On a discounted notion of strict dissipativity [⋆]

Lars Grüne [∗] **Christopher M. Kellett** [∗∗] **Steven R. Weller** [∗∗]

[∗] *Mathematical Institute, University of Bayreuth, 95440 Bayreuth, Germany (e-mail: lars.gruene@uni-bayreuth.de)*
[∗∗] *School of Electrical Engineering and Computer Science at the University of Newcastle, Callaghan, New South Wales 2308, Australia (e-mail: chris.kellett, steven.weller@newcastle.edu.au)*

Abstract: Recent results in the literature have provided connections between the so-called turnpike property, near optimality of closed-loop solutions using model predictive control schemes, and strict dissipativity. An important feature of these results is that strict dissipativity provides a checkable condition for the other two properties. These results relate to optimal control problems with undiscounted stage cost. Motivated by applications in economics, we consider optimal control problems with *discounted* stage cost and define a notion of discounted strict dissipativity. As in the undiscounted case, we show that discounted strict dissipativity provides a checkable condition for various properties of the solutions of the optimal control problem associated with the appropriately defined discounted available storage function.

Keywords: Nonlinear Systems, Dissipativity, Optimal Control

1. INTRODUCTION

Since its introduction by Willems (1972), dissipativity has become one of the most widely used concepts in mathematical systems theory. Recent research has established close connections between strict dissipativity and the stability and near-optimality of closed-loop solutions of model predictive control schemes, see Angeli et al. (2012); Grüne and Stieler (2014); Grüne and Panin (2015). In this paper, we extend this research to discrete-time optimal control problems with discounted stage cost, wherein the value function incorporates, at each time $k \in \mathbb{N}_0$, a multiplicative term β^k where $0 < \beta < 1$ is called the *discount factor*.

Our consideration of discounted optimal control problems is motivated by applications in economics, where discounting is pervasive. In the Ramsey–Cass–Koopmans (RCK) model of neoclassical economic growth, for example, policies are chosen so as to maximize a social welfare function consisting of a discounted sum of aggregate economic utility (Ramsey (1928); Cass (1965); Koopmans (1965); Brock and Mirman (1972)). In this framework, the discount factor reflects the weighting attached to the economic utility enjoyed by different generations (Nordhaus and Sztorc (2013)).

One specific application of the RCK framework that is prominent in climate change economics is the DICE (Dynamic Integrated model of Climate and the Economy) integrated assessment model (IAM) of Nordhaus (2014). In DICE, trajectories of anthropogenic carbon dioxide (CO_2) emissions reflect an optimal tradeoff between reduced economic consumption today and economically harmful climate change in the future. In this context, the choice of discount factor plays a central role in determining the conclusions of the IAM-based optimal abatement analyses, e.g. Nordhaus (2014) and Stern (2007). The policy-relevance of DICE (see, e.g., the Interagency Working Group on Social Cost of Carbon, United States Government (2013)) therefore provides strong motivation for an optimal control framework which incorporates discounting. Moreover, model predictive control appears to be ideally suited to analyze the behavior of this model under uncertainty, see Weller et al. (2015), which motivates extending the study of near-optimality of model predictive control schemes to the discounted setting. We expect that the discounted version of strict dissipativity presented in this paper will provide an important building block for this study.

The paper is organized as follows. Section 2 defines the system class we are considering and provides background information on (non-discounted) strict dissipativity. Section 3 defines the newly proposed notion of discounted strict dissipativity and shows its relation to a positive definite auxiliary system. Section 4 studies the particular case of affine linear systems with convex cost while Section 5 extends the concept of available storage to the discounted case using a concept of robustly optimal equilibria. Section 6 shows that discounted strict dissipativity implies a continuity property of optimal trajectories and Section 7 concludes the paper.

2. SETTING AND PRELIMINARIES

2.1 System class and notation

We consider discrete time nonlinear systems of the form

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \qquad (1)$$

for a map $f : X \times U \to X$, where X and U are normed spaces. We also write (1) briefly as $x^+ = f(x, u)$. We impose the constraints $(x, u) \in \mathbb{Y} \subseteq X \times U$ on the state x and the input u and define $\mathbb{X} := \{x \in X \mid \exists u \in U : (x, u) \in \mathbb{Y}\}$ and $\mathbb{U} := \{u \in U \mid \exists x \in X : (x, u) \in \mathbb{Y}\}$. A control sequence $u \in \mathbb{U}^N$ is called admissible for $x_0 \in \mathbb{X}$ if $(x(k), u(k)) \in \mathbb{Y}$ for $k = 0, \dots, N-1$ and $x(N) \in \mathbb{X}$. In this case, the corresponding trajectory $x(k)$ is also called admissible. The set of admissible control sequences is denoted by $\mathbb{U}^N(x_0)$. Likewise, we define $\mathbb{U}^\infty(x_0)$ as the set of all control sequences $u \in \mathbb{U}^\infty$ with $(x(k), u(k)) \in \mathbb{Y}$ for all $k \in \mathbb{N}_0$. In order to keep the presentation technically simple, we assume that $\mathbb{X}$ is controlled invariant, i.e., that $\mathbb{U}^\infty(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{X}$. We expect that our results remain true if one restricts the initial values under consideration to the viability kernel $\mathbb{X}_\infty := \{x_0 \in \mathbb{X} \mid \mathbb{U}^\infty(x_0) \neq \emptyset\}$, however, the technical details of this extension are beyond the scope of this paper. The trajectories of (1) are denoted by $x_u(k, x_0)$ or simply by $x(k)$ if there is no ambiguity about x_0 and u.

We will make use of the function classes $\mathcal{K}$ and $\mathcal{K}_\infty$. Recall that $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies $\alpha \in \mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. Additionally, if $\alpha \in \mathcal{K}$ is unbounded, then $\alpha \in \mathcal{K}_\infty$.

2.2 A brief summary of undiscounted strict dissipativity

Our goal in this paper is to derive a notion of strict dissipativity with discounting. To this end, we first recall the classical notion of strict dissipativity introduced by Willems (1972) in continuous time and by Byrnes and Lin (1994) in the discrete time setting of this paper. To this end, we recall that $(x^e, u^e) \in \mathbb{Y}$ is an equilibrium of (1) if $f(x^e, u^e) = x^e$.

Definition 1. Given an equilibrium (x^e, u^e), the system (1) is called *strictly dissipative* with respect to the *supply rate* $s : \mathbb{Y} \to \mathbb{R}$ if there exists a *storage function* $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and a function $\alpha \in \mathcal{K}_\infty$ such that

$$s(x, u) + \lambda(x) - \lambda(f(x, u)) \geq \alpha(\|x - x^e\|) \qquad (2)$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$.

One of the most useful theorems in dissipativity theory states that strict dissipativity holds for a given supply rate s if and only if

$$\lambda(x_0) := \sup_{K \in \mathbb{N}_0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} -\Big(s(x(k), u(k))$$
$$- \alpha(\|x(k) - x^e\|) \Big) < \infty \,(3)$$

holds for each $x_0 \in \mathbb{X}$, see Willems (1972) in continuous time and Byrnes and Lin (1994) in discrete time [1]. The function λ defined in (3) is then a storage function, called the *available storage*. One of the goals of our discounted generalization of strict dissipativity will be to allow for a similar notion of available storage.

The notion of dissipativity has a long history in systems and control theory, dating back to the work of Willems

(1972). Dissipativity theory now underpins a wide range of application domains, including distributed model predictive control, plant-wide control of chemical processes, control of cyberphysical systems, power electronics and mechanical systems, and for establishing input–output stability of adaptive control systems, switched systems, and nonlinear H_∞ control systems; see for example van der Schaft (1996); Brogliato et al. (2007); Moylan (2014) and the references therein.

By comparison, applications of *strict* dissipativity have appeared less frequently in the literature. Recent research, however, has established connections between strict dissipativity and the behavior of optimal trajectories via the so-called turnpike property. It is this connection that provides the motivation for this paper. Indeed, if we consider the optimal control problem

$$\min_{u \in \mathbb{U}^N(x_0)} J_N(x_0, u) \ \text{ with } \ J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$
$$(4)$$

with *stage cost* $\ell : \mathbb{Y} \to \mathbb{R}$ and subject to (1), then under an appropriate reachability condition on x^e it is known that the optimal trajectories most of the time stay in a neighborhood of the equilibrium x^e if the system is strictly dissipative with supply rate $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and bounded storage function. This property, known as the turnpike property, is due to the fact that the optimal trajectories of (4) are qualitatively similar to those of (4) when ℓ is replaced by

$$\tilde{\ell}(x, u) := \ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u)). \quad (5)$$

Strict dissipativity then implies that $\tilde{\ell}$ is a positive definite stage cost [2] with respect to x^e at (x^e, u^e), which means that it penalizes the deviation of x from x^e and thus forces the optimal trajectory to stay near x^e most of the time. For details we refer to (Grüne, 2013, Theorem 5.6). The turnpike property, in turn, allows for making rigorous statements about the near optimality of closed loop solution of model predictive control schemes (Grüne and Stieler (2014)).

The aforementioned connection between the turnpike property and behavior of closed-loop solutions of model predictive control schemes has recently been extended to discounted optimal control problems, i.e., to problems of the type

$$\min_{u \in \mathbb{U}^\infty(x_0)} J_\infty(x_0, u) \ \text{ with } \ J_\infty(x_0, u) = \sum_{k=0}^{\infty} \beta^k \ell(x(k), u(k)),$$
$$(6)$$

see Grüne et al. (2015). Herein, the number β with $0 < \beta < 1$ is called the *discount factor*. With

$$V_\infty(x_0) := \min_{u \in \mathbb{U}^\infty(x_0)} J_\infty(x_0, u)$$

we denote the *optimal value function* of (6). We remark that in the discounted case it is often possible to directly consider the infinite horizon problem because discounting ensures the convergence of the infinite sum in (6) under much more mild conditions than for the undiscounted

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[1] In both references this result is formulated and proved for a non-strict notion of dissipativity. The modifications for the strict dissipativity notion discussed here are, however, straightforward.

[2] Positive definiteness of $\tilde{\ell}$ with respect to x^e at (x^e, u^e) is defined as $\tilde{\ell}(x^e, u^e) = 0$ and $\tilde{\ell}(x, u) \geq \alpha(\|x - x^e\|)$ for some $\alpha \in \mathcal{K}_\infty$ and all $(x, u) \in \mathbb{Y}$.

problem (4). Working directly with the infinite horizon problem simplifies some of the considerations in this paper.

Since discounted optimal control problems play an important role particularly in economic applications, it is of great interest to adapt the results outlined above to the discounted case. From extensions of the results in Gaitsgory et al. (2015) which are currently under development, it follows that asymptotic stability (for the infinite horizon problem (6)) or the turnpike property (for the finite horizon counterpart of (6)), respectively, can under reasonable conditions be expected, provided the stage cost is positive definite (see also the discussion at the end of Example 13, below). Therefore, our "guideline" for deriving a discounted version of strict dissipativity will be that it should allow for a definition of a modified stage cost $\tilde{\ell}$ analogous to (5), which is equivalent in the sense that the infinite horizon discounted optimal trajectories corresponding to ℓ and to $\tilde{\ell}$ are identical.

3. DISCOUNTED STRICT DISSIPATIVITY

Following the motivation just discussed, we propose the following definition of discounted strict dissipativity. The subsequent proposition shows that for the particular supply rate $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ it indeed yields an equivalent positive definite stage cost.

Definition 2. Given a discount factor $0 < \beta < 1$, we say that the system (1) is *discounted strictly dissipative* w.r.t. an equilibrium (x^e, u^e) with supply rate $s : \mathbb{Y} \to \mathbb{R}$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below with $\lambda(x^e) = 0$ and a class $\mathcal{K}_\infty$-function α such that the inequality

$$s(x, u) + \lambda(x) - \beta\lambda(f(x, u)) \geq \alpha(\|x - x^e\|) \qquad (7)$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$.

Proposition 3. Consider the discounted optimal control problem (6) with discount factor $0 < \beta < 1$ and assume the system (1) is discounted strictly dissipative with supply rate $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and bounded storage function λ. Then the optimal trajectories of (6) coincide with those of the problem

$$\min_{u \in \mathbb{U}^\infty(x_0)} \widetilde{J}_\infty(x_0, u) \text{ with } \widetilde{J}_\infty(x_0, u) := \sum_{k=0}^\infty \beta^k \tilde{\ell}(x(k), u(k))$$

(8)

with stage cost

$$\tilde{\ell}(x, u) = \ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \beta\lambda(f(x, u))$$

which is positive definite w.r.t. x^e at (x^e, u^e).

Proof. A straightforward calculation shows that

$$\widetilde{J}_\infty(x_0, u) = J_\infty(x_0, u) - \frac{\ell(x^e, u^e)}{1 - \beta}$$
$$+ \lambda(x_0) - \lim_{k \to \infty} \beta^k \lambda(x_u(k)). \qquad (9)$$

Since λ is bounded and $0 < \beta < 1$, the last limit exists and is equal to 0. Hence, the objectives differ only by expressions which are independent of u, from which the identity of the optimal trajectories immediately follows. The positive definiteness of $\tilde{\ell}$ follows from (7) and the fact that $\lambda(x^e) = 0$ implies $\ell(x^e, u^e) = 0$. $\square$

Remark 4. The requirement that $\ell(x^e, u^e) = 0$ is the reason for imposing $\lambda(x^e) = 0$ as a condition in Definition 2. Note that in the undiscounted case $\ell(x^e, u^e) = 0$ can be assumed without loss of generality, since if λ is a storage function then $\lambda + c$ is a storage function for all $c \in \mathbb{R}$. In the discounted case, this invariance with respect to addition of constants does no longer hold.

Remark 5. Boundedness of λ is typically a rather mild condition if the state constraint set $\mathbb{X}$ is bounded, but it may be restrictive in case $\mathbb{X}$ is unbounded. For instance if λ is an affinely linear function as in the setting discussed in Theorem 6, below. In this case, other conditions ensuring $\lim_{k\to\infty} \beta^k \lambda(x_u(k)) = 0$ could be imposed in Proposition 3. For instance, one could assume boundedness of λ on bounded sets and boundedness of near optimal trajectories for both (6) and (8).

4. THE AFFINE LINEAR AND CONVEX CASE

In the non-discounted setting it is known that strict dissipativity holds for finite dimensional affine dynamics $f(x, u) = Ax + Bu + c$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; i.e., $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$, and $c \in \mathbb{R}^n$, and strictly convex stage cost ℓ, see Diehl et al. (2011) or (Damm et al., 2014, Proposition 4.3). The proof of this fact relies on the necessary optimality conditions for an optimal equilibrium.

In the discounted case, these optimality conditions read

$$x^e = f(x^e, u^e) \qquad (10)$$

$$p^e = -\frac{\partial}{\partial x}\ell(x^e, u^e) + \beta p^e \frac{\partial}{\partial x} f(x^e, u^e) \qquad (11)$$

$$0 = -\frac{\partial}{\partial u}\ell(x^e, u^e) + \beta p^e \frac{\partial}{\partial u} f(x^e, u^e), \qquad (12)$$

cf. Becker et al. (2007), where the n-dimensional row vector p^e denotes the co-state (or Lagrange multiplier) at the optimal equilibrium.

The following theorem shows that these conditions imply strict dissipativity also in the discounted case.

Theorem 6. Consider the optimal control problem (6) with $0 < \beta < 1$, $\mathbb{X} \subseteq \mathbb{R}^n$ bounded, $\mathbb{U} \subseteq \mathbb{R}^m$, affine dynamics f, and strictly convex stage cost ℓ. Assume there is an equilibrium $(x^e, u^e) \in \mathbb{Y}$ and $(p^e)^T \in \mathbb{R}^n$ satisfying (10)–(12). Then the system is discounted strictly dissipative with supply rate $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and storage function $\lambda(x) = p^e(x - x^e)$.

Proof. By definition and boundedness of $\mathbb{X}$, λ satisfies $\lambda(x^e) = 0$ and is bounded from below. Strict convexity of ℓ and affine linearity of f together with the linearity of λ imply that

$$\tilde{\ell}(x, u) := \ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \beta\lambda(f(x, u))$$

is strictly convex. Moreover, from

$$\frac{\partial}{\partial x}\tilde{\ell}(x, u) = \frac{\partial}{\partial x}\ell(x, u) + p^e - \beta p^e \frac{\partial}{\partial x} f(x, u)$$

$$\frac{\partial}{\partial u}\tilde{\ell}(x, u) = \frac{\partial}{\partial u}\ell(x^e, u^e) - \beta p^e \frac{\partial}{\partial u} f(x, u)$$

and (11), (12) it follows that $D\tilde{\ell}(x^e, u^e) = 0$, which by strict convexity of $\tilde{\ell}$ implies that (x^e, u^e) is the

unique strict minimum of this function. This implies that $\inf_{u \in U} \tilde{\ell}(x, u) > \tilde{\ell}(x^e, u^e) = 0$ for all $x \neq x^e$, which by exploiting strict convexity of $\tilde{\ell}$ and boundedness of $\mathbb{X}$ implies the existence of $\alpha \in \mathcal{K}_\infty$ with (7). $\square$

The following example illustrates that this theorem indeed provides a constructive way to check discounted strict dissipativity.

Example 7. We consider a basic growth model in discrete time which goes back to Brock and Mirman (1972). Cost function and dynamics are given by

$$\ell(x, u) = -\ln(Ax^\alpha - u) \quad \text{and} \quad x(n+1) = u(n).$$

Herein, Ax^α is a production function with constants $A > 0$, $0 < \alpha < 1$, capital stock x and control variable u. The difference between output and next period's capital stock (given by u) is consumption. The exact solution to this problem is known (see Santos and Vigo-Aguiar (1998)) and is given by

$$V_\infty(x) = B + C \ln x$$

with

$$C = \frac{\alpha}{1 - \alpha\beta} \quad \text{and} \quad B = \frac{\ln((1 - \alpha\beta)A) + \frac{\beta\alpha}{1 - \beta\alpha}\ln(\alpha\beta A)}{1 - \beta}.$$

From this it is straightforward to check that the unique optimal equilibrium for this example is given by $x^e = 1/\sqrt[\alpha-1]{\beta\alpha A}$.

Since f is linear and ℓ is strictly convex, Theorem 6 can be applied. In order to verify discounted strict dissipativity and to compute the corresponding storage function λ (and in order to show how to verify optimality of x^e without using the knowledge of the exact solution), we solve the equations (10)–(12). Here, the equations read

$$x^e = u^e \tag{13}$$

$$p^e = \frac{\alpha A (x^e)^{\alpha-1}}{A(x^e)^\alpha - u^e} \tag{14}$$

$$0 = -\frac{1}{A(x^e)^\alpha - u^e} + \beta p^e. \tag{15}$$

Inserting $p^e = \frac{1}{\beta(A(x^e)^\alpha - u^e)}$ from (15) and $u^e = x^e$ from (13) into (14) yields again $x^e = 1/\sqrt[\alpha-1]{\beta\alpha A}$. From this we obtain

$$\lambda(x) = p^e(x - x^e) \quad \text{with} \quad p^e = \frac{\sqrt[\alpha-1]{\beta\alpha A}}{\frac{1}{\alpha} - \beta}$$

as a storage function which is bounded on every bounded interval $\mathbb{X} \subseteq \mathbb{R}_{>0}$ containing x^e.

5. AVAILABLE STORAGE AND ROBUST OPTIMALITY

Incorporating the discount factor in the available storage formula (3) is reasonably straightforward and using a dynamic programming argument it is relatively easy to see that the resulting function — if it assumes finite values — satisfies the discounted strict dissipativity inequality (7) (the details are provided in the proof of Theorem 11, below). However, in order to adapt the concept of the available storage to the discounted setting, we have to make sure that the appropriate modification of (3) leads to a storage function satisfying $\lambda(x^e) = 0$. In order to

accomplish this, it is beneficial to replace the $\sup_K$ in the non-discounted available storage formula (3) by an infinite sum. That is, we consider the discounted available storage defined by

$$\lambda(x_0) := \sup_{u \in \mathbb{U}^\infty(x_0)} \sum_{k=0}^\infty -\beta^k \Big(s(x(k), u(k)) - \alpha(\|x(k) - x^e\|) \Big) \tag{16}$$

As we will see, the equality $\lambda(x^e) = 0$ is closely linked to the optimality of the equilibrium (x^e, u^e). To clarify this relation we need the following definitions.

Definition 8. Consider the optimal control problem (6) with $0 < \beta < 1$.

(i) An equilibrium $(x^e, u^e) \in \mathbb{Y}$ is called optimal if $V_\infty(x^e) = \ell(x^e, u^e)/(1 - \beta)$.

(ii) An equilibrium $(x^e, u^e) \in \mathbb{Y}$ is called robustly optimal if there is $\sigma \in \mathcal{K}_\infty$ such that (x^e, u^e) is optimal for the optimal control problem (6) with stage cost $\hat{\ell}(x, u) := \ell(x, u) - \sigma(\|x - x^e\|)$.

It is immediate that robust optimality of an equilibrium implies optimality of this equilibrium. Moreover, it is easy to see that an equilibrium is optimal if and only if the corresponding (constant) trajectory is an optimal trajectory. The next two lemmas clarify certain relations of these optimality concepts to positive definiteness of ℓ and to strict dissipativity.

Lemma 9. If the stage cost of the optimal control problem is positive definite w.r.t. an equilibrium x^e at (x^e, u^e), then this equilibrium is optimal.

Proof. Positive definiteness of ℓ implies $V_\infty(x^e) \geq 0$ and the constant control $u \equiv u^e$ yields $V_\infty(x^e) \leq J_\infty(x^e, u) = 0$. This yields $V_\infty(x^e) = 0 = \ell(x^e, u^e)/(1 - \beta)$. $\square$

Lemma 10. Discounted strict dissipativity of (1) with $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and bounded storage function λ implies that the equilibrium (x^e, u^e) is robustly optimal.

Proof. Let α be the $\mathcal{K}_\infty$ function from discounted strict dissipativity (7) and define $\sigma \in \mathcal{K}_\infty$ by $\sigma := \alpha/2$. Then the cost function

$$\hat{\ell}(x, u) := \ell(x, u) - \sigma(\|x - x^e\|)$$

satisfies

$$\hat{\ell}(x, u) - \hat{\ell}(x^e, u^e) + \lambda(x) - \beta\lambda(f(x, u))$$
$$= \ell(x, u) - \sigma(\|x - x^e\|) - \ell(x^e, u^e)$$
$$+ \lambda(x) - \beta\lambda(f(x, u))$$
$$\geq -\sigma(\|x - x^e\|) + \alpha(\|x - x^e\|) = \sigma(\|x - x^e\|).$$

Hence, the problem with stage cost $\hat{\ell}$ is discounted strictly dissipative (with $\mathcal{K}_\infty$ function σ) and thus the equivalent problem (8) has a stage cost which is positive definite w.r.t. x^e at (x^e, u^e). Hence, by Lemma 9 (x^e, u^e) is an optimal equilibrium. Since the optimal trajectories of (8) coincide with that of the original problem, i.e., of that with stage cost $\hat{\ell}$, (x^e, u^e) is also an optimal equilibrium for stage cost

$\hat{\ell}$ and thus a robustly optimal equilibrium with respect to the stage cost ℓ. □

The following main theorem of this section now shows that — under appropriate boundedness assumptions — the discounted available storage (16) is a storage function in the sense of Definition 2 if and only if x^e is robustly optimal.

Theorem 11. Let $\mathbb{X}$ be bounded and ℓ be bounded on $\mathbb{Y}$. Let $(x^e, u^e) \in \mathbb{Y}$ be an equilibrium of (1) and consider the discounted optimal control problem (6) with $0 < \beta < 1$. Then discounted strict dissipativity with $s(x,u) = \ell(x,u) - \ell(x^e, u^e)$ and bounded storage function holds if and only if (x^e, u^e) is robustly optimal.

Proof. "⇒" This follows directly from Lemma 10.

"⇐" From boundedness of $\mathbb{X}$ and ℓ it follows that

$$\lambda(x_0) := \sup_{u \in \mathbb{U}^\infty(x_0)} \sum_{k=0}^\infty -\beta^k \Big(\ell(x(k), u(k))$$
$$- \ell(x^e, u^e) - \alpha(\|x(k) - x^e\|) \Big)$$

is a bounded function in x_0. We claim that λ is a discounted storage function for the system. From robust optimality of (x^e, u^e) it follows that $\lambda(x^e) = 0$. In order to prove the dissipation inequality (7), let $(x,u) \in \mathbb{Y}$ with $x^+ = f(x,u) \in \mathbb{X}$. Given $\varepsilon > 0$, consider $u_\varepsilon \in \mathbb{U}^\infty(x^+)$ such that

$$\lambda(x^+) \le \sum_{k=0}^\infty -\beta^k \Big(\ell(x_{u_\varepsilon}(k, x^+), u_\varepsilon(k))$$
$$- \ell(x^e, u^e) - \alpha(\|x_{u_\varepsilon}(k, x^+) - x^e\|) \Big) + \varepsilon.$$

Then for the control sequence $\hat{u} = (u, u_\varepsilon(0), u_\varepsilon(1), \ldots)$ we obtain $x_{\hat{u}}(k, x) = x_{u_\varepsilon}(k-1, x^+)$ for all $k \ge 1$ and

$$\lambda(x) \ge \sum_{k=0}^\infty -\beta^k \Big(\ell(x_{\hat{u}}(k, x), \hat{u}(k)) - \ell(x^e, u^e)$$
$$- \alpha(\|x_{\hat{u}}(k, x) - x^e\|) \Big)$$
$$= -\ell(x_{\hat{u}}(0, x), \hat{u}(0)) + \ell(x^e, u^e) + \alpha(\|x_{\hat{u}}(0, x) - x^e\|)$$
$$+ \sum_{k=1}^\infty -\beta^k \Big(\ell(x_{\hat{u}}(k, x), \hat{u}(k)) - \ell(x^e, u^e)$$
$$- \alpha(\|x_{\hat{u}}(k, x) - x^e\|) \Big)$$
$$= -\ell(x, u) + \ell(x^e, u^e) + \alpha(\|x - x^e\|)$$
$$+ \beta \sum_{k=0}^\infty -\beta^k \Big(\ell(x_{u_\varepsilon}(k, x^+), u_\varepsilon(k)) - \ell(x^e, u^e)$$
$$- \alpha(\|x_{u_\varepsilon}(k, x^+) - x^e\|) \Big)$$
$$\ge -\ell(x, u) + \ell(x^e, u^e) + \alpha(\|x - x^e\|)$$
$$+ \beta\lambda(f(x,u)) - \beta\varepsilon.$$

This shows the desired strict dissipation inequality (7) for supply rate $s(x,u) = \ell(x,u) - \ell(x^e, u^e)$ since $\varepsilon > 0$ was arbitrary. □

6. CONTINUITY OF OPTIMAL TRAJECTORIES NEAR THE EQUILIBRIUM

It was shown in (Grüne, 2013, Lemma 6.3) that in the non-discounted setting, strict dissipativity (along with other assumptions) implies that optimal trajectories starting near x^e stay near x^e for a certain number of time steps. Our last result in this paper shows that the same is true for our proposed discounted notion of strict dissipativity.

Theorem 12. Consider the discounted optimal control problem (6) with $0 < \beta < 1$ and assume system (1) is discounted strictly dissipative with $s(x,u) = \ell(x,u) - \ell(x^e, u^e)$ and bounded storage function λ. Assume, moreover, that V_∞ and λ are continuous in the equilibrium x^e. Then for each $K \in \mathbb{N}$ there exists $\eta_K \in \mathcal{K}_\infty$ such that the optimal trajectories x^* satisfy

$$\|x^*(k) - x^e\| \le \eta_K(\|x_0 - x^e\|)$$

for all $k = 0, \ldots, K$, where $x_0 = x^*(0)$.

Proof. It is sufficient to show the property for the equivalent optimal control problem (8). Since V_∞ and λ are continuous in x^e, it follows from (9) that $\widetilde{V}_\infty$ is continuous in x^e, too. Hence, since positive definiteness of $\tilde{\ell}$ implies $\widetilde{V}_\infty(x^e) = 0$, by continuity there is $\rho \in \mathcal{K}_\infty$ with

$$\widetilde{V}_\infty(x) \le \rho(\|x - x^e\|). \tag{17}$$

Given $K \in \mathbb{N}$, we claim that the assertion holds for

$$\eta_K(r) := \alpha^{-1}(\rho(r)/\beta^K)$$

with $\alpha \in \mathcal{K}_\infty$ from the strict dissipativity property.

Indeed, assume there is $k \in \{0, \ldots, K\}$ with $\|x^*(k) - x^e\| > \eta_K(\|x_0 - x^e\|)$. Then from discounted strict dissipativity we obtain

$$\tilde{\ell}(x^*(k), u(k)) > \alpha(\eta_K(\|x_0 - x^e\|)) = \rho(\|x_0 - x^e\|)/\beta^K.$$

Thus, since $\tilde{\ell} \ge 0$ we obtain

$$\widetilde{V}_\infty(x_0) \ge \beta^k \tilde{\ell}(x^*(k), u(k)) > \rho(\|x_0 - x^e\|)$$

contradicting (17). □

Our final example shows that the statement of Theorem 12 is in general wrong for $K = \infty$, i.e., that discounted strict dissipativity does not necessarily imply stability of the optimal equilibrium x^e.

Example 13. Consider the discounted linear quadratic optimal control problem with

$$f(x,u) = 2x + u, \qquad \ell(x,u) = x^2 + u^2$$

and $x \in \mathbb{R}$, $u \in \mathbb{R}$. Since ℓ is bounded from below by $\alpha(\|x - x^e\|)$ with $\alpha(r) = r^2$ and $x^e = 0$, it is straightforward to see that the system is (discounted) strictly dissipative at $(x^e, u^e) = (0,0)$ for all $0 < \beta \le 1$ with supply rate $s(x,u) = \ell(x,u) - \ell(x^e, u^e)$ and bounded storage function $\lambda \equiv 0$.

A brief computation reveals that the discounted optimal control can be obtained by solving the discrete time algebraic Riccati equation with $\sqrt{\beta}A$ and $\sqrt{\beta}B$ in place of A and B. For $\beta = 0.3$, the resulting controller is $K \approx -0.9201$, leading to the closed loop system $x^+ = (A + BK)x \approx 1.0799x$. The origin $x^e = 0$ of this system is obviously unstable, hence the assertion of Theorem 12 does not hold for $K = \infty$.

We note that the instability of the closed loop is consistent with the continuous time result in Gaitsgory et al. (2015)

which only ensures asymptotic stability for positive definite stage cost for discount rate sufficiently close to 0, which translates to a discrete time discount factor β sufficiently close to 1. Indeed, by computing the controller for various β indicates that for $\beta \geq \beta_0 \approx 1/3$ the discounted optimal control problem yields a stabilizing control for this example.

Remark 14. In the linear quadratic and unconstrained setting of Example 13, the assertion of Theorem 12 could also be concluded from the Lipschitz continuity of the right hand side of the optimally controlled closed loop system. However, in general — and in particular in the presence of nonlinearities and constraints — optimal controls do not need to depend continuously on the initial value, which makes the assertion of Theorem 12 nontrivial.

7. CONCLUSIONS

Prior work in the literature demonstrated a close connection between strict dissipativity, available storage, the turnpike property, and the near optimality of closed-loop solutions of model predictive control schemes. These classical notions of dissipativity and available storage are related to an optimal control problem with an undiscounted stage cost. In this paper, we modified these classical notions for application to optimal control problems with a discounted stage cost. We subsequently showed that affine linear systems with a strictly convex stage cost are discounted strictly dissipative (Theorem 6). We further demonstrated that discounted strict dissipativity is equivalent to a form of robust optimality (Theorem 11) and that discounted strict dissipativity implies a certain continuity of trajectories near an optimal equilibrium (Theorem 12). These results are required as a prerequisite to demonstrating an equivalence between discounted strict dissipativity, turnpike properties, and near optimality of closed loop solutions of model predictive control schemes based on optimal control problems with discounted stage costs. This is the subject of future work and is motivated by the application of model predictive control schemes in economics where the stage cost is often discounted (e.g., Weller et al. (2015)). Depending on the application, the discount factor is frequently close to unity, and a consequent question is: does strict dissipativity for $\beta = 1$ imply strict dissipativity for $\beta < 1$, when β is sufficiently close to one?

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