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VALUE ITERATION CONVERGENCE OF ϵ-MONOTONE SCHEMES FOR STATIONARY HAMILTON-JACOBI EQUATIONS

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Abstract. We present an abstract convergence result for the fixed point approximation of stationary Hamilton–Jacobi equations. The basic assumptions on the discrete operator are invariance with respect to the addition of constants, ε-monotonicity and consistency. The result can be applied to various high-order approximation schemes which are illustrated in the paper. Several applications to Hamilton–Jacobi equations and numerical tests are presented.

1. Introduction.

The numerical approximation of Hamilton-Jacobi equations (HJ) plays a crucial role in many fields of application including optimal control, image processing, fluid dynamics, robotics and geophysics. This has motivated a number of different contributions where the main effort has been concentrated on the construction of schemes in multidimensional domains and on the conditions ensuring convergence to the weak solution (to be understood in this framework as the unique viscosity solution). It is well known (see e.g. [6, 7]) that viscosity solution are typically nonsmooth, so the difficulty is to have a good resolution around the singularities and a good accuracy in the domains where the solution is regular.

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The theory of approximation schemes for viscosity solutions has been developed starting from the huge literature existing for the numerical solution of conservation laws in one dimension. In fact, this seems quite natural since in one dimension there is a strong link between the viscosity solution of an evolutive Hamilton-Jacobi equation with convex hamiltonian $\mathcal{H}(u_x)$ and the corresponding conservation law with convex flux $\mathcal{H}(u)$. More precisely, the viscosity solution can be written as the space integral of the corresponding entropy solution (see e.g. [23]) and this relation can be applied to the construction of numerical schemes (see the pionnering work [24]). In order to pass from a scheme for conservation laws to a scheme for the Hamilton-Jacobi equation one has to integrate in space the original scheme. This approach is valid only in one dimension but, in practice, it has been extended to multidimensional problems using a dimensional splitting ([39, 42], see also [38]).

As we mentioned, the literature dealing with the correct approximation for conservation laws is huge and a typical result concerns the convergence of monotone schemes (like the celebrated Godunov scheme) in the L^1-norm; note that this is the natural norm for this class of problems since entropy solutions may have jumps. The rate of convergence of monotone schemes has been shown to be at most 1 with respect to the discretization parameters Δt and Δx (which are linked by the stability CFL condition). This bound has motivated new efforts to develop high-order approximation schemes based on different ideas and exploiting the fact that entropy solution are TVD (Total Variation Diminishing) in time. Starting from the work of van Leer [46] many authors have proposed new schemes trying to improve the rate of convergence and to avoid oscillations around the discontinuities by making use of special types of local interpolation techniques like ENO (Essentially Non-Oscillatory) and WENO (Weighted Essentially Non-Oscillatory) methods, for which the interested reader is referred, e.g., to [45, 36, 37]. Another important idea that emerged is that one has to reduce the numerical diffusion in the schemes and to this end anti-diffusive flux corrections have to be considered in the approximation. The above methods are essentially based on finite difference or finite volume methods and general convergence results can be found in [21, 22] or the textbook [32].

Passing to Hamilton–Jacobi equation, we mention that the interest for high-order numerical methods is also motivated by the fact that, given the accuracy, they allow to reduce the number of nodes using coarser grids with respect to monotone schemes. This can be a crucial point when the dimension of the state space is high (as in Hamilton–Jacobi equations related to optimal control). Clearly, a number of different numerical approaches and techniques have been applied to HJ equations. Some of them, like Finite Differences, use structured grids and are strictly related to the above mentioned schemes for conservation laws. Other methods, like Finite Volume, Discontinuous Galerkin and semi-Lagrangian schemes can easily work on unstructured grids and are based on different ideas, e.g., on the Hopf-Lax representation formula. Also in this framework, monotonicity has an important role in proving convergence to the viscosity solution and a general result for monotone scheme applied to second order fully nonlinear equations has been proved by Barles and Souganidis in [8]. Although a complete list of the contributions to numerical methods for HJ equations goes beyond the scopes of this paper, let us quote the application of Godunov/central schemes [1, 2, 3, 15], antidissipative and SuperBee/UltraBee [14, 13, 12], MUSCL [43], Discontinuous Galerkin [40], adaptive [33, 10] and sparse grid [11] semi-Lagrangian, WENO [47, 16]. In particular, later in this paper we will examine more in detail the case of filtered schemes and of semi-Lagrangian (SL) schemes. High-order SL schemes for HJ equations have been first considered for a semi-discretization in time in [27] and for the fully discrete scheme in [28]. A convergence analysis based on the condition $\Delta x = O(\Delta t^2)$ is carried out in [30]. The adaptation of the theory to weighted ENO reconstructions is presented in [20], along with a number of numerical tests comparing the various high-order versions of the scheme. Other numerical tests, mostly in higher dimension and concerned with applications to front propagation and optimal control, are presented in [19]. Filtered schemes have been analyzed in [43].

In this paper, we consider high-order approximations to stationary Hamilton–Jacobi equations. We prove an abstract convergence result for high-order methods relaxing the monotonicity assumption to ε-montonicity and show how some known schemes fit into this theory. Consider, as a prototype problem, the following Hamilton–Jacobi–Bellman PDE

$$\delta v(x) + \sup_{u\in U}\{-Dv(x)\cdot f(x,u) - \ell(x,u)\} = 0, \tag{1}$$

for $x \in \Omega$, which corresponds to an infinite horizon discounted optimal control problem. Here, $\Omega \subset \mathbb{R}^n$ is assumed to be a compact set which is optimally invariant for the dynamics or at whose boundary suitable boundary conditions are imposed. For simplicity, we develop our ideas for this equation but the following considerations also apply to the Kruzhkov-transformed minimum-time problem (see Section 2 for more details) or to the regularized Zubov equation [17].
In a large number of situations, a numerical approximation of (1) can be performed by regarding its solution as the asymptotic state of an evolutive problem of the form

$$v_t + \delta v + \sup_{u\in U}\{-Dv\cdot f(x,u) - \ell(x,u)\} = 0. \tag{2}$$

Looking for a *numerical* asymptotic state for equation (2) corresponds to the so-called *time-marching* schemes. When applied to (1), these schemes are of the form

$$v = T(v) \tag{3}$$

where T is an appropriate operator (examples will be given in Sections 4 and 5 of this paper) while when applied to (2), the schemes lead to the iteration

$$v^{j+1} = T(v^j), \tag{4}$$

the so called *value iteration*. Convergence of value iterations of this form to a fixed point of (3) (and, next, convergence of the numerical solution to the exact one) are well known in the case of monotone schemes, for which the operator T is typically a contraction. However, the problem has not yet been studied for high-order schemes, for which numerical evidence exists that value iterations may fail to converge to a fixed point. This is the gap we are trying to close with this paper, which is organized as follows.

In Section 2 we illustrate two examples coming from optimal control and differential games problems. They lead to stationary Hamilton-Jacobi equation in the form that fits into the general theory presented in Section 3. It is important to note that this theory allows to weaken the monotonicity assumption to ε-monotonicity and this is the crucial point to use it for high-order schemes. In Section 4 we deal with semi-Lagrangian schemes and we prove some error bounds for fully discrete schemes. Section 5 is devoted to the analysis of some special cases including filtered schemes and high-order Finite Volume methods, and also includes numerical experiments illustrating the behavior of the schemes.

2. **Some motivating examples.**

As we mentioned in the introduction, one motivating example for the equations considered in this paper comes from control theory and is related to the infinite horizon problem. More generally, stationary equation similars to (1) appear in the characterization of optimal control problems and pursuit-evasion games which we will briefly describe in this section.

Infinite horizon optimal control. Consider a finite dimensional control system with dynamics given by

$$\begin{cases} \dot{y}(t) = f(y(t), u(t)) & \text{for } t > 0 \\ y(0) = x \end{cases} \tag{5}$$

where $y \in \mathbb{R}^n$ is the state, $u : [0, +\infty) \to U$ is the control and $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is the controlled vector field. To get a unique trajectory for every initial condition and a given control function we will always assume that f is continuous with respect to both variables, Lipschitz continuous with respect to the state space (uniformly in u). Moreover, we will assume that the set $\mathcal{U}$ of control functions consists of measurable functions u of time so that we can apply the Caratheodory theorem for the Cauchy problem (5). We want to minimize the functional

$$J_x(u) = \int_0^{+\infty} \ell(y(s), u(s)) e^{-\delta s} ds \tag{6}$$

over the set of control functions u. Here $\ell : \mathbb{R}^n \times U \to \mathbb{R}$ is the running cost and y solves (5). Note that the presence of the exponential discount term $e^{-\delta s}$ implies that current values of ℓ are more important that future values, since the parameter δ is positive, so the contribution of the costs corresponding to future times will be increasingly reduced. Via dynamic programming (see, e.g., [6]) one can prove that the value function of this problem, i.e.,

$$v(x) = \inf_{u \in \mathcal{U}} J_x(u) \tag{7}$$

is the unique viscosity solution of the Hamilton-Jacobi equation (for sufficiently large $\delta > 0$ in case ℓ is unbounded)

$$\delta v(x) + \sup_{u \in U} \{-Dv(x) \cdot f(x, u) - \ell(x, u)\} = 0, \qquad x \in \mathbb{R}^n. \tag{8}$$

In a completely analogous way, the corresponding maximizing optimal control problem $v(x) = \sup_{u \in \mathcal{U}} J_x(u)$ leads to the equation

$$\delta v(x) + \inf_{u \in U} \{-Dv(x) \cdot f(x, u) - \ell(x, u)\} = 0, \qquad x \in \mathbb{R}^n. \tag{9}$$

Minimum time problem and pursuit-evasion games. For the *minimum time problem* we consider the same dynamics (5) as in the infinite horizon problem and we want to minimize the time of arrival at a given target $\mathcal{C}$. So the cost will be given by

$$t(x, u) = \inf\{t \geq 0 \, : \, y_x(t; u) \in \mathcal{C}\} \tag{10}$$

with the convention $\inf \emptyset = \infty$. By dynamic programming one can prove that the minimum time functions

$$\mathcal{T}(x) := \inf_{u \in \mathcal{U}} t(x,u) \tag{11}$$

safisfies the Bellman equation

$$\sup_{u \in U} \{-D\mathcal{T}(x) \cdot f(x,a)\} = 1 \tag{12}$$

in the domain where $\mathcal{T}$ is finite (the so-called reachable set). Introducing the Kruzhkov transformation

$$v(x) := \frac{1}{\mu}(1 - e^{-\mu \mathcal{T}(x)}) \tag{13}$$

using the convention $e^{-\infty} = 0$, where μ is a free positive parameter to be suitably chosen, one can characterize $\mathcal{T}$ as the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \mu v(x) + \sup_{u \in U} \big\{ - Dv(x) \cdot f(x,a) \big\} = 1 & \text{for } x \in \mathbb{R}^n \setminus \mathcal{C} \\ v(x) = 0 & \text{for } x \in \partial \mathcal{C}. \end{cases} \tag{14}$$

Another example comes from the dynamic programming approximation of the Hamilton-Jacobi-Isaacs equations related to *pursuit-evasion games* (see [6, 26] for more details). Player a (the *pursuer*) wants to catch Player b (the *evader*) who is escaping and the controlled dynamics for each player is known. To simplify the notations we will denote by $y(t) = (y_P(t), y_E(t))$ the state of the system where $y_P(t)$ and $y_E(t)$ are the positions at time t of the pursuer and of the evader both belonging to $\mathbb{R}^n$ and by $f : \mathbb{R}^{2n} \times A \times B \to \mathbb{R}^{2n}$ the dynamics of the system. Here the dynamics depends on the controls of both players denoted by $a(\cdot) \in \mathcal{A}$ and $b(\cdot) \in \mathcal{B}$ respectively, where $\mathcal{A}$ denotes the set of measurable functions $a : [0, \infty) \to A$ and $\mathcal{B}$ the set of measurable functions $b : [0, \infty) \to B$. The payoff is clearly the time of capture, but, in order to have a fair game, we need to restrict the strategies of the players to the so-called *non-anticipating strategies* (i.e., strategies that cannot exploit the knowledge of the future strategy of the opponent). These strategies will be denoted respectively by $\alpha[\cdot] \in \Delta$ and $\beta[\cdot] \in \Gamma$. If Player a plays using strategy $\alpha[\cdot]$, while Player b plays with the control $b(\cdot)$, we can define the corresponding time of capture as

$$t_x(\alpha[b], b) := \inf \big\{ t \geq 0 \,:\, y_P(t) = y_E(t) \big\}.$$

Again we use the convention $t_x(\alpha[b], b) = +\infty$ if there is no capture. Then we can define the lower time of capture as

$$\mathcal{T}(x) = \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} t_x(\alpha[b], b),$$

and again $\mathcal{T}$ can be infinite if there is no way to catch the evader from the initial position of the system x. In order to get a fixed point problem and to deal with finite values, it is useful to again use the Kruzhkov transformation (13) which corresponds to the payoff

$$J_x(a,b) = \int_0^{t_x(a,b)} e^{-\mu t} dt \equiv \frac{1}{\mu}(1 - e^{-\mu t_x(a,b)}).$$

The rescaled minimal time will be given by

$$v(x) = \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} J_x(\alpha[b], b).$$

Similarly, reversing the order of inf and sup and letting Player b play strategies, we can define the *upper time of capture* as

$$\widetilde{\mathcal{T}}(x) = \sup_{\beta \in \Gamma} \inf_{a \in \mathcal{A}} t_x(a, \beta[a]),$$

getting for v the following relation

$$\widetilde{v}(x) = \sup_{\beta \in \Gamma} \inf_{a \in \mathcal{A}} J_x(a, \beta[a]).$$

Note that lower and upper value differ in general, but if they coincide, i.e., if $\mathcal{T} = \widetilde{\mathcal{T}}$ or $v = \widetilde{v}$, we say that *the game has a value.* Since both lower and upper value satisfy a Dynamic Programming Principle we can characterize them by an Hamilton-Jacobi-Isaacs equations, which for the lower value is

$$\min_{b \in B} \max_{a \in A} \{-D\mathcal{T}(x) \cdot f(x, a, b)\} = 1,$$

Similarly, for the upper value we have

$$\max_{a \in A} \min_{b \in B} \{-D\widetilde{\mathcal{T}}(x) \cdot f(x, a, b)\} = 1.$$

Those equations are complemented by the homogeneous boundary condition on the target where $\mathcal{T}(x) = 0$ (resp. $\widetilde{\mathcal{T}}(x) = 0$). Finally, if the transformed optimal value function $v(\cdot)$ is continuous, then v is a viscosity solution in $\mathbb{R}^n \setminus \mathcal{C}$ of the Dirichlet problem

$$\begin{cases} \mu v + \min_{b \in B} \max_{a \in A} \{-Dv(x) \cdot f(x, a, b)\} = 1 & \text{on } \mathbb{R}^n \setminus \mathcal{C} \\ v(x) = 0 & \text{on } \partial\mathcal{C}. \end{cases} \tag{15}$$

3. Abstract results for ε-monotone schemes.

3.1. Approximate convergence of the value iteration.

We start by setting up a general abstract framework for analysing the behaviour of value iterations, requiring neither strict monotonicity nor a contraction property for an abstract operator T. To this end, we denote the space of bounded real valued functions on $\Omega \subset \mathbb{R}^n$ by $B(\Omega)$. Note that $B(\Omega)$ is a Banach space when equipped with the supremum norm $\|\cdot\|_\infty$. In this section we provide abstract results for fixed point equations of the form (3) with $T : B(\Omega) \to B(\Omega)$. Hereafter, for any $w_1, w_2 \in B(\Omega)$ we will write $w_1 \geq w_2$ if $w_1(x) \geq w_2(x)$ for all $x \in \Omega$.

In the following theorem we first show an equivalence result between the ε-monotone property and a quasi-Lipschitz property of an operator T.

Theorem 3.1. *Consider an operator $T : B(\Omega) \to B(\Omega)$, and ε a positive constant. Let $A \subseteq B(\Omega)$ be a nonempty subset such that $w + c \in A$ holds for all $w \in A$ and $c \in \mathbb{R}$. Assume furthermore that there exists a constant $\beta > 0$ such that*

$$T(w + c) = T(w) + \beta c \tag{16}$$

for all $w \in A$ and all $c \in \mathbb{R}$.

Then, the following properties are equivalent:

1. *For all $w_1, w_2 \in A$, with $w_1 \leq w_2$,*

$$T(w_1) \leq T(w_2) + \varepsilon \tag{17}$$

2. *For all $w_1, w_2 \in A$,*

$$\|T(w_1) - T(w_2)\|_\infty \leq \beta \|w_1 - w_2\|_\infty + \varepsilon \tag{18}$$

Proof: The result and its proof are essentially a slight adaptation of Proposition 2 in [25], but we repeat here the arguments for completeness.

We first prove that (i) implies (ii). Let $w_1, w_2 \in A$ and consider the function

$$\widetilde{w} = w_1 + \|(w_2 - w_1)^+\|_\infty,$$

which lies in A and for which it is clear that $\widetilde{w} \geq \sup(w_1, w_2)$. Using now (16) and (17), we have

$$\begin{aligned} (T(w_1) - T(w_2))^+ &\leq T(\widetilde{w}) - T(w_2) + \varepsilon \\ &\leq T(w_1) + \beta\|(w_2 - w_1)^+\|_\infty - T(w_1) + \varepsilon \\ &= \beta\|(w_2 - w_1)^+\|_\infty + \varepsilon. \end{aligned}$$

It is clear that $w_2 \leq w_1 + \|(w_2 - w_1)^+\|_\infty$, and furthermore $w_1 + \|(w_2 - w_1)^+\|_\infty$ lies in A. Using now (16) and (17), we have

$$\begin{aligned} T(w_2) &\leq T(w_1 + \|(w_2 - w_1)^+\|_\infty) + \varepsilon \\ &\leq T(w_1) + \beta\|(w_2 - w_1)^+\|_\infty + \varepsilon, \end{aligned}$$

and therefore

$$(T(w_2) - T(w_1))^+ \leq \beta\|(w_2 - w_1)^+\|_\infty + \varepsilon.$$

Interchanging the roles of w_1 and w_2, we obtain the reverse inequality

$$(T(w_1) - T(w_2))^+ \leq \beta\|(w_1 - w_2)^+\|_\infty + \varepsilon,$$

and hence, (18).

To prove that (ii) implies (i), assume now that $w_1 \geq w_2$ and set

$$r = \|(w_1 - w_2)^+\|_\infty = \|w_1 - w_2\|_\infty.$$

We have then:

$$\begin{aligned} \|T(w_2) - T(w_1) + \beta r\|_\infty &= \|T(w_2 + r) - T(w_1)\|_\infty \\ &\leq \beta\|w_2 - w_1 + r\|_\infty + \varepsilon \\ &\leq \beta r + \varepsilon, \end{aligned}$$

which in turn implies that

$$T(w_2) - T(w_1) \leq \varepsilon.$$

The second result states the existence of a fixed point, and in what sense the iteration $w^{j+1} = T(w^j)$ approximates such a fixed point. In what follows, the assumption of working on a finite-dimensional space is crucial in order to apply Schauder's fixed point theorem. It is justified by the idea of treating numerical solutions.

Theorem 3.2. *Assume that $T : B(\Omega) \to B(\Omega)$ is continuous, and that furthermore $T(A) \subset A$ where A is a finite dimensional subspace of $B(\Omega)$ and such that $\forall w \in A$, $\forall c \in \mathbb{R}$, $w + c \in A$*

Assume moreover that (18) *holds for some $\varepsilon > 0$ and $0 \leq \beta < 1$ (which is implied by* (16) *and* (17)*). Then, the following holds:*

1. *The fixed point equation*

$$w = T(w)$$

has a solution $w^* \in A$ satisfying furthermore the bound

$$\|w^*\|_\infty \leq \frac{\|T(0)\|_\infty + \varepsilon}{1 - \beta}. \tag{19}$$

2. *Any two fixed points* $w_1^*, w_2^* \in A$ *of* T *satisfy*
$$\|w_1^* - w_2^*\|_\infty \le \frac{\varepsilon}{1-\beta}.$$
3. *For any sequence of the form* $w^{j+1} = T(w^j)$ *with* $w^0 \in A$*, any fixed point* $w^* \in A$ *of* T *and any constant* $c > 1$*, there exists a* $j^* \in \mathbb{N}$ *such that*
$$\|w^j - w^*\|_\infty \le \frac{c\varepsilon}{1-\beta}, \quad \forall j \ge j^*. \tag{20}$$
For $\varepsilon/(1-\beta) \to 0$*,* j^* *can be chosen to be of the order*[1]
$$j^* \sim -\frac{\log(\frac{\varepsilon}{1-\beta})}{\log \beta}. \tag{21}$$

Proof: (i) We show that there exists a closed ball $\overline{\mathcal{B}}_R(0)$ in $B(\Omega)$ with radius R such that $\overline{\mathcal{B}}_R(0) \cap A$ is mapped into itself by T, i.e.,
$$\|w\|_\infty \le R \quad \Rightarrow \quad \|T(w)\|_\infty \le R. \tag{22}$$
Then, by Schauder's fixed point theorem we can conclude the existence of a fixed point in the compact and convex set $\overline{\mathcal{B}}_R(0) \cap A$.

By (18), we have
$$\|T(w) - T(0)\|_\infty \le \beta\|w\|_\infty + \varepsilon \le \beta R + \varepsilon,$$
which gives
$$\|T(w)\|_\infty \le \|T(0)\|_\infty + \beta R + \varepsilon$$
and (22) is satisfied a fortiori if
$$\|T(0)\|_\infty + \beta R + \varepsilon \le R,$$
that is, as soon as
$$R \ge \frac{\|T(0)\|_\infty + \varepsilon}{1-\beta}.$$
Existence of a fixed point then follows, along with the bound (19).

(ii) Let w_1^*, $w_2^* \in A$ be two fixed points. Then, on the one hand, the fixed point property implies
$$\|T(w_1^*) - T(w_2^*)\|_\infty = \|w_1^* - w_2^*\|_\infty$$
while on the other hand (18) implies
$$\|T(w_1^*) - T(w_2^*)\|_\infty \le \beta\|w_1^* - w_2^*\|_\infty + \varepsilon.$$
Together this yields
$$\|w_1^* - w_2^*\|_\infty \le \beta\|w_1^* - w_2^*\|_\infty + \varepsilon,$$
implying
$$\|w_1^* - w_2^*\|_\infty \le \frac{\varepsilon}{1-\beta}.$$
(iii) The definition of w^j, the fixed point property of w^* and inequality (18) imply
$$\|w^{j+1} - w^*\|_\infty = \|T(w^j) - T(w^*)\|_\infty \le \beta\|w^j - w^*\|_\infty + \varepsilon.$$
Then by simple recursion we get the estimate
$$\begin{aligned} \|w^j - w^*\|_\infty &\le \beta^j\|w^0 - w^*\|_\infty + \varepsilon(1 + \beta + \cdots + \beta^{j-1}) \\ &\le \beta^j\|w^0 - w^*\|_\infty + \frac{\varepsilon}{1-\beta}. \end{aligned}$$

[1] We write $a(\varepsilon) \sim b(\varepsilon)$ for $\varepsilon \to 0$ if $b(\varepsilon) \neq 0$ for all sufficiently small $\varepsilon > 0$ and $\lim_{\varepsilon\to 0} a(\varepsilon)/b(\varepsilon) = 1$.

Let us chose j^* as the smallest integer such that

$$\beta^{j^*}\|w^0 - w^*\|_\infty \le \frac{(c-1)\varepsilon}{1-\beta},$$

which means in particular that j^* can be taken of the order of

$$j^* \sim \frac{\log(\frac{\varepsilon}{1-\beta})}{\log\beta} + \frac{\log(c-1) - \log(\|w^0 - w^*\|)}{\log\beta}. \tag{23}$$

If $\varepsilon/(1-\beta) \to 0$ and c is fixed, the term $(\log(c-1) - \|w^0 - w^*\|_\infty)/\log(\beta)$ is neglectable and we obtain (21). Then, for all $j \ge j^*$ we get the desired result

$$\|w^j - w^*\|_\infty \le \frac{c\varepsilon}{1-\beta}.$$

3.2. **The Barles-Souganidis theorem for ε-monotone schemes.** Proving that (3) admits fixed point solutions (and that value iterations converge to some neighborhood of any such fixed point) ensures that a solution of the scheme, at least up to some uncertainty, might be computed via the iteration (4). A second step of the analysis is then to study the convergence of such numerical solution to the exact solution of (1) if T is obtained from a discretization of (1). Conventionally, we will use in the sequel two discretization steps h and k to account for respectively time and space discretization, cf. Section 4, but everything applies to a different number of discretization parameters. For the moment these discretization parameters are regrouped in an abstract parameter denoted $\rho = (h, k)$, and $\rho \to 0$ means that each discretization parameter goes to 0.

The classical Barles-Souganidis theorem provides a first, relatively simple framework for a convergence analysis of ε-monotone schemes. Among the various versions of this theorem, we refer to [3] and [29]. Here, we apply the theory to schemes which may present some defect of monotonicity, provided its magnitude is "small", in a sense to be made precise. In order to formulate the theorem, we will need to impose assumptions on the asymptotic behavior of ε and β as $\rho \to 0$.

Let us assume that v is the unique viscosity solution of the abstract problem

$$\mathcal{F}(x, v(x), Dv(x)) = 0, \quad x \in \Omega. \tag{24}$$

Here we assume that (24) is well posed, and in particular that it satisfies a strong comparison principle (see [6] for details).

We will use the almost contraction property which, in view of Theorem 3.1, is similar to ε-monotonicity. From now on, the set S_ρ will typically contain the numerical solution and will depend on ρ. We assume that:

Assumption (H)

(H1) For all ρ, T_ρ is continuous.

(H2) For all ρ, there exists $S_\rho \subseteq B(\Omega)$ such that $T_\rho(S_\rho) \subset S_\rho$, where S_ρ is a finite dimensional subspace of $B(\Omega)$, and with $\forall w \in S_\rho$, $\forall c \in \mathbb{R}$, $w + c \in S_\rho$.

(H3) There exists $0 < \beta < 1$ and $\varepsilon \ge 0$ (depending on ρ), such that

$$\forall w_1, w_2 \in S_\rho, \quad \|T_\rho(w_1) - T_\rho(w_2)\|_\infty \le \beta\|w_1 - w_2\|_\infty + \varepsilon$$

and, in the limit for suitable sequences[2] $\rho \to 0$,

$$\lim_{\rho\to 0} \frac{\varepsilon}{1-\beta} = 0. \tag{25}$$

[2]For instance, for $\rho = (h, k)$ one may require $h \to 0$ and $k/h \to 0$.

(H4) We require the uniform bound, for some constant $M_v \geq 0$ independent of ρ:

$$\forall \rho, \quad \frac{\|T_\rho(0)\|_\infty}{1-\beta} \leq M_v. \tag{26}$$

(H5) The scheme is consistent with (24) in the sense that there exists some constant $c > 0$, independent of ρ, such that, $\forall x \in \Omega$,

$$\lim_{\rho\to 0,\ y\to x,\ \xi\to 0} \frac{(\varphi(y)+\xi) - T_\rho(\varphi+\xi)(y)}{c(1-\beta)} = \mathcal{F}(x,\, \varphi(x)),\, D\varphi(x)) \tag{27}$$

for all $\varphi \in C^1(\mathbb{R}^n)$.

Theorem 3.3. *Let v be the viscosity solution of* (24)*. We consider an iterative scheme of the form $w^{j+1} := T_\rho(w^j)$, where $T_\rho : B(\Omega) \to B(\Omega)$ is an operator depending on a parameter ρ, and satisfying Assumption (H). Then*

(i) *There exists a solution $w^*_\rho \in S_\rho$ of $w^*_\rho = T_\rho(w^*_\rho)$, such that $\|w^*_\rho\|_\infty \leq M_v$, and $w^*_\rho \to v$ uniformly on compact subsets of $\mathbb{R}^n$.*

(ii) *If $\varepsilon > 0$, considering any index j_ρ such that*

$$j_\rho \geq j^*_\rho := \frac{1}{\log(\beta)}\left(\log(\frac{\varepsilon}{1-\beta}) - \log(\|w^0\|_\infty + 2M_v)\right)$$

it holds

$$\|w^{j_\rho} - w^*_\rho\|_\infty \leq \frac{2\varepsilon}{1-\beta}.$$

(iii) *If $\varepsilon = 0$, for any $K_\rho > 0$ and for any j_ρ such that*

$$j_\rho \geq j^*_\rho := \frac{1}{\log(\beta)}\left(\log(K_\rho) - \log(\|w^0\|_\infty + M_v)\right)$$

it holds $\|w^{j_\rho} - w^*_\rho\|_\infty \leq K_\rho$.

In particular $\lim_{\rho\to 0} w^{j_\rho}_\rho = v$ uniformly on compact subsets of $\mathbb{R}^n$ for all sequences $\rho \to 0$ satisfying (25)*.*

Proof. (i)-(ii) By Theorem 3.2, there exists $w^*_\rho : B(\Omega)$ such that $w^*_\rho = T(w^*_\rho)$ and

$$\|w^*_\rho\|_\infty \leq \frac{\|T_\rho(0)\|_\infty + \varepsilon}{1-\beta} \leq 2M_v \tag{28}$$

as $\rho \to 0$.

For $x \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $\varphi \in B(\Omega)$, let

$$S_\rho(x, r, \varphi) := \frac{r - T_\rho(\varphi)(x)}{c(1-\beta)},$$

so that $w = T_\rho(w)$ can be written equivalently as $S_\rho(x, w(x), w) = 0$, and w^*_ρ is one solution of $S_\rho = 0$. The map S_ρ is ε-monotone in the sense that for all $x, r, \varphi_1, \varphi_2$:

$$\varphi_1 \leq \varphi_2 \quad \Rightarrow \quad S_\rho(x, r, \varphi_1) \geq S_\rho(x, r, \varphi_2) - \frac{\varepsilon}{c(1-\beta)},$$

where, by assumption, $\lim_{\rho\to 0} \frac{\varepsilon}{1-\beta} \to 0$. Hence, by following exactly the same arguments as in the proof of the Barles-Souganidis theorem [8] and using that ε-monotonicity is sufficient for convergence as remarked in Augoula and Abgrall [4], we obtain $\lim_{\rho\to 0} w^*_\rho = v$ uniformly on compact subsets of $\mathbb{R}^n$. (The consistency assumption is simpler than the one of [8] because we do not need to deal with boundaries here).

Therefore it remains to estimate $w_\rho^j - w_\rho^*$ for some well chosen j index. As in (23) (in the proof of Theorem 3.2) and using the constant $c = 2$, for $\varepsilon > 0$, we have obtained that $\|w^j - w_\rho^*\|_\infty \leq \frac{2\varepsilon}{1-\beta}$ for any index j such that

$$j \geq \frac{\log(\frac{\varepsilon}{1-\beta})}{\log(\beta)} - \frac{\log(\|w^0 - w_\rho^*\|_\infty)}{\log(\beta)}.$$

By using the uniform bound (28) we deduce the desired result.

The proof of (iii) is similar.

The final result is obtained by using that $\lim_{\rho\to 0} \frac{\varepsilon}{1-\beta} = 0$ if $\varepsilon > 0$, or choosing $K_\rho \to 0$ in the case of $\varepsilon = 0$. □

4. **The semi-Lagrangian case.** In this section, we develop our results for the equation

$$\delta v(x) + \sup_{u\in U}\{-Dv(x)\cdot f(x,u) - \ell(x,u)\} = 0, \quad x \in \Omega, \tag{29}$$

where $\Omega \equiv \mathbb{R}^n$, which corresponds to the problem (6)-(7). Note that the extension to equations of type (14) or (15) is straightforward. For simplicity, we do not explicitly treat boundary conditions here and remark that they could be included into our analysis in a straightforward way. We impose the following assumptions:

- $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous function, Lipschitz continuous with respect to the first variable x, uniformly in $u \in U$ (with Lipschitz constant L_f);
- $\ell : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a bounded, continuous function, Lipschitz continuous with respect to the first variable x and uniformly in $u \in U$.

4.1. **Setting.** In a semi-Lagrangian discretization, the first step is to discretize (29) in time. The most simple way to do this is by using a first order discretization with time step $h > 0$ which leads to the semi-discrete equation

$$w(x) = \min_{u\in U}\{(1-\delta h)w(x + hf(x,u)) + h\ell(x,u)\}. \tag{30}$$

We will hereafter assume that $h < \dfrac{1}{\delta}$ so that the following parameter

$$\beta := 1 - \delta h$$

belongs to $]0,1[$. Solving (30) amounts to finding a fixed point v_h of the equation

$$v_h = T_h(v_h) \tag{31}$$

where

$$T_h(w)(x) := \min_{u\in U}\{(1-\delta h)w(x + hf(x,u)) + h\ell(x,u)\}. \tag{32}$$

It is straightforward to prove that T_h is a contraction on $B(\Omega)$ w.r.t. the norm $\|\cdot\|_\infty$ with contraction constant β, i.e.,

$$\forall w_1, w_2 \in B(\Omega), \quad \|T_h(w_1) - T_h(w_2)\|_\infty \leq \beta\|w_1 - w_2\|_\infty.$$

Since $B(\Omega)$ equipped with the supremum norm $\|\cdot\|_\infty$ is a Banach space, the Banach fixed point theorem implies the existence of a unique fixed point $v_h \in B(\Omega)$ of (31) follows. Then, for any $w^0 \in B(\Omega)$ the value iteration (4) for T_h will converge for $j \to \infty$ with $\lim_{j\to\infty} w^j = v_h$.

One easily checks that T_h is consistent in the sense of (27) with $c = 1/\delta$ (so that $c(1-\beta) \equiv h$) and that all other assumptions of Theorem 3.3 are satisfied, too (with $\varepsilon = 0$). Hence, according to this theorem the fixed point v_h of T_h converges to the exact solution v as $h \to 0$.

Remark 4.1. *Under suitable conditions, convergence estimates for $h \to 0$ can be established, like, e.g., the estimate*

$$\|v_h - v\|_\infty \le C\, h^\gamma \tag{33}$$

with $\gamma = \frac{1}{2}$, where C depends of the Lipschitz constants of f and ℓ, see [18]*. Moreover, in specific cases* (33) *holds with $\gamma = 1$.*

As the second step we now discretize (31) in space. To this end, we fix a function space $S_k \subset B(\Omega)$ in which $k > 0$ is a discretization parameter. For instance, S_k could be the space of continuous and piecewise linear functions on a triangulation with k denoting the maximal diameter of a grid element Alternatively, S_k could be a higher order finite element space on Ω or, in case $n = 1$, the space of cubic splines on Ω, etc.

Then let $\Pi_k : B(\Omega) \to S_k$ denote a map from $B(\Omega)$ to S_k. One possible way to construct such a map is via a grid mesh denoted (x_i), and an operator $\Pi_k : B(\Omega) \to S_k$ satisfying

- $\forall w \in B(\Omega)$, $\Pi_k(w)$ depends only on the values of $(w(x_i))$, i.e.:

$$\forall w, \tilde{w} \in B(\Omega), \quad \Big(\forall i,\ w(x_i) = \tilde{w}(x_i)\Big) \Rightarrow \Pi_k(w) \equiv \Pi_k(\tilde{w}).$$

- $\forall w \in B(\Omega),\ \forall i, \quad \Pi_k(w)(x_i) = w(x_i)$.
- $\Pi_k w = w$ for all $w \in S_k$.

The latest relation states that Π_k is a projection, i.e., $\Pi_k \circ \Pi_k \equiv \Pi_k$. We remark that our analysis is not restricted to maps Π_k based on grid mesh values $w(x_i)$; for an alternative way of defining Π_k see Section 5.3.

In the simplest case, the operator Π_k is linear w.r.t. its argument, and is explicitly defined by

$$\Pi_k(w)(x) := \sum_{i=0}^{n} \lambda_i(x) w(x_i), \tag{34}$$

for a basis $\{\lambda_i\}$ of cardinal functions such that

$$\lambda_i(x_j) = \delta_{ij}.$$

However in the remainder of the paper, we do not necessarily assume this linearity.

An important case of interpolation operator occurs when the basis functions λ_i are piecewise linear functions in x. In this situation, the x_i in (34) are the vertices of a grid simplex containing x and the coefficients $\lambda_i(x)$ are uniquely determined by the equation $\sum_{i=0}^n \lambda_i(x) x_i = x$. This results in a convex combination of the values w_i, and implies therefore monotonicity of the operator Π_k. In one space dimension, this procedure gives the well-known piecewise linear interpolation

$$\Pi_k^1 w(x) = w_i + \frac{x - x_i}{x_{i+1} - x_i}(w_{i+1} - w_i) \quad \text{for } x \in [x_i, x_{i+1}] \tag{35}$$

(we have chosen to give the specific notation Π_k^1 to this operator as it will play a special role in the subsequent theoretical analysis).

Using this framework we can now define a value iteration in S_k: we pick an arbitrary $w^0 \in S_k$ and iterate, for $j \ge 0$,

$$w^{j+1} = \Pi_k \circ T_h(w^j). \tag{36}$$

Alternatively, one may consider the iteration $w^{j+1} = T_h \circ \Pi_k(w^j)$, however, due to the fact that the iterates of (36) always lie in the finite dimensional function space S_k, (36) is easier to analyse.

Different from the value iteration (4) for $T = T_h$, it is in general not clear whether the value iteration (36) converges to a fixed point $w_\rho \equiv w_{h,k} \in A_k$. On the one hand, it is quite easy to see that if Π_k is linear and monotone, then $T_\rho := \Pi_k \circ T_h$ is a contraction (with same contraction constant β as T_h) and convergence of (36) again follows from Banach's fixed point theorem. This is usually enough to prove convergence for first order approximation schemes. On the other hand, however, numerical experiments in, e.g., [9] (cf. also Section 5 of this paper) show that for non-monotone interpolation operators Π_k convergence does not necessarily hold, as the iteration may end up in a limit cycle. This is the main difficulty when one tries to prove value iteration convergence for high-order methods.

In the following section we give conditions under which "almost" convergence can be proved.

4.2. **Results for fully discrete schemes.** The interplay of the following properties will play a role in our analysis. These properties are defined on subsets A of the space of bounded functions $B(\Omega)$. This is necessary because for many interpolation methods the ε in the ε-monotonicity depends on suitable regularity properties of the function w, e.g., bounds on Lipschitz constants. The set A then consists of all functions with these properties. In the analysis of a particular scheme, a difficult part is to show that A is invariant under the value iteration. An example of such an analysis can be found in Section 5.1.

Hence we will need

- a space $A \subset B(\Omega)$ (typically the set of L-Lipschitz functions for a given $L \geq 0$), that will contain all numerical solutions independently of the discretisation parameter k,
- a space S_k, typically finite dimensional, that corresponds to the image of Π_k.

Definition 4.2. *The interpolation operator* Π_k *is called* invariant w.r.t. addition of constants *if*

$$\Pi_k(w + c) = \Pi_k(w) + c$$

holds for all $w \in B(\Omega)$ *and all* $c \in \mathbb{R}$ *(identifying* c *with the constant function).*

Definition 4.3. *The interpolation operator* Π_k *is called* ε-monotone *on a set* $A \subseteq B(\Omega)$ *if for all* $w_1, w_2 \in A$ *with* $w_1 \leq w_2$ *the inequality*

$$\Pi_k(w_1) \leq \Pi_k(w_2) + \varepsilon$$

holds.

Remark 4.4. *(i) Any interpolation operator based on polynomials, like piecewise polynomial or spline interpolation is invariant w.r.t. addition of constants, because if the polynomial* p *interpolates* w *then* $p + c$ *interpolates* $w + c$.

(ii) Any interpolation method maintaining an interpolation error $\|\Pi_k(w) - w\|_\infty \leq \varepsilon_k$ *for all* $w \in A$ *is* ε*-monotone on* A *with* $\varepsilon = 2\varepsilon_k$ *because* $w_1 \leq w_2$ *then implies*

$$\Pi_k(w_2) - \Pi_k(w_1) \quad \leq \quad (w_2 + \varepsilon_k) - (w_1 - \varepsilon_k) \quad \leq \quad w_2 - w_1 + 2\varepsilon_k.$$

(iii) Even if Π_k *is not monotone we will show how to set back the interpolation into a monotone interpolation plus a small perturbation in Section 5.2.*

Lemma 4.5. *If Π_k is invariant w.r.t. addition of constants and ε-monotone on a space $A \subset B(\Omega)$, then for all $w_1, w_2 \in A$ the inequality*

$$\|\Pi_k \circ T_h(w_1) - \Pi_k \circ T_h(w_2)\|_\infty \le \beta \|w_1 - w_2\|_\infty + \varepsilon$$

holds with $\beta = 1 - \delta h$.

Proof: One easily proves that $T_\rho = \Pi_k \circ T_h$ satisfies the properties (16) and (17). Then, Theorem 3.1 yields the assertion.

Using Lemma 4.5 and Theorem 3.2 we obtain the convergence of the sequence generated by (36) to a ball around a fixed point of this equation, leading to the following theorem.

Theorem 4.1. *Let S_k be a finite dimensional subspace of $B(\Omega)$ and assume that $\Pi_k : B(\Omega) \to S_k$ is continuous, invariant w.r.t. addition of constants and ε-monotone on a space $A \subseteq B(\Omega)$ with $T_h(A) \subseteq A$. Let $w^0 \in S_k \cap A$ and consider the sequence generated by* (36)*. Then, there exists a fixed point $w_{h,k} \in S_k \cap A$ of the equation $\Pi_k \circ T_h(w) = w$ and for each $c > 1$ the relation $w^j \in B_{c\varepsilon/(\delta h)}(w_{h,k})$ holds for all sufficiently large j.*

Theorem 4.1 does not make any statement about the distance of w^j to the fixed point v_h of T_h. In order to make such a statement, the following consistency property is needed.

Definition 4.6. *The projection Π_k is called consistent of order $\varepsilon_c(k)$ on a set $A \subset B(\Omega)$ if there exists a function $\varepsilon_c : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{k\to 0} \varepsilon_c(k) = 0$ and*

$$\forall w \in A, \quad \|w - \Pi_k w\|_\infty \le \varepsilon_c(k).$$

Remark 4.7. *In case v_h is L-Lipschitz continuous for some $L \ge 0$, independent of h, we can deduce that for the monotone interpolation operator Π_k^1 the estimate*

$$\|v_h - \Pi_k^1 v_h\|_\infty \le CLk$$

holds, where $C \ge 0$ is a constant independent of v_h and k. Hence, in this case the projection is consistent with $\varepsilon_c(k) = CLk$, i.e., with first order in k, on the set of L-Lipschitz functions.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold and assume in addition that Π_k is consistent of order $\varepsilon_c(k)$ on the set A, and that $v_h \in A$. Then, for any $c > 1$, the relation $w^j \in B_{c(\varepsilon+\varepsilon_c(k))/(\delta h)}(v_h)$ holds for all sufficiently large j.*

Proof: For all $j \in \mathbb{N}$, Lemma 4.5 and consistency on the set A imply

$$\begin{aligned} \|w^{j+1} - v_h\|_\infty &= \|w^{j+1} - \Pi_k(v_h) \ + \ \Pi_k(v_h) - v_h\|_\infty \\ &= \|\Pi_k \circ T_h(w^j) - \Pi_k \circ T_h(v_h) + \ \Pi_k(v_h) - v_h\|_\infty \\ &\le \beta \|w^j - v_h\|_\infty + \varepsilon + \varepsilon_c(k). \end{aligned}$$

Now the assertion follows as in the proof of Theorem 3.2(iii) with $\Pi_k v_h$ in place of w^* and $\varepsilon + \varepsilon_c(k)$ in place of ε.

Remark 4.8. *As in the proof of Theorem 3.2(iii), one sees that the statement of the theorem is true for all $j \ge j^*$ where j^* is of the order of (as $h \to 0$ and $\frac{\varepsilon+\varepsilon_c(k)}{h} \to 0$)*

$$j^* \sim \log\left(\frac{\varepsilon + \varepsilon_c(k)}{1-\beta}\right) \Big/ \log \beta \ \sim \ -\log\left(\frac{\varepsilon + \varepsilon_c(k)}{\delta h}\right) \Big/ \delta h$$

Remark 4.9. *Together with the fact that T_h satisfies the assumption of Theorem 3.3 this estimate yields that the numerical value iteration will end up in a neighborhood of the exact solution v whose diameter shrinks to 0 as $h \to 0$, $\varepsilon_c(k)/h \to 0$ and $\varepsilon/h \to 0$.*

5. Examples of ε-monotone schemes, error estimates, and numerical illustrations.

5.1. **In-depth analysis of a simplified case.** In order to analyse the convergence of solutions obtained using a high-order interpolation operator in the SL scheme (31)–(36), we place ourselves in the simplified setting of the parallel analysis for the time-dependent problem, carried out in [30]. Consider therefore the problem

$$\delta v(x) + H(Dv(x)) = g(x) \tag{37}$$

posed on the whole of $\mathbb{R}$, with $\delta > 0$ and a strictly convex Hamiltonian $H : \mathbb{R} \to \mathbb{R}$. This problem fits our framework by setting $\ell(x,u) := H^*(u) + g(x)$, $f(x,u) := u$ and $U := \mathbb{R}$, where H^* denotes the Legendre transform of H.

The scheme then may be put in the form (36), with T_h defined by

$$T_h(w)(x) := \min_{u\in\mathbb{R}}\{\beta w(x+hu) + h(H^*(u) + g(x))\} \tag{38}$$

and where $\beta = 1 - \delta h$.

We consider a uniform grid mesh on the whole of $\mathbb{R}$:

$$x_i = ki, \quad i \in \mathbb{Z}.$$

Let Π_k^1 be the monotone interpolation operator defined in (35). For a given $L \geq 0$, let $Lip_L(\Omega)$ be the set of functions $w \in B(\Omega)$ which are L-Lipschitz, i.e.,

$$Lip_L(\Omega) := \left\{ w \in B(\Omega), \quad \sup_{x\neq y} \frac{|w(x) - w(y)|}{|x-y|} \leq L \right\}. \tag{39}$$

The central assumption we will need on the interpolation operator Π_k is, following [30], to assume that for any $L > 0$ there exists $C_L > 0$ such that

$$w \in Lip_L(\Omega) \quad \Rightarrow \quad \|\Pi_k(w) - \Pi_k^1(w)\|_\infty \leq C_L\ k. \tag{40}$$

Inequality (40) and the fact that Π_k^1 is monotone implies ε-monotonicity with $\varepsilon = 2C_L k$ on the set of functions $Lip_L(\Omega)$.

Now we aim to give a framework in which (40) can be proved.

We make the following basic assumptions on the data of our problem and on the scheme:

- Uniformly convex Hamiltonian: $H''(p) \geq m_H > 0$. Note that this also implies the dual inequality
$$0 < {H^*}''(p) \leq \frac{1}{m_H}. \tag{41}$$
- As for (29), a Lipschitz continuous source term
$$|g(x) - g(y)| \leq L_g|x-y|. \tag{42}$$
- There exists a constant $C \geq 0$, with $C < 1$, such that for any Lipschitz function w,
$$\left|\Pi_k(w)(x) - \Pi_k^1(w)(x)\right| \leq C \max_{x_{i-1},x_i,x_{i+1}\in\mathcal{S}(x)} |w_{i+1} - 2w_i + w_{i-1}| \tag{43}$$

where $\mathcal{S}(x)$ is a neighborhood of x containing at least all the points used for computing the interpolation operator. We also assume that this neighborhood is bounded, i.e., that the interpolation is computed on the basis of local values.

Remark 5.1. *Note that* (43) *holds true, for some constant $C > 0$, for a large class of interpolations. The stronger requirement that* (43) *holds true with $C < 1$ is proved in* [30] *for symmetric Lagrange or WENO interpolation up to degree 9, and for finite element or ENO interpolations up to degree 5.*

The numerical solution w^j of the iterative scheme can be identified with the corresponding sequence of values $(w_i^j)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, and the scheme is written in the more convenient form:

$$w_i^{j+1} = \min_{u\in\mathbb{R}} \left\{ \beta\Pi_k \left(w^j\right)(x_i + uh) + h(H^*(u) + g(x_i)) \right\} \tag{44}$$

which is completely equivalent to the previous formulation if $w^0 \in S_k$.

The proof of convergence relies on a slight adaptation of the following lemma from [30]:

Lemma 5.2. *Consider the scheme* (44)*, and denote by u_i^j a minimizing value for its right-hand side. If* (41) *holds, then, for any $l \in \mathbb{Z}$ and $j \geq 1$,*

$$w_{l+1}^j - 2w_l^j + w_{l-1}^j \leq \frac{2k^2}{m_H h}. \tag{45}$$

Moreover, if (43) *holds, then, for any $i \in \mathbb{Z}$ and $j \geq 1$,*

$$\max_{x_{l-1},x_l,x_{l+1}\in\mathcal{S}(x_i+u_i^j h)} \left| w_{l+1}^j - 2w_l^j + w_{l-1}^j \right| \leq \bar{C}\frac{k^2}{h} \tag{46}$$

for some positive constant $\bar{C}$ depending on C, $\mathcal{S}$, and m_H.

Note that the meaning of this lemma is that the second increments of numerical solutions satisfy a global one-sided bound, which becomes two-sided at the arrival points $x_i + u_i^j h$ of characteristics (optimal trajectories).

Proof. We only sketch the main modifications with respect to the proof given in [30, Lemma 2.1]. First, we have the upper bound (which holds for $j \geq 0$ and $k \leq (2m_H L_g)^{-1}$, and parallels estimate (2.5) in [30]):

$$\begin{aligned} w_{l+1}^{j+1} - 2w_l^{j+1} + w_{l-1}^{j+1} &\leq h\left[H^*\left(u_l^j - \frac{k}{h}\right) - 2H^*(u_l^j) + H^*\left(u_l^j + \frac{k}{h}\right)\right] \\ &\quad + h\left[g(x_{l+1}) - 2g(x_l) + g(x_{l-1})\right] \\ &\leq h\left(\frac{k}{h}\right)^2\left(\frac{1}{m_H} + 2kL_g\right) \\ &\leq \frac{2k^2}{m_H h} \end{aligned} \tag{47}$$

where we have used the suboptimal control values $u_l^j - \frac{k}{h}$ for w_{l+1}^{j+1} (resp. $u_l^j + \frac{k}{h}$ for w_{k-1}^{l+1}), the convexity assumption (41) and the Lipschitz continuity of g. Hence we obtain (45). To prove the reverse bound (46), note that the upper bound is the essential point on which the original proof relies. Therefore, by carefully retracing the whole proof of the Lemma given in [30], and except for a change in the constant $\bar{C}$, it is possible to prove an analogous double-sided bound in the form (46) for the second increment of the numerical solution at the feet of characteristics. □

The convergence result is provided by the following

Proposition 5.3. *Assume* (41), (42)*, and let* Π_k *be an interpolation operator satisfying* (43)*. Consider the iterates* w^j *of the scheme* (44)*, initialized with a Lipschitz continuous function* w^0*. Assume, as* $h, k \to 0$*, that*

$$k = O(h^2). \tag{48}$$

Then, there exists $L \geq 0$ *such that:*

(i) *The* w^j *are uniformly Lipschitz:* $w^j \in Lip_L(\Omega)$*,* $\forall j \geq 1$*, and therefore* (40) *holds.*
(ii) *The projection* Π_k *is* ε*-monotone with* $\varepsilon = C_L k$ *for some* $C_L \geq 0$*.*
(iii) *The sequence* w^j *converges uniformly to* v *on compact subsets.*
(iv) *For* j *sufficiently large, the estimate*

$$\|w^j - v_h\|_\infty \leq C\frac{k}{h} \tag{49}$$

holds for some constant $C \geq 0$*, where* v_h *is the solution of* (31)*.*

Remark 5.4. *A sharper information can be recovered from Theorem 4.2, which provides, for* j *large enough, the estimate*

$$\|w^j - v\|_\infty \leq C\left(h^\gamma + \frac{k}{h}\right), \tag{50}$$

for some $\gamma > 0$*, once taken into account the error bound* $\|v_h - v\|_\infty$ *for the time-discrete approximation* v_h *(see Remark 33 and* [18]*).*

Proof. Let us first check the consistency of the scheme, assuming that the ε-monotonicity holds true with $\varepsilon = Ck$. Under the condition (40), we have $\|\Pi_k(w) - \Pi_1(w)\|_\infty \leq C_L\, k$ for any regular function $w \in Lip_L(\Omega)$. Furthermore, $\|\Pi_1(w) - w\|_\infty \leq Lk$. Hence $\|\Pi_k \circ T_h(w) - T_h(w)\|_\infty \leq \beta(C_L + L)k$. Then, as soon as $h, k, \frac{k}{h} \to 0$ it holds

$$\begin{aligned} \frac{w(x) - \Pi_k \circ T_h(w)(x)}{h} &= \frac{w(x) - T_h(w)(x)}{h} + O(\frac{k}{h}) \\ &\to \delta w(x) + \mathcal{H}(x, Dw(x)) \end{aligned} \tag{51}$$

(where $\mathcal{H}(x, Dw(x)) := H(Dw(x)) - g(x)$). We then deduce that the iterative scheme based on $T_\rho := \Pi_k \circ T_h$ is consistent with the PDE (37) in the sense of (27) (here using $c = \frac{1}{\delta}$ so that $c(1-\beta) = h$).

In order to bound the discrete Lipschitz constant

$$L_j := \sup_i \frac{\left|w^j_{i+1} - w^j_i\right|}{k}.$$

Since u_i^j is a suboptimal control for w_{i+1}^{j+1}, we have:

$$\begin{aligned}
\frac{w_{i+1}^{j+1}-w_i^{j+1}}{k} &\le \frac{1}{k}\Big(\beta\Pi_k\left(w^j\right)(x_{i+1}+u_i^j h)+hH^*(u_i^j)+hg(x_{i+1}) \\
&\qquad -\beta\Pi_k\left(w^j\right)(x_i+u_i^j h)-hH^*(u_i^j)-hg(x_i)\Big) \\
&\le \frac{\beta}{k}\Big(\Pi_k\left(w^j\right)(x_{i+1}+u_i^j h)-\Pi_k\left(w^j\right)(x_i+u_i^j h)\Big)+hL_g \\
&\le \frac{\beta}{k}\Big|\Pi_k^1\left(w^j\right)(x_{i+1}+u_i^j h)-\Pi_k^1\left(w^j\right)(x_i+u_i^j h)\Big|+C\frac{k}{h}+hL_g \\
&\le \beta L_j+C\frac{k}{h}+hL_g \\
&\le \beta L_j+Ch. \qquad (52)
\end{aligned}$$

where we have used (43), the fact that Π_k^1 is nonexpansive in the Lipschitz norm, and the relationship $k=O(h^2)$. By the reverse inequality (which can be proved with the same ideas), we obtain

$$L_{j+1}\le \beta L_j+Ch.$$

Then, iterating the estimate for all $j\ge 0$, we get the uniform bound

$$\begin{aligned}
L_j &\le \beta^j L_0+\frac{Ch}{1-\beta} \\
&\le L_0+\frac{C}{\delta}=: L.
\end{aligned}$$

To prove (ii), we now use (43) and get $\|\Pi_k(w^j)-\Pi_k^1(w^j)\|_\infty\le 2CL\,k$. Hence the projection is ε-monotone with $\varepsilon=4CL\,k$.

(iii)–(iv) The convergence now follows from Theorem 4.2. The estimate is obtained following the previous arguments. □

Example 5.5. *Consider the 1d Hamilton–Jacobi–Bellman equation of type* (9) *on* $\Omega=[0,3]$ *with*

$$\begin{aligned}
\ell(x,u) &= au^\sigma-k\beta x^2 \\
f(x,u) &= u(t)-\mu x(t)+\frac{mx(t)^\rho}{n^\rho+x(t)^\rho}.
\end{aligned}$$

These functions correspond to an infinite horizon optimal control problem modelling a lake management problem, cf. [34]. *We specify the parameters* $a=2$, $\sigma=\beta=k=\frac{1}{2}$, $m=n=1$, $\rho=2$, $\mu=0.55$, $U=[0,0.4]$ *and discount rate* $\delta=0.1$. *The solution of the equation is depicted in Figure 1. Observe that the solution is nonsmooth, i.e., it has a kink at* $\bar{x}\approx 0.7$. *This is precisely the reason why the value iteration does not converge for the high-order interpolation, cf.* [9].

We have performed a value iteration in the form (4) *for the SL scheme with respectively a cubic Lagrange and a cubic spline space reconstruction: the first scheme fits the convergence framework of Subsection 5.1, while the second does not, in particular due to the nonlocal nature of the spline interpolation (while it provides an* ε*-monotone scheme due to Remark 4.4 we cannot control the Lipschitz constant* L*). The values* $\|w^{j+1}-w^j\|_\infty$ *have been plotted in Fig. 2 up to 4000 iterations for meshes of 51, 101 and 201 nodes, with* $h\sim k^{1/2}$. *While the behaviour of a pure contraction operator* T *would be an exponential convergence (a straight line in a linear-log plot), we see that the effect of* ε*-monotonicity is to make the convergence*

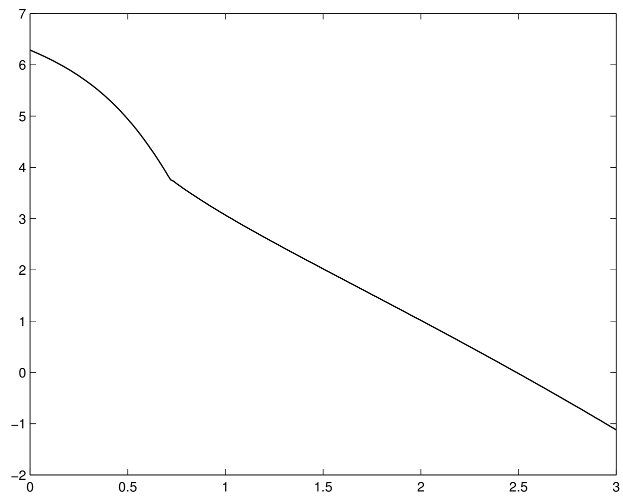

FIGURE 1. Solution of the Hamilton–Jacobi PDE of Example 5.5

history more complicate, although refining the mesh reduces ε and ultimately the convergence error.

Note that the lack of monotonicity has a less apparent effect on the cubic Lagrange scheme, i.e., the scheme behaves better than predicted by the theory in Section 4. To explain this behaviour, observe that oscillations of the scheme are caused by the kink at $\bar{x}$, and characteristics (i.e., optimal trajectories) collapse into this singularity. In the cubic Lagrange scheme, the space reconstruction is only sensitive to the local *regularity, and therefore oscillations are restricted to a small neighbourhood of the kink and feet of characteristics propagate the solution from points at which the monotonicity defect is "small", while spline reconstruction causes oscillations at a relatively large distance from the singularity. In this latter case the situation is as described in Section 4: in the first 4000 iterations, and after the first region of regular convergence, the difference $\|w^{j+1} - w^j\|_\infty$ remains roughly below 0.15 with 51 nodes, below 0.026 with 101 nodes and below 0.013 with 201 nodes.*

5.2. **Froese's and Oberman's filtering scheme.** In previous section 5.1, we have used a high-order interpolation and have been able to prove that it is ε-monotone in some particular cases. Here, we will consider a general type of high-order interpolation that is not *a priori* ε-monotone, and show how to modify it in order to obtain an ε-monotone interpolation.

In [31], Froese and Oberman proposed a general way to mix a first order, monotone scheme with a high-order (non-monotone) scheme. The coupling, in the framework of finite difference approximation (and applied to second order elliptic problem), is called *filtered scheme*. By using an ε-monotonicity property of the scheme and Barles–Souganidis [8] theorem, a convergence result can be proved.

In our context, we shall define a filtered interpolation in a similar way. Let Π_k^1 denote a standard first order (monotone) interpolation operator on a given grid mesh, and let Π_k^A denote an interpolation operator, not necessarily monotone, of

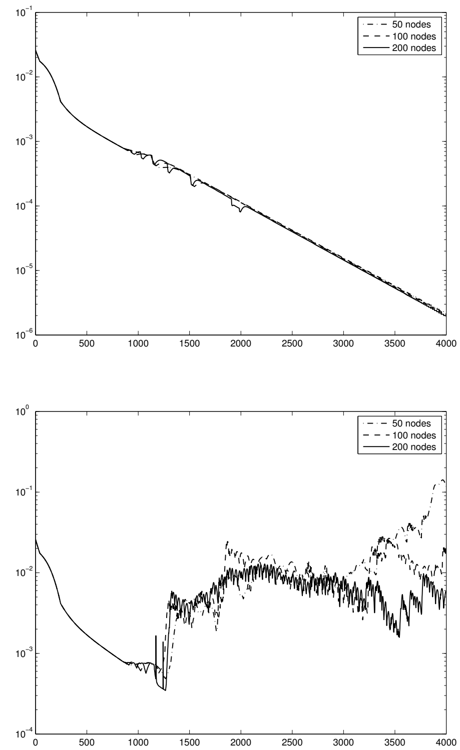

FIGURE 2. Evolution $\|w^{j+1}-w^j\|$ during the value iteration for the SL scheme with cubic Lagrange (left) and cubic spline (right) space reconstruction, and a varying space-time mesh for Example 5.5

higher order in the case of regular functions: there exist integers $r \geq 2$, $m \geq 1$, such that if w is C^r regular on the grid interval $[x_j - m\Delta x, x_j + m\Delta x]$ then for all $x \in [x_i, x_{i+1}]$,

$$\left|\Pi_k^A(w)(x) - w(x)\right| \leq Ck^r \tag{53}$$

for some constant $C \geq 0$. Since the value iteration method may not converge when using a non-monotone interpolation, the idea introduced in [31] consists in using a filtered interpolation

$$\Pi_k^F(w)(x) := \Pi_k^1(w)(x) + \varepsilon F\left(\frac{\Pi_k^A(w)(x) - \Pi_k^1(w)(x)}{\varepsilon}\right), \tag{54}$$

where F is the "filtering function":

$$F(x) := \text{sign}(x)\max\left(1 - \big||x| - 1\big|, 0\right) \equiv \begin{cases} x & \text{if } |x| \leq 1 \\ \text{sign}(x)(2 - |x|) & \text{if } 1 \leq |x| \leq 2 \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

for some $\varepsilon > 0$. The parameter ε may depend of k and h and will be fixed later on.

Let us emphasize that the filtered interpolation is *not*, in general, a convex combination of two different types of interpolation.

By using the fact that $|F(x)| \leq 1$ it is easily seen that Π_F is an ε-monotone scheme in the sense of definition (4.3). A particular nice feature of the filtering scheme is that ε-monotonicity holds for all $w \in B(\Omega)$, i.e., we can choose $A = B(\Omega)$ which is trivially invariant under the value iteration. From Theorem 4.2, one would now expect that the value iteration converges to smaller and smaller neighborhoods of a fixed point for $\varepsilon \to 0$. We illustrate this by re-considering Example 5.5.

Example 5.6. *We consider again the problem of Example 5.5 and use the filtering scheme in which Π_k^A was chosen as the cubic spline interpolation already presented in Example 5.5, cf. Figure 2 (right). The numerical parameters were chosen as space and time step $k = h = 0.06$, resulting in 51 nodes, and the minimum in* (30) *was computed over a discrete set of controls U discretizing the interval* $[0, 0.4]$ *with 51 equidistant values.*

Figure 3 shows the evolution of the value iteration plotting the difference $\|w^{j+1} - w^j\|$ depending on j for different filtering parameter ε. One clearly observes that the iteration converges to increasingly smaller sets for shrinking filtering parameter, i.e., for increasing weight on the first order monotone scheme. Obviously, the filtering significantly improves the convergence behavior of the value iteration.

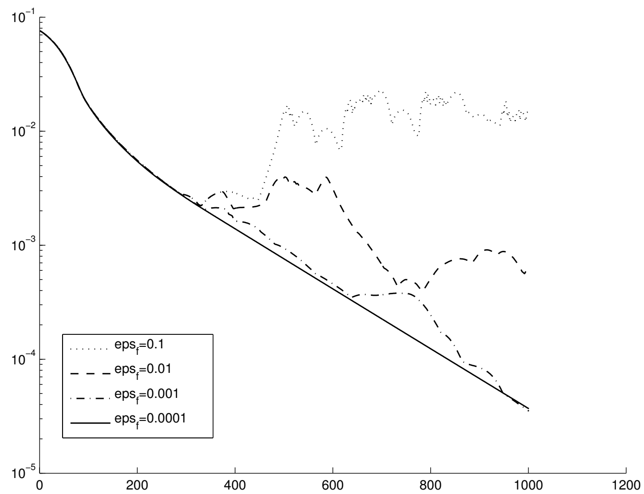

FIGURE 3. Evolution $\|w^{j+1} - w^j\|$ during the value iteration for filtering scheme with varying filter parameter ε for Example 5.5

In order to illustrate the benefit of the filtering approach compared to the plain first order approximation in terms of accuracy, we derive a convergence estimate which will also explain how to tune the filtering parameter ε. To this end, note that in a region where w is a C^2 function it holds that $|\Pi_k^A(w)(x) - \Pi_k^1(w)(x)| \leq Ck^2$ (since both $\Pi_k^A[w](x)$ and $\Pi_k^1(w)(x)$ are equal to $u(x)$ up to an error of order less or equal to $O(k^2)$).

Therefore as soon as $Ck^2 \leq \varepsilon$, by using that $F(x) = x$ for $|x| \leq 1$, it holds $\Pi_k^F(w)(x) = \Pi_k^A(w)(x)$ which means that the filtered interpolation is a high order interpolation. It also means that the filtered scheme should be typically used with $k^2 = o(\varepsilon)$ (as $k, \varepsilon \to 0$), i.e., $\varepsilon := C_1 k^2$, where $C_1 \geq 0$ is a sufficiently large constant.

Then the following error estimate (and therefore convergence result) holds:

Proposition 5.7. *We consider the value iteration algorithm* $w^{j+1} = \Pi^F \circ T_h \Pi_k^F(w^j)$ *for a given* w^0*, where* Π_k^F *is the filtered interpolation operator* (54)*. Consider*

$$\varepsilon := C_1 k^p, \quad \text{with } 1 \leq p \leq 2, \tag{55}$$

where $C_1 > 0$ *(and furthermore, in the case* $p = 2$*,* C_1 *is sufficiently large). Then it holds, for* j^* *large enough,*

$$\|w^j - w_h\|_\infty \leq C \frac{k}{\delta h}, \quad \forall j \geq j^*,$$

for some constant $C \geq 0$*. If furthermore* $\frac{\varepsilon}{h} \to 0$*, then* j^* *can be taken to be of order*

$$j^* \sim -\frac{\log(k/h)}{\delta h}.$$

Proof. Using the fact that for Π_1 we have the consistency error $\|w_h - \Pi_k^1 w_h\|_\infty \leq CLk \equiv \varepsilon_c(k)$, it follows from Theorem 4.2 that

$$\|w^j - w_h\|_\infty \quad \leq \quad C \frac{\varepsilon + \varepsilon_c(k)}{\delta h} \quad \leq \quad C \frac{k^p + k}{\delta h} \leq C \frac{k}{\delta h} \tag{56}$$

for $j \geq j^*$ large enough and for some constant C. By Remark 4.8, assuming $\frac{k}{h} \to 0$ we deduce the desired estimate for j^*.

□

By using the error estimate (33) for $\|w - w_h\|_\infty$ we can immediately conclude that the following holds for the exact solution v.

Corollary 5.8. *Under the assumptions of Proposition 5.7 it holds*

$$\|w^j - w\| \leq C \frac{k}{\delta h} + C h^\gamma, \quad \forall j \geq j^*.$$

where $\gamma > 0$ as commented in remark 4.1.

Example 5.9. *We consider the following PDE*

$$v(x) + \max_{a = \pm 1} \Big(a f(x) v_x + \ell(x) \Big) = 0, \quad x \in \Omega = (-1, 1)$$

with periodic boundary conditions on $(-1, 1)$*, and where*

$$f(x) := 0.2 \sin(\pi x) \quad \text{and} \quad \ell(x) = 1 - \sin(\sin(2\pi x)).$$

We will consider, for a given $h > 0$*, the approximation* v_h*, solution of*

$$w_h(x) + \max_{a = \pm 1} \left(\frac{1}{h}(w_h(x) - w_h(x - a f(x) h)) + \ell(x) \right) = 0, \quad x \in (-1, 1)$$

with periodic boundary conditions on $(-1, 1)$*. A plot of the exact solution is given in Figure 4. A straightforward second order interpolation (hereafter named* Π_k^2*) is used: for* $x \in [x_i, x_{i+1}]$*,*

$$\Pi_k^A(w)(x) \equiv \Pi_k^2(w)(x) := a_i w(x_i) + b_i w(x_{i+1}) + c_i w(x_{i-1}),$$

where $q = (x - x_i)/k$*,* $a_i = 1 - q^2$*,* $b_i = (q^2 + q)/2$ *and* $c_i = (q^2 - q)/2$*. Note that* c_i *may be negative and* Π_k^2 *is not a monotone operator. However* (53) *holds with* $r = 3$ *and* $m = 1$*.*

Table 1 shows L^∞ errors for a fixed value of h ($h = \frac{1}{150}$) and varying mesh sizes M and corresponding mesh step $k := \frac{2}{M}$ (a reference value is computed using $M = 12800$). Here we compare the errors for several schemes: Π_k^1 stands for scheme using the monotone first order interpolation, Π_k^2 for the second order interpolation, and $\Pi_k^{2,F}$ stands for the filtered interpolation.

We observe that the Π_k^1 scheme is first order convergent in k, as expected. We also observe that the value iteration based on the Π_k^2 interpolation is not stable and diverges.

In contrast to this, the filtered scheme $\Pi_k^{2,F}$ based on (54) with $\Pi_k^A := \Pi_k^2$ is convergent and has a better behavior (in the sense that the L^∞ error decreases as M increases). According to Proposition 5.7, the parameter ε has been choosen as

$$\varepsilon := 10k^2.$$

The global errors using the $\Pi_k^{2,F}$ interpolation are better than the ones obtained with Π_k^1 interpolation for mesh sizes $M \geq 100$. Furthermore, in the last two columns of Table 1 errors and corresponding orders are computed away from the singularities (local errors are computed on the mesh points x_i such that $d(x_i, \Gamma) \geq 0.02$ where the singularity set is $\Gamma = \{-1,\ -0.1789,\ 0,\ 0.8211,\ 1.0\}$). An order of convergence of 2 is roughly observed, for local errors.

Finaly Figure 5 also shows the errors $\|v^{j+1} - v^j\|$ with respect to the iteration number j, for different ε parameters, ranging from 10^{-2} to 10^{-5}, and $M = 200$ mesh points (corresponding to $k = 2/M = 1/100$). In that case the convergence rate does not depend very much on the choice of the parameter ε.

L^∞ error	Π_1		Π_2		$\Pi_{2,F}$		$\Pi_{2,F}$ (local errors)	
M	error	order	error	order	error	order	error	order
25	8.17E-02	-	∞	-	1.46E-01	-	1.46E-01	-
50	4.53E-02	0.85	∞	-	6.48E-02	1.17	3.79E-02	1.95
100	2.51E-02	0.85	∞	-	5.54E-03	3.55	3.31E-03	3.52
200	1.32E-02	0.93	∞	-	4.02E-03	0.46	1.19E-03	1.48
400	6.65E-03	0.99	∞	-	6.12E-04	2.71	1.83E-04	2.70
800	3.21E-03	1.05	∞	-	2.29E-04	1.42	4.68E-05	1.96

TABLE 1. Error table for Example 5.9, with variable number of mesh points M for a fixed h: first order scheme using Π_1, higher order schemes using Π_2 or the filtered interpolation $\Pi_{2,F}$.

5.3. **A high-order WENO/Finite Volume scheme for differential games.**

In this section, we consider *WENO/Finite Volume*-based reconstruction operators for spatial discretization in a semi-Lagrangian setting. In particular, we show how this approach fits the theory developed in the previous sections. We illustrate this class of schemes for an Isaacs equation of the form (15), for which the operator T_h in the semi-discrete fixed point problem (31) becomes

$$T_h(w) := \begin{cases} \beta \min_{b\in B} \max_{a\in A}\{w(x + hf(x,a,b))\} + 1 - \beta & \text{in } \mathbb{R}^n \setminus \mathcal{T} \\ 0 & \text{on } \partial\mathcal{T}\,, \end{cases},$$

with $\beta = e^{-h}$. In order to obtain a fully discrete scheme, we need to consider a discretization in space defined by an interpolation operator Π_k. The finite volume

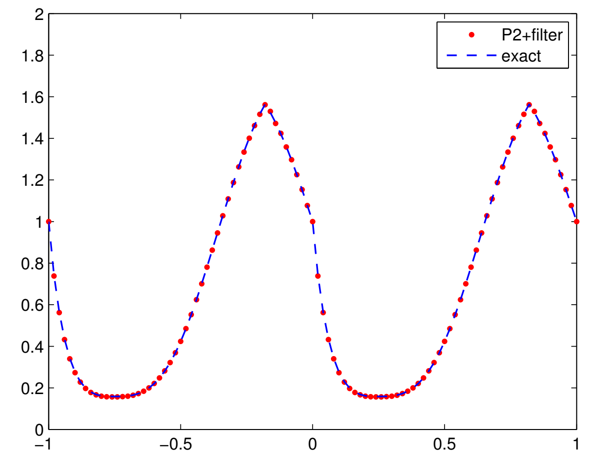

FIGURE 4. Graph of v_h and its approximation with the $\Pi_{2,F}$ filtering scheme and 100 mesh points, for Example 5.9 (using $\varepsilon = 10k^2$).

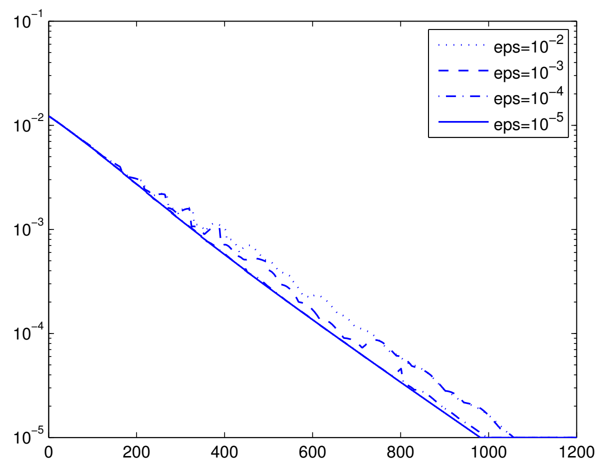

FIGURE 5. Behavior of the error $\|v^{j+1} - v^j\|$ with respect to iteration j for the filtering scheme and $M = 200$, Example 5.9.

scheme presented here differs from the setting presented in Section 4 in the sense that the finitely many values w_i are not given by the node values $w_i = w(x_i)$ but rather by averaged values. In a one-dimensional setting, given a mesh width $k > 0$, and a set of nodes $\{x_i\}_{i=1}^N$, the domain is discretized into a set of cells $\Omega_i = [x_i - k/2, x_i + k/2]$. The solution w is then represented by its local *average* *value*

$$w_i = \frac{1}{k}\int_{x_i-k/2}^{x_i+k/2} w(x)\,dx\,,$$

which we abstractly write as the mapping

$$E_k : B(\mathbb{R}) \to \mathbb{R}^N, \quad E_k(w) = (w_1, \dots, w_N)^T \text{ with } w_i = \frac{1}{k}\int_{\Omega_i} w(x)dx.$$

In practice, E_k is implemented using a quadrature rule

$$E_{k,i}(w) = \frac{1}{k}\sum_j \gamma_j w(y_j) \tag{57}$$

where y_j and γ_j are Gauss points and weights inside the i-th cell Ω_i. Based on the N values $E_k(w)$, a function $w \in S_k$ is reconstructed by a WENO (weighted essentially non-oscillatory) interpolation procedure $I_k : \mathbb{R}^N \to S_k$ whose details we describe below. The resulting projection operator then becomes

$$\Pi_k = I_k \circ E_k : B(\mathbb{R}) \to S_k \subset B(\mathbb{R}).$$

WENO reconstruction and related numerical schemes date back to the work of [44, 41], in the context of numerical methods for conservation laws, as a way of circumventing Godunov's Barrier Theorem by considering nonlinear (on the data) reconstruction procedures for the construction of high-order accurate schemes. As it has been shown in [30], the use of a WENO interpolation procedure can be considered as a building block in high-order, semi-Lagrangian schemes for time-dependent HJ equations. Here we introduce an application to static Isaacs equations, which is justified in the framework of ε-monotone schemes.

Given a sufficiently smooth function w its averaged values $W = (w_1, \dots, w_N)^T = E_k(w)$ and a polynomial degree r, the WENO reconstruction procedure yields a set of polynomials $P = \{p_i\}_{i=1}^N$ of degree r defined on Ω_i and satisfying

$$w_i = \frac{1}{k}\int_{\Omega_i} p_i(x)\,dx\,, \quad w(x) = p_i(x) + o(k^r) \;\; \forall x \in \Omega_i,\; i = 1, \dots, N, \tag{58}$$

and an *essentially non-oscillatory condition* [35]. In general, such an interpolant is built by considering a set of stencils per cell, and weighting them according to some smoothness indicator. Several variations of this procedure can be found in the literature; for illustration purposes, we restrict ourselves to the reconstruction procedure presented in [5] in one space dimension and with degree $r = 2$. In this case, given a vector of averaged values W, the reconstruction procedure seeks, for every cell, a local quadratic expansion upon a linear combination of Legendre polynomials rescaled in local coordinates $\xi = [-1/2, 1/2]$ expressed in the form

$$p(\xi) = v_0 + v_\xi p_1(\xi) + v_{\xi\xi} p_2(\xi),$$

with

$$p_1(\xi) = \xi \qquad p_2(\xi) = \xi^2 - \frac{1}{12}.$$

We assign the subscript "0" to the cell where we compute the coefficients, other values indicating location and direction with respect to v_0 (note that the notation is coherent with the fact that the first coefficient in the expansion v_0 coincides with the averaged value, i.e., $v_0 = v_i$). Next, for this particular problem we define three stencils

$$S^1 = \{v_{-2}, v_{-1}, v_0\}\,, \quad S^2 = \{v_{-1}, v_0, v_1\}\,, \quad S^3 = \{v_0, v_1, v_2\}\,,$$

and in every stencil we compute a polynomial of the form

$$p^{(i)}(\xi) = v_0^{(i)} + v_\xi^{(i)} p_1(\xi) + v_{\xi\xi}^{(i)} p_2(\xi) \qquad i = 1, 2, 3.$$

Imposing the conservation condition (58), the coefficients are given by

$$\begin{aligned} S^1 \quad &: \quad v_\xi^{(1)} = -2v_{-1} + v_{-2}/2 + 3v_0/2, \quad v_{\xi\xi}^{(1)} = (v_{-2} - 2v_{-1} + v_0)/2\,, \\ S^2 \quad &: \quad v_\xi^{(2)} = (v_1 - v_{-1})/2, \quad v_{\xi\xi}^{(2)} = (v_{-1} - 2v_0 + v_1)/2\,, \\ S^3 \quad &: \quad v_\xi^{(3)} = -3v_0/2 + 2v_1 - v_2/2, \quad v_{\xi\xi}^{(3)} = (v_0 - 2v_{-1} + v_2)/2. \end{aligned}$$

For every polynomial we calculate a smoothness indicator defined as

$$IS^{(i)} = \sum_{l=1}^{r} \int_{\Omega_0} k^{2l-1} \left(\frac{\partial^l p^{(i)}}{\partial x^l} \right)^2 dx\,,$$

where r is the polynomial reconstruction degree (in our case $r = 2$), and which in our case evaluates to

$$IS^{(i)} = \left(v_\xi^{(i)} \right)^2 + \frac{13}{3} \left(v_{\xi\xi}^{(i)} \right)^2 .$$

The smoothness indicator is then used in order to compute the WENO weights

$$\omega^{(i)} = \frac{\alpha^{(i)}}{\sum_{i=1}^{3} \alpha^{(i)}}\,, \quad \alpha^{(i)} = \frac{\lambda^{(i)}}{(\epsilon + IS^{(i)})^q}\,,$$

where ϵ is a parameter introduced in order to avoid division by zero (usually $\epsilon = 10^{-12}$). The scheme is in general rather insensitive to the tuning parameter q, which we set to $q = 5$. The parameters $\lambda^{(i)}$ are usually computed in an optimal way to increase the accuracy of the reconstruction at certain points; here we opt for a centered approach instead, thus $\lambda^{(1)} = \lambda^{(3)} = 1$, while $\lambda^{(2)} = 100$. The 1d reconstructed polynomial on Ω_i is then given by

$$p_i(\xi) = \omega^{(1)} p^{(1)}(\xi) + \omega^{(2)} p^{(2)}(\xi) + \omega^{(3)} p^{(3)}(\xi). \tag{59}$$

To summarize, this interpolation $I_j(W)$ procedure generates, upon the averaged data $W = E_k(v)$, a set of N polynomials $p_i(x)$ defined locally in every cell. This operator, together with (57), allows us to define the fully-discrete fixed point iteration (36) with

$$\begin{aligned} \Pi_k \circ T_h(w) \quad &= \quad I_k \circ E_k \circ T_h(w) \qquad\qquad (60) \\ &= \quad I_k \left(\left(\frac{1}{k} \sum_j \gamma_j \left(\beta \min_{b \in B} \max_{a \in A} \{ w(y_j + hf(y_j, a, b)) \} + 1 - \beta \right) \right)_{i=1}^{N} \right) \end{aligned}$$

on $\mathbb{R} \setminus \mathcal{T}$.

Remark 5.10. *(i) The fixed-point operator* (60) *can be interpreted as a variation of the classical REA (Reconstruct-Evolve-Average) Finite Volume setting, as at the beginning and at the end of every iteration, the available data corresponds to a piecewise polynomial function defined upon the grid. The high-order data is then evolved and averaged, yielding a piecewise constant function over which a reconstruction procedure is performed, concluding the iteration.*

(ii) As in Section 5.1, in order to prove ε-monotonicity, a first step is to establish (40)*. To this end, one may rely on the results on high-order semi-Lagrangian/WENO schemes for HJB equations from* [20]*, where property* (40) *is proven for reconstructions up to order 9. The key idea is to express the WENO interpolant as a convex*

combination of Lagrange polynomials, for which the required interpolation properties have been proved in [30]*. In our particular case, we can sketch a more direct proof as follows. We begin by considering a first-order monotone,* minmod*-like interpolant locally defined as*

$$[I_1(\xi)]_i := w_i + \frac{\xi}{2}\Phi(\Delta w_L, \Delta w_R)\,, \quad \Delta w_L = w_i - w_{i-1}\,, \quad \Delta w_R = w_{i+1} - w_i\,, \tag{61}$$

$$\Phi(a,b) := sign(ab)(\omega a + (1-\omega)b)\,, \quad \omega = \begin{cases} 1 & \text{if } |a| < |b| \\ 0 & \text{if } |b| \le |a| \end{cases} . \tag{62}$$

Next, note that our WENO interpolant puts a large weight on the central polynomial. For illustration purposes, wee can focus on this stencil in order to obtain a bound of the type (40)*. Without loss of generality, we can assume that $\omega = 1$ and a positive slope sign. For every cell, it holds*

$$|I_k - I_1| = \left| w_i + w_\xi p_1(\xi) + w_{\xi\xi} p_2(\xi) - w_i + \frac{\xi}{2}\Phi(\Delta w_L, \Delta w_R) \right| \tag{63}$$

$$= \left| w_\xi p_1(\xi) + w_{\xi\xi} p_2(\xi) - \frac{\xi}{2}\Delta w_L \right| \tag{64}$$

$$= \left| (w_{i+1} - w_{i-1})\frac{\xi}{2} + (w_{i-1} - 2w_i + w_{i+1})(\xi^2 - \frac{1}{12}) - (w_i - w_{i-1})\frac{\xi}{2} \right| \tag{65}$$

$$= \left| (w_{i+1} - w_i)\frac{\xi}{2} + (w_{i-1} - 2w_i + w_{i+1})(\xi^2 - \frac{1}{12}) \right| \tag{66}$$

$$\le c_1 k + c_2\,|w_{i-1} - 2w_i + w_{i+1}| \le c_3 k\,, \tag{67}$$

the last inequality being a consequence of assumption (39)*. The following step is to prove that the fixed point iteration will generate uniformly Lipschitz discrete solutions. For this purpose, it would be necessary to derive estimates of the form* (52)*, which in our case are nontrivial, due to the nonconvex character of the Hamiltonian. For the sake of brevity we postpone this analysis to future research.*

The resulting scheme yields a fully-discrete, high-order in space and ϵ-monotone approximation of the Isaacs equation (15). We illustrate its convergence and capabilities in a numerical example related to pursuit-evasion games.

Numerical example Consider a 1D pursuit-evasion game with dynamics given by

$$\dot{x}_P = v_P a$$
$$\dot{x}_E = v_E b\,,$$

where v_P and v_E denote the velocity of the pursuer and the evader respectively; $a \in [0,1]$ and $b \in [-1,1]$ are control variables. By defining the reduced coordinate $x = x_E - x_p$, the game is written as

$$\dot{x} = v_E b - v_P a.$$

The solution of this game is obtained by performing the fixed point iteration (36) using Π_k and T_h just defined.

If we consider the target set $\mathcal{T} = B(0, R)$, the exact solution is given by

$$v(x) = \begin{cases} 1 - \exp(-|x + R|) & \text{if } x < R \\ 0 & \text{if } x \in (-R, R) \\ 1 & \text{if } x > R\,. \end{cases}$$

We implement our WENO/semi-Lagrangian scheme for reconstruction degree $r = 2$; the results are shown in Figure 6. One clearly sees the non-monotone convergence behavior of the convergence, which — as expected for $\varepsilon = o(k^2)$ — is particularly pronounced for larger values of k. Similar to Figure 2 and in contrast to Figure 3, despite the non-monotonocity all iterations eventually converge to a fixed point up to machine accuracy, i.e., they show a better convergence behavior than the worst case scenarios in Theorems 4.1 and 4.2 which only predict convergence to a neighborhood of the fixed point proportional to ε. Like in the cubic interpolation in Example 5.5, this is probably due to the interplay of the particular type of non-smoothness and the chosen interpolation method, which in the case of the WENO scheme damps the oscillations efficiently enough to eventually achieve convergence.

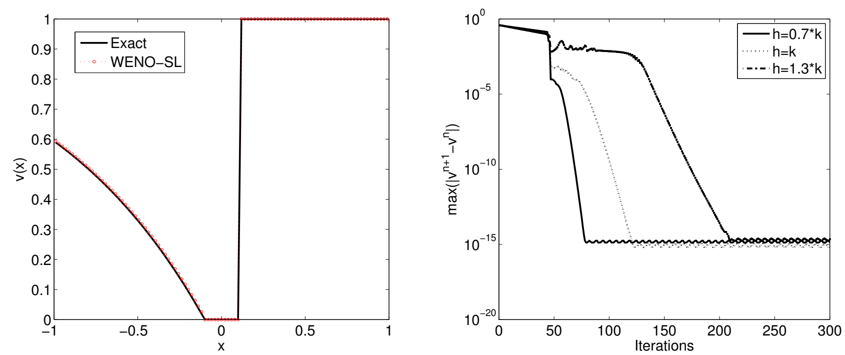

FIGURE 6. WENO-SL scheme for a 1d differential game. Left: exact and approximated solution. Right: ϵ-monotone convergence evolution for different values of h.

Concluding remarks. We have developed convergence results for fixed-point operators arising in high-order approximations of static HJ equations. By suitably defining the concept of ε-monotonicity, we characterize, both theoretically and numerically, the type of convergence behavior that is observed when high-order discretizations are combined with fixed-point iterations for approximating HJ equations. From a theoretical perspective, we derive a convergence result in the framework of viscosity solutions by using a generalized version of the Barles-Souganidis theorem for ϵ-monotone schemes. As a direct consequence of this result, the convergence of high-order semi-Lagrangian schemes, as well as filtered schemes, can be embedded within the proposed convergence framework. In general, the presented numerical experiments are in line with the presented theoretical developments, as convergence is observed in a non-monotone way, with an oscillatory behavior which is possible to control upon the discretization parameters. Although this article is focused on HJ equations related to optimal control, the core of the presented results relates to a wider class of nonlinear problems covering, for instance, differential games, for which a detailed analysis needs to be developed.

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