mathbf{P}(\mathcal{O}_D \oplus N_{D/X}) \to D$, the projective completion of the normal bundle of D.

- $\Delta(D)(m)$, the union of m copies of $\Delta(D)$ by identifying the zero section $D_0 = \mathbf{P}(\mathcal{O}_D \oplus 0) \hookrightarrow \Delta(D)$ of the $i+1$-th copy $\Delta(D)$ with the ∞-section $D_\infty = \mathbf{P}(0 \oplus N_D/Y)$ of the i-th copy $\Delta(D)$. Denote the k-th section in this digeneration by D_k.

- $X[m] = X \cup \Delta(D)(m)$. The m-fold degeneration of X along the divisor.

- $(\mathbf{C}^\times)^m$ acts on $\Delta(D)(m)$, the action is trivially on the divisor.

- $\pi[m] : X[m] \to X$, the natural projection which is equivariant under the torus action.

- $\beta \in \mathrm{H}_2(X, \mathbf{Z})$, an effective curve class.

- $d = \displaystyle\int_\beta c_1(\mathcal{O}_D) \geq 0$, the intersection number of β with the divisor.

- μ, a partition of d and let $\ell(\mu)$ be the length of the partition. This keeps track of the intersection type of β with the divisor.

- C, the source curve, let $\{x_i\}_{i=1}^{\ell(\mu)}$ be the marked points on C that are mapped to be the intersection points of β and D, let y be another free marked point.

$$\begin{tikzcd} & C \arrow[ld] \arrow[d, "f"] \\ X \arrow[r, shift left, hook, "i"] & X[m] \arrow[l, shift left, "\pi[m]"] \end{tikzcd}$$

Now we can give the definition of $\overline{\mathcal{M}}_{g,1}(X; D|\mu)$, the moduli space of relative stable morphism, if the divisor D is fixed, we also denote it by $\overline{\mathcal{M}}_{g,1}(X, \mu)$.

Definition 4.1.1. $\overline{\mathcal{M}}_{g,1}(X; D|\mu)$ is the moduli space of morphisms

$$f : (C; \{x_i\}_{i=1}^{\ell(\mu)}; y) \to X[m]$$

with the conditions that

- $(C; \{x_i\}_{i=1}^{\ell(\mu)}; y)$ is a connected prestable curve of arithmetic genus g with $1 + \ell(\mu)$ marked point.

- $(\pi[m] \circ f)_*[C] = \beta$.

- $f^{-1}(D_m) = \sum_{i=1}^{\ell(\mu)} \mu_i x_i$ as a Cartier divisor. In other words, the partition actually denotes the intersection type of f with the last copy of the divisor D in the degeneration.

- $f^{-1}(D_i)$ are nodes of C for $0 \le i \le m-1$, that is, except the last divisor in the degeneration, the intersection points of f and D_i are all nodes. Moreover, if $x \in f^{-1}(D_i)$ $(i \neq m)$ is the intersection of two irreducible components C_1, C_2 of C, then $f|_{C_1}$ and $f|_{C_2}$ have the same contact order to D_i.

- Two morphisms are identified up to the torus action on the target.

- $|\mathrm{Aut}(f)|$ is finite, which takes into consideration of the torus action above.

It's shown in [LLZ07] that $\overline{\mathcal{M}}_{g,1}(X; D|\mu)$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of dimension

$$\int_\beta c_1(TX) + (1-g)(\mathrm{dim}X - 3) + 1 + (\ell(u) - |\mu|).$$

For example, if $X = \mathbf{P}^1 \times \mathbf{P}^1$, $\beta = aH + bV$, then $\dim[\overline{\mathcal{M}}_{g,1}(\mathbf{P}^1 \times \mathbf{P}^1; \mu, \nu)^{\mathrm{vir}}] = -2a - 2b + g + \ell(\mu) + \ell(\nu) - |\mu| - |\nu|$. Similarly, for possibly disconnected stable relative morphisms, we can define $\overline{\mathcal{M}}^\bullet_{g,1}(X; D|\mu)$, which is also a separated, proper Deligne-Mumford stack with a perfect obstruction theory of dimension

$$\int_\beta c_1(TX) + (1-g)(\mathrm{dim}X - 3) + 1 + \sum_{k=1}^{r} (\ell(u_r) - |\mu_r|),$$

where r denote the number of irreducible components and g is the arithmetic genus in the sense that $2-2g=\chi=\sum_{k=1}^{r}(2-2g_k)$, g_k is the genus for the k-th component. For more details, see [LLZ07].

Example 4.1.2. If $X=\mathbf{P}^1\times\mathbf{P}^1$, $D^1=0\times\mathbf{P}^1$, $D^2=\infty\times\mathbf{P}^1$ we have

$$\Omega_X(\sum_{\alpha=1}^{2}\log D^{\alpha}_{(m^{\alpha})})\cong\Omega_{\mathbf{P}^1\times\mathbf{P}^1}\boxtimes p^*\Omega_{\mathbf{P}^1}(\log 0+\log\infty)=\Omega_{\mathbf{P}^1\times\mathbf{P}^1}\boxtimes\varnothing_{\mathbf{P}^1},$$

where p is the projection to the first factor. Similarly, if $X=T^*\mathbf{P}^1\times\mathbf{P}^1$, $D_1=T^*\mathbf{P}^1\times 0$, $D_2=T^*\mathbf{P}^1\times\infty$

$$\Omega_X(\sum_{\alpha=1}^{2}\log D^{\alpha}_{(m^{\alpha})})\cong\Omega_{\mathbf{P}^1\times\mathbf{P}^1}\boxtimes p^*\Omega_{\mathbf{P}^1}(\log 0+\log\infty)=\Omega_{T^*\mathbf{P}^1\times\mathbf{P}^1}\boxtimes\varnothing_{\mathbf{P}^1},$$

where p is the projection to the second factor.

Example 4.1.3. Let $\beta=dV+mH\in\mathrm{H}_2(\mathbf{P}^1\times\mathbf{P}^1,\mathbf{Z})$ be an effective curve class. The Euler characteristics of the restriction to β of $\Omega_{T^*\mathbf{P}^1\times\mathbf{P}^1}$ and $\Omega_{\mathbf{P}^1\times\mathbf{P}^1}$ are well defined. Let $\pi:T^*\mathbf{P}^1\to\mathbf{P}^1$ be the natural projection. Since we have the exact sequence

$$0\to\pi^*\Omega_{\mathbf{P}^1}\to\Omega_{T^*\mathbf{P}^1}\to\Omega_{T^*\mathbf{P}^1/\mathbf{P}^1}\to 0.$$

$\chi(\Omega_{T^*\mathbf{P}^1/\mathbf{P}^1})=\chi(\Omega_{T^*\mathbf{P}^1})-\chi(\pi^*\Omega_{\mathbf{P}^1})$ can be computed in the following way: $T^*\mathbf{P}^1$ can be constructed by gluing two copies of $\mathbf{C}^2$, let (x,u), (y,v) be the coordinates respectively. The transition function is given by $(x,u)\leftrightarrow(y,v)=(\frac{1}{x},\frac{1}{x^2}u)$. Thus $dy\wedge dv=\frac{-1}{x^2}dx\wedge(-2x^{-3}udx+x^{-2}du)=-\frac{1}{x^4}dx\wedge du$. We also know a general differential on $\mathbf{P}^1$ is given by $\frac{-1}{x^2}dx$. Restrict everything to β, we have $c_1(\Omega_{T^*\mathbf{P}^1\times\mathbf{P}^1}|_\beta)=(0)(dV+mH)=0$, $c_1(\Omega_{\mathbf{P}^1\times\mathbf{P}^1})=(-2H)(dV+mH)=-2d$. Thus the Euler characteristic of $\Omega_{T^*\mathbf{P}^1\times\mathbf{P}^1/\mathbf{P}^1\times\mathbf{P}^1}|_\beta$ is given by the Grothendieck-Riemann-Roch formula $(1-g)\mathrm{rk}+(-4d-(-2d))=-2d-g+1$

Example 4.1.4. Let $X=T^*\mathbf{P}^1\times\mathbf{P}^1$, $Y=\mathbf{P}^1\times\mathbf{P}^1$ and denote the natural projections as $T^*\mathbf{P}^1 \times \mathbf{P}^1 \xrightarrow{\pi} \mathbf{P}^1 \times \mathbf{P}^1 \xrightarrow{p} \mathbf{P}^1$. In the case of degeneration, we have

$$\Omega_{X[m_1,m_2]}(\sum_{\alpha=1}^{2} \log D^{\alpha}_{(m^\alpha)}) = \Omega_X \otimes \pi^* p^* \Omega_{\mathbf{P}^1[m_1,m_2]}(\log 0 + \log \infty)$$
$$\Omega_{Y[m_1,m_2]}(\sum_{\alpha=1}^{2} \log D^{\alpha}_{(m^\alpha)}) = \Omega_Y \otimes p^* \Omega_{\mathbf{P}^1[m_1,m_2]}(\log 0 + \log \infty).$$

Therefore, tensoring the relative differential sequence above with $\pi^* p^* \Omega_{\mathbf{P}^1[m_1,m_2]}(\log 0 + \log \infty)$, we get

$$0 \to \pi^* \Omega_{Y[m_1,m_2]}(\sum_{\alpha=1}^{2} \log D^{\alpha}_{(m^\alpha)}) \to \Omega_{X[m_1,m_2]}(\sum_{\alpha=1}^{2} \log D^{\alpha}_{(m^\alpha)})$$

$$\to \Omega_{T^*\mathbf{P}^1/\mathbf{P}^1} \otimes \pi^* p^* \Omega_{\mathbf{P}^1[m_1,m_2]}(\log 0 + \log \infty) \to 0.$$

The invariants we'll care about later is the equivariant Euler characteristic of the last term. Let $\beta = dV + mH$, it's straightforward to check that $\langle c_1(\pi^* p^* \Omega_{\mathbf{P}^1[m_1,m_2]}(\log 0 + \log \infty)), \beta \rangle = 0$. It's a line bundle, so it doesn't affect the rank, in other words, the Euler characteristic of the last term in the short exact sequence is exactly the same as in the previous example, that is $-2d - g + 1$.

4.1.2 The virtual fundamental class

We explain the tangent-obstruction spaces at a point in $\overline{\mathcal{M}}_{g,1}(X, \beta; \mu, \nu)$ following [LLZ07]. First we need some notations.

- $R = z + \sum_{i=1}^{\ell(\mu)} x_i + \sum_{i=1}^{\ell(\nu)} y_i$ is the divisor on C formed by those marked points.
- $n_k^\alpha = \#\{q | q \in f^{-1}(D_k^\alpha)\}$, the number of nodes in the fibre over the intersection divisors in the degeneration.
- $\mathrm{H}^0_{\mathrm{et}}(\mathbf{R}_k^{\alpha\bullet}) = \bigoplus_{q \in f^{-1}(\Delta(D^\alpha)_k)} T_q(f^{-1}(\Delta(D^\alpha)_k)) \otimes T_q^*(f^{-1}(\Delta(D^\alpha)_k)) \cong \mathbf{C}^{\oplus n_k^\alpha}$.

- L_k^α are line bundles on D_k^α defined by

$$L_k^\alpha = \begin{cases} N_{D^\alpha/X} \otimes N_{D_0^\alpha/\Delta(D^\alpha)_1} & k=0, \\ N_{D_k^\alpha/\Delta(D^\alpha)_k} \otimes N_{D_{k+1}^\alpha/\Delta(D^\alpha)_{k+1}} & 1 \le k \le m^\alpha - 1. \end{cases}$$

- $\mathrm{H}^1_{\mathrm{et}}(\mathbf{R}_k^{\alpha\bullet}) \cong \mathrm{H}^0(D_k^\alpha, L_k^\alpha)^{\oplus n_k^\alpha}/\mathrm{H}^0(D_k^\alpha, L_k^\alpha)$, where $\mathrm{H}^0(D_k^\alpha, L_k^\alpha)$ is viewed as a subgroup of $\mathrm{H}^0(D_k^\alpha, L_k^\alpha)^{\oplus n_k^\alpha}$ via the diagonal embedding.

Let $[f:(C,\{x_i\}_{i=1}^{\ell(\mu)},\{y_i\}_{i=1}^{\ell(\nu),z}) \to X[m_1,m_2]]$ be a relative stable morphism, the tangent space T^1 and the obstruction space T^2 is given by the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^0(\Omega_C(R),\mathcal{O}_C) \arrow[r] & \mathrm{H}^0(\mathbf{D}^\bullet) \arrow[r] & T^1 \arrow[dll, out=-10, in=170] \\ & \mathrm{Ext}^1(\Omega_C(R),\mathcal{O}_C) \arrow[r] & \mathrm{H}^1(\mathbf{D}^\bullet) \arrow[r] & T^2 \arrow[r] & 0, \end{tikzcd}$$

where the terms $\mathrm{H}^0(\mathbf{D}^\bullet)$ and $\mathrm{H}^1(\mathbf{D}^\bullet)$ can be computed from the following long exact sequence:

$$\mathrm{H}^0(C, f^*(\Omega_{X[m^1,m^2]}(\sum_{\alpha=1}^{2}\log D_{m^\alpha}^\alpha))^\vee) \to \mathrm{H}^0(\mathbf{D}^\bullet) \to \bigoplus_{\alpha=1}^{2}\bigoplus_{k=0}^{m^\alpha-1}\mathrm{H}^0_{\mathrm{et}}(\mathbf{R}_k^{\alpha\bullet})$$

$$\to \mathrm{H}^1(C, f^*(\Omega_{X[m^1,m^2]}(\sum_{\alpha=1}^{2}\log D_{m^\alpha}^\alpha))^\vee) \to \mathrm{H}^1(\mathbf{D}^\bullet) \to \bigoplus_{\alpha=1}^{2}\bigoplus_{k=0}^{m^\alpha-1}\mathrm{H}^1_{\mathrm{et}}(\mathbf{R}_k^{\alpha\bullet}).$$

All terms out of the range shown above are zeros. Specially, in either exact sequence, the first arrow is an injection, the last arrow is a surjection. We explain the six terms that are important for our computation of the perfect obstruction theory by

- $B_1 = \mathrm{Ext}^0(\Omega_C(R),\mathcal{O}_C)$ is the space of infinitesimal automorphisms of the domain curve (C,R).

- $B_2 = \mathrm{H}^0(C, f'^*(\Omega_{T^*\mathbf{P}^1} \boxtimes \mathcal{O}_{\mathbf{P}^1}))$ is the space of infinitesimal deformations of the map with the domain curve (C,R) fixed.

- $B_4 = \mathrm{Ext}^1(\Omega_C(D), \Omega_C)$ is the space of infinitesimal deformation of the domain curve (C, R).

- $B_5 = \mathrm{H}^1(C, f^*(\Omega^\vee_{T^*\mathbf{P}^1} \boxtimes \mathcal{O}_{\mathbf{P}^1}))$ is the space of obstructions to the deformations of the map with the domain curve (C, R) fixed.

- $B_3 = \displaystyle\bigoplus_{\alpha=1}^{2} \bigoplus_{\ell=0}^{m^\alpha-1} \mathrm{H}^0_{\mathrm{et}}(\mathbf{R}^{\alpha\bullet}_\ell)$ corresponds to the smoothing of the nodes in $f^{-1}(D^\alpha_k)$.

- $B_6 = \displaystyle\bigoplus_{\alpha=1}^{2} \bigoplus_{\ell=0}^{m^\alpha-1} \mathrm{H}^1_{\mathrm{et}}(\mathbf{R}^{\alpha\bullet}_\ell)$ corresponds to the obstructions to the smoothing of the nodes in $f^{-1}(D^\alpha_k)$.

The virtual fundamental class $\overline{\mathcal{M}}_{g,1}(X, \beta; \mu, \nu)$ is defined to be the one associated to the perfect obstruction theory $T_1 - T_2$.

4.1.3 Localization

We assume all relative divisors are T-equivariant. Thus T acts on $\Delta(D^\alpha)$ and $X[m_1, m_2]$ naturally. T acts on the moduli spaces $\overline{\mathcal{M}}_{g,1}(X; D; \beta|\mu, \nu)$ by moving the image. The T-fixed point of the moduli space is a disjoint union of combinatorial configurations parametrized by certain types of graphs, we denote one such type of fixed points by $\mathcal{F}_\Gamma$. Let $p \in \mathcal{F}_\Gamma$ and consider the two exact sequences defining the tangent-obstruction spaces T^1, T^2 at this point. Let $T^{i,f}$ and $T^{i,m}$ be the submodules of trivial T-weight and nontrivial T-weights respectively. Then $T^{1,f} - T^{2,f}$ defines a perfect obstruction theory on $\mathcal{F}_\Gamma$ and $T^{1,m} - T^{2,m}$ define the virtual normal bundle $N^{\mathrm{vir}}_{\mathcal{F}_\Gamma}$ of $\mathcal{F}_\Gamma$ in the corresponding moduli space of relative stable morphisms. More precisely, we have

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})} = \frac{e_T(T^{2,m})}{e_T(T^{1,m})} = \frac{e_T(B_1^m)e_T(B_5^m)e_T(B_6^m)}{e_T(B_2^m)e_T(B_4^m)}.$$

The virtual localization theorem in [GP99, GV05] is applicable in our case. That is, the T-equivariant virtual fundamental class is a summation of pushforward of virtual fundamental classes on the fixed loci after localization. In the case of a compact target X, it reads

$$[\overline{\mathcal{M}}_X]^{\mathrm{vir}}_T = \sum_{\Gamma \in G_{g,1}} (i_{\mathcal{F}_\Gamma})_* (\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})}).$$

In the case of a noncompact target (e.g. $X = T^*\mathbf{P}^1 \times \mathbf{P}^1$), the ordinary virtual fundamental class vanishes, we have to use the reduced virtual fundamental class which shows up as the $\hbar$-coefficient of the construction of the T-equivariant virtual fundamental class. The localization theorem in this case is given by

$$\hbar \cdot [\overline{\mathcal{M}}_X]^{\mathrm{red}}_T = \sum_{\Gamma \in G_{g,1}} (i_{\mathcal{F}_\Gamma})_* (\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})}).$$

As we mention above, $[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T$ is the virtual fundamental class from the perfect obstruction theory $T^{1,f} - T^{2,f}$, the equivariant Euler class of the virtual normal bundle $e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})$ is given by $T_{1,m} - T_{2,m}$. Similarly for possibly disconnected invariants, we have

$$[\overline{\mathcal{M}}^\bullet_X]^{\mathrm{vir}}_T = \sum_{\Gamma \in G^\bullet_{g,1}} (i_{\mathcal{F}_\Gamma})_* (\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})}).$$

4.1.4 Reduced classes for absolute Gromov-Witten theory

In this section, we briefly recall the construction of the reduced virtual fundamental class for $\overline{M}_{g,\beta}(X = T^*\mathbf{P}^1, \beta)$, via this we can understand the vanishing of naive nonreduced invariants and the fact that non-equivariant reduced invariants are encoded in the $(t_1 + t_2)$-linear part of the equivariant invariants. Similar phenomenon appears for relative Gromov-Witten invariants.

For a fixed nodal curve C of genus g as the domain, let $M_C(X, \beta)$ denote the moduli space of stable morphisms form C to X of degree $\beta \neq 0$. The ordinary obstruction theory for $M_C(X, \beta)$ is given by

$$R\pi_*(\mathrm{ev}^* T_X)^\vee \to L_{M_C}$$

where L_{M_C} is the cotangent complex of $M_C(X, \beta)$ and

$$\mathrm{ev} : C \times M_C(X, \beta) \to X$$

$$\pi : C \times M_C(X, \beta) \to M_C(X, \beta).$$

Denote the holomorphic symplectic form on X by ω, ω is induced from the standard 2-form $dx \wedge dy$ on $\mathbf{C}^2$. As the discussion in section 3.2.1, the torus weight of ω is $-(t_1 + t_2)$. Let Ω_π and ω_π be the sheaf of relative differentials and the relative dualizing sheaf. We have the canonical map

$$\mathrm{ev}^*(\Omega_X) \to \Omega_\pi \to \omega_\pi$$

and the symplectic pairing

$$T_X \to \Omega_X \otimes (\mathbf{C}\omega)^\vee$$

together gives a bundle map

$$\mathrm{ev}^*(T_X) \to \omega_\pi \otimes (\mathbf{C}\omega)^\vee.$$

This induces a map of complexes

$$R\pi_*(\omega_\pi)^\vee \otimes (\mathbf{C}\omega)^\vee \to R\pi_*(\mathrm{ev}^*(T_X)^\vee)$$

The truncation

$$\tau_{\leq -1} R\pi_*(\omega_\pi)^\vee \otimes (\mathbf{C}\omega)^\vee \to R\pi_*(\mathrm{ev}^*(T_X)^\vee)$$

is a trivial line bundle with nontrivial equivariant weight $-(t_1 + t_2)$. [Mau09], this is precisely the modified obstruction theory in the definition of the reduced virtual fundamental class. Moreover, since all maps are compatible with the torus action, this also gives the equivariant reduced virtual fundamental class. The reduced equivariant perfect obstruction theory is obtained by varying the domain curve. In summary, the new obstruction theory differs from the standard one only by the trivial line bundle $(\mathbf{C}\omega)^\vee$ which have weight t_1+t_2. Thus the dimension of the virtual fundamental class is $1+m+(g-1)$. The standard virtual class is divisible by t_1+t_2, i.e.

$$[\overline{M}_{g,m}(X,\beta)]^{\text{vir}} = c_1((\mathbf{C}\omega)^\vee)[\overline{M}_{g,m}(X,\beta)]^{\text{red}} = (t_1+t_2)[\overline{M}_{g,m}(X,\beta)]^{\text{red}}.$$

In other words, the standard equivariant Gromov-Witten invariants of X with $\beta \neq 0$ are all divisible by (t_1+t_2). Nonequivariant reduced invariants are encoded in the coefficient of (t_1+t_2) of the standard equivariant theory. If $X = T^*\mathbf{P}^1$, we recall the following lemma

Proposition 4.1.5 ([Mau09]). *For $d>0$, we have an identification of moduli spaces $\overline{M}_{g,m}(X,dE) = \overline{M}_{g,m}(\mathbf{P}^1,d)$, moreover we have a linear relation between the virtual fundamental classes*

$$[\overline{M}_{g,m}(X,dE)]^{\text{red}} = c_{2d+g-2}(R\pi_*\text{ev}^*\mathcal{O}(-2))[\overline{M}_{g,m}(\mathbf{P}^1,d)]^{\text{vir}}$$

Proof. See [Mau09, Corollary 2.2]. □

4.2 A Fock space approach to Severi degrees

In this section, we study the Severi degree problem on surfaces via Gromov-Witten theory, and present a Fock space approach to these invariants following [CP17b].

4.2.1 Severi degrees as Gromov-Witten invariants

The classical Severi degree problem studies the number of algebraic curves in $\mathbf{P}^2$ of geometric genus g and degree d pass through $3d+g-1$ general points. This problem can be reformulated for any nonsingular projective surface X via Gromov-Witten theory. Let $\beta \in \mathrm{H}_2(X, \mathbf{Z})$ be en effective curve class. Given cohomology classes $\gamma_1, \ldots, \gamma_n$, the n-pointed genus g Gromov-Witten invariant is defined by

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{M}^{\bullet}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{k=1}^{n} \mathrm{ev}_k^*(\gamma_k),$$

where $\mathrm{ev}_k : \overline{M}_{g,n}(X, \beta) \to X$ is the evaluation map associated to the k-th marked point and the integration is against the virtual fundamental class $\overline{M}^{\bullet}_{g,n}(X, \beta)$ of the moduli space of genus g **possibly disconnected** stable morphism along the pushforward map

$$\mathrm{H}^*(\overline{M}^{\bullet}_{g,n}(X, \beta)) \to \mathrm{H}^*(\mathrm{pt}).$$

The virtual fundamental class has dimension

$$\dim[\overline{M}^{\bullet}_{g,n}(X, \beta)]^{\mathrm{vir}} = \int_\beta c_1(X) + (\dim X - 3)(1 - g) + n.$$

Thus when X is a surface, the virtual dim is $\int_\beta c_1(X) + g - 1 + n$, inserting a point is a codimention 2 condition, thus if $n = \int_\beta c_1(X) + g - 1$, we expect to get numerical invariants. The Severi invariants is defined to be

$$N^{\bullet}_{g,\beta} := \int_{[\overline{M}^{\bullet}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{k=1}^{\int_\beta c_1(X)+g-1} \mathrm{ev}^*(\mathrm{pt}).$$

If $n < 0$, $N^{\bullet}_{g,\beta}$ vanishes by definition.

For the simple surface $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $\beta = d_1 H + d_2 V$, where H denote the horizontal divisor class and V denote the vertical divisor class. Then$\int_\beta c_1(X) + g - 1 = 2d_1 + 2d_2 + g - 1$. The Severi degree invariants can be encoded in the following generating function

$$Z^{\mathbf{P}^1\times\mathbf{P}^1} = 1 + \sum_{g\in\mathbf{Z}} u^{g-1} \sum_{(d_1,d_2)\neq(0,0)} N^{\bullet}_{g,(d_1,d_2)} \frac{t^{2d_1+2d_2+g-1}}{(2d_1+2d_2+g-1)!} H^{d_1}V^{d_2}.$$

4.2.2 Degeneration: absolute to relative

The absolute Gromov-Witten invariants can be computed by relatives one via degeneration [IP03, LR01, Li02]. To compute the disconnected Severi degrees $N^{\bullet}_{g,(d_1,d_2)}$ counting genus g curves passing $n = 2d_2 + 2d_2 + g - 1$ points on $\mathbf{P}^1 \times \mathbf{P}^1$, we can degenerate the horizontally to get $C \times \mathbf{P}^1$, where C is a chain of $n+2$ rational curves E_k, such that $E_k \cap E_\ell = \mathrm{pt}$ if and only $|k-\ell| = 1$. In other words, we get $n+1$ components of $\mathbf{P}^1 \times \mathbf{P}^1$ intersect adjacently, denote them by X_k for $0 \leq k \leq n+1$. According to section 4.1, for X_k, $1 \leq k \leq n$, we can consider the moduli space $\overline{\mathcal{M}}_k := \overline{\mathcal{M}}^{\bullet}_{g_k,1}(\mathbf{P}^1 \times \mathbf{P}^1, (d_1, d_2^k)|\mu_k^*, \mu_{k-1})$ of relative stable morphisms, $\overline{\mathcal{M}}_0$ only has relative condition over the ∞-divisor, $\overline{\mathcal{M}}_{n+1}$ only has relative condition over the 0-divisor. Note that μ is actually a cohomology weighted partition (i.e. of the form $\mu = |\mu,\nu\rangle$ and $|\mu,\nu\rangle^* = |\nu,\mu\rangle$). Denote $m(\mu) = \prod_{k}^{\ell(\mu)} \mu_k \prod_{k=1}^{\ell(\nu)} \nu_k$. Then by degeneration we have

Proposition 4.2.1 (Degeneration). *The generating function of Severi degrees on $\mathbf{P}^1 \times \mathbf{P}^1$ can be computed by relative invariants, i.e*

$$\begin{aligned} Z^{\mathbf{P}^1\times\mathbf{P}^1} = 1 &+ \sum_{g,d_1,d_2} \left(H^{d_1}V^{d_2}u^{g-1}\frac{t^n}{n!}\right) \times \\ &\sum_{d_2^k,\mu_k} \left(\int_{\overline{\mathcal{M}}_0} \mathbf{1}\right) \prod_{k=1}^{n} \left[\left(\int_{\overline{\mathcal{M}}_k} \mathrm{ev}^*(\mathrm{pt})\right) \frac{m(\mu_{k-1})}{\mathrm{Aut}(\mu_{k-1})}\right] \left(\int_{\overline{\mathcal{M}}_{n+1}} \mathbf{1}\right) \frac{m(\mu_n)}{\mathrm{Aut}(\mu_n)}. \end{aligned}$$

The summation is over all degree splittings $\sum_k d_2^k = d_2$, relative conditions $\{\mu_k\}_{k=0}^n$ and all compatible graph types which combined forms a genus g curve, possibly disconnected. From the degeneration, the most important invariants for the computation is $\int_{\overline{\mathcal{M}}_k} \mathrm{ev}^*(\mathrm{pt})$. The disconnected invariants can be derived from connected invariants by taking products. Denote $\overline{\mathcal{M}} := \overline{\mathcal{M}}^{\circ}_{g_k,1}(\mathbf{P}^1 \times \mathbf{P}^1, (a,b)|\mu, \mu')$, where $\mu = |\mu, \nu\rangle$. From [CP17b], the only nonzero contributions to the degeneration formula can be described by

Proposition 4.2.2. *The integral* $\int_{\overline{\mathcal{M}}} \mathrm{ev}^*(\mathrm{pt}) = 1$ *if*

Type A: $g = 0,\ b = 0,\ \ell(\mu) = \ell(\mu') = 1.$

Type B: $g = 0,\ b = 1,\ \ell(\mu) = \ell(\mu') = 0,\ \ell(\mu), \ell(\mu') \neq 0.$

Type C: $g = 0,\ b = 0,\ \ell(\mu) = \ell(\mu') = 0,\ \ell(\nu) = \ell(\nu') = 0.$

All other cases vanish.

Proof. See [CP17b, Page 10, 11]. □

4.2.3 Fock Space formalism

To start, we know that $\mathrm{H}^\bullet(\mathbf{P}^1, \mathbf{Q}) = \mathbf{Q} \cdot \mathbf{1} \oplus \mathbf{Q} \cdot \mathrm{pt}$. Consider the Heisenberg algebra $\mathcal{H}$ generated over the field $\mathbf{Q}$ by a central element c, $\{\alpha_k(\gamma)\}$ for $\gamma \in \mathrm{H}^\bullet(\mathbf{P}^1)$ and $k \in \mathbf{Z} \setminus \{0\}$. The Lie algebra structure of $\mathcal{H}$ is given by

$$[\alpha_k(\gamma_1), \alpha_\ell(\gamma_2)] = -k\delta_{k+l}\langle \gamma_1, \gamma_2 \rangle c$$
$$[c, \alpha_k(\gamma)] = 0.$$

Note that the only nonvanishing Lie bracket is essentailly $[\alpha_k(\mathrm{pt}), \alpha_{-k}(\mathbf{1})] = kc$.

The Fock space $\mathcal{F}$ is freely generated over $\mathbf{Q}$ by the commutation relations of $\alpha_{-k}(\gamma)$ on the vacuum vector $v_\emptyset$. $\mathcal{F}$ has a natural grading induced by defining $\deg(v_\emptyset) = 0$ and $\deg(\alpha_{-k}(\gamma)) = k$.

A natural basis of $\mathcal{F}$ can be described by cohomology-weighted partitions $\overrightarrow{\mu} = \{(\mu_i, \gamma_i)\}_{i=1}^{\ell(\mu)}$, where $\{\mu_i\}_{i=1}^{\ell(\mu)}$ is a partition and $\gamma_i = \mathbf{1}$ or pt. A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu,\nu\rangle = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \prod_{i=1}^{\ell(\mu)} \alpha_{-k}(\mathbf{1}) \prod_{j=1}^{\ell(\nu)} \alpha_{-\nu_j}(\mathrm{pt}) v_{\emptyset}$$

where

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} \mu_i.$$

We also denote this basis element by $\overrightarrow{\mu}$ if it's clear in the context. We can define a nondegenerate pairing in this basis is given by

$$\langle \overrightarrow{\mu}, \overrightarrow{\nu} \rangle = \frac{u^{-\ell(\overrightarrow{\mu})}}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \delta_{\overrightarrow{\mu}\,\overrightarrow{\nu}}.$$

Note that if we write $|\overrightarrow{\mu}\rangle = |\mu,\nu\rangle$, then $\ell(\overrightarrow{\mu}) = \ell(\mu) + \ell(\nu)$ and $\delta_{\overrightarrow{\mu}\,\overrightarrow{\nu}} = \delta_{\mu\nu'}\delta_{\mu'\nu}$. The cohomological degree of $|\overrightarrow{\mu},\nu\rangle$ is $|\mu| + |\nu| - \ell(\mu)$.

Now we recall the main results in [CP17b]. Consider the operator M_S on the Fock space $\mathcal{F}$

$$M_S(u,Q) = \sum_{k>0} \alpha_{-k}(\mathrm{pt})\alpha_k(\mathrm{pt}) + Q \sum_{|\mu|=|\nu|>0} u^{\ell(\mu)-1} \alpha_{-\mu}(\mathbf{1})\alpha_{\nu}(\mathbf{1}).$$

We also denote $M_S^{\mathrm{cl}}(u,Q) = \sum_{k>0} \alpha_{-k}(\mathrm{pt})\alpha_k(\mathrm{pt})$, and

$$M_S^{\mathrm{q}}(u,Q) = Q \sum_{|\mu|=|\nu|>0} u^{\ell(\mu)-1} \alpha_{-\mu}(\mathbf{1})\alpha_{\nu}(\mathbf{1}).$$

Remark 4.2.3. We want to explain two convention differences that will be important for the actual matching in chapter 5. First is that the genus in $Z^{\mathbf{P}^1\times\mathbf{P}^1}$ is parametrized by u^{g-1}, in the quantum cohomology of Hilbert schemes, it corresponds to u^{2g-2} in the expansion of the q-term. Secondly, in the definition of the nondegenerate pairing in the Fock space, we have a term $u^{\ell(\mu)}$, this goes into the definition of $M_S(u,Q)$. In other words, if we use the convention that is compatible with the quantum cohomology section, $M_S^{\mathrm{q}}(u,Q)$ would be $Qu^{-1}\sum_{|\mu|=|\nu|>0}\alpha_{-\mu}\alpha_{\nu}$. This form is the one we'll going to match with the quantum multiplication operator in chapter 5.

Let $v=\sum_{d_1\geq 0}|(1)^{d_1},\emptyset\rangle=\sum_{d_1\geq 0}\alpha_{-1}^{d_1}(\mathrm{pt})$. Cooper and Pandahripande prove the following theorem

Theorem 4.2.4 ([CP17a]). *The partition function for Severi degrees of* $\mathbf{P}^1\times\mathbf{P}^1$ *is*

$$Z^{\mathbf{P}^1\times\mathbf{P}^1}=e^{\frac{tQ_2}{u}}\langle v|Q_1^{|\cdot|}M_S(u,Q_2)|v\rangle$$

Chapter 5

Proof of the matching

In this chapter, we study the relative Gromov-Witten theory on $T^*\mathbf{P}^1 \times \mathbf{P}^1$ and show in section 5.1 that certain equivariant limits give the relative invariants on $\mathbf{P}^1 \times \mathbf{P}^1$. In section 5.2, we match the classical multiplication operator in [MO09] with $M_S^{\mathrm{cl}}(u, Q)$ in section 4.2.3. In section 5.3, by formulating the quantum multiplications on $\mathrm{Hilb}(T^*\mathbf{P}^1)$ [MO09] as vertex operators and computing the product expansion, we demonstrate how to get the insertion operator computed by Yaim Cooper and Rahul Pandharipande in the equivariant limits. Lastly in section 5.4, we apply a result in [AO17] to write the eigenvalues of the quantum multiplication operator in terms of Bethe equations.

5.1 Matching the invariants

In this subsection, by comparing all terms appear in the localization formula, we make precise the intuitively obvious observation that the only difference between the relative Gromov-Witten theory of $T^*\mathbf{P}^1 \times \mathbf{P}^1$ and that of $\mathbf{P}^1 \times \mathbf{P}^1$ come from the deformation in the fibre direction of the natural projection $T^*\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$. Let $T = (\mathbf{C}^\times)^2$ acts on the first component of $T^*\mathbf{P}^1 \times \mathbf{P}^1$ as in section 3.2.1 and arbitrarily on the second component. We can prove the following analogue of proposition 4.1.5.

Proposition 5.1.1. *All relative Gromov-Witten invariants of* $\mathbf{P}^1 \times \mathbf{P}^1$ *appear as the coefficients of* $(t_1+t_2)^{2d+g-1}$*-terms of the equivariant relative Gromov-Witten invariants of* $T^*\mathbf{P}^1 \times \mathbf{P}^1$.

Proof. The relative divisors are naturally T-equivariant. Thus T acts on $\Delta(D^\alpha)$ and $X[m_1, m_2], Y[m_1, m_2]$ naturally. T acts on the moduli spaces $\overline{\mathcal{M}}_{g,1}(X; D; \beta|\mu, \nu)$ and $\overline{\mathcal{M}}_{g,1}(X; D; \beta|\mu, \nu)$ by moving the image. From now on, we denote two moduli spaces by $\overline{\mathcal{M}}_X$ and $\overline{\mathcal{M}}_Y$ respectively. The T-fixed point of both moduli spaces are the same—a disjoint union of combinatorial configurations parametrized by certain types of graphs, we denote one such type of fixed points by $\mathcal{F}_\Gamma$. Let $p \in \mathcal{F}_\Gamma$ and consider the two exact sequences defining the tangent-obstruction spaces T^1, T^2 at this point. Then every term in the exact sequences can be view as a T-module.

$$B_1 = \mathrm{Ext}^0(\Omega_C(R), \mathcal{O}_C),\ B_2 = \mathrm{H}^0(C, {f'}^*(\Omega_{T^*\mathbf{P}^1} \boxtimes \mathcal{O}_{\mathbf{P}^1}))$$
$$B_3 = \bigoplus_{\alpha=1}^{2} \bigoplus_{\ell=0}^{m^\alpha-1} \mathrm{H}^0_{\mathrm{et}}(\mathbf{R}_\ell^{\alpha\bullet}),\ B_4 = \mathrm{Ext}^1(\Omega_C(D), \Omega_C)$$
$$B_5 = \mathrm{H}^1(C, f^*(\Omega^\vee_{T^*\mathbf{P}^1} \boxtimes \mathcal{O}_{\mathbf{P}^1})),\ B_6 = \bigoplus_{\alpha=1}^{2} \bigoplus_{\ell=0}^{m^\alpha-1} \mathrm{H}^1_{\mathrm{et}}(\mathbf{R}_\ell^{\alpha\bullet})$$

Let $T^{i,f}$ and $T^{i,m}$ be the submodules of trivial T-weight and nontrivial T-weights respectively. Then $T^{1,f}-T^{2,f}$ defines a perfect obstruction theory on $\mathcal{F}_\Gamma$ and $T^{1,m}-T^{2,m}$ define the virtual normal bundle $N^{\mathrm{vir}}_{\mathcal{F}_\Gamma}$ of $\mathcal{F}_\Gamma$ in the corresponding moduli space of relative stable morphisms. More precisely, we have

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})} = \frac{e_T(T^{2,m})}{e_T(T^{1,m})} = \frac{e_T(B_1^m)e_T(B_5^m)e_T(B_6^m)}{e_T(B_2^m)e_T(B_4^m)}.$$

T. Graber and R. Pandharpande prove a localization theorem in [GP99] which is applicable in our case. That is, the T-equivariant virtual fundamental class is a summation of pushforward of virtual fundamental classes on the fixed loci after localization. In the case of $Y = \mathbf{P}^1 \times \mathbf{P}^1$, it reads

$$[\overline{\mathcal{M}}_Y]^{\mathrm{vir}}_T = \sum_{\Gamma \in G_{g,1}} (i_{\mathcal{F}_\Gamma})_* \left(\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})}\right).$$

In the case of $X = T^*\mathbf{P}^1 \times \mathbf{P}^1$, the ordinary virtual fundamental class vanishes, we have to use the reduced virtual fundamental class which shows up as the $\hbar$-coefficient of the construction of the T-equivariant virtual fundamental class. The localization theorem in this case is given by

$$\hbar \cdot [\overline{\mathcal{M}}_X]^{\mathrm{red}}_T = \sum_{\Gamma \in G_{g,1}} (i_{\mathcal{F}_\Gamma})_* \left(\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})}\right).$$

As we mention above, $[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T$ is the virtual fundamental class from the perfect obstruction theory $T^{1,f} - T^{2,f}$, the equivariant Euler class of the virtual normal bundle $e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})$ is given by $T_{1,m} - T_{2,m}$. The relative Gromov-Witten invariants on $\mathbf{P}^1 \times \mathbf{P}^1$ needed for the computation of Severi degree in [CP17b] is the point insertion $\int_{\overline{\mathcal{M}}_Y} \mathrm{ev}_Y^*(\mathrm{pt})$. Consider the following diagram

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_X \arrow[r, "\mathrm{ev}_X"] & X \arrow[d, "\pi"] & \\
\mathcal{M}_\Gamma \arrow[ur, hook] \arrow[dr, hook] & & & \mathrm{pt} \arrow[ul, hook] \arrow[dl, hook] \\
 & \overline{\mathcal{M}}_Y \arrow[uu, "i"] \arrow[r, "\mathrm{ev}_Y"] & Y &
\end{tikzcd}$$

.

Apply the localization theorem we have

$$\begin{aligned}\int_{\overline{\mathcal{M}}_Y} \mathrm{ev}_Y^*(\mathrm{pt}) &= \sum_{\Gamma\in G_{g,1}} \int_{\mathcal{M}_\Gamma} \frac{i_Y^*\mathrm{ev}_Y^*(\mathrm{pt})}{e_T(N_{\Gamma/Y}^{\mathrm{vir}})} \\ &= \sum_{\Gamma\in G_{g,1}} \int_{\mathcal{M}_\Gamma} \frac{i_X^*\mathrm{ev}_X^*(\iota_*\omega)}{e_T(N_{\Gamma/Y}^{\mathrm{vir}})} \\ &= \sum_{\Gamma\in G_{g,1}} \int_{\mathcal{M}_\Gamma} \frac{i_X^*\mathrm{ev}_X^*(\iota_*\omega)}{e_T(N_{\Gamma/X}^{\mathrm{vir}})} \frac{e_T(N_{\Gamma/X}^{\mathrm{vir}})}{e_T(N_{\Gamma/Y}^{\mathrm{vir}})}.\end{aligned}$$

The second identity is because when restricted to the fixed loci, the constrain of inserting a point of the base is the same as the curve to pass the intersection of $\iota\omega$ with the base $\mathbf{P}^1$. To compare the Euler class of the two virtual normal bundles, we go back to the exact sequences defining them. $B_1 = \mathrm{Ext}^0(\Omega_C(R), \mathcal{O}_C)$ and $B_4 = \mathrm{Ext}^1(\Omega_C(R), \mathcal{O}_C)$ only depend on the source curve, they're the same in both cases. For $B_6 = \bigoplus_{\alpha=1}^{2} \bigoplus_{\ell=0}^{m^\alpha-1} \mathrm{H}^1_{\mathrm{et}}(\mathbf{R}_\ell^{\alpha\bullet})$, where $\mathrm{H}^1_{\mathrm{et}}(\mathbf{R}_\ell^{\alpha\bullet}) = \mathrm{H}^0(D_{(k)}^{\alpha,L_k^\alpha})^{n_k^\alpha}/\mathrm{H}^0(D_k^\alpha, L_k^\alpha)$. In our case, L_k^α viewed as a line bundle on D_k^α is just the trivial bundle, the global functions are constants, which are not affected by the torus action, thus $B_6^m = 0$ and $e_T(B_6^m) = 1$ in both cases. By the formula for the virtual normal bundle above, the difference boils down to $\frac{e_T(B_5^m)}{e_T(B_2^m)}$, which is just the T-equivariant Euler characteristics of $f^*(\Omega_{X[m_1,m_2]}(\sum_{\alpha=1}^2 \log D_{(m^\alpha)}^\alpha)^\vee$. Now, we can apply Grothendieck-Riemman-Roch theorem to compute $\frac{e_T(N_{\Gamma/X}^{\mathrm{vir}})}{e_T(N_{\Gamma/Y}^{\mathrm{vir}})}$.

$$\begin{aligned}\frac{e_T(N_{\Gamma/X}^{\mathrm{vir}})}{e_T(N_{\Gamma/Y}^{\mathrm{vir}})} &= \frac{e_T(C, f^*(\Omega_{X[m_1,m_2]}(\sum_{\alpha=1}^2 \log D_{(m^\alpha)}^\alpha)))^\vee}{e_T(C, f^*(\Omega_{Y[m_1,m_2]}(\sum_{\alpha=1}^2 \log D_{(m^\alpha)}^\alpha)))^\vee} \\ &= (t_1+t_2)^{-2d-g+1}\end{aligned}$$

$\mathbf{P}^1 \times \mathbf{P}^1$ is compact, thus all the relative invariants are numbers. By the localization theorem, We also have $\langle\mu|\mathrm{pt}|\nu\rangle_Y = (t_1+t_2)^{-2d-g+1}\langle\mu|\iota_*\omega|\nu\rangle_X$. Therefore, all relative invariants of $\mathbf{P}^1\times\mathbf{P}^1$ appear as the coefficients of $(t_1+t_2)^{2d+g-1}$-terms of the equivariant relative invariants of $T^*\mathbf{P}^1 \times \mathbf{P}^1$. $\square$

Next we specialize the relative Gromov-Witten invariants computed by Maulik [Mau09] to the case of $T^*\mathbf{P}^1 \times \mathbf{P}^1$ and illustrate the invariant level comparison in proposition 5.1.1.

Proposition 5.1.2 ([MO09])**.** *If μ, ν are partitions of $m > 0$ and the cohomology classes labelling μ,ν are divisors, then we have*

$$u^{\ell(\mu)+\ell(\nu)-1} Z^{\circ}_{\beta\neq 0}(T^*\mathbf{P} \times \mathbf{P}^1)_{\overrightarrow{\mu},(2),\overrightarrow{\nu}} = \frac{d}{du}\Theta^{\circ}(\overrightarrow{\mu},\overrightarrow{\nu})$$
$$u^{\ell(\mu)+\ell(\nu)} Z^{\circ}_{\beta\neq 0}(T^*\mathbf{P} \times \mathbf{P}^1)_{\overrightarrow{\mu},(1,\omega),\overrightarrow{\nu}} = s\frac{d}{ds}\Theta^{\circ}(\overrightarrow{\mu},\overrightarrow{\nu}).$$

where $\Theta^{\circ}(\overrightarrow{\mu},\overrightarrow{\nu})$ above is given by

$$\Theta^{\circ}(\overrightarrow{\mu},\overrightarrow{\nu}) = \frac{t_1+t_2}{|\mathrm{Aut}(\overrightarrow{\mu})||\mathrm{Aut}(\overrightarrow{\nu})|}\sum_{d=1}^{\infty}(du)^{\ell(\mu)+\ell(\nu)-2}\frac{\prod_{k=1}^{\ell(\mu)} S(d\mu_k u)\prod_{k=1}^{\ell(\nu)} S(d\nu_k u)}{dS(du)^2}s^d.$$

Otherwise, these invariants vanish.

Recall that $S(u) = \frac{\sin(\frac{u}{2})}{\frac{u}{2}}$ and $S(du)^2$ in the denominator means $(S(du))^2$. Now we compute examples of the relative Gromov-Witten invariants of $\mathbf{P}^1 \times \mathbf{P}^1$ by taking the described limits of the formulae in the case of $T^*\mathbf{P}^1 \times \mathbf{P}^1$.

Example 5.1.3 ($\mu = \nu = (1)$)**.** When μ, ν are given by the trivial partition of 1, we have $Z_{dE\neq 0,H} = (t_1+t_2)u^{-2}\sum_{d=1}^{\infty} s^d$. By our discussion above, only the $u^{2g-2}s^d(t_1+t_2)^{2d+g-1}$-coefficient contributes to the $\mathbf{P}^1 \times \mathbf{P}^1$ invariants. Since the exponent of t_1+t_2 must be 1, it forces $g = 0$ and $d = 1$. That is $Z^{\circ}(\mathbf{P}^1 \times \mathbf{P}^1)_{dV\neq 0,H} = [u^{-2}s^1(t_1+t_2)]Z^{\circ}_{dE\neq 0,H} = 1$.

Example 5.1.4 ($\mu = \nu = (n)$)**.** This relates to the Type-A curve counting in [CP17b].

In these cases, we have $|\mathrm{Aut}(\mu)| = |\mathrm{Aut}(\nu)| = 1$, $\ell(\mu)+\ell(\nu)-2=0$. We thus have

$$\Theta((n),(n)) = (t_1+t_2)\sum_{d=1}^{\infty}\frac{S(dnu)S(dnu)}{dS^2(du)}s^d = (t_1+t_2)\sum_{d=1}^{\infty}\frac{s^d}{d}(\frac{S(dnu)}{S(du)})^2.$$

Proposition 5.1.2 specializes to

$$Z_{dE\neq 0,nH}(T^*\mathbf{P}^1\times\mathbf{P}^1)_{(n),\omega,(n)} = u^{-2}(t_1+t_2)\sum_{d=1}^{\infty}s^d(\frac{S(dnu)}{S(du)})^2.$$

Apply the same argument as above and notice that $\dfrac{\sin(nu)}{\sin(u)} = 1 - \dfrac{(ndu)^2}{6} + \dfrac{(ndu)^4}{120} +$ higher order terms. We recover the Type-B computation in [CP17b], that is

$$Z^\circ(\mathbf{P}^1\times\mathbf{P}^1)_{dV\neq 0,nH} = [u^{-2}s^1(t_1+t_2)]Z^\circ_{dE\neq 0,nH} = 1.$$

To go from the invariants to the operators, we have the following

Proposition 5.1.5. *If μ, ν are partitions of $m>0$ and the cohomology classes labelling μ,ν are divisors, we have*

$$\langle\mu|\tau_1[F]\nu\rangle^\circ_{g,\beta} = \langle\mu,(2),\nu\rangle^\circ_{g,\beta}$$
$$\langle\nu|\tau_0(\iota_*\omega)|\nu\rangle^\circ_{g,\beta} = \langle\mu,\omega_k,\nu\rangle^\circ_{g,\beta}$$

Proof. See [MO09, Proposition 4.3, Page 1759]. □

Next, we match the classical and purely quantum parts of the quantum multiplication operator $M_{(1,\omega)}$ with the parts of operator $M_S(u,Q)$ in [CP17a].

5.2 Matching the classical multiplication

Lemma 5.2.1. *The weights of the standard torus action on the tautological bundle $\mathcal{V}_k$ on* $\mathrm{Hilb}(\mathcal{A}_n)$*,* $0 \leq k \leq n$*, restricted at a fixed point* $p = |\lambda_1, \ldots, \lambda_n\rangle$*,* $1 \leq \ell \leq n+1$ *are given by*

$$\mathcal{V}_k|_p = \sum_{\ell=1}^{k} \left[(n+1-k)t_1 + \sum_{(i,j)\in\mu_\ell} (i-1)w_i^R + (j-1)w_i^L \right] + \sum_{\ell=k+1}^{n} \left[kt_1 + \sum_{(i,j)\in\mu_\ell} (i-1)w_i^R + (j-1)w_i^L \right]$$

Proof. Note that $\mathcal{V}_k|_Z = \mathrm{H}^0(\mathcal{A}_n, \mathscr{V}_k \otimes \mathcal{O}_Z)$. At a fixed point $p = |\lambda_1, \ldots, \lambda_n\rangle$, $\mathrm{H}^0(\mathcal{O}_Z)$ contributes the part

$$\sum_{(i,j)\in\mu_\ell} (i-1)w_i^R + (j-1)w_i^L.$$

The lemma is equivalent to the property that as a one-dimensional torus module

$$\mathscr{V}_k|_{p_\ell} = \begin{cases} (n+1-k)t_1, \ \ell \leq k \\ kt_2, \ \ell > k \end{cases},$$

where p_ℓ is a fixed point on $\mathcal{A}_n$. The framed quiver variety $\mathrm{Hilb}(\mathcal{A}_n)$ 3.2.6 can be realized as ordinary quiver variety by the work of King and Grawley-Bovey[Gin12] by extending the stability condition to include $\theta_\infty = -\sum_k \theta_k \dim V_k$ to remove the framing at W_0. We use the extended stability condition $(\theta; \theta_\infty) = (n+\epsilon, 1, 1, \ldots, 1; -\epsilon)$. For a nontrivial subrepresentation $(V_0', \ldots, V_n', W_0')$ of the extented quiver to be stable, we need $-\epsilon \dim W_0' + \sum_k \theta_k \dim V_k' > 0$ and $\dim V_0' = 0$ or 1 since the dimension vector $v_0 = (1, \ldots, 1)$. Observe that for a subrepresentation to be fixed by the torus action, it can't contain any loop since such a loop carries nontrivial weights that cannot be canceled by the structure group G_V action. Thus a fixed subrepresentation must be of the form of a tree, the stability condition tells us W_0 must generate all other V_k's. In other words, all fixed subrepresentations are parametrized by directed subgraphs of the quiver with a root at the framing point, i.e. all other vertex can be reached by following nonzero arrows. At a fixed point p_ℓ, the torus weight is given by $\mathrm{wt}(p_\ell) = (n+2-\ell)t_1 + (n-\ell)t_1 + \cdots + t_1 + 0 + t_2 + \ldots (\ell-1)t_2$, the $\mathbf{Z}/(n+1)\mathbf{Z}$ weight k part is given by kt_1 if $\ell - 1 \geq k$, otherwise it's given by $(n+1-k)t_1$, this is exactly the weight of $\mathscr{V}_k$. $\square$

Proposition 5.2.2. *The divisor $(1,\omega)$ is the difference of the first Chern classes of the tautological bundle $\mathcal{V}_1$ and $\mathcal{V}_0$ and the cup product acts diagonally in the fixed point basis. More precisely,*

$$(1,\omega) = c_1(\mathcal{V}_1) - c_1(\mathcal{V}_0)$$
$$(1,\omega) \cup |\lambda,\mu\rangle = (t_1|\lambda| + t_2|\mu|)|\lambda,\mu\rangle.$$

Proof. By Lehn's formula[Leh99], for a line bundle L on X the Chern classes of the tautological bundle $L^{[n]}$ is given by

$$\sum_{n\geq 0} c(L^{[n]})z^n = \exp(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\alpha_{-k}(c(L))z^m)v_\emptyset.$$

We have $c_1(\mathcal{O}_X) = 1$, $c_1(\mathcal{O}_X(\omega)) = 1 + (1,\omega)$. By the construction in [Kuz07], we know that $\mathcal{V}_0 = \mathcal{O}_X^{[n]}$ and $\mathcal{V}_1 = \mathcal{O}_X((1,\omega))^{[n]}$. Compute the expansion in Lehn's formula and note that

$$\mathrm{codim}(\alpha_{-\mu}(\overrightarrow{\gamma})v_\emptyset) = \sum_i \mathrm{codim}(\gamma_i) + \sum_i(\mu_i - 1) = \sum_i \mathrm{codim}(\gamma_i) + |\mu| - \ell(\mu).$$

That is to get a codimension 2 cycle, we can either let one part of the partition to be

2 or we let one the the part to have the u insertion.

$$c_1(\mathcal{V}_0) = \frac{1}{(n-2)!}\alpha_{-2}(1)\alpha_{-1}(-1)\ldots\alpha_{-1}(1)v_\emptyset$$
$$c_1(\mathcal{V}_1) = \frac{1}{(n-1)!}\alpha_{-1}(w)\alpha_{-1}(1)\ldots\alpha_{-1}(1)v_\emptyset + c_1(\mathcal{V}_0).$$

Recall that the first term in $c_1(\mathcal{V}_1)$ is exactly $(1,\omega)$ in our definition. Thus $(1,\omega) = c_1(\mathcal{V}_1) - c_1(\mathcal{V}_0)$. Now by Nakajima's result[Nak], and the fact that at a length n subscheme Z, $V_k = \mathrm{H}^0(T^*\mathbf{P}^1, \mathscr{V}_k \otimes \mathcal{O}_Z)$. Since at a fixed point $|\mu,\nu\rangle$, $\mathcal{O}_Z$ is generated by monomials $x^i y^j$

$$\begin{aligned} c_1(\mathcal{V}_0) \cup |\lambda,\mu\rangle &= \Big[\sum_{(i,j)\in\lambda}((i-1)(t_2-t_1)+(j-1)(2t_1)) \\ &\quad + \sum_{(i,j)\in\mu}((i-1)(t_1-t_2)+(j-1)(2t_2)\Big]|\lambda,\mu\rangle \\ c_1(\mathcal{V}_1) \cup |\lambda,\mu\rangle &= \sum_{(i,j)\in\lambda}(t_1+(i-1)(t_2-t_1)+(j-1)(2t_1)) \\ &\quad + \sum_{(i,j)\in\mu}(t_2+(i-1)(t_1-t_2)+(j-1)(2t_2)|\lambda,\mu\rangle. \end{aligned}$$

In summary, we get

$$(1,\omega)\cup|\mu,\nu\rangle = (\sum_{(i,j)\in\mu} t_1 + \sum_{(i,j)\in\nu} t_2)|\lambda,\mu\rangle = (t_1|\mu| + t_2|\nu|)|\mu,\nu\rangle.$$

□

Proposition 5.2.3. $\mathrm{val}_{t_1+t_2} M^{\mathrm{cl}}_{(1,\omega)} = M^A_S(u,Q)$.

Proof. By the invariant level comparison, we need to take $(t_1+t_2)u^{-2}s$ part of the operator.

- $t_1|\mu| + t_2|\lambda|$ is divisable by t_1+t_2 if and only if $|\mu| = |\nu|$.

- The leading coefficient is $u^{-\ell(\mu)-\ell(\nu)}s$. Thus we only contribution comes from $\ell(\mu)=\ell(\nu)=1$.

Combined we know the only possibility is that $\mu=\nu=|\mu|=|\nu|$, i.e. they are partitions of only one part of the same size. This is exactly the type-A curve counting in [CP17b]. Thus we know the valuation of $M^{\text{cl}}_{(1,\omega)}$ at (t_1+t_2) matches with the type-A operator in [CP17b] □

5.3 Matching the purely quantum multiplication

Now we consider certain relevant vertex operators and match the purely quantum part of M in [MO09] with the type-B curve counting operator in [CP17b]. For any $\gamma\in Q$ and a complex variable z. The 2-cocycle in the $\widehat{\mathfrak{sl}}_2$ case is given by $\epsilon(m\alpha,n\alpha)=(-1)^{mn}$. Denote $e(z)=\sum_{k\in\mathbf{Z}}e(k)z^{-k-1}$, $f(z)=\sum_{k\in\mathbf{Z}}f(k)z^{-k-1}$. Then $e(z)$ and $f(z)$ acts on L_Λ by vertex operators:

$$e(z)\mapsto\Gamma_+(z)=X(\alpha,z)=\exp\Big(\sum_{k\geq1}\frac{a^k}{k}\alpha(-k)\Big)\exp\Big(-\sum_{k\geq1}\frac{z^{-k}}{k}\alpha(k)\Big)qz^{2q\frac{\partial}{\partial q}}c_\alpha$$

$$f(z)\mapsto\Gamma_-(z)=X(-\alpha,z)=\exp\Big(-\sum_{k\geq1}\frac{a^k}{k}\alpha(-k)\Big)\exp\Big(\sum_{k\geq1}\frac{z^{-k}}{k}\alpha(k)\Big)q^{-1}z^{-2q\frac{\partial}{\partial q}}c_{-\alpha},$$

where $c_{\pm\alpha}(f\otimes e^{n\alpha})=(-1)^n$ is a special case of $\epsilon(\alpha,n\alpha)$, and $z^{\pm2q\frac{\partial}{\partial q}}=z^{\pm2n}q^n$.

Proposition 5.3.1. *The qs-coefficient of Ω_+ in [MO09] is the operator corresponding to the type-B curve counting in [CP17b]. More precisely,*

$$[qs]\Omega_+=\sum_{k\neq0}:f(k)e(-k):=\sum_{|\mu|=|\nu|>0}\alpha_{-\mu}\alpha_\nu.$$

They're vertex operators of $\widehat{\mathfrak{sl}}_2$.

Proof. Note that $\sum_{k\neq0}:f(k)e(-k):$ is just the z^0-coefficient of $:\Gamma_+(z)\Gamma_-(z):$. We use the formula in [KR87, page 309],

$$\Gamma_\alpha(z)\Gamma_\beta(w) = (1-\frac{w}{z})^{(\alpha|\beta)} z^{(\alpha|\beta)}\epsilon(\alpha,\beta)$$
$$\times \exp(\sum_{k\geq 1}\frac{\alpha_{-k}(z^k-w^k)}{k})\exp(-\sum_{k\geq 1}\frac{\alpha_k(z^{-k}-w^{-k})}{k}) \times e^{\alpha+\beta}z^{\alpha_0}w^{\alpha_0}c_\alpha c_\beta$$

In our case, we have $\alpha=\alpha, \beta=-\alpha$, $\langle\alpha,-\alpha\rangle=-2$, $\epsilon(\alpha,-\alpha)=-1$. By the previous identification, we only act on the q^0 piece of the basic representation, thus

$$e^{\alpha+\beta}z^{\alpha(0)}w^{-\alpha(0)}c_\alpha c_{-\alpha}=1$$

The operator product is simplified to be

$$\Gamma_+(z)\Gamma_-(w) = \frac{-1}{(1-\frac{w}{z})^2z^2}\exp(\sum_{k\geq 1}\frac{\alpha_{-k}(z^k-w^k)}{k})\exp(-\sum_{k\geq 1}\frac{\alpha_k(z^{-k}-w^{-k})}{k}).$$

The normal ordering means taking the z^0 of the the regular part of the expansion above, thus

$$\sum_{k\neq 0} : f(k)e(-k) : = \mathrm{Res}_{z=0}z^{-1}(\frac{1}{(z-w)^2}+\Gamma_+(z)\Gamma_-(z))$$
$$= [z^0](\sum_{k\geq 1}\frac{\alpha(-k)}{k}(1-(\frac{w}{z})^k)z^k + (\sum_{k\geq 1}\frac{\alpha(-k)}{k}(1-(\frac{w}{z})^k)z^k)^2+\dots)$$
$$\times(\sum_{k\geq 1}\frac{\alpha(k)}{k}(-1+(\frac{w}{z})^{-k})z^{-k} + (\sum_{k\geq 1}\frac{\alpha(k)}{k}(1-(\frac{w}{z})^{-k})z^{-k})^2+\dots).$$

The position of a term means the length of the partition, the product gives all possible combinations of two partitions, the z^0-coefficient condition means the two partitions have to have the same size. Also note that

$$\lim_{\frac{w}{z}\to 1}\frac{(1-(\frac{w}{z})^k)(-1+(\frac{w}{z})^{-\ell})}{(1-\frac{w}{z})^2} = -k\ell,$$

which kills the denominators. As a consequence, we get the desired matching

$$\sum_{|\mu|=|\nu|>0} \alpha_{-\mu}\alpha_{\nu} = \sum_{k\neq 0} : f(k)e(-k) : .$$

$\square$

5.4 Bethe equations and eigenvalues

Now the relative Gromov-Witten theory of $\mathbf{P}^1 \times \mathbf{P}^1$ and $T^*\mathbf{P}^1 \times \mathbf{P}^1$ can be viewed as shades of the quantum cohomology theory of the Nakajima quiver variety $\mathrm{Hilb}(T^*\mathbf{P}^1)$, which is well-understood as in [MO12]. Moreover, the K-theoretic version of the story is developed in [AO17]. Eigenvalues of quantum multiplications of any tautological class are encoded in certain Bethe equations. To be more precise, if X is a Nakajima quiver variety, the equivariant K-theoretic class of the tangent bundle is given by $TX = T(T^*\mathrm{Rep}(v,w)) - \sum_i (1+\hbar^{-1})\mathrm{End}(V_i)$. As we shall see in the $\mathrm{Hilb}(T^*\mathbf{P}^1)$ example, it's a Laurent polynomial in terms of the Chern roots $x_{i,k}$ of the tautological bundles and the equivariant parameters. The elliptic genus $\widehat{a}$ is defined to be $\widehat{a}(\sum n_i\chi_i) = \prod(\chi_i^{\frac{1}{2}} - \chi_i^{-\frac{1}{2}})^{n_i}$, where χ_i are weights of $T \times \prod GL(V_i)$. In general, we have

Proposition 5.4.1 ([AO17])**.** *The eigenvalues of M are $\sum_{i,k}(-1)^k x_{i,k}$, where $x_{i,k}$ are the roots of the Bethe equations*

$$\widehat{a}(x_{i,k}\frac{\partial}{\partial x_{i,k}} TX) = z_i.$$

The cohomological Bethe equations can be extracted from the K-theoretic version by taking the linear term. We work this computation out in the $\mathrm{Hilb}(T^*\mathbf{P}^1)$ case. To simplify the notation a little bit, we denote the Chern roots of $\mathcal{V}_1^*$ by x_i and those of

$\mathcal{V}_2^*$ by y_j. Then in $K_{\mathbf{C}_a^\times \times \mathbf{C}_\hbar^\times}(\mathrm{Hilb}(T^*\mathbf{P}^1))$, we have

$$
\begin{aligned}
TX = &\sum_i ax_i + \hbar^{-1}\sum_i \frac{1}{ax_i} \\
&+ \sum_{i,j}\frac{y_j}{x_i} + \hbar^{-1}\sum_{i,j}\frac{x_i}{y_j} + \sum_{i,j}\frac{x_i}{y_j} + \hbar^{-1}\sum_{i,j}\frac{y_j}{x_i} \\
&- (1+\hbar^{-1})\sum_{i,j}(\frac{x_i}{x_j} + \frac{y_i}{y_j}).
\end{aligned}
$$

The partial derivative w.r.t x_i is

$$
\begin{aligned}
x_i\frac{\partial}{\partial x_i}TX =& ax_i - \hbar^{-1}\frac{1}{ax_i} \\
&- \sum_j \frac{y_j}{x_i} + \hbar^{-1}\sum_j \frac{x_i}{y_j} + \sum_j \frac{x_i}{y_j} - \hbar^{-1}\sum_j \frac{y_j}{x_i} \\
&- (1+\hbar^{-1})\sum_j(\frac{x_i}{x_j} - \frac{x_j}{x_i}).
\end{aligned}
$$

Take the Euler class of the expression above, we get the Bethe equation

$$
\begin{aligned}
q &= \frac{a+x_i}{-\hbar - a - x_i}\prod_j \frac{(x_i-y_j)(x_i-y_j-\hbar)}{(y_j-x_i)(y_j-x_i-\hbar)}\prod_{j\neq i}\frac{x_j-x_i}{x_i-x_j}\frac{x_j-x_i-\hbar}{x_i-x_j-\hbar} \\
&= \frac{a+x_i}{a+x_i+\hbar}\prod_j \frac{x_i-y_j-\hbar}{y_j-x_i-\hbar}\prod_{j\neq i}\frac{x_j-x_i-\hbar}{x_i-x_j-\hbar}.
\end{aligned}
$$

Similar for y_i and s (the curve degree corresponding to $(1,\omega)$). In summary, we get

Corollary 5.4.2. *The eigenvalues of $M_{(1,\omega)}$ are given by $\sum_i (x_i - y_i)$, where x_i, y_i are the roots of the Bethe equations*

$$
\begin{aligned}
q &= \frac{a+x_i}{a+x_i+\hbar}\prod_j \frac{x_i-y_j-\hbar}{y_j-x_i-\hbar}\prod_{j\neq i}\frac{x_j-x_i-\hbar}{x_i-x_j-\hbar} \\
s &= \frac{a+y_i}{a+y_i+\hbar}\prod_j \frac{y_i-x_j-\hbar}{x_j-y_i-\hbar}\prod_{j\neq i}\frac{y_j-y_i-\hbar}{y_i-y_j-\hbar}.
\end{aligned}
$$

Remark 5.4.3. To be compatible with our previous torus action, just let $\hbar = t_1 + t_2$ and $a = t_2 - t_1$.

Chapter 6

Perverse sheaves and weights

6.1 Perverse sheaves

6.1.1 Fonctions-faisceaux dictionary

Let X be a scheme over a finite field $\mathbf{F}_q$, according to Grothendieck's "fonctions-faisceaux dictionary", instead of considering $\overline{\mathbf{Q}}_\ell$-valued functions on $X(\mathbf{F}_q)$—the set of $\mathbf{F}_q$-points, the "correct" geometric object is the notion of complex of ℓ-adic sheaves. Given an ℓ-adic sheaf $\mathcal{F}$ on X and a morphism $f : X' \to X$, the group of automorphisms of f acts on the $f^*\mathcal{F}$. Specially, let $x \in X$ be a $\mathbf{F}_q$-point and $\overline{x}$ the $\mathbf{F}_q$ point corresponding to an inclusion $\mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_q$. By pulling back along the composition $\overline{x} \to x \to X$, we get a sheaf $\mathcal{F}_{\overline{x}}$ on $\overline{x}$, which is just the fibre of $\mathcal{F}$ at $\overline{x}$, which is a $\overline{\mathbf{Q}}_\ell$-vector space. $\mathbf{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) \cong \widehat{\mathbf{Z}}$, as the group of automorphisms of the morphism $\overline{x} \to x$, acts naturally on $\mathcal{F}_{\overline{x}}$. In particular, the geometric Frobenius element $\mathrm{Fr}_{\overline{x}}$ in the Galois group acts on the $\overline{\mathbf{Q}}_\ell$-vector space $\mathcal{F}_{\overline{x}}$. Then we can construct a function on $X(\mathbf{F}_q)$ by taking the trace of the Frobenius element,

$$f_{\mathcal{F}}(x) = \mathbf{Tr}(\mathrm{Fr}_{\overline{x}}, \mathcal{F}_{\overline{x}}).$$

More generally if we have a complex $\mathcal{C}$ of ℓ-adic sheaves, the associated function is obtained by taking the alternating sums of the Frobenius traces

$$f_{\mathcal{C}}(x) = \sum_k (-1)^k \mathbf{Tr}(\mathrm{Fr}_{\overline{x}}, \mathcal{C}^k_{\overline{x}}) = \sum_k (-1)^k \mathbf{Tr}(\mathrm{Fr}_{\overline{x}}, \mathrm{H}^k_{\overline{x}}(\mathcal{C})),$$

where $\mathcal{C}^k$ is the degree k component of $\mathcal{C}$ and $\mathrm{H}^k(\mathcal{C})$ is the k-th cohomology of the complex $\mathcal{C}$. The construction $\mathcal{C} \to f_{\mathcal{C}}$ preserves most intuitions on the function side, for example, pull-back of sheaves corresponds to pull-back of functions, tensor product of sheaves corresponds to product of functions, push-forward of sheaves corresponds to integration of functions along the fibres.

The construction only depends on the alternating sum of the cohomologies, two homotopic complexes give the same function, this leads us to the consideration the derived category of sheaves. Even so, the map $\mathcal{C} \to f_{\mathcal{C}}$ is not injective, since two complexes not isomorphic in the derived category might have the same alternating sum of cohomologies, therefore the same associated functions. A trivial example is the zero complex and $0 \to \mathcal{F} \xrightarrow{d=0} \mathcal{F} \to 0$, their assicated functions are the same, namely, the zero function, although the second has non-trivial cohomologies and thus not a zero complex in the derived category. For these reasons, it would be very nice if we can construct an abelian subcategory $\mathcal{A}$ of the derived category of sheaves such that the restriction to $\mathcal{A}$ of the sheaf-function construction gives rise to an injective map from the Grothendieck group of $\mathcal{A}$ to the space of functions on X. Surely, we have many different candidates for $\mathcal{A}$, for example, the category of sheaves, which can be viewed as complexes concentrated all on degree 0. However, this subcategory doesn't have many good properties, for instance, it's not stable under the Verdier duality. The invention of perverse sheaves fixes these problems and realizes the "fonctions-faisceaux dictionary" in an injective and faithful way.

6.1.2 Preliminary on perverse sheaves

Definition 6.1.1 (Local system, locally constant sheaf)**.** A locally constant sheaf on a topological space X is a sheaf $\mathcal{F}$ on X such that for each x in X, there is an open neighborhood U of x such that the restriction $\mathcal{F}|_U$ is a constant sheaf on U. It is also called a local system

Example 6.1.2 (local system, locally constant sheaf)**.** Let $X = \mathbf{C}^2 \setminus \{0\}$, and define the n^{th} root sheaf $\underline{\mu}_n$

$$\underline{\mu}_n(U) = \{f \in C^\infty(U) | f^n = x\}.$$

Note that if the monodromy of U is 0(which means $\mathrm{im}(\pi_1(U) \to \pi_1(\mathbf{C}^*)) = 0)$, $\underline{\mu}_n(U) \cong \mathbf{Z}/n\mathbf{Z}$ consisting of the n functions $\{\zeta_n^i x^{\frac{1}{n}}\}_{i=0}^{n-1}$, otherwise $\underline{\mu}_n(U) = 0$.

Definition 6.1.3 (Constructible sheaf)**.** A sheaf $\mathcal{F}$ is called constructible if there exists a finite partition $X = \bigsqcup_{S \in \mathcal{S}} S$ of X as a union of locally closed subschemes, such that for each subscheme S, the sheaf $\mathcal{F}|_S = i_S^*\mathcal{F}$ is a finite locally constant sheaf.

Example 6.1.4 (Weierstrass family of elliptic curves)**.** Consider the family of degenerating elliptic curves over $\mathbf{C}$ $\pi : E \to \mathbf{C}$ given by $E_t = V(y^2 - x(x-1)(x-t))$. E_t is a nodal curve if $t = 0$ or 1, otherwise it's an elliptic curve. Then $\mathbf{R}^0\pi_*(\underline{\mathbf{Q}}_X) \cong \mathbf{R}^2\pi_*(\underline{\mathbf{Q}}_X) \cong \underline{\mathbf{Q}}_{\mathbf{C}}$ and $\mathbf{R}^1\pi_*(\underline{\mathbf{Q}}_X) \cong \mathcal{L}_{\mathbf{C}\setminus\{0,1\}} \oplus \underline{\mathbf{Q}}_{\{0,1\}}$, where the stalks of the local system $\mathcal{L}_{\mathbf{C}\setminus\{0,1\}}$ are isomorphic to $\mathbf{Q}^2$.

Example 6.1.5 (Constructible but not locally constant)**.** Consider the skyscraper $\mathcal{F}$ sheaf at a point on $X = \mathbf{P}^1$, the point is that although every point in X has a neighborhood U, such that $\mathcal{F}(U) = 0$ or k. But, locally constant means it has to be a constant sheaf when restricted to some U, not just the section being constant.

Let $D^b(X, k)$ denote the bounded derived category of sheaves of k-vector spaces. For any complex $\mathcal{C} \in D^b(X, k)$, we denote the k-th cohomology sheaf by $\mathrm{H}^k(\mathcal{C})$. We denote

Strata	$\dots$	$-d$	$-d+1$	$\dots$	-1	0
S_d	0	*	0	0	0	0
S_{d-1}	0	*	*	0	0	0
$\dots$	0	$\dots$	$\dots$	$\dots$	0	0
S_1	0	*	*	*	*	0
S_0	0	*	*	*	*	*

Figure 5: Perverse sheaf

$D^b_c(X,k)$ the full subcategory of $D^b(X,k)$ with objects $\mathcal{C}$ such that all its cohomology sheaves $\mathcal{H}^k(\mathcal{C})$ are constructible.

The category $P(X,k)$ of (constructible) perverse sheaves is the full subcategory of $D^b_c(X,k)$ consists of objects $\mathcal{F} \in D^b_c(X,k)$, constructible with respect to certain stratification $\mathcal{S}$, such that

1. $\forall S_\alpha \in \mathcal{S}$, $i^*_{S_\alpha}\mathcal{F}$ is concentrated in degrees $\leq -\mathrm{dim} S_\alpha$, and

2. $\forall S_\alpha \in \mathcal{S}$, $i^!_{S_\alpha}\mathcal{F}$ is concentrated in degrees $\geq -\mathrm{dim} S_\alpha$.

We note that the Verdier dual $\mathbf{D}$ functor interchanges condition 1. and 2., thus the category of perverse sheaves is stable under the Verdier duality. Let S_k denote the union of strata of dimension k, then by induction one shows that $\mathcal{F}|_{S_k}$ only have non-trivial stalks for degree at least $-k$, together with condition 1., we know that $\mathcal{F}$ is a perverse sheaf if and only if the cohomology sheaves of both $\mathcal{F}$ and $\mathbf{D}\mathcal{F}$ have the form in Figure 5 Namely, restriction of $\mathcal{F}$ to the dimension k strata S_k has non-trivial cohomology sheaves only from degree $-d = -\dim X$ to $-k$.

Strata	$-d$	$\dots$	$-d_S$	$\dots$	-1	0
S_d	0	0	0	0	0	0
$\dots$	0	0	0	0	0	0
S_{d_S}	0	0	$i_{S*}\mathcal{L}$	0	0	0
$\dots$	0	0	$*$	0	0	0
$\dots$	0	0	$*$	$*$	0	0
S_0	0	0	$*$	$*$	$*$	0

Figure 6: Intersection cohomology complex $\mathrm{IC}(\overline{S}, \mathcal{L})$

6.1.3 The intersection cohomology complex

The category of perverse sheaves is an abelian category, it's important to understand the simple objects. They are given by the so-called IC-sheaves (or intersection cohomology complex) $\mathrm{IC}(\overline{S}, \mathcal{L})$ associated to a pair $(S, \mathcal{L})$, where $S \in \mathcal{S}$ is a strata and $\mathcal{L}$ is an irreducible local system on S. $\mathrm{IC}(\overline{S}, \mathcal{L})$ is characterized by the following conditions

1. $i_S^* \mathrm{IC}(\overline{S}, \mathcal{L}) = \mathcal{L}[d_S]$,
2. $\mathrm{IC}(\overline{S}, \mathcal{L})$ is supported on $\overline{S}$,
3. $\forall$ strata $T \subset \overline{S}, T \neq S$, $i_T^* \mathrm{IC}(\overline{S}, \mathcal{L})$ is concentrated in degrees $< -\dim(T)$,
4. $\forall$ strata $T \subset \overline{S}, T \neq S$, $i_T^! \mathrm{IC}(\overline{S}, \mathcal{L})$ is concentrated in degrees $> -\dim(T)$.

Thus intersection cohomology complex is a special perverse sheaf, the corresponding cohomology sheaves have the form in Figure 6.

The most important result in terms of intersection complex is arguably the decomposition theorem [BBD82] proved by Beilinson, Bernstein, Deligne and Gabber. An

IC-sheaf looks like a block, the decomposition theorem tells us we can construct perverse sheaves by taking the direct sum of shifted IC sheaves. We define an object $\mathcal{K}$ to be semi-simple if

$$\mathcal{K} \cong \oplus P_{S,\mathcal{L}_S} \cdot \mathrm{IC}(\overline{S}, \mathcal{L}).$$

where $P_{S,\mathcal{L}_S}$ are some Laurent polynomials with $\mathbf{Z}$ coefficients, i.e. $P_{S,\mathcal{L}_S} = \sum_i a_i t^i$, then

$$P_{S,\mathcal{L}_S} \cdot \mathcal{F} := \oplus \mathcal{F}[i]^{\oplus a_i}.$$

Some authors like to use another essentially equivalent description

$$\pi_* \underline{k}_X[\dim(X)] \cong \bigoplus_\beta \mathrm{IC}(\overline{S_\beta}, \mathcal{L}) \otimes_k V_\beta,$$

where V_β is a graded finite dimensional k-vector space.

Theorem 6.1.6 (Decomposition theorem[BBD82])**.** *Let $\pi : X \to Y$ is a proper morphism with X smooth, then*

- $\pi_* \underline{k}_X[\dim(X)]$ *is semisimple.*
- *If π is semismall, then $\pi_* \underline{k}_X[\dim(X)]$ is a perverse sheaf.*
- *If π is small, then $\pi_* \underline{k}_X[\dim(X)]$ is an intersection cohomology complex.*

In general the decomposition theorem fails in positive characteristics. The decomposition theorem actually gives us a way to compute the intersection cohomology sheaves in practice. We give some examples here.

Example 6.1.7 (Nilpotent cone in $\mathfrak{sl}_2$)**.** The nilpotent cone in $\mathfrak{sl}_2$ is given by all the traceless 2-by-2 matrices with vanishing determinant, namely

$$\mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} | x^2 + yz = 0 \right\} \subset \mathfrak{sl}_2 \cong \mathbb{A}^3.$$

The Springer resolution in this situation is just the blow up of the cone

$$\pi : Bl_{\{0\}}\mathcal{N} \cong \mathrm{Tot}(\mathcal{O}_{\mathbf{P}^1}(-2)) = \mathrm{Tot}(T^*\mathbf{P}^1) \to \mathcal{N}.$$

On the regular locus $\mathcal{O}_{reg} = \mathcal{N} \setminus \{0\}$, this is an isomorphism, the fiber over 0 is $\mathbf{P}^1_k$, we thus get $\pi_* \underline{k}_{T^*\mathbf{P}^1}[2]$

	-2	-1	0
$S_2 = \mathcal{O}_{reg}$	k	0	0
$S_0 = \{0\}$	k	0	k

In the case that $\mathrm{char}(k) = 0$, by the decomposition theorem we have $\pi_* \underline{k}_{T^*\mathbf{P}^1}[2] = \mathrm{IC}(\mathcal{N}, k) \oplus \mathrm{IC}(\{0\}, k)$, where $\mathrm{IC}(\{0\}, k)$ is just the skyscraper sheaf at the point in degree 0, and $\mathrm{IC}(\mathcal{N}, k) \cong \underline{k}_{T^*\mathbf{P}^1}[2]$.

6.1.4 Intermediate extension

We then review the construction of intersection cohomology complexes via the so-called intermediate extensions. From now on we abuse notation:

$$f_* = \mathbf{R}f_*, f_! = \mathbf{R}f_!, f^* = \mathbf{R}f^*, \mathcal{H}om = \mathbf{R}\mathcal{H}om.$$

Note that, if $f : X \to \mathrm{pt}$ is the projection to a point, then $f_* = \mathbf{R}\Gamma(X, -)$ and $f_! = \mathbf{R}\Gamma_c(X, -)$. We have a natural morphism of functors $f_! \to f_*$.

Definition 6.1.8 (Intermediate extension)**.** Suppose X is purely of dimension n, and let $j : U \to X$ be the inclusion of the smooth locus U of X in U. Then the intermediate extension functor is

$$D^b_c(U, k) \to D^b_c(X, k)$$

$$j_{!*}\mathcal{F} := \mathrm{Im}(j_!\mathcal{F} \to j_*\mathcal{F})$$

The intermediate extension has several properties, for example $j_{!*}$ has no subobject or subquotient supported on the complement of U[BBD82]. It turns out this construction gives us intersection cohomology complexes. Depending on the field k we're working with, we can define the IC-sheaves accordingly, for latter use, we define the ℓ-adic intersection complex.

Definition 6.1.9 (ℓ-adic intersection complex)**.** Suppose X is purely of dimension n, and let $j : U \to X$ be the inclusion of the smooth locus U of X in U. Then the ℓ-adic intersection complex is

$$\mathbf{IC}^{\bullet}(X) := (j_{!*}\overline{\mathbf{Q}}_{\ell,U}[n])[-n],$$

where $\overline{\mathbf{Q}}_{\ell,U}$ is the constant sheaf $\overline{\mathbf{Q}}_{\ell}$ on U. The ℓ-adic intersection cohomology is the cohomology of $\mathbf{IC}^{\bullet}(X)$, it's denoted by $\mathbf{IH}^{\bullet}(X, \overline{\mathbf{Q}}_{\ell})$.

When we have more strata, Deligne gives an inductive construction of the intermediate extension by **cohomological truncation**. First let $X_k = S_{\geq k} := \bigcup\limits_{d \geq k} S_d$, the union of all strata of dimension at least k. Then we have a sequence of inclusions

$$X_d \xhookrightarrow{j_{d-1}} X_{d-1} \hookrightarrow \dots \overset{j_1}{\hookrightarrow} X_1 \overset{j_0}{\hookrightarrow} X_0 = X.$$

Then the construction could be thought as first build the highest dimension d_S part, and then build the degree $d_S - 1$, so on so forth, and stops at the dimension 0 part. In formula,

$$\mathrm{IC}(\overline{S}, \mathcal{L}) := (\tau_{\leq -1} \circ j_{0,*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S - 1,*})(\mathscr{L}[d_S]),$$

where $\tau_{\leq k}$ means truncation by cohomological degree. Note that if $S_{d_S} \neq S$, $\mathcal{L}[d_S]$ means the extension by 0. Moreover, if X is pure of dimension d, U is open in X, Z is the compliment. $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$, we have

$${}^{p}j_{!}\mathcal{F} = \tau^{Z}_{\leq d-3} j_{*}\mathcal{F}, \; j_{!*}\mathcal{F} = \tau^{Z}_{\leq d-2} j_{*}\mathcal{F}, \; {}^{p}j_{*}\mathcal{F} = \tau^{Z}_{\leq d-1} j_{*}\mathcal{F}.$$

Example 6.1.10 (Deligne's Cohomological truncation)**.** Consider $* \overset{i}{\hookrightarrow} \mathbf{C} \overset{j}{\hookleftarrow} \mathbf{C}^\times$.

Then the stalks of $j_*\mathbf{Q}[1]$ are

	-1	0
$\mathbf{C}^\times$	$\mathbf{Q}$	0
$*$	$\mathbf{Q}$	$\mathbf{Q}$

By Deligne's cohomological truncation construction, the stalks of ${}^p j_! \mathcal{F} = \tau^Z_{\leq -2} j_* \mathcal{F}$ are

	-1	0
$\mathbf{C}^\times$	$\mathbf{Q}$	0
$*$	0	0

, and the stalks of $j_{!*}\mathcal{F} = \tau^Z_{\leq -1} j_* \mathcal{F}$ are

	-1	0
$\mathbf{C}^\times$	$\mathbf{Q}$	0
$*$	$\mathbf{Q}$	0

. That is $j_{!*}\mathbf{Q}[1] = \mathbf{Q}_X[1]$ as expected.

6.2 Review of weights

Definition 6.2.1 (Weight of a complex, [Del80])**.** $\mathcal{F} \in D^b_c(X, \overline{\mathbf{Q}}_\ell)$ is said to be mixed of weight $\leq w$ if for any $k \in \mathbf{Z}$, the cohomology sheaf $\mathcal{H}^k(\mathcal{F})$ is mixed of weight $\leq w + i$. $\mathcal{F}$ is said to be pure of weight w if $\mathcal{F}$ is mixed of weight $\leq w$ and its Verdier dual $\mathbf{D}_{X/k}\mathcal{F}$ is mixed of weight $\leq -w$.

Remark 6.2.2. If $\mathcal{F}$ is mixed of weight $\leq w$, $\mathcal{F}[n]$ is mixed of weight $\leq w + n$. Namely shift operator in the derived category also shifts the Frobenius weights.

Definition 6.2.3 (Trace function)**.** For an object $\mathcal{F}$ in $D^b_c(X, \overline{\mathbf{Q}}_\ell)$, its trace function is the $\overline{\mathbf{Q}}_\ell$-valued function on pairs (a finite field extension E/k, a point $x \in X(E)$) defined by

$$\mathbf{Tr} : (E, x) \mapsto \sum_k (-1)^i \mathrm{Trace}(\mathbf{Frob}_{E,x} | \mathcal{H}^k(\mathcal{F})).$$

Trace functions of ℓ-adic perverse sheaves can be thought of analogues of character functions of representations. Actually, Katz proves the orthogonality theorem in this context.

Theorem 6.2.4 (The orthogonality theorem, [Kat05]). *Let $\mathcal{F}$ and $\mathcal{G}$ on X/k be perverse sheaves and both pure of weight 0. Write the pullbacks $\mathcal{F}_{\overline{X}}$ and $\mathcal{G}_{\overline{X}}$ of $\mathcal{F}$ and $\mathcal{G}$ to $\overline{X} = X \otimes_k \overline{k}$ as sums of irreducible perverse sheaves with multiplicities, say*

$$\mathcal{F}_{\overline{X}} = \sum_k m_k V_k,\ \ \mathcal{G}_{\overline{X}} = \sum_k n_k V_k,$$

with $\{V_k\}_k$ a finite set of pairwise non-isomorphic irreducible perverse sheaves on $\overline{X}$, and with non-negative integer coefficients m_k, n_k [BBD82][5.3.8].

- *For any $n \in \mathbf{Z}_+$, denoting by k_n/k the extension field of degree n, we have*

$$\sum_k m_k n_k = \limsup_{E/k_n} |\sum_{x \in X(E)} \mathbf{Tr}_{\mathcal{F}}(E,x)\overline{\mathbf{Tr}_{\mathcal{G}}(E,x)}|,$$

  the $\limsup$ *is taken over all finite extensions E/k_n.*

- *If $\sum_k m_k n_k = 0$, i.e. if $\mathcal{F}_{\overline{X}}$ and $\mathcal{G}_{\overline{X}}$ have no common irreducible components, then for variable finite extensions E/k, we have*

$$\sum_{x \in X(E)} |\mathbf{Tr}_{\mathcal{F}}(E,x)\overline{\mathbf{Tr}_{\mathcal{G}}(E,x)}| = O((\#E)^{-\frac{1}{2}}).$$

- *The following two conditions are quivalent*

  – *For variable finite extensions E/k, we have*

$$\sum_{x \in X(E)} |\mathbf{Tr}_{\mathcal{F}}(E,x)|^2 = 1 + O((\#E)^{-\frac{1}{2}}).$$

  - *$\mathcal{F}$ is geometrically irreducible, i.e., $\mathcal{F}_{\overline{X}}$ is an irreducible perverse sheaf on $\overline{X}$.*

6.3 Weight truncation

Let X be a quasi-separated scheme of finite type over a finite field $\mathbf{F}_q$. We have defined the category of mixed ℓ-adic complexes $D^b_m(X, \overline{\mathbf{Q}}_\ell)$ on X. As discussed in previous sections [BBD82, 5], we get a subcategory $P_m(X)$ of mixed ℓ-adic perverse sheaves. Moreover $P_m(X)$ admits a canonical weight truncation operators $w_{\leq a}$, such that $\{w_{\leq a}K\}_{a\in\mathbf{Z}}$ gives a filtration of K, where each $w_{\leq a}K$ is a perverse subsheaf of K of weight $\leq a$, $K/w_{\leq a}K$ is of weight $> a$.

The weight truncation operator $w_{\leq a}$ doesn't extend to $D^b_m(X, \overline{\mathbf{Q}}_\ell)$. However, according to [Mor08], we can consider ${}^wD^{\leq a}(X)$, the full subcategory of $D^b_m(X, \overline{\mathbf{Q}}_\ell)$ whose objects are complexes K such that the k-th perverse cohomology ${}^p\mathrm{H}^kK$ is of weight $\leq a$ for any $k \in \mathbf{Z}$. Then the inclusion ${}^wD^{\leq a}(X) \hookrightarrow D^b_m(X, \overline{\mathbf{Q}}_\ell)$ admits a right adjoint functor that extends the previous $w_{\leq a}$, we still denote it by the same notation. Similarly, we can define ${}^wD^{\geq a}(X) \hookrightarrow D^b_m(X, \overline{\mathbf{Q}}_\ell)$, which admits a left adjoint functor that extends the previous $w_{\geq a} : K \mapsto K/w_{a-1}K$ functor. We denote it by $w_{\geq a}$.

Morel's main idea is that instead of viewing the intersection complex $\mathbf{IC}^\bullet(X)$ as a truncation of $j_*\overline{\mathbf{Q}}_{\ell,U}$ by cohomological degree, we want to view it as a truncation by Frobenius weights.

Theorem 6.3.1 ([Mor08])**.** *Let $j : U \to X$ a nonempty open subset of X and K a pure perverse sheaf of weight a on U. Then there are canonical isomorphisms:*

$$j_{!*}K \cong w_{\leq a}j_*K \cong w_{\geq a}j_!K.$$

More generally, let $\{S_k\}_{k=0}^n$ be a partition of X as above, and define $T_k = \bigcup_{r=\dim X-k} S_r$, the union of codimension exactly k strata, we also require T_k is open in $X - \bigcup_{l<k} T_k$. By abuse of notation, we denote by i_k the embedding $T_k \hookrightarrow X$. Then Morel proves the following

Theorem 6.3.2 ([Mor08])**.** *Let $\mathcal{F}$ be a perverse sheaf on T_0 pure of weight $a \in \mathbf{Z}$. We have an equivalence in the Grothendieck group of the derived category $D^b_m(X, \mathbf{Q}_\ell)$ of bounded complex of $\mathbf{Q}_\ell$-sheaves with mixed weights:*

$$[i_{0!*}\mathcal{F}] = \sum_{1 \le n_1 < n_2 \cdots < n_r < n} (-1)^r [i_{n_r!} w_{\le a} i^!_{n_r} \dots i_{n_1!} w_{\le a} i^!_{n_1} i_{0!}\mathcal{F}]$$
$$+ \sum_{1 \le n_1 < n_2 \cdots < n_r = n} (-1)^r [i_{n_r!} w_{< a} i^!_{n_r} i_{n_{r-1}!} w_{\le a} i^!_{n_{r-1}} \dots i_{n_1!} w_{\le a} i^!_{n_1} i_{0!}\mathcal{F}]$$

Remark 6.3.3. If no sequence satisfies $1 \le n_1 < \cdots < n_r < n$, the first summation is **NOT** empty, it's $i_{0!}\mathcal{F}$ instead.

Example 6.3.4 (Weight truncation)**.** Consider $X = \mathbf{A}^1_{\mathbf{F}_p} = \mathbf{Spec}(\mathbf{F}_p[t])$ and $U = \mathbf{Spec}(\mathbf{F}_p[t, t^{-1}])$. $X = U \cup \{0\}$, let $j : U \to X$ be the inclusion. The theorem above says that $[j_{!*}\mathbf{Q}_{\ell,U}[1]] = [j_!\mathbf{Q}_{\ell,U}[1]] - [i_! w_{<0} i^! j_! \mathbf{Q}_{\ell,U}[1]]$. The stalks of $j_!\mathbf{Q}_{\ell,U}[1]$ are

	-1	0
U	$\mathbf{Q}_\ell(-1)$	0
$*$	0	0

. Apply the excision sequence, we have $[i^! j_! \mathbf{Q}_{\ell,U}[1] = [\mathbf{Q}_\ell \to \mathbf{Q}_\ell(1)]$, which is in degree 0 and 1 and it's nontrivial only over the point $*$. Thus we have the stalks of its dual

	-1	0
$*$	$\mathbf{Q}_\ell(-1)$	$\mathbf{Q}_\ell$

. Now apply the weight truncation $w < 0$ and pushforward (here $i_* = i_!$) to X, the stalks of the intermediate extension are

	-1	0
U	$\mathbf{Q}_\ell(-1)$	0
$*$	$\mathbf{Q}_\ell(-1)$	0

. This recovers the fact that $j_{!*}\mathbf{Q}_{\ell,U}[1] = \mathbf{Q}_X[1]$, since X is smooth of dimension 1.

Remark 6.3.5. Note that in Morel's definition of the intersection complex, we have to shift the degree back, i.e. $\mathbf{IC}_{\overline{X}} = (j_{!*}\mathbf{Q}_\ell[\dim X])[-\dim X]$, as a consequence, it's **NOT** a perverse sheaf. Some authors define the intersection complex just as $\mathbf{IC}_{\overline{X}} = j_{!*}\mathbf{Q}_\ell[\dim X]$. This has the advantage that an intersection complex is a perverse sheaf, however, this is not what goes into the Kazhdan-Lusztig theorem, we have to take the cohomology sheaf of intersection complex with support in non-negative degrees.

For latter use, we have another way to encode the trace function. Let $\mathcal{T}(X)$ denote the subcategory of $D^b_m(X, \mathbf{Q}_\ell)$ generated by objects isomorphic to $\mathbf{Q}_\ell(m)$, $m \in \mathbf{Z}$. Then the Grothendieck group $K(\mathcal{T}(X))$ is isomorphic to $\mathbf{Z}[t, t^{-1}]$, t can be identified with $\mathbf{Q}_\ell(1)$. Actually, this isomorphism is given by the Frobenius trace function:

$$\phi : \mathcal{T}(X) \to \mathbf{Z}[t, t^{-1}]$$
$$\phi([K])(q^k) = \mathbf{Tr}(\mathbf{Frob}^{k*}, i_x^* K), \ \forall x \in X(\mathbf{F}_{q^k}).$$

Thus give an object in $\mathcal{T}(X)$, we can always get a Laurent polynomial, this paves the way to go from intersection cohomology complex $j_{!*}\mathbf{Q}_\ell$ to the R-polynomials as we shall see in the coming chapter.

Lemma 6.3.6. *Let X be smooth and connected over $\mathbf{F}_q$. For any object $K \in \mathcal{T}(X)$ and $a \in \mathbf{Z}$, we have*

$$\phi([w_{\leq a} K]) = w_{\leq a - \dim(X)}(\phi(K)).$$

Proof. See [Mor08]. □

Similarly, the weight truncation operators have simple incarnations in the Laurent polynomial ring.

Definition 6.3.7 (Truncation operator on Larent polynomials)**.** The truncation operator $\tau_{\leq d}$ on the Larent polynomials is defined to be the $\mathbf{Q}$-linear endormorphism of the Laurent polynomial ring $\mathbf{Q}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$,

$$\tau_{\leq d}\left(\sum_{k \in \mathbf{Z}} a_k q^{\frac{k}{2}}\right) = \sum_{k \leq d} a_k q^{\frac{k}{2}}.$$

In other words, q has degree 2, this minor point sometimes causes confusions in computation.

Chapter 7

Nonrecursive formula for Kazhdan-Lusztig polynomials

7.1 Geometry of G/P

Let (W, S) be a general crystallographic Coxeter group, J a subset of S. Let W_J be the subgroup generated by $J \subset S$. W^J, the set of minimal coset representatives of W/W^J. we have a notion of "standard" parabolic subgroup P_J associated to $J \subset S$. P_J is the subgroup of G generated by the Borel subgroup B and U_J^- (the one-parameter subgroups $\{U_\alpha | \alpha \in \Phi^-, \alpha \in \mathrm{Span}(J)\}$). Let W_J be the subgroup of W generated by J and U_J be the subgroup of U generated by $\{U_\alpha | \alpha \in \Phi^+, \alpha \in \mathrm{Span(J)}\}$. Moreover, we denote $B_J = HU_J$ and $B_J^- = HU_J^-$. One then has a decomposition of P_J and analogues of Bruhat and Birkoff decompositions.

Proposition 7.1.1. *In the settings as above, P_J and G can be decomposed into affine spaces,*

- $P_J = \bigcup_{w \in W_J} BwB = \bigcup_{w \in W_J} U_J wB = \bigcup_{w \in W_J} U_J^- wB.$

- $G = \bigcup_{w \in W^J} BwP = \bigcup_{w \in W^J} B^- wP.$

This proposition can be used to describe the geometry of $\pi : G/B \to G/P$. Let $\sigma \in W^J$, then the restriction to $B\sigma B$ to $B\sigma P$ is an isomorphism, moreover $\pi^{-1}(B\sigma P) \to B\sigma P$ is smooth fibre bundle over the affine base with fibre $P_J/B = \bigcup_{w \in W_J} BwB/B$, the fibre is given by the Bruhat cells associated to elements in $W_J \subset W$. We denote the Bruhat cell BwP/P (resp. $B^- wP/P$) by X_w (resp. X^w).

Example 7.1.2 (Affine Grassmannian of PGL_2)**.** The affine Grassmannian $\mathrm{Gr} = PGL_2((t))/PGL_2[[t]]$ has two irreducible components since $\pi_1(PGL_2(\mathbf{C})) \cong \mathbf{Z}/2\mathbf{Z}$. We denote them by $\mathrm{Gr}^{(0)}$ and $\mathrm{Gr}^{(1)}$. In fact $\mathrm{Gr}^{(0)}$is isomorphic to $SL_2((t))/SL_2[[t]]$, the affine Grassmannian associated to $SL_2(\mathbf{C})$.

The Bruhat decomposition of Gr are given by $PGL_2[[t]]$-orbits, they are parametrized by dominant weights of SL_2, the dual group of PGL_2. These dominants thus can be identified with $\mathbf{Z}_+$. More precisely, we denote the n-th orbit by

$$\mathrm{Gr}_n = PGL_2[[t]] \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} /PGL_2[[t]].$$

For example, $\mathrm{Gr}_0 \cong \mathrm{pt}$, $\mathrm{Gr}_1 \cong \mathbf{CP}^1$. When $n = 2k$, Gr_n is isomorphic to the Bruhat cells in the affine Grassmannian of SL_2,

$$\mathrm{Gr}_n \cong SL_2[[t]] \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} /SL_2[[t]].$$

Then $Gr^{(0)} = \bigsqcup_{n=2k} \mathrm{Gr}_n$ and $\mathrm{Gr}^{(1)} = \bigsqcup_{n=2k+1} \mathrm{Gr}_n$. The closure $\overline{\mathrm{Gr}}_n$ is the union of all Gr_m for m with the same parity as n. It is in general singular, the IC-sheaf on it is just the constant sheaf with cohomological dimension $-2n$. The analogue of the opposite Bruhat decomposition in this case is given by the $N((t))$-orbits of Gr, where N is the upper triangular unipotent subgroup. We denote them by

$$S_m = N((t)) \begin{pmatrix} t^m & 0 \\ 0 & 1 \end{pmatrix} PGL_2[[t]]/PGL_2[[t]].$$

Then $\overline{\mathrm{Gr}}_n$ is the union of S_m for $|m| \leq n$ and has the same parity as n. In this case

$$\overline{\mathrm{Gr}}_n \cap S_m = \Big\{ \left(\begin{pmatrix} 1 & \sum_{\frac{n-m}{2}}^{n-1} a_i t^i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \right), a_i \in \mathbf{C} \Big\} \cong \mathbf{C}^{\frac{n+m}{2}}.$$

All other $\overline{\mathrm{Gr}}_n \cap S_m$ are empty. Note that this example belongs to the G/P case, instead of G/B. To be more precise, let I be the Iwahori subgroup $\pi^{-1}(B)$, where $\pi : G[[t]] \to G$ is the reduction by t. We have a natural projection $G((t))/I \to G((t))/G[[t]]$. The Weyl group of the affine flag variety is $W \ltimes X_*(T)$. The subgroup corresponding to the affine Grassmannian is the finite weyl group W, a set of minimal length representatives can be identified with the dominant weights $P_+ = W \backslash W \ltimes X_*(T)/W$.

Two other geometric inputs are important for later proof of the main theorem

1. Refined Bruhat decomposition of the Schubert cell X_w.

2. Local structure of $\overline{X}_w$ along X^v for $v < w$.

1. will be used to give a geometric interpretation of the so-called R-polynomials, and 2. is essential in the computation of $i_v^! j_{w!} \mathbf{Q}_\ell$.

We first discuss a refined Bruhat decomposition of Schubert cells first studied by [Deo87]. To describe the combinatorics of this decomposition, we define the so-called J-distinguished expressions.

Definition 7.1.3 (J-distinguished expressions)**.** Let W_J be the subgroup generated by $J \subset S$. W^J, the set of minimal coset representatives of W/W^J. Let $\sigma \in W^J$. We fix a reduced expression $\sigma = s_1 \dots s_r$. The set of J-expressions $\Gamma = \{(\theta_1, \dots, \theta_{r+1}) \in$

$(W^J)^{r+1}|\theta_{r+1} = 1,\ \theta_p \in \{\theta_{p+1}, s_p\theta_{p+1}\}\ \forall\ 1 \leq p \leq r\}$. Let $\mathscr{D}^J$ denote the set of J-expressions such that $\ell(s_p\theta_{p+1}) \geq \ell(\theta_{p+1})$ if $\theta_p = \theta_{p+1}$. The natural projection $\pi : \mathscr{D}^J \to W^J$ is given by $\pi((\theta_1, \dots, \theta_{r+1})) = \theta_1$.

We also define certain "statistics" on J-distinguished expressions

$$n_1(\theta) = \#\{p|\theta_p = \theta_{p+1} \text{ and } s_p\theta_{p+1} \in W^J\}$$
$$n_2(\theta) = \#\{p|\theta_p = \theta_{p+1} \text{ and } s_p\theta_{p+1} \notin W^J\}$$
$$m(\theta) = \#\{p|\theta_p = s_p\theta_{p+1} \text{ and } \ell(\theta_p) \leq \ell(\theta_{p+1})\}$$

The following refined Bruhat decomposition was first stated without proof in [Deo87], we give a proof following Morel's method [Mor11]in the G/B-case.

Proposition 7.1.4 ([Deo87]). *The Schubert cell $X_\sigma = B\sigma \cdot P$ decomposes canonically into a disjoint union of locally closed subvarieties:*

$$B\sigma \cdot P = \bigcup_{\theta \in \mathscr{D}^J} D_\theta.$$

Moreover $D_\theta \cong \mathbf{A}^{m(\theta)+n_2(\theta)} \times \mathbf{G}_m^{n_1(\theta)}$. For $\tau \in W^J$, we have

$$B\sigma \cdot P \cap B^-\tau \cdot P = \bigcup_{\theta \in \mathscr{D}^J, \pi(\theta)=\tau} D_\theta.$$

Proof. Consider the composition of the Bott-Samelson resolution of singularity $p : P_{s_1} \times_B \cdots \times_B P_{\alpha_r}/B \to \overline{BwB/B}$ and $\pi : G/B \to G/P$. If $w \in W^J$, the image of $\pi \circ p$ is just $\overline{BwP/P} \in G/P$. We let $T \subset B$ act on the Bott-Samelson variety by multiplication from the left on the first factor and act on G/P by multiplication on the left. The T-fixed points on $(G/P)^T$ are precisely $\{wP\}_{w \in W^J}$, for v, $w \in W^J$, $BwP \cap B^-vP \neq \emptyset$ iff $v \leq w$ [Kum02]. The T-fixed points on the Bott-Samelson variety are parametrized by the set $\Gamma = \{1, s_1\} \times \cdots \times \{1, s_r\}$. For $(\gamma_1, \dots, \gamma_k) \in \Gamma$, consider the morphism $u_\lambda = (p_{\gamma_1(-\alpha_1)}, \dots, p_{\gamma_r(-\alpha_r)}) : \mathbf{A}^r \to BS$, where $p_\alpha : \mathbf{A}^1 \to U_\alpha \subset G$ is the one-parameter subgroup associated to the root α. Let U_λ be the image, then we have $U_{(s_1,\ldots,s_r)} = (\pi \circ p)^{-1}(BwP/P) = p^{-1}(BwB/B)$. Moreover, for any $(\gamma_1, \ldots, \gamma_r) \in \Gamma$, $U_\lambda \cap U_{(s_1,\ldots,s_r)} = u_\lambda(\{(x_1, \ldots, x_r) \in \mathbf{A}^r | x_i \neq 0 \text{ if } \gamma_i = 1\})$. [Har04] also shows that the cell of contraction associated to the fixed point $[\gamma_1, \ldots, \gamma_r]$ is

$$C_\gamma = u_\gamma(\{(x_1, \ldots, x_r) \in \mathbf{A}^r, x_i = 0 \text{ if } i \notin J(\gamma)\}).$$

Similarly the cell of repelling associated to the fixed point $[\gamma_1, \ldots, \gamma_r]$ is giving by the complimentary directions, that is

$$C^\gamma = u_\gamma(\{(x_1, \ldots, x_r) \in \mathbf{A}^r, x_i = 0 \text{ if } i \in J(\gamma)\}).$$

Since the Bott-Samelson variety is a disjoint union of locally closed subvarieties C^γ, for $\gamma \in \Gamma$, we know $p : BS \to \overline{BwB/B}$ is an isomorphism on BwB/B, hence

$$X_w = \bigcap_{\gamma \in \Gamma} U_{(s_1,\ldots,s_r)} \cap C^\gamma$$

. Moreover, we have [Deo87, Lemma 4.3]

$$B\sigma \cdot P \cap B^-\tau \cdot P = \sum_{\tau w_J \leq \sigma} \pi(B\sigma \cdot B \cap B^{-1}\tau w_J \cdot B).$$

By the discussion above, $B\sigma \cdot P \cap B^-\tau \cdot P$ is $u_\lambda(\{x_1, \ldots, x_r\} \in \mathbf{A}^r)$ under the following conditions

- $x_p = 0$ if $\gamma_1 \ldots \gamma_p(-\alpha_p) \in \Phi^+$, this is possible if and only if $\theta_p = s_p\theta_{p+1}$ and $\ell(\theta_p) > \ell(\theta_{p+1})$. For a component to contribute, the index p must lie in the compliment of $\theta_p = s_p\theta_{p+1}$ and $\ell(\theta_p) > \ell(\theta_{p+1})$, which is $\{p|\theta_p = s_p\theta_{p+1} and \ell(\theta_p) \leq \ell(\theta_{p+1})\} \bigcup \{p|\theta_p = \theta_{p+1}\}$, its cardinality is exactly $m(\theta) + n_1(\theta) + n_2(\theta)$
- $x_p \neq 0$, if $\theta_p = \theta_{p+1}$ and $s_p\theta_{p+1} \in W^J$. This is because if $s_p\theta_{p+1} \notin W^J$, we can

always find another sequence λ in Γ, such that $U_\lambda \cap U_{(s_1,\ldots,s_r)}$ contains the image of $x_p = 0$ and under the projection, it goes to the same locus parametrized by the θ-sequence. Thus $x_p \neq 0$ if and only if $p \in \{p|\theta_p = \theta_{p+1}$ and $s_p\theta_{p+1} \in W^J\}$.

In Summary, we have the desired decomposition

$$B\sigma \cdot P \cap B^-\tau \cdot P = \bigcup_{\theta \in \mathscr{D}^J, \pi(\theta)=\tau} D_\theta.$$

$\square$

Remark 7.1.5. The Bott-Samelson resolution can also be described as $p : P_{\alpha_1} \times \cdots \times P_{\alpha_r}/B^r$ where the product group acts on $P_{\alpha_1} \times \ldots P_{\alpha_r}$ from the right by

$$(p_1, \ldots, p_r)(b_1, \ldots, b_r) = (p_1b_1, b_1^{-1}p_2b_2, \ldots, b_{n-1}p_nb_n).$$

Lemma 7.1.6. *For any $v, w \in W^J$ with $v \leq w$, the varieties $X_w^J \cap B_J^v$ and $X_w^J \cap vB_J^e$ are both affine.*

Proof. See [Kum02, Lemma 7.3.5]. $\square$

Now we briefly describe the local structure of $X_v \hookrightarrow X_w$ for $v \leq w$.

Lemma 7.1.7. *For v, $w \in W^J$ with $v \leq w$, the map*

$$\theta_{v,w} : U_v \times (X_w^J \cap B_Y^v) \to X_w^Y \times vB_Y^e$$
$$(g, x) \mapsto gx$$

is a biregular T-equivariant isomorphism, where T-acts by conjugation on U_v and by left-multiplication on the other two factors.

Proof. See [Kum02, Lemma 7.3.10]. $\square$

Remark 7.1.8. Note that U_v is a set of representatives of $Uv \cdot P$ in our notation, namely, $\theta_{v,w}$ actually defines an affine open neighborhood of $Uv \cdot P$ in $Uw \cdot P$. When G is finite dimensional, what we're saying is that

- $U \cap vU^{-}v^{-1}$ is an affine space of dimension $\ell(v)$. Because U and U^{-} are affine spaces viewed subspaces of $\mathbb{A}^{n^2}$. Conjugate by a given v is a linear map on $\mathbb{A}^{n^2}$. The intersection is just a intersection of two affine subspaces.
- $Bv \cdot B \cong Bv \cdot P \cong U \cap vU^{-1}v^{-1}$.

7.2 Kazhdan-Lusztig theory of G/P

Let W be a Coxeter group and let S be the set of simple reflections. The associated Hecke algebra $\mathcal{H}$ is the $\mathbf{Z}[q^{\pm\frac{1}{2}}]$-algebra generated by T_w for $w \in W$, with the relations:

$$T_w T_{w'} = T_{ww'}, \text{ if } \ell(ww') = \ell(w) + \ell(w')$$
$$(T_s + 1)(T_s - q) = 0$$

$\mathcal{H}$ has an $\mathbf{Z}$-linear involution as defined in [KL80], given by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$, and $T^{-1}_{w^{-1}}$.

We first recall the definition of generalized R-polynomial. Let M^J be a free $\mathbf{Z}[q^{\pm\frac{1}{2}}]$-module generated by $\{m^J_v | \sigma \in W^J\}$. Let u be a root of $(u+1)(u-q)$ i.e. $u = 1$ or q. For $s \in S$, $L(s) \in \mathrm{End}_{\mathbf{Z}[q^{\pm\frac{1}{2}}]}(M^J)$ is defined by

$$L(s)(m^J_w) = \begin{cases} qm^J_{sw} + (q-1)m^J_w, \text{ if } \ell(sw) \leq \ell(w) \\ qm^J_{sw}, \text{ if } \ell(sw) \geq \ell(w) \text{ and } sw \in W^J \\ um^J_w, \text{ if } \ell(sw) \geq \ell(w) \text{ and } sw \notin W^J. \end{cases}$$

The $\mathcal{H}$-module structure on M^J is defined by the map $\phi_J : \mathcal{H} \mapsto M^J$, $T_s \mapsto u^{\ell(w_J)}L(s)$, where w_J comes from the decomposition $s = \sigma w^J$ with $\sigma \in W^J$, $w_J \in W_J$. M^J inherits a $\mathbf{Z}$-linear involution, that is, for $w \in W^J$, $\overline{m_w^J} = \overline{\phi_J(T_w)} := \phi_J(\overline{T_w}) = \phi_J(T_{w^{-1}}^{-1})$. Then the generalized R-polynomial is defined as follows

Definition 7.2.1. $R_{v,w}^J \in \mathbf{Z}[q^{\pm\frac{1}{2}}]$ are defined to be

$$\overline{m}_w^J = \sum_{v \in W^J} (-1)^{\ell(v)+\ell(w)} q^{-\ell(w)} R_{v,w}^J m_v^J.$$

The generalized Kazhdan-Lusztig polynomials $P_{v,w}^J$ can be characterized by the involution invariant elements $\{C_w^J\}$ [Deo87] in M^J. We have

Proposition 7.2.2. *For any $w \in W^J$, the involution invariant element C_w^J can be written as*

$$C_w^J = \sum_{v \in W^J} (-1)^{\ell(v)+\ell(w)} q^{\frac{-(\ell(w)-\ell(v))}{2}} P_{v,w}^J \overline{m}_v^J,$$

where $P_{v,w}^J \in \mathbf{Z}[q]$ are characterized by the formula above and the conditions

- $P_{v,v}^J = 1$,
- $\deg_q(P_{v,w}^J) \leq \frac{\ell(w)-\ell(v)-1}{2}$, *for all $v \leq w$.*

Example 7.2.3. Let s, t be two different simple reflections, then we have

$$C_1 = 1$$
$$C_s = t^{-1}(T_s + 1)$$
$$C_{st} = t^{-2}(T_{st} + T_s + T_t + 1)$$

Then by the definition, we have $P_{1,s}(q) = P_{s,st}(q) = P_{1,st} = 1$.

Example 7.2.4 (SL_4)**.** Let s_1, s_2, s_3 be the simple reflections associated to the simple roots in SL_4, one can check that $P_{s_2,s_2s_1s_3s_2} = 1 + q$. We'll explain the geometric meaning

The geometry of G/P in section 7.1 together with works of Kazhdan-Lusztig [KL79, KL80], we can give $P^J_{v,w}$ and $R^J_{v,w}$ explicit geometric meaning.

Theorem 7.2.5 ([Deo87])**.** *Let $\sigma \in W^J$. The sheaf of cohomology $\mathcal{H}^k(IC_{\overline{X}_\sigma})$ is zero if k is odd. If k is even and $B' \in X_\sigma$ is stable under $\mathbf{Frob}^r$, the power of the Frobenius action $\mathbf{Frob}$. Then the eigenvalues of $(\mathbf{Frob}^r)^*$ on the fibre $\mathcal{H}^k(IC_{\overline{X}_\sigma})_{B'}$ are all equal to $q^{\frac{ik}{2}}$. Moreover, for any $\tau \leq \sigma$, we have*

$$P^J_{\tau,\sigma}(t) = \sum_{k\geq 0} \dim \mathcal{H}^{2k}((\mathrm{IC}_{\overline{X}_\sigma})_{\tau\cdot P})t^i.$$

Corollary 7.2.6. *For all $v, w \in W^J$ with $v \leq w$, we have*

$$P^J_{v,w}(q) = \phi([i^*_{v,w}\mathrm{IC}_{\overline{X}_w}]).$$

Proof. Note that since X_v is a T-equivariant orbit, $\phi([i^*_{v,w}\mathrm{IC}_{\overline{X}_w}])$ can be computed at the fixed point x_v. By the purity of Frobenius action in Theorem 7.2.5, $\mathcal{H}^{2k}((\mathrm{IC}_{\overline{X}_w})_{v\cdot P})$ is exactly the weight k part of the stalk at x_v. We're done. □

Corollary 7.2.7. *For $v, w \in W^J$ with $v \leq w$, we have*

$$R^J_{v,w}(q^k) = \#(X_w \cap X^v)(\mathbf{F}_{q^k})$$

Proof. This is [Deo87, Proposition 4.2], with the notation in the previous section, we have

$$R^J_{v,w} = \sum_{\theta\in\mathscr{D}^J,\pi(\theta)=v} (q-1)^{n_1(\theta)+n_2(\theta)} q^{m(\theta)}.$$

Thus the result follows from the refined Bruhat decomposition of $X_w \cap X^v$ in Proposition 7.1.4. □

7.3 Proof of the theorem and examples

7.3.1 Proof of the main theorem

Lemma 7.3.1. *$i_{v,w}\mathrm{IC}_{\overline{X}_w}$ is in $T(X_v)$.*

Proof. See [Mor11]. ☐

Lemma 7.3.2. *For all $v, w \in W^J$ with $v \leq w$, for any $K \in \mathcal{T}(X_w)$, the complex $i^!_{v,w}j_{w!}K$ is in $\mathcal{T}(X_v)$ and there's a $\mathbf{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$-equivariant isomorphism*

$$i^!_{v,w}j_{w!}K \cong R\Gamma_c((X_w \cap X^v)_{\overline{\mathbf{F}}_q}, K|_{X_w \cap X^v}).$$

Moreover $\phi(i^!_{v,w}j_{w!}K) = \phi(K)R^J_{v,w}(t)$.

Proof. We have the following commuting diagram,

$$\begin{tikzcd}
 & \overline{X}_w & X_w \arrow[l, "j_w"'] \\
X_v \arrow[ur, "i_{v,w}"] \arrow[r, "(id,x_v)"'] & X_v \times (\overline{X}_w \times X^v) \arrow[u] & X_v \times (X_w \cap X^v). \arrow[l, "(id,j)"] \arrow[u]
\end{tikzcd}$$

Thus

$$\begin{aligned}
i^!_{v,w}j_{w!}K &= (id, x_v)^!(id, j)_!(\mathbf{Q}_{\ell,X_v} \boxtimes \mathbf{Q}_{\ell,X_w\cap X^v}) \\
&= \mathbf{Q}_\ell \boxtimes (x^!_v j_! \mathbf{Q}_\ell(m)).
\end{aligned}$$

This implies that

$$(i^!_{v,w}j_{w!}K)_{x_v} = x^!_v j_! \mathbf{Q}_\ell(m) = x^!_v j_{w!}(\mathbf{Q}_\ell|_{X_w \cap X^v}).$$

Since the torus action contracts $X_w \cap X^v$ to x_v, according to [Mor11][Sous-lemme 1], we have a $\mathbf{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$-isomorphism

$$x_v^!(j_!\mathbf{Q}_\ell(m)) \cong R\Gamma_c((X_w \cap X^v)_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell(m)|_{X_w\cap X^v}).$$

Now we can compute the value of the Laurent polynomial $\phi([i_{v,w}^! j_{w!}K])$ at $t = q^k$ by the Grothendieck-Lefschetz trace formula

$$\begin{aligned}
\phi([i_{v,w}^! j_{w!}K]) &= \mathbf{Tr}(\mathbf{Frob}^{k*}, R\Gamma_c((X_w \cap X^v)_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell(m)|_{X_w\cap X^v})) \\
&= \sum_{x\in(X_w\cap X^v)(\mathbf{F}_{q^k})} \mathbf{Tr}(\mathbf{Frob}_x^*, \mathbf{Q}_\ell(m))) \\
&= q^{-km}\#(X_w \cap X^v)(\mathbf{F}_{q^k}) \\
&= q^{-km} R_{v,w}^J(q^k) \\
&= \phi([K])R_{v,w}^J(q^k).
\end{aligned}$$

$\square$

Theorem 7.3.3 ([WZ19])**.** *For any τ, $\sigma \in W^J$, the generalized Kazhdan-Lusztig polynomial can be computed from the generalized R-polynomials:*

$$P_{\tau,\sigma}^J = \tau_{\ell(\sigma)-\ell(\tau)-1} \sum_{\tau=v_1<\cdots<v_r<\sigma} (-1)^r (T_1 \circ \cdots \circ T_{r-1} \circ T_r)\mathbf{1},$$

where $\mathbf{1}$ is the constant polynomial 1 and

$$T_r(f) = \tau_{\ell(\sigma)-\ell(v_{r+1})}(R_{v_{r+1},\sigma}^J \cdot f)$$

Proof. We specialize Theorem 6.3.2 to the case $\mathcal{F} = j_{!*}\mathbf{Q}_\ell[n]$. Apply ϕ to both sides, the left-hand side gives us $P_{\tau,\sigma}^J$ by 7.2.6. Then the result follows from Lemma 7.3.2 and Lemma 6.3.6. $\square$

7.3.2 Examples: $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ partial flag varieties

Example 7.3.4 (SL_3)**.** Let $G = SL_3$, $B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, the standard Borel subgroup. $P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$, the standard parabolic subgroup corresponding to the simple root $\{\alpha_2 - \alpha_3\}$. Then G/B is the variety of complete flags in $\mathbf{C}^3$, G/P parametrizes lines in $\mathbf{C}^3$, in other words, $G/P \cong \mathbf{P}^2$. The G-action on $\mathbf{P}^2$ is just the matrix multiplication on column vectors $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$ from the left. $W_J = \langle (2,3) \rangle \cong \mathbf{Z}/2\mathbf{Z}$, the right cosets and minimal length representatives are given as follows:

$$
\begin{aligned}
()W_J &= \{(), (2,3)\} \\
(1,2)W_J &= \{(1,2), (1,2,3)\} \\
(1,3,2)W_J &= \{(1,3), (1,3,2)\}.
\end{aligned}
$$

Note that although $(1,3)$ is a simple permutation, it has the longest length 3 in the Bruhat order. The geometry of the fibration is encoded in the SL_3 Bruhat order graph and the subgraph associated to the partial flag variety, in our case, it's just $\mathbf{P}^2$. See Fig 7. The pink line segments encodes the cell structure of $G/P \cong \mathbf{P}^2$, the green, blue and purple line segments are the corresponding $\mathbf{P}^1$-fibration over each strata.

First we lift elements in $W^J = \{(), (1,2), (1,3,2)\}$ to G for later computation.

$$
() = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (1,3,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

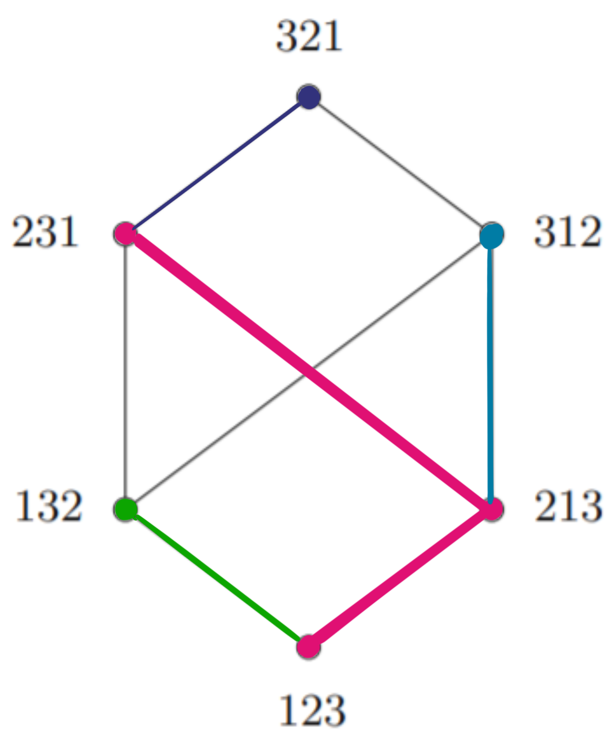


Figure 7: SL_3 Bruhat graph and fibration

The corresponding points are given by the image of the action these elements on $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in$ $\mathbf{P}^2$. The Bruhat decomposition is given by the B-orbits:

$$X_{()} = \{x_1 = x_2 = 0\} \cong \text{pt}$$
$$X_{(1,2)} = \{x_1 \neq 0, x_2 = 0\} \cong \mathbf{A}^1$$
$$X_{(1,3,2)} = \{x_2 \neq 0\} \cong \mathbf{A}^2.$$

The opposite Bruhat decomposition is given by the B^--orbits,

$$X^{()} = \{x_0 \neq 0\} \cong \mathbf{A}^2$$
$$X^{(1,2)} = \{x_0 = 0, x_1 \neq 0\} \cong \mathbf{A}^1$$
$$X^{(1,3,2)} = \{x_0 = x_1 = 0\} \cong \text{pt}.$$

Next we demonstrate the key local structure of $\overline{X}_w$ along X_v has a nice product description. For example $X_{(1,2)} \hookrightarrow \overline{X}_{(1,3,2)} = \mathbf{P}^2$. It's given by the following multiplication

$$X_{(1,2)} \times X^{(1,2)} \to \mathbf{P}^2$$
$$(bv \cdot P, [gP]) \mapsto [bgP].$$

Namely, the first component is viewed as elements in the group, the second component $X^{(1,2)} = B^-(1,2)$-orbits of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is viewed as points in $G/P \cong \mathbf{P}^2$, the map is induced from the group action on the left. In our case, it's given by

$$(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ y \end{pmatrix}) \mapsto \begin{pmatrix} bx+cy \\ dx+ey \\ fy \end{pmatrix},$$

where $a, d, f, x \neq 0$. $dx = (dx + ey) - \frac{e}{f}(fy)$, thus we know the image is isomorphic to $\mathbf{A}^2$. To be more clear, the morphism defined above has some ambiguity in the first factor, actually, we have to choose representatives in $X_{(1,2)} \cong U \cap (1,2)U_J^-(1,2)$, where $U_J^- = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$. That is $X_{(1,2)} \cong \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus in the map above, we're safe to assume that $a = d = f = 1, c = e = 0$, the image is given by $\begin{pmatrix} bx \\ dx \\ y \end{pmatrix}$, since $d, x \neq 0$, this is just $\mathbf{A}^2 \cong \{x_1 \neq 0\} \subset \mathbf{P}^2$, which is surely an open neighborhood of $X_{(1,2)} = \{x_1 \neq 0, x_2 = 0\}$, even more down to earth, the local picture of $X_{(1,2)}$ is just the second axis embedded in a plane. This is essential for our computation of $i_v^! j_{w!}\overline{\mathbf{Q}}_\ell$.

Next, we try to compute all the relative R-polynomials and relative P-polynomials. By the geometric interpretation of R-polynomials, we have

$R^J_{u,v}$	()	(12)	(132)
()	1	$q-1$	$q(q-1)$
(12)		1	$q-1$
(132)			1

On the other hand, $G/P \cong \mathbf{P}^2$ is smooth in our case,the Bruhat decomposition agrees with the ordinary cell decomposition of $\mathbf{P}^2$, all the associated intermediate extension complexes are constant $\mathbf{Q}_\ell$ sheaves, thus we have

$P^J_{u,v}$	()	(12)	(132)
()	1	1	1
(12)		1	1
(132)			1

We demonstrate what does the weight truncation theorem says in this case.

- $v = ()$ and $w = (12)$, then $\ell(v) = 0$ and $\ell(w) = 1$, we have one and only one possible path from () to (12) in the Bruhat order. Then the main theorem 7.3.3 says

$$\begin{aligned} P^J_{(),(12)} &= \tau_{\leq 0}(-1)^1 \tau_{\leq 1} R^J_{(),(12)}(q) \\ &= \tau_{\leq 0}(-1)^1 \tau_{\leq 1}(q-1) \\ &= \tau_{\leq 0}(-1)^1(-1) \\ &= 1. \end{aligned}$$

- $v = (12)$ and $w = (132)$, $\ell(v) = 1$ and $\ell(w) = 2$, exactly similar to the case above, the main theorem says that

$$\begin{aligned} P^J_{(12),(132)} &= \tau_{\leq 0}(-1)^1 \tau_{\leq 1} R^J_{(12),(132)}(q) \\ &= \tau_{\leq 0}(-1)^1 \tau_{\leq 1}(q-1) \\ &= \tau_{\leq 0}(-1)^1(-1) \\ &= 1. \end{aligned}$$

- $v = ()$ and $w = (132)$, $\ell(v) = 0$ and $\ell(w) = 2$. Now we have two different paths from () to (132) in the Bruhat order, $() \to (12) \to (132)$ and $() \to (132)$. The main theorem says

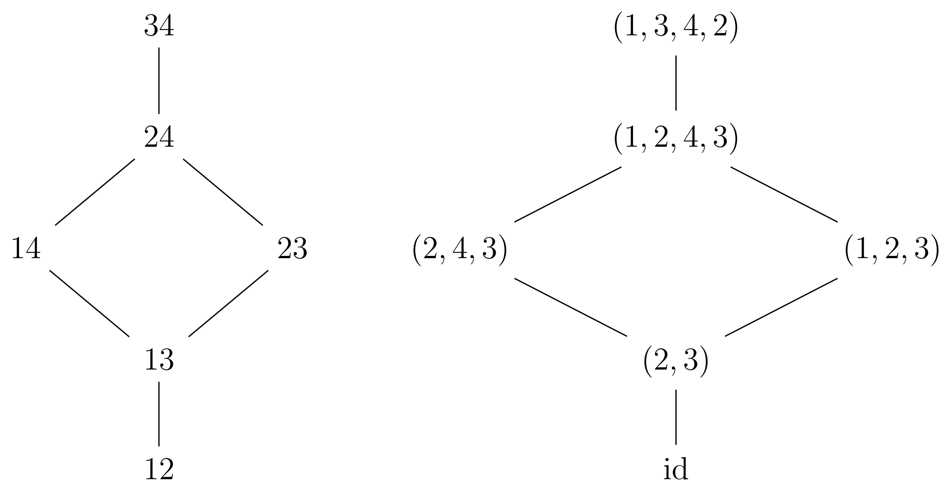
$$\begin{aligned}
P^J_{(),(132)} &= \tau_{\le 1}(-1)^2 R^J_{(),(12)}\tau_{\le 1}R^J_{(12),(132)}(q) + \tau_{\le 1}(-1)^1\tau_{\le 2}R^J_{(),(132)}(q) \\
&= \tau_{\le 1}(-1)^2(q-1)\tau_{\le 1}(q-1) + \tau_{\le 1}(-1)^1\tau_{\le 2}q(q-1) \\
&= \tau_{\le 1}(-1)^2(q-1)(-1) + \tau_{\le 1}(-1)^1(-q) \\
&= 1+0 \\
&= 1.
\end{aligned}$$

We thus conclude that in the SL_3 case we have fully checked the validity of the main theorem for G/P.

Remark 7.3.5. Note that $\deg(q) = 2$.

Example 7.3.6 (SL_4 and Grassmannian)**.** Take a maximal parabolic subgroup P of $SL(n)$, we can get a relation between the Kazhdan-Lusztig polynomials of Schubert varieties in Grassmannian and the R-polynomials. For example, $G = SL_4$, the corresponding parabolic subgroup is given by the isotropic group of the standard flag, $P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$. In the language of Coxeter groups, we have $W = S_4$,

$J = \{\alpha_1 - \alpha_2, \alpha_3 - \alpha_4\}$. $W_J \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, $|W^J| = 6$. The Bruhat order is given by



W^J, the set of minimal length representatives of W/W_J is on the right-hand side. More precisely, the corresponding cosets are

$$\begin{aligned}
()W_J &= \{(), (3,4), (1,2), (1,2)(3,4)\} \\
(2,3,4)W_J &= \{(2,3), (2,3,4), (1,3,2), (1,3,4,2)\} \\
(2,4,3)W_J &= \{(2,4,3), (2,4), (1,4,3,2), (1,4,2)\} \\
(1,2,3)W_J &= \{(1,2,3), (1,2,3,4), (1,3), (1,3,4)\} \\
(1,2,4,3)W_J &= \{(1,2,4,3), (1,2,4), (1,4,3), (1,4)\} \\
(1,3)(2,4)W_J &= \{(1,3)(2,4), (1,3,2,4), (1,4,2,3), (1,4)(2,3)\}.
\end{aligned}$$

The geometry of the fibration $\pi : G/B \to G/P$ is shown in the SL_4 Bruhat order graph and the subgraph associated to the corresponding partial flag variety $G/P \cong \mathrm{Gr}(2,4)$. See Figure 8. The bold pink line segments give the cell structure of G/P, the 4-gons over each vertex on the pink subgraph means the $\mathbf{P}^1 \times \mathbf{P}^1$-fibration.

Now we compute some of the R-polynomials and Kazhdan-Lusztig polynomials. We first collect some information of W_J and reduced expressions of elements for elements in W^J. If we denote the simple reflections by $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$,

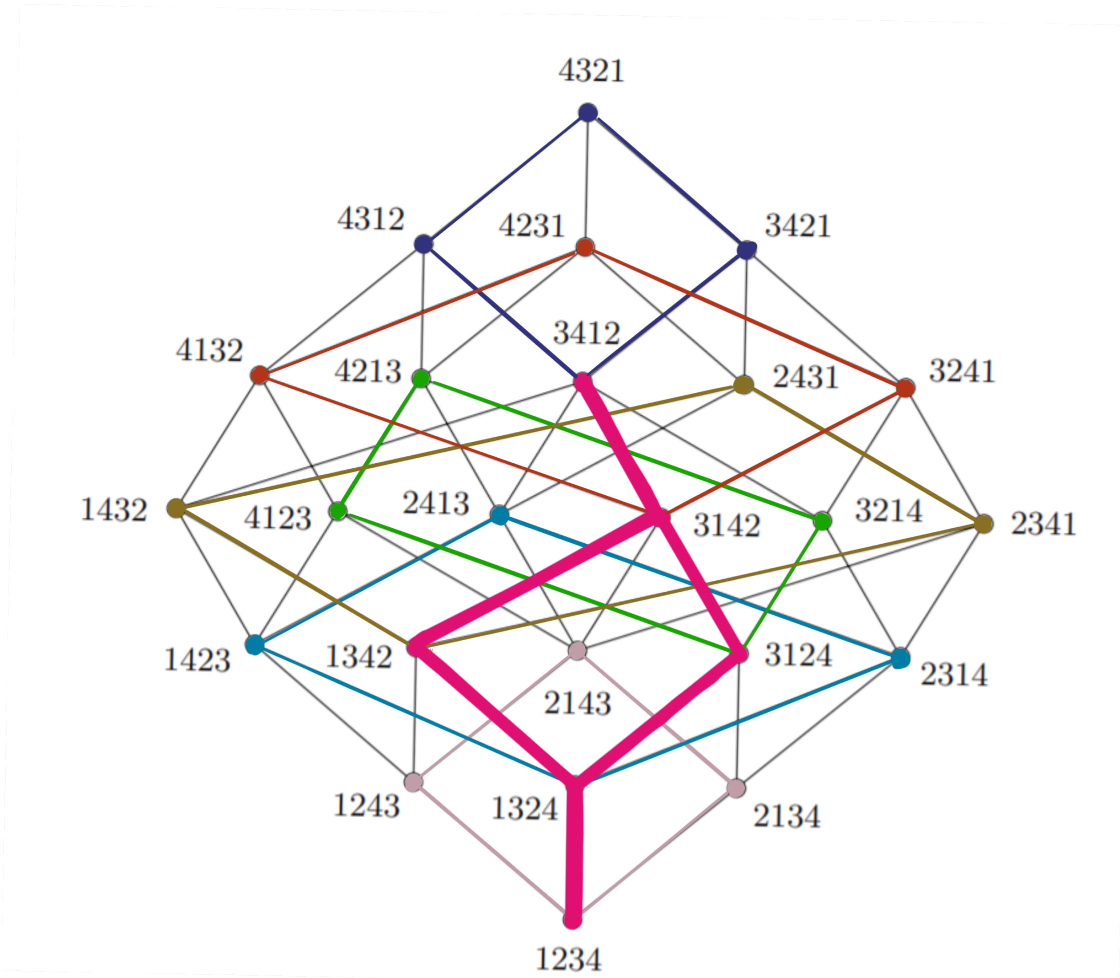


Figure 8: SL_4 Bruhat graph and fibration

then we have $W_J = \{1, s_1, s_3, s_1s_3\}$, s_1s_3 is the longest element in W_J. $W^J = \{1, s_2, s_3s_2, s_1s_2, s_3s_1s_2, s_2s_3s_1s_2\}$. We can compute the generalized R-polynomials and Kazhdan-Lusztig polynomials similarly as in the previous examples. The results are given in the following two tables, note that an empty cell means 0.

$P^J_{u,v}$	1	s_2	s_3s_2	s_1s_2	$s_3s_1s_2$	$s_2s_3s_1s_2$
1	1	1	1	1	$q+1$	1
s_2		1	1	1	1	1
s_3s_2			1		1	1
s_1s_2				1	1	1
$s_3s_1s_2$					1	1
$s_2s_3s_1s_2$						1

$R^J_{u,v}$	1	s_2	s_3s_2	s_1s_2	$s_3s_1s_2$	$s_2s_3s_1s_2$
1	1	$q-1$	q^2-q	q^2-q	q^3-q^2	$q^4-q^3-q^2+q$
s_2		1	$q-1$	$q-1$	q^2-2q+1	q^3-q^2
s_3s_2			1		$q-1$	q^2-q
s_1s_2				1	$q-1$	q^2-q
$s_3s_1s_2$					1	$q-1$
$s_2s_3s_1s_2$						1

We only demonstrate the most non-trivial case when $u = 1$ ((1234) in the Bruhat graph) and $v = s_3s_1s_2$ ((3142) in the Bruhat graph). We have 6 path in total, we compute the contribution one by one

- (1234) $\to$ (3142). $\tau_{\leq 2}(-1)R_{1,s_3s_1s_2} = \tau_{\leq 2}(q^2 - q^3) = 0$
- (1234) $\to$ (1324) $\to$ (3142). $\tau_{\leq 2}(-1)^2 R_{1,s_2}\tau_{\leq 2}R_{s_2,s_3s_1s_2} = \tau_{\leq 2}(q-1)\tau_{\leq 2}(q^2 - 2q + 1) = 3q - 1$
- (1234) $\to$ (1342) $\to$ (3142). $\tau_{\leq 2}(-1)^2 R_{1,s_3s_2}\tau_{\leq 1}R_{s_3s_2,s_3s_1s_2} = \tau_{\leq 2}(q^2 - q)\tau_{\leq 1}(q - 1) = q$

- $(1234) \to (3124) \to (3142)$. $\tau_{\leq 2}(-1)^2 R_{1,s_1s_2}\tau_{\leq 1}R_{s_1s_2,s_3s_1s_2} = \tau_{\leq 2}(q^2-q)\tau_{\leq 1}(q-1) = q$

- $(1234) \to (1324) \to (1342) \to (3142)$. $\tau_{\leq 2}(-1)^3 R_{1,s_2}\tau_{\leq 2}R_{s_2,s_3s_2}\tau_{\leq 1}R_{s_3s_2,s_3s_1s_2} = -\tau_{\leq 2}(q-1)\tau_{\leq 2}(q-1)\tau_{\leq 1}(q-1) = -2q+1$

- $(1234) \to (1324) \to (3124) \to (3142)$. $\tau_{\leq 2}(-1)^3 R_{1,s_2}\tau_{\leq 2}R_{s_2,s_1s_2}\tau_{\leq 1}R_{s_1s_2,s_3s_1s_2} = -\tau_{\leq 2}(q-1)\tau_{\leq 2}(q-1)\tau_{\leq 1}(q-1) = -2q+1$.

In total, we get

$$3q-1+q+q+(-2q+1)+(-2q+1) = q+1,$$

which is exactly $P_{1,s_3s_1s_2} = q+1$!

Bibliography

[AB09] Marian Aprodu and Vasile Brînzănescu, *Beilinson type spectral sequences on scrolls*, Moduli spaces and vector bundles, London Math. Soc. Lecture Note Ser., vol. 359, Cambridge Univ. Press, Cambridge, 2009, pp. 426–436. MR 2537076 26

[AO17] Mina Aganagic and Andrei Okounkov, *Quasimap counts and Bethe eigenfunctions*, Mosc. Math. J. **17** (2017), no. 4, 565–600. MR 3734654 5, 6, 53, 64

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966 71, 72, 74, 76, 77

[BBR15] Claudio Bartocci, Ugo Bruzzo, and Claudio L. S. Rava, *Monads for framed sheaves on Hirzebruch surfaces*, Adv. Geom. **15** (2015), no. 1, 55–76. MR 3300710 26

[Bre98] Francesco Brenti, *Lattice paths and Kazhdan-Lusztig polynomials*, J. Amer. Math. Soc. **11** (1998), no. 2, 229–259. MR 1460390 6

[Buc87] N. P. Buchdahl, *Stable 2-bundles on Hirzebruch surfaces*, Math. Z. **194** (1987), no. 1, 143–152. MR 871226 26, 27, 28

[CP17a] Yaim Cooper and Rahul Pandharipande, *A Fock space approach to Severi degrees*, Proc. Lond. Math. Soc. (3) **114** (2017), no. 3, 476–494. MR 3653237 3, 4, 52, 58

[CP17b] ———, *A Fock space approach to Severi degrees*, Proc. Lond. Math. Soc. (3) **114** (2017), no. 3, 476–494. MR 3653237 5, 8, 21, 47, 50, 51, 55, 57, 58, 62

[Del80] Pierre Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252. MR 601520 75

[Deo85] Vinay V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. Math. **79** (1985), no. 3, 499–511. MR 782232 7

[Deo87] ———, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra **111** (1987), no. 2, 483–506. MR 916182 6, 7, 82, 83, 84, 87, 88

[EG00] Geir Ellingsrud and Lothar Göttsche, *Hilbert schemes of points and Heisenberg algebras*, School on Algebraic Geometry (Trieste, 1999), ICTP Lect. Notes, vol. 1, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000, pp. 59–100. MR 1795861 10

[EHX97] Tohru Eguchi, Kentaro Hori, and Chuan-Sheng Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. B **402** (1997), no. 1-2, 71–80. MR 1454328 2

[FK81] I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980/81), no. 1, 23–66. MR 595581 4, 19

[Fuk86] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986, Translated from the Russian by A. B. Sosinskiĭ. MR 874337 110

[Gin12] Victor Ginzburg, *Lectures on Nakajima's quiver varieties*, Geometric methods in representation theory. I, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 145–219. MR 3202703 59

[Giv01] Alexander B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, vol. 1, 2001, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary, pp. 551–568, 645. MR 1901075 2

[GP99] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518. MR 1666787 44, 54

[Gro96] I. Grojnowski, *Instantons and affine algebras. I. The Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996), no. 2, 275–291. MR 1386846 23

[GV05] Tom Graber and Ravi Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130** (2005), no. 1, 1–37. MR 2176546 44

[Har04] Martin Harterich, *The T-equivariant cohomology of Bott-Samelson varieties*. 84

[IP03] Eleny-Nicoleta Ionel and Thomas H. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. (2) **157** (2003), no. 1, 45–96. MR 1954264 49

[Kat05] Nicholas M. Katz, *Moments, monodromy, and perversity: a Diophantine perspective*, Annals of Mathematics Studies, vol. 159, Princeton University Press, Princeton, NJ, 2005. MR 2183396 76

[KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184. MR 560412 6, 88

[KL80] ———, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203. MR 573434 6, 7, 86, 88

[KMPS90] S. Kass, R. V. Moody, J. Patera, and R. Slansky, *Affine Lie algebras, weight multiplicities, and branching rules. Vols. 1, 2*, Los Alamos Series in Basic and Applied Sciences, vol. 9, University of California Press, Berkeley, CA, 1990. MR 1117679 15

[Kon92] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23. MR 1171758 2

[KR87] V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematical Physics, vol. 2, World Scientific Publishing Co., Inc., Teaneck, NJ, 1987. MR 1021978 9, 12, 14, 63

[Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1923198 83, 85

[Kuz07] Alexander Kuznetsov, *Quiver varieties and Hilbert schemes*, Mosc. Math. J. **7** (2007), no. 4, 673–697, 767. MR 2372209 30, 60

[Leh99] Manfred Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), no. 1, 157–207. MR 1681097 60

[Li02] Jun Li, *A degeneration formula of gw-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293. 49

[LLZ07] Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou, *A formula of two-partition Hodge integrals*, J. Amer. Math. Soc. **20** (2007), no. 1, 149–184. MR 2257399 38, 40, 41, 42

[LR01] An-Min Li and Yongbin Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), no. 1, 151–218. MR 1839289 49

[Mau09] Davesh Maulik, *Gromov-Witten theory of $\mathscr{A}_n$-resolutions*, Geom. Topol. **13** (2009), no. 3, 1729–1773. MR 2496055 3, 36, 46, 47, 57

[MO09] Davesh Maulik and Alexei Oblomkov, *Quantum cohomology of the Hilbert scheme of points on $\mathscr{A}_n$-resolutions*, J. Amer. Math. Soc. **22** (2009), no. 4, 1055–1091. MR 2525779 3, 4, 8, 21, 24, 31, 33, 34, 37, 53, 57, 58, 62

[MO12] Davesh Maulik and Andrei Okounkov, *Quantum Groups and Quantum Cohomology*. 64

[Mor08] Sophie Morel, *Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel*, J. Amer. Math. Soc. **21** (2008), no. 1, 23–61. MR 2350050 7, 77, 78, 79

[Mor11] ———, *Note sur les polynômes de Kazhdan-Lusztig*, Math. Z. **268** (2011), no. 1-2, 593–600. MR 2805449 6, 7, 8, 83, 89

[Nak] Hiraku Nakajima, *Jack polynomials and hilbert schemes of points on surfaces*, alg-geom/9610021. 61

[Nak99] ———, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999. MR 1711344 10, 11, 23, 26, 29, 30

[NS09] Nikita A. Nekrasov and Samson L. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, Nuclear Phys. B Proc. Suppl. **192/193** (2009), 91–112. MR 2570974 5

[NS10] ———, *Quantization of integrable systems and four dimensional gauge theories*, XVIth International Congress on Mathematical Physics, World Sci. Publ., Hackensack, NJ, 2010, pp. 265–289. MR 2730782 5

[OP06] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), no. 1, 47–108. MR 2208418 2

[OP09] ———, *Gromov-Witten theory, Hurwitz numbers, and matrix models*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 325–414. MR 2483941 2

[OP10] ———, *Quantum cohomology of the Hilbert scheme of points in the plane*, Inventiones mathematicae **179** (2010), no. 3, 523–557. 3

[Pop74] V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fiberings*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 294–322. MR 0357399 16

[Wan19] Shuai Wang, *Relative gromov-witten theory and vertex operators.* 1, 5

[Wit91] Edward Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310. MR 1144529 2

[WZ19] Shuai Wang and Yihang Zhu, *Non-recursive formulas for Kazhdan-Lusztig polynomials of finite and affine G/P.* 1, 6, 90

Appendix

Cohomology of infinite dimensional Lie algebras

In this extra section, explain the relation between non-trivial class in $\mathrm{H}^2(\mathfrak{g}, \mathbb{C})$ and central extension of the Lie algebra $\mathfrak{g}$. This clarifies certain aspects of the central extension in the construction of Kac-Moody Lie algebras.

Central extension and $\mathrm{H}^2(\mathfrak{g}, \mathbf{C})$

Recall that a central extension of a Lie algebra $\mathfrak{g}$ is defined to be an exact sequence of Liealgebras

$$0 \to \mathbf{C} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

such that the image of $\mathbf{C}$ is contained in the center of $\widetilde{\mathfrak{g}}$. For a cohomology class $c \in \mathrm{H}^2(\mathfrak{g}, \mathbf{C})$, the corresponding one-dimensional central extension is

$$0 \to \mathbf{C} \xrightarrow{\lambda \mapsto (\lambda, 0)} \widetilde{\mathfrak{g}} \xrightarrow{(\lambda, g) \mapsto g} \mathfrak{g} \to 0,$$

where the Lie bracket if given by

$$[(\lambda, g), (\mu, h)] = (c(g, h), [g, h]).$$

The Jacobi identity of this bracket is equivalent to the cocycle condition of c in Lie algebra cohomology, namely $0 = dc(g_1, g_2, g_3) = c([g_1, g_2], g_3) - c([g_1, g_3], g_2) + c([g_2, g_3], g_1)$. The boundary condition is equivalent to the trivial direct sum extension, note that two central extensions $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}'$ are equivalent if there exist a Lie algebra morphism f such that the following diagram commutes

$$\begin{tikzcd}
 & & \widetilde{\mathfrak{g}} \arrow[dd, dashed, "\exists f"] \arrow[dr, "(\lambda,g)\mapsto g"] & & \\
0 \arrow[r] & \mathbf{C} \arrow[ur, "\lambda\mapsto(\lambda,0)"] \arrow[dr, "\lambda\mapsto(\lambda,0)"'] & & \mathfrak{g} \arrow[r] & 0 \\
 & & \widetilde{\mathfrak{g}}'. \arrow[ur, "(\lambda,g')\mapsto g'"'] & &
\end{tikzcd}$$

The commuting condition forces the morphism between $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}'$ to be of the form $f : (\lambda, g) \mapsto (\lambda + \mu(g), g)$, where μ is a linear map. Then if f is an isomorphism between the associated central extension and direct sum extension, we have

$$\begin{aligned}
f[(\lambda_1, g_1), (\lambda_2, g_2)] &= f((c(g_1, g_2), [g_1, g_2])) \\
&= (c(g_1, g_2) + \mu([g_1, g_2]), [g_1, g_2]) \\
&= (0, [g_1, g_2])(= [f((\lambda_1, g_1)), f((\lambda_2, g_2))] \text{ in the trivial extension}).
\end{aligned}$$

This means exactly $c = d\mu$. In conclusion, we know $\mathrm{H}^2(\mathfrak{g}, \mathbf{C})$ is one-to-one correspond to equivalent classes of central extensions of the Lie algebra $\mathfrak{g}$.

Remark 7.3.7. It's straightforward from the Lie algebra (co)homology definition, we have

$$\begin{aligned}
\mathrm{H}_1(\mathfrak{g}) &= \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \\
\mathrm{H}^1(\mathfrak{g}) &= (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])'
\end{aligned}$$

.

Example 7.3.8 (Virasoro algebra)**.** The Lie algebra cohomology ring of $\mathrm{Vect}(S^1)$—the Lie algebra of complex polynomial vector fields on S^1 is a tensor product of the algebra of polynomial generated by a single element x in degree 2 and the exterior algebra generated by a single element y in degree 3

$$\mathrm{H}^\bullet(\mathrm{Vect}(S^1), \mathbf{C}) = \mathbf{C}[x] \otimes \wedge^\bullet y.$$

Specially, $\mathrm{H}^k(\mathfrak{g}, \mathbf{C}) = \mathbf{C}$ if $k \neq 1$, and $\mathrm{H}^1 = 0$. Thus we know the Virasoro algebra is the unique nontrivial central extension of $\mathrm{Vect}(S^1)$, though we use a specific two-cocycle $a(m,n) = \frac{m^3-m}{12}$. $\mathrm{Vect}(S^1)$ is semisimple, it's obvious from the definition, it's good to also have a cohomological interpretation that $\mathrm{H}^1(\mathrm{Vect}(S^1)) = 0$.

Example 7.3.9 ($\widehat{\mathfrak{sl}}_2$)**.** By [Fuk86, Page 194], we have

$$\dim \mathrm{H}_k(\mathfrak{sl}_2[t, t^{-1}], \mathbf{C}) = \begin{cases} 0, & \text{for } k = 0 \\ 1, & \text{otherwise.} \end{cases}$$

The central extension of $\mathfrak{sl}_2[t^{\pm 1}]$ contained in $\widehat{\mathfrak{sl}_2}$ is essentially the unique non-trivial extension of $\mathfrak{sl}_2[t^{\pm 1}]$ that we can think of.