On Stock Trading Via Feedback Control When Underlying Stock Returns Are Discontinuous

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Abstract

A lot of work was done on feedback trading. There, it is shown that for so-called simultaneously long short strategies, gains are positive for continuously differentiable prices and expected ones are positive for geometric Brownian motion prices. But, both models are jump-less. This work shows that if the price is governed by Merton's jump diffusion model the expected gain is still positive and depends neither on intensity nor on kind or size of the jumps.

Index Terms

Feedback-based Stock Trading, Technical Trading Rules, Simultaneously Long Short Strategy, Merton's Jump Diffusion Process

I. Introduction and Literature Review

This paper analyzes a control-based and model-free[1] trading method – the simultaneously long short (SLS) trading strategy introduced inter alia in [1] – if the stock price is governed

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[1]A trading rule is called model-free if there is no price model assumend for constructing it.

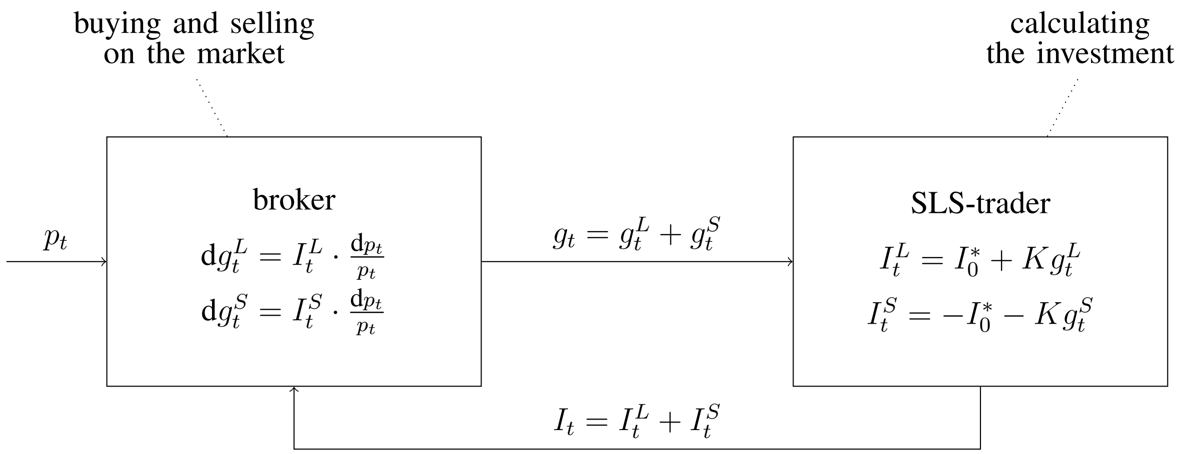

Fig. 1. Schematical interaction between broker and SLS-trader

by Merton's jump diffusion process, see [2]. In the last decade, there have been a lot of papers published concerning feedback-based trading strategies[2]. This shows that this topic holds enormous potential. Basically, a so-called feedback trader f is a trader that treats financial markets like machines, this means, he tries to control the output of the machines – his gain g_t^f – using the input variable – his investment I_t^f – whereby the input is calculated as a function of the output, i.e., $I_t^f = h(g_t^f)$ for some function h. The price process $p_t > 0$ can be seen as a disturbance variable and is used only indirectly for calculating I_t^f, since

$$g_t^f = \int_0^t I_\tau^f \cdot \frac{\mathrm{d}p_\tau}{p_\tau}. \tag{1}$$

Since feedback traders do not consider any fundamental value of the stock, but instead the price (indirectly), it follows that they are chartists and not fundamentalists.

One basic trading strategy is the so-called linear feedback long[3] trader L with investment rule

$$I_t^L := I_0^* + K g_t^L,$$

[2]See for example [3] and [4] for control-based trading when stock prices are continuously differentiable and [1], [5], and [6] (with non-constant trend) when stock prices are governed by geometric Brownian motions. [7] uses the Capital Asset Pricing Model (CAPM). [8] and [9] give an overview over existing results and the latter one over future questions. In [10] it is explained how to simulate feedback traders. [11] quantifies some technical trading rules and [12] deals with the effects of feedback trading. In [13] a controler using an integral is introduced. [14] and [15] links control-strategies to option pricing and hedging. For feedback trading in a binomial tree-model see [16]. In [17] and in [18] it is dealt with the problem of skewed gain distributions obtained for feedback trading.

[3]An investment I_t is called "long" if $I_t > 0$ and "short" if $I_t < 0$.

where $I_0^* > 0$ is the start investment and $K > 0$ is the so-called feedback parameter. This means, the trader starts with I_0^* and then he adds K-times his gain (the gain of this strategy, the so-called long side) to his initial investment. Note that $g_0^L = 0$ and that t is in continuous time. If the price process is continuous this trader is a long trader and therefore a trend follower, too. Analogously, one can construct a linear feedback short trader S

$$I_t^S := -I_0^* - Kg_t^S,$$

with g_t^S is the short side's gain. These two strategies are analyzed in literature for two reasons. Both, the investment formulae are easy to handle and one can construct more complex strategies using these two linear strategies. The simultaneously long short trading rule

$$I_t = I_t^L + I_t^S$$

schematically pictured in Fig. 1 is an example for the combination of the linear feedback long and short trader[4], with the same K and I_0^*. Usually, this strategy is analyzed in idealized markets with different assumptions: continuously differentiable prices, geometric Brownian motion (GBM), or the Cox Ross Rubinstein (CRR) model. The GBM is given through the stochastic differential equation $\frac{\mathrm{d}b_t}{b_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$ with the solution

$$b_t = b_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t},$$

where b_0 is the start price, μ the so-called trend, σ the volatility, and W_t a Wiener process. We want to consider the SLS rule, too, because there are some astonishing results in related work. In [3] it could be shown that the trading gain is positive if prices are continuously differentiable and in [1] it is proven, that the expected gain is positive for prices following a GBM (for non-zero drifts). Also for real market data a modified SLS-rule performs very well (see [7]). In these papers idealized markets are used as so-called proving grounds. This means that in a first step a strategy has to perform well on a modeled market and then, in a second step, it will be tested on real market data. But neither the continuously differentiable prices nor the GBM allow for price jumps. But since it could be shown in the context of option pricing[5] that jumps have a high influence on hedging, namely, that markets become incomplete the question arises: *What happens if there are jumps in the model?* We use Merton's jump diffusion model (see Fig. 2) to

[4]To provide readability we write I_t and g_t instead of I_t^{SLS} and g_t^{SLS}, respectively.

[5]For this issue have a look at [2] and [19]. See also [20].

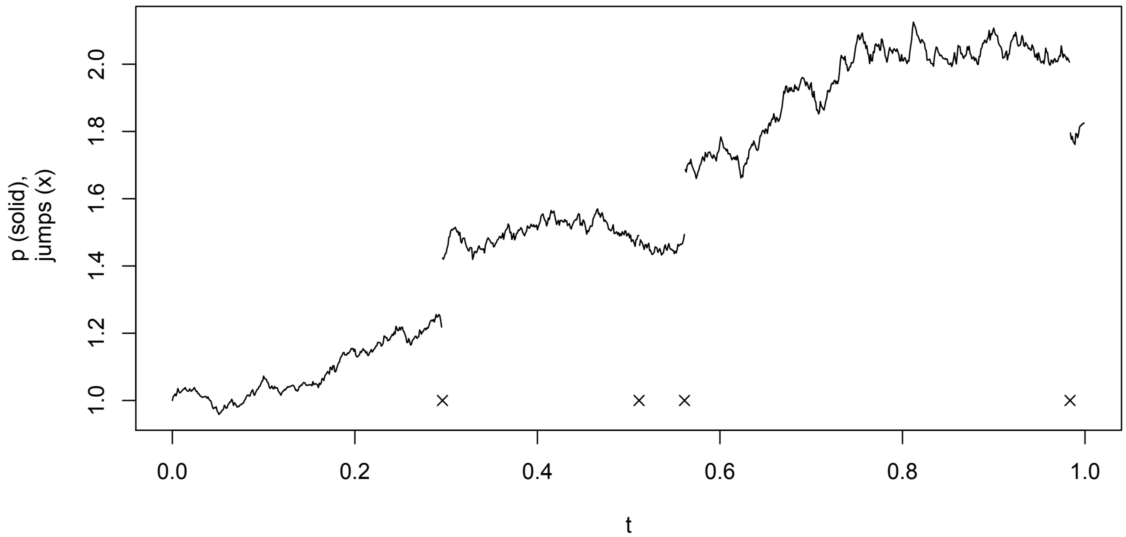

Fig. 2. A simulation of a price process following Merton's jump diffusion model. The $\times$-signs mark the jumps and like for all simulations, the jumps are lognormal distributed. Parameters: $t = 1$, increment $\tau = 0.001$, $p_0 = 1$, $\alpha = 0.1$, $\sigma = 0.2$, $\lambda = 3$, $\mu_{Y_i} = -0.1$, $\sigma_{Y_i} = 0.2$.

examine the influence of jumps. We deduce formulae for the gain/loss and examine the expected gain and the standard deviation of the gain analytically. The relative price change in Merton's model is given through

$$\frac{\mathrm{d}p_t}{p_t} = (\alpha - \lambda\kappa)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}N_t, \tag{2}$$

where W_t is a Wiener process, N_t is a Poisson-driven[6] process[7] with jump intensity $\lambda > 0$ and jumps $(Y_i - 1)$ i.i.d. with existing first moment (this is equivalent to $\mathbb{E}[Y_i - 1] < \infty$) and $Y_i > 0$ for the reason of positive prices. We define $\kappa := \mathbb{E}[Y_i - 1]$. Parameter $\alpha > -1$ denotes the jumpless trend and $\sigma > 0$ the volatility. Note that the $Y_i > 0$ are random variables without explicitly given distribution[8]. It could be shown (using Itô's lemma and its generalization for Poisson-driven processes, see [2], [21], [22], [23]) that the solution of (2) is

$$p_t = p_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \prod_{i=1}^{N} Y_i = b_t \prod_{i=1}^{N} Y_i,$$

[6]A Poisson process is a special Poisson-driven process with $Y_i - 1 \equiv 1$.

[7]Note that generally $\int_0^t \mathrm{d}N_\tau$ is discontinuous.

[8]One possibility is $\ln Y_i \sim \mathcal{N}(\mu_{Y_i}, \sigma_{Y_i})$.

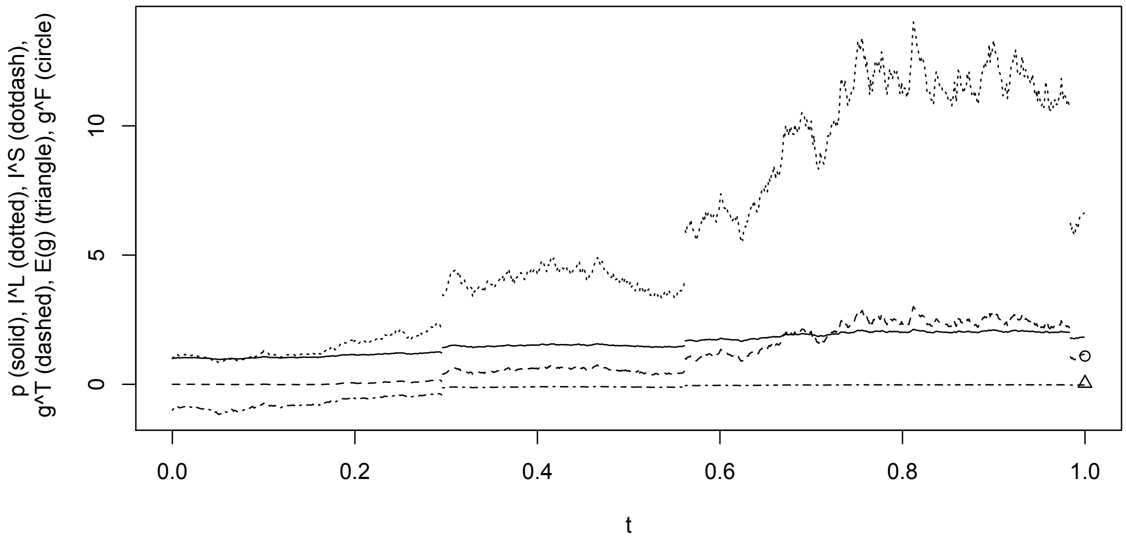

Fig. 3. SLS-Trading in discrete time on the price process of Fig. 2. The circle marks the gain at $t = 1$ for continuous SLS-trading and the triangle the expected gain for this case. Further parameters: $I_0^* = 1$, $K = 4$.

where $\mu := \alpha - \lambda\kappa, p_0 > 0, N \sim \mathcal{P}ois(\lambda t)$ and W_t, Y_i, and N all are independently distributed. The term "$-\lambda\kappa$" in (2) is added for a risk-neutral growth and it is set $b_0 = p_0$.

In the work at hand a formula for the gain/loss function of the SLS-trading rule is obtained for stock returns following Merton's jump diffusion process. In addition, formulae for the expected gain and for the standard deviation of the gain are stated. It is shown that – independent of intensity and kind and size of the jumps – the expected gain is still positive. To conclude the paper, simulations to illustrate all results are performed. For an illustration of SLS-trading on Merton's jump diffusion model see Fig. 3.

II. Market requirements

Before starting our analytical work, we introduce some market requirements:

We want to allow *Continuous Trading*, i.e., at every point of time $t > 0$ the trader knows his gain, the price, and is able to buy and sell stocks to adjust his investment. Although in real markets traders actually cannot trade continuously, in times of high-frequency-trading one might call this quasi-continuous trading. Further, the market lacks trading costs (*Costless Trading*), which is approximately true for big trading companies. The trader also has *Adequate Resources*, i.e., the trader has always enough money for trading. Admittedly this could be problematic in real life. But if the investment is not too high and the trading company is big enough this assumption might be plausible. We furthermore see the trader as a so-called *Price-Taker*, that means, the trader's actions do not have any influence on the market, especially not on the stock price. This is approximately true if again the investment is not too high. In [12] it is discussed what happens if this assumption is relaxed. *Perfect Liquidity* means that there is no gap between bid price and ask price and that the trader can arbitrarily buy and sell stocks.

The main assumption, that is really new in the work at hand, is that we consider a *Stock Price Governed by Merton's Jump Diffusion Process* (see [2]). This is a stochastic process with a countably infinite number of jumps. The jumps are given through the multiplication of random variables $Y_i > 0$ i.i.d. with existing first moment. The time between two jumps is independently and identically exponentially distributed with parameter $\lambda > 0$. One can show that the number of jumps which occurred up to at time t is Poisson distributed with parameter λt. Between every two jumps the process follows a GBM with "jump-adjusted" trend.

III. Gain/Loss

Now we want to derive a formula for g_t in Merton's jump diffusion model.

Theorem 1. *For the SLS trading strategy and a stock price following* (2)*, it holds*

$$\begin{aligned} g_t &= \frac{I_0^*}{K}(\left(\frac{b_t}{b_0}\right)^K e^{\frac{(K-K^2)\sigma^2 t}{2}} \prod_{i=1}^N (1 + K(Y_i - 1)) \\ &\quad + \left(\frac{b_t}{b_0}\right)^{-K} e^{\frac{-(K+K^2)\sigma^2 t}{2}} \prod_{i=1}^N (1 - K(Y_i - 1)) - 2) \\ &= \frac{I_0^*}{K}(\left(\frac{p_t}{p_0}\right)^K e^{\frac{(K-K^2)\sigma^2 t}{2}} \prod_{i=1}^N (Y_i^{-K} + KY_i^{-K}(Y_i - 1)) \\ &\quad + \left(\frac{p_t}{p_0}\right)^{-K} e^{\frac{-(K+K^2)\sigma^2 t}{2}} \prod_{i=1}^N (Y_i^K - KY_i^K(Y_i - 1)) - 2). \end{aligned} \tag{3}$$

Proof. At first we decompose $g_t = g_t^L + g_t^S$, with g_t^L and g_t^S referring to (1). Firstly, the long side's gain is considered. The change of the gain is given through

$$\mathrm{d}g_t^L = I_t^L \cdot \frac{\mathrm{d}p_t}{p_t} = (I_0^* + Kg_t^L)((\alpha - \lambda\kappa)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}N_t).$$

With $f_t := I_0^* + Kg_t^L$ it follows

$$\frac{\mathrm{d}f_t}{f_t} = (K\alpha - \lambda K\kappa)\mathrm{d}t + K\sigma \mathrm{d}W_t + K\mathrm{d}N_t$$

$$= (K\alpha - \lambda K\kappa)\mathrm{d}t + K\sigma \mathrm{d}W_t + \mathrm{d}\tilde{N}_t.$$

We remark that $\tilde{N}_t$ is again a Poisson-driven process with jump intensity $\lambda > 0$ but with jumps $(X_i^L - 1)$ with $X_i^L := 1 + K(Y_i - 1)$. It also holds $\mathbb{E}[X_i^L - 1] = K\kappa$ and $f_0 = I_0^*$. It then follows[9]

$$f_t = f_0 e^{(K\alpha - \lambda K\kappa - \frac{K^2\sigma^2}{2})t + K\sigma W_t} \prod_{i=1}^{N} X_i^L.$$

The resubstitution of f_t leads to

$$\begin{aligned} g_t^L &= \frac{I_0^*}{K}(e^{(K\mu - \frac{K\sigma^2}{2})t + K\sigma W_t} e^{\frac{(K-K^2)\sigma^2 t}{2}} \prod_{i=1}^{N} X_i^L - 1) \\ &= \frac{I_0^*}{K}(\left(\frac{b_t}{b_0}\right)^K e^{\frac{(K-K^2)\sigma^2 t}{2}} \prod_{i=1}^{N}(1 + K(Y_i - 1)) - 1) \\ &= \frac{I_0^*}{K}(\left(\frac{p_t}{p_0}\right)^K e^{\frac{(K-K^2)\sigma^2 t}{2}} \prod_{i=1}^{N}(Y_i^{-K} + KY_i^{-K}(Y_i - 1)) - 1). \end{aligned}$$

Now let us consider $X_i^S := 1 - K(Y_i - 1)$ and note $\mathbb{E}[X_i^S - 1] = -K\kappa$. Substituting K and I_0^* by $-K$ and $-I_0^*$, respectively, for the short side's gain leads to

$$\begin{aligned} g_t^S &= \frac{I_0^*}{K}(\left(\frac{b_t}{b_0}\right)^{-K} e^{\frac{-(K+K^2)\sigma^2 t}{2}} \prod_{i=1}^{N} X_i^S - 1) \\ &= \frac{I_0^*}{K}(\left(\frac{p_t}{p_0}\right)^{-K} e^{\frac{-(K+K^2)\sigma^2 t}{2}} \prod_{i=1}^{N}(Y_i^K - KY_i^K(Y_i - 1)) - 1). \end{aligned}$$

Together, the theorem's statements hold. □

This is the first one of our desired results. The formula tells us that the gain does not depend on the diffusion part (the GBM part) of the price process. In (3) only b_t at time t and jumps $(Y_i)_{i=1,\dots,N}$ are of importance. In this formula only a countably infinite number of random variables is present since $(b_t)_t$ is not used but just b_t. Next, we want to analyze what can be expected for the gain at arbitrary time t.

IV. Expected Gain

Next, we want to focus on the expected gain. We obtain

Theorem 2. *The expected gain of the SLS trading strategy with a stock price following* (2) *is*

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}(e^{K\alpha t} + e^{-K\alpha t} - 2). \tag{4}$$

[9] For the solution of the stochastic differential equation see [2], [21], [22], [23].

Proof. In order to calculate the expected gain $\mathbb{E}[g_t]$ we consider equation (3). With basic rules for the calculation of expected values and remembering that b_t, N, and $(Y_i)_i$ are all independent and $\ln b_t \sim \mathcal{N}(\mu - \frac{1}{2}\sigma^2, \sigma^2)$ we can transform[10]

$$\begin{aligned}\mathbb{E}[g_t] = &\frac{I_0^*}{K}(\mathbb{E}[\left(\frac{b_t}{b_0}\right)^K]e^{\frac{(K-K^2)\sigma^2 t}{2}}\mathbb{E}[\prod_{i=1}^N(1+K(Y_i-1))] \\ &+ \mathbb{E}[\left(\frac{b_t}{b_0}\right)^{-K}]e^{\frac{-(K+K^2)\sigma^2 t}{2}}\mathbb{E}[\prod_{i=1}^N(1-K(Y_i-1))] - 2) \\ = &\frac{I_0^*}{K}(e^{K\mu t}\mathbb{E}[\prod_{i=1}^N(1+K(Y_i-1))] \\ &+ e^{-K\mu t}\mathbb{E}[\prod_{i=1}^N(1+K(Y_i-1))] - 2).\end{aligned}$$

The next step makes use of the theorem of Fubini-Tonelli. Since[11]

$$\begin{aligned}&\int_{\Omega_N}\int_{\Omega_Y}\prod_{i=1}^N(1+2K+K(Y_i-1))\mathrm{d}P_Y\mathrm{d}P_N \\ &= \sum_{n=0}^\infty \frac{(\lambda t)^n}{n!}e^{-\lambda t}\int_{\Omega_Y}\prod_{i=1}^n(1+2K+K(Y_i-1))\mathrm{d}P_Y \\ &= \sum_{n=0}^\infty \frac{(\lambda t)^n}{n!}e^{-\lambda t}\mathbb{E}[1+2K+K(Y_i-1)]^n \\ &= e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t(1+2K+K\kappa))^n}{n!} = e^{\lambda K t(\kappa+2)} < \infty\end{aligned}$$

we can apply Fubini-Tonelli for calculating the expected values[12]:

$$\begin{aligned}&\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^N(1+K(Y_i-1))\mathrm{d}(P_N\otimes P_Y) \\ &= \int_{\Omega_N}\int_{\Omega_Y}\prod_{i=1}^N(1+K(Y_i-1))\mathrm{d}P_Y\mathrm{d}P_N \\ &= \sum_{n=0}^\infty \frac{(\lambda t)^n}{n!}e^{-\lambda t}\int_{\Omega_Y}\prod_{i=1}^n(1+K(Y_i-1))\mathrm{d}P_Y = e^{\lambda t K\kappa}\end{aligned}$$

[10] If Z is a random variable with $\ln Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ it holds: $\mathbb{E}[Z^K] = e^{K\mu_Z + \frac{1}{2}K^2\sigma_Z^2}$

[11] We assume that Y_i is defined on Ω_{Y_i}, N is defined on Ω_N, $\Omega_Y := \Omega_{Y_1}\times\Omega_{Y_2}\times\Omega_{Y_3}\times\ldots$, and $Y := Y_1\otimes Y_2\otimes Y_3\otimes\ldots$.

[12] Because it holds $1+2K+K(Y_i-1) \geq \max\{|1+K(Y_i-1)|, |1-K(Y_i-1)|\}$.

and analogously

$$\int_{\Omega_N \times \Omega_Y} \prod_{i=1}^{N} (1 - K(Y_i - 1)) \mathrm{d}(P_N \otimes P_Y) = e^{-\lambda t K \kappa}.$$

It follows

$$\begin{aligned}\mathbb{E}[g_t] &= \frac{I_0^*}{K}(e^{K\mu t} e^{\lambda t K \kappa} + e^{-K\mu t} e^{-\lambda t K \kappa} - 2) \\ &= \frac{I_0^*}{K}(e^{K\alpha t} + e^{-K\alpha t} - 2)\end{aligned}$$

for all $\lambda > 0, (Y_i)_{i \in \mathbb{N}} > 0$. □

This means the expected value is independent of the jumps' specifications.

V. Positive Expectation Value

At first, we want to show that the expected gain (4) is positive if $\alpha \neq 0$ and $t > 0$. Therefore, we note $e^x + e^{-x} - 2 > 0$ for all $\mathbb{R} \ni x \neq 0$ which implies

Corollary 3. *The expected gain of the SLS trading strategy with underlying* (2) *is in general positive, i.e.,*

$$\mathbb{E}[g_t] > 0. \tag{5}$$

This holds for all $t > 0$, $\alpha \neq 0$ and for all $\lambda > 0$, $(Y_i)_{i \in \mathbb{N}} > 0$. Thus, inequality (5) neither depends on the jumps' intensity λ nor on the jumps' distribution Y_i (if $\mathbb{E}[Y_i - 1] < \infty$).

We want to mention that this is not an arbitrage possibility in the classical sense[13]. But it holds that the start investment is zero ($I_0 = K(g_0^L - g_0^S) = 0$) and that the discounted[14] net gain is positive in expectation ($e^{-\alpha t}\mathbb{E}[g_t] = \frac{I_0^*}{K}(e^{-\alpha t} - 1)^2 > 0$ $(\alpha \neq 0, t > 0)$). In [6] this is called a "remarkable property".

VI. Variance

After having studied the expected value we now want to look at the variance $\mathbb{V}[g_t]$ or, equivalently, the standard deviation $\mathbb{S}(g_t)$.

[13] An arbitrage strategy is a strategy π with $\pi_0 \leq 0$ and $\pi_t \geq 0$ and $\mathcal{P}(\pi_t > 0) > 0$.

[14] We use $e^{-\alpha t}$ for discounting.

Theorem 4. *The standard deviation of the gain of the SLS trading strategy with a stock price governed by* (2) *is*

$$\mathbb{S}[g_t] = \frac{I_0^*}{K}((e^{2Kt\alpha} + e^{-2Kt\alpha})(e^{K^2t(\sigma^2+\lambda\zeta)} - 1) + 2(e^{-K^2t(\sigma^2+\lambda\zeta)} - 1))^{1/2}.$$

Proof. We set $\zeta := \mathbb{E}[(Y_i-1)^2]$ and assume $\zeta < \infty$. Analogous to the calculation of the expected values in Section V it can be shown that $\int_{\Omega_N}\int_{\Omega_Y}\prod_{i=1}^N((1+4K+4K^2)+(2K+4K^2)(Y_i-1)+K^2(Y_i-1)^2)\mathrm{d}P_Y\mathrm{d}P_N < \infty$. This allows for using Fubini-Tonelli[15]. It holds $\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^N((1+2K(Y_i-1)+K^2(Y_i-1)^2))\mathrm{d}(P_N\otimes P_Y) = e^{\lambda t(2K\kappa+K^2\zeta)}$, $\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^N((1-2K(Y_i-1)+K^2(Y_i-1)^2))\mathrm{d}(P_N\otimes P_Y) = e^{\lambda t(-2K\kappa+K^2\zeta)}$, and $\int_{\Omega_N\times\Omega_Y}\prod_{i=1}^N((1-K^2(Y_i-1)^2))\mathrm{d}(P_N\otimes P_Y) = e^{-\lambda tK^2\zeta}$. A well-known transformation for the variance is

$$\mathbb{V}[g_t] = \mathbb{E}[g_t^2] - (\mathbb{E}[g_t])^2.$$

Thus, we first want to calculate the second moment of the gain (given through (3)):

$$\begin{aligned}
\mathbb{E}[g_t^2] &= \frac{I_0^{*2}}{K^2}\mathbb{E}[(\left(\frac{b_t}{b_0}\right)^K e^{\frac{(K-K^2)\sigma^2 t}{2}}\prod_{i=1}^N(1+K(Y_i-1)) \\
&\quad + \left(\frac{b_t}{b_0}\right)^{-K} e^{\frac{-(K+K^2)\sigma^2 t}{2}}\prod_{i=1}^N(1-K(Y_i-1)) - 2)^2] \\
&= \frac{I_0^{*2}}{K^2}(\mathbb{E}[\left(\frac{b_t}{b_0}\right)^{2K}]e^{(K-K^2)\sigma^2 t}\mathbb{E}[\prod_{i=1}^N(1+K(Y_i-1))^2] \\
&\quad + \mathbb{E}[\left(\frac{b_t}{b_0}\right)^{-2K}]e^{-(K+K^2)\sigma^2 t}\mathbb{E}[\prod_{i=1}^N(1-K(Y_i-1))^2] + 4 \\
&\quad - 4\mathbb{E}[\left(\frac{b_t}{b_0}\right)^{K}]e^{\frac{(K-K^2)\sigma^2 t}{2}}\mathbb{E}[\prod_{i=1}^N(1+K(Y_i-1))] \\
&\quad - 4\mathbb{E}[\left(\frac{b_t}{b_0}\right)^{-K}]e^{\frac{-(K+K^2)\sigma^2 t}{2}}\mathbb{E}[\prod_{i=1}^N(1-K(Y_i-1))] \\
&\quad + 2e^{-K^2\sigma^2 t}\mathbb{E}[\prod_{i=1}^N(1-K^2(Y_i-1)^2)]) \\
&= \frac{I_0^{*2}}{K^2}(e^{2\mu tK+\sigma^2K^2t}\mathbb{E}[\prod_{i=1}^N(1+2K(Y_i-1)+K^2(Y_i-1)^2)]
\end{aligned}$$

[15]Note: $(1+4K+4K^2)+(2K+4K^2)(Y_i-1)+K^2(Y_i-1)^2 \geq \max\{|1+2K(Y_i-1)+K^2(Y_i-1)^2|, |1-2K(Y_i-1)+K^2(Y_i-1)^2|, |1-K^2(Y_i-1)^2|\}$

$$
\begin{aligned}
&+ e^{-2\mu K t+\sigma^2 K^2 t}\mathbb{E}[\prod_{i=1}^{N}(1-2K(Y_i-1)+K^2(Y_i-1)^2)] \\
&+ 4 - 4e^{K\alpha t} - 4e^{-K\alpha t} \\
&+ 2e^{-K^2\sigma^2 t}\mathbb{E}[\prod_{i=1}^{N}(1-K^2(Y_i-1)^2)]]) \\
= &\frac{I_0^{*2}}{K^2}(e^{K^2 t(\sigma^2+\lambda\zeta)}(e^{2Kt\alpha}+e^{-2Kt\alpha}) \\
&+ 2e^{-K^2 t(\sigma^2+\lambda\zeta)} - 4(e^{K\alpha t}+e^{-K\alpha t}-1))
\end{aligned}
$$

With this and (4) it follows

$$
\begin{aligned}
\mathbb{V}[g_t] = &\frac{I_0^{*2}}{K^2}(e^{K^2 t(\sigma^2+\lambda\zeta)}(e^{2Kt2\alpha}+e^{-2Kt\alpha}) \\
&+ 2e^{-K^2 t(\sigma^2+\lambda\zeta)} - (e^{2K\alpha t}+e^{-2K\alpha t}+2))
\end{aligned}
$$

and the proposition about $\mathbb{S}[g_t]$. □

Note that for using Fubini-Tonelli not only the first but also the second moment of Y_i must exist. One interesting conspicuity is that the variance of g_t depends on the jump intensity and the second moment of the jumps, but not on the first moment, i.e., the expected height of the jumps.

VII. Simulations and Plots

In Fig. 4 the dependency of the expected gain and of the standard deviation on several parameters is illustrated. Because in Fig. 4 the influence of the feedback parameter K which is chosen by the trader on the expected gain cannot be seen adequately Fig. 5 was inserted. In the four graphs, one of the parameters K, α, σ, and λ was varied whereas all others remain fixed. Even if the expectation does not depend on “the jump parameters” λ, μ_{Y_i}, and σ_{Y_i}, the standard deviation does. Note that for creating the graphs no stochastic process was simulated and no random number generated. At the end of this section let us have a look at Fig. 6. These two graphs show two interesting facts. On the one hand, one can see that the trading results obtained by “real” trading on stochastic processes with discrete time and the results calculated via the formula (with continuous time) do not differ very much. On the other hand, the histograms show that the gains are highly skewed (what is in line with [17] and [18]), especially, the gains have

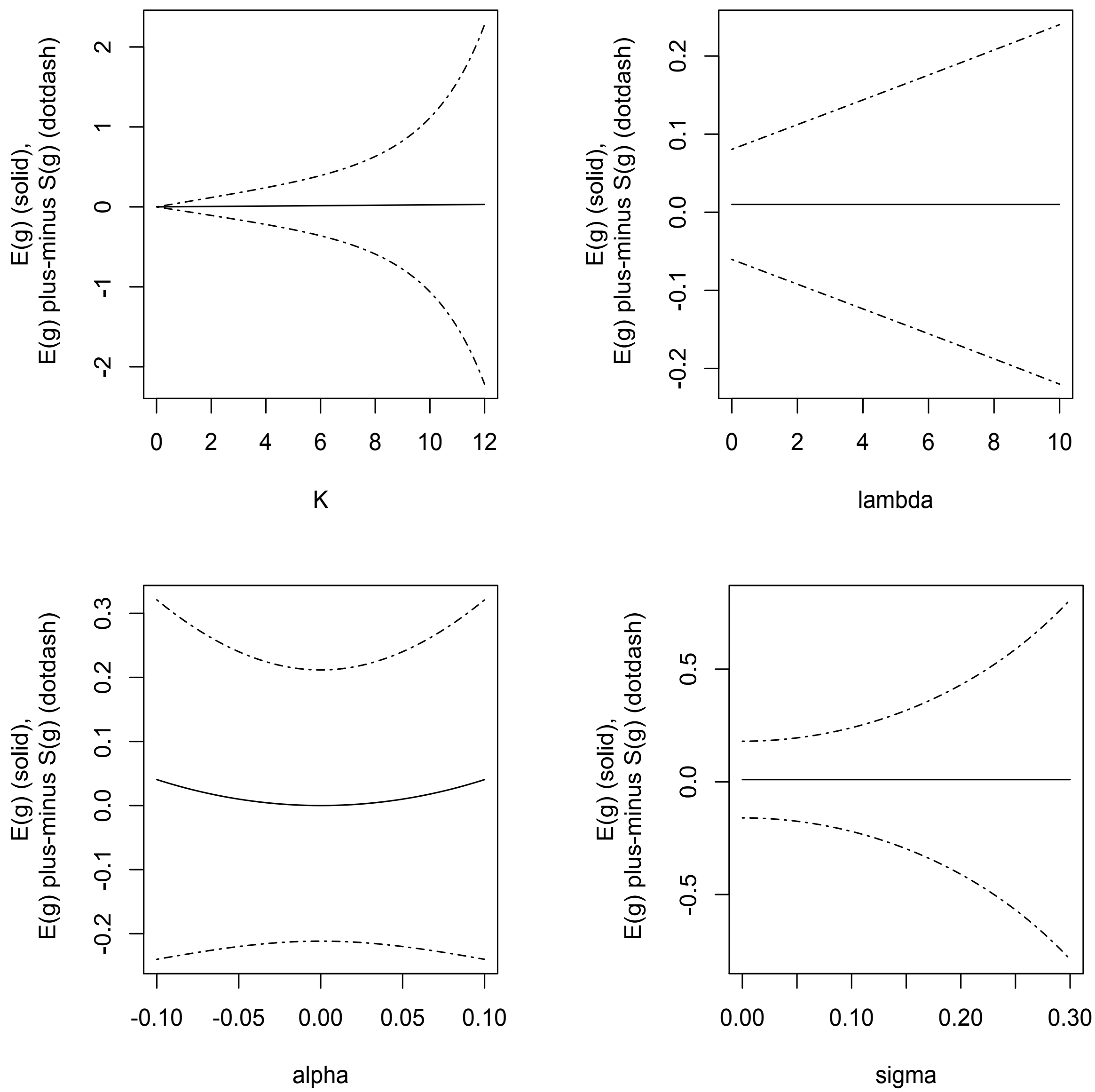

Fig. 4. Dependency of the expected gain and of the standard deviation on K ($\in [0.01, 12]$), λ ($\in [0, 10]$), α ($\in [-0.1, 0.1]$), and σ ($\in [0, 0.3]$). All other parameters respectively: $I_0 = 1$, $K = 4$, $\alpha = 0.05$, $\sigma = 0.1$, $t = 1$, $\lambda = 10$, $\mu_{Y_i} = 0.01$, and $\sigma_{Y_i} = 0.05$.

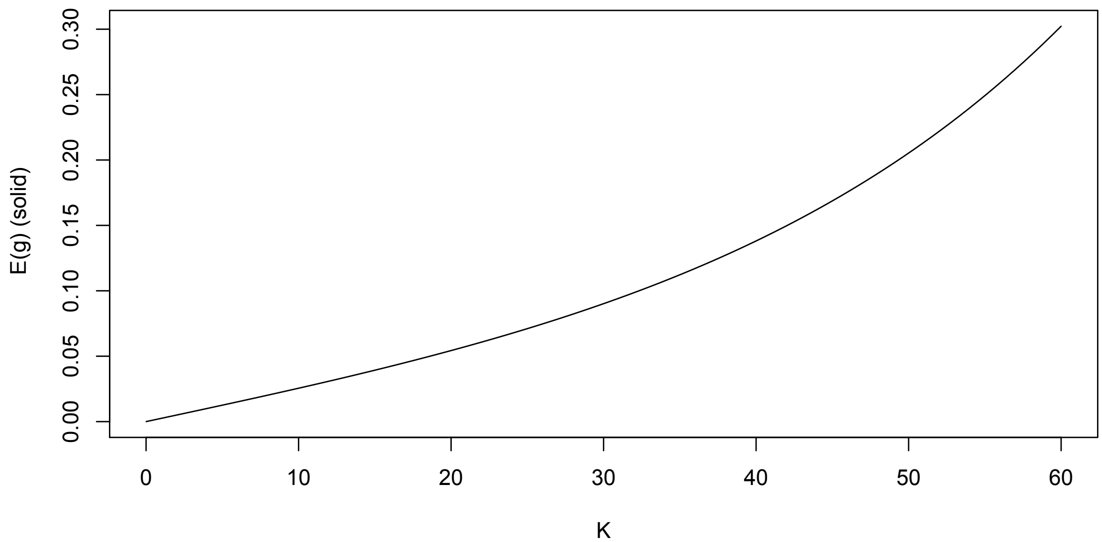

Fig. 5. Dependency of the expected gain on K ($\in [0.01, 20]$). All other parameters: $I_0 = 1$, $\alpha = 0.05$, $\sigma = 0.1$, $t = 1$, $\lambda = 10$, $\mu_{Y_i} = 0.01$, and $\sigma_{Y_i} = 0.05$.

a so-called fat tail to the positive side. I.e., it is likely that the gain is negative and small, but sometimes it is positive and rather high. This leads to a positive mean. Note that p_t, N, and Y_i were the same for both graphs. For all simulations the jumps are lognormally distributed, like it is recommended in [2] and used in [24], with $\ln Y_i \sim \mathcal{N}(\mu_{Y_i}, \sigma^2_{Y_i})$.

VIII. Conclusion

We tried to prove whether the remarkable results concerning SLS trading obtained in related work (for example [1]) also hold if jumps may occur in the market/price model. For this, we analyzed SLS-trading (with zero start investment) on Merton's jump diffusion model: We could deduce formulae for the gain/loss function g_t, for the expected gain/loss $\mathbb{E}[g_t]$ with the astonishing result

$$I_0 = 0 \quad \& \quad e^{-\alpha t}\mathbb{E}[g_t] > 0$$

for all $t > 0$, $\alpha \neq 0$, and for the standard deviation of the gain/loss function. The two findings concerning the expected gain are independent of intensity, kind (that means, the distribution of Y_i), and size of the jumps. One could say that the expected gain is robust against jumps. Maybe, some economists might be interested in the results of this paper, too, since crises and related price jumps could make funds loosing money.

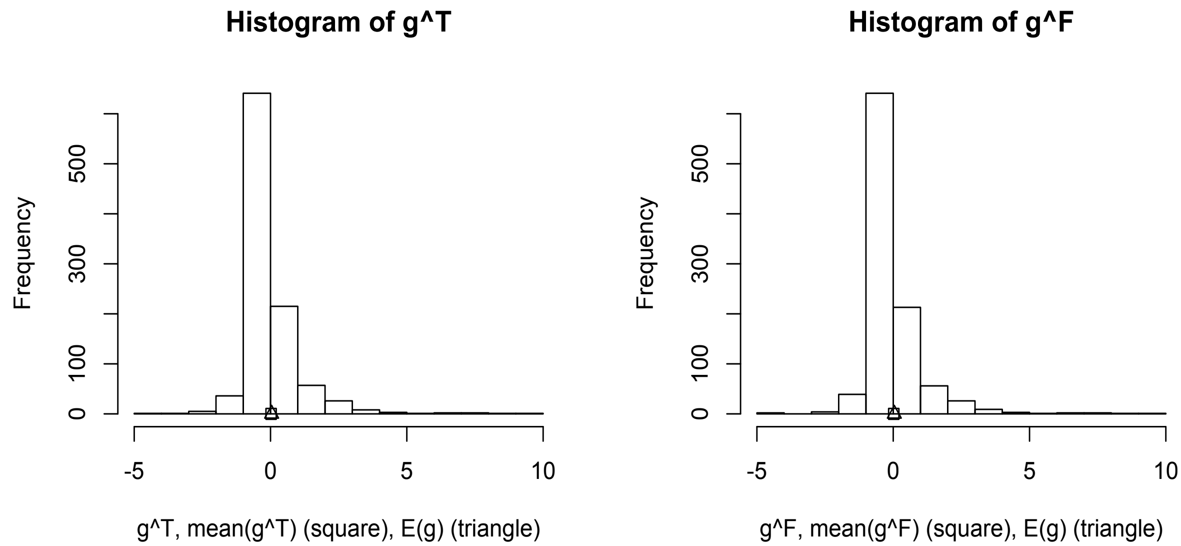

Fig. 6. Histograms of 1000 gains. On the left hand side obtained with discrete trading, that means, 1000 stochastic processes were simulated (g^T; "Trading"). On the right hand side obtained via the formula for g_t, i.e., continuous trading was assumed (g^F; "Formula"). One can see, that these figures do not differ too much and that both are highly skewed. Parameters: $p_0 = 1$, $I_0 = 1$, $K = 4$, $\alpha = 0.05$, $\sigma = 0.1$, $t = 1$, $\lambda = 10$, $\mu_{Y_i} = 0.01$, $\sigma_{Y_i} = 0.05$, and increment $\tau = 0.001$.

IX. Ongoing Research

An extended analysis of feedback trading in more economical/game-theoretical settings is one of the topics for future research as well as a variation of the investigated control strategy. Maybe the target of constructing another feedback trading rule could be to obtain a non-skewed gain. For practical trading, parameter estimation, e.g., maximum likelihood on training data, would be important and is thus worth future research. Settings similar to that one presented in this paper, that means, SLS and jump processes, are still interesting for future work.

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