Nominal Model Predictive Control

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Abstract: Model Predictive Control is a controller design method which synthesizes a sampled data feedback controller from the iterative solution of open loop optimal control problems. We describe the basic functionality of MPC controllers, their properties regarding feasibility, stability and performance and the assumptions needed in order to rigorously ensure these properties in a nominal setting.

1. INTRODUCTION

Model predictive control (MPC) is a method for the optimization based control of linear and nonlinear dynamical systems. While the literal meaning of "model predictive control" applies to virtually every model based controller design method, nowadays the term commonly refers to control methods in which pieces of open loop optimal control functions or sequences are put together in order to synthesize a sampled data feedback law. As such, it is often used synonymously with "receding horizon control".

The concept of MPC was first presented in Propoĭ (1963) and was re-invented several times already in the 1960s. Due to the lack of sufficiently fast computer hardware, for a while these ideas did not have much of an impact. This changed during the 1970s when MPC was successfully used in chemical process control. At that time, MPC was mainly applied to linear systems with quadratic cost and linear constraints, since for this class of problems algorithms were sufficiently fast for real time implementation — at least for the typically relatively slow dynamics of process control systems. The 1980s have then seen the development of theory and increasingly sophisticated concepts for linear MPC, while in the 1990s nonlinear MPC (often abbreviated as NMPC) attracted the attention of the MPC community. After the year 2000 several gaps in the analysis of nonlinear MPC without terminal constraints and costs were closed and increasingly faster algorithms were developed. Together with the progress in hardware, this has considerably broadened the possible applications of both linear and nonlinear MPC.

In this article we explain the functionality of nominal MPC along with its most important properties and the assumptions needed to rigorously ensure these properties. We also give some hints on the underlying proofs. The term nominal MPC refers to the assumption that the mismatch between our model and the real plant is sufficiently small to be neglected in the following considerations. If this is not the case, methods from robust MPC must be used [reference to robust MPC]. We describe all concepts for nonlinear discrete time systems, noting that the basic results outlined in this article are conceptually similar for linear and for continuous time systems.

2. MODEL PREDICTIVE CONTROL

In this article we discuss MPC for discrete time control systems of the form

$$x_{\mathbf{u}}(j+1) = f(x_{\mathbf{u}}(j), u(j)), \quad x_{\mathbf{u}}(0) = x_0 \qquad (1)$$

with state $x_{\mathbf{u}}(j) \in X$, initial condition $x_0 \in \mathbb{X}$ and control input sequence $\mathbf{u} = (u(0), u(1), \dots)$ with $u(k) \in U$, where the state space X and the control value space U are normed spaces. For control systems in continuous time, one may either apply the discrete time approach to a sampled data model of the system. Alternatively, continuous time versions of the concepts and results from this article are available in the literature, see, e.g., Findeisen and Allgöwer (2002) or Mayne et al. (2000).

The core of any MPC scheme is an optimal control problem of the form

$$\text{minimize } J_N(x_0, \mathbf{u}) \qquad (2)$$

w.r.t. $\mathbf{u} = (u(0), \dots, u(N-1))$ with

$$J_N(x_0, \mathbf{u}) := \sum_{j=0}^{N-1} \ell(x_{\mathbf{u}}(j), u(j)) + F(x_{\mathbf{u}}(N)) \qquad (3)$$

subject to the constraints

$$\begin{aligned} & u(j) \in \mathbb{U}, x_{\mathbf{u}}(j) \in \mathbb{X} \quad \text{for} \quad j = 0, \dots, N-1 \\ & x_{\mathbf{u}}(N) \in \mathbb{X}_0, \end{aligned} \qquad (4)$$

for control constraint set $\mathbb{U} \subseteq U$, state constraint set $\mathbb{X} \subseteq X$ and terminal constraint set $\mathbb{X}_0 \subseteq X$. The function $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is called stage cost or running cost, the function $F : \mathbb{X} \to \mathbb{R}$ is referred to as terminal cost. We assume that for each initial value $x_0 \in \mathbb{X}$ the optimal control problem (2) has a solution and denote the corresponding minimizing control sequence by $\mathbf{u}^\star$. Algorithms for computing $\mathbf{u}^\star$ are discussed in [reference to Optimization Algorithms for MPC and Explicit MPC].

The key idea of MPC is to compute the values $\mu_N(x)$ of the MPC feedback law μ_N from the open loop optimal control sequences $\mathbf{u}^\star$. To formalize this idea, consider the closed loop system

$$x_{\mu_N}(k+1) = f\Big(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))\Big). \tag{5}$$

In order to evaluate μ_N along the closed loop solution, given an initial value $x_{\mu_N}(0) \in \mathbb{X}$ we iteratively perform the following steps.

Basic MPC loop:

(i) set $k := 0$

(ii) solve (2)–(4) for $x_0 = x_{\mu_N}(k)$; denote the optimal control sequence by $\mathbf{u}^\star = (u^\star(0), \ldots, u^\star(N-1))$

(iii) set $\mu_N(x_{\mu_N}(k)) := u^\star(0)$, compute $x_{\mu_N}(k+1)$ according to (5), set $k := k+1$ and go to (1)

Due to its ability to handle constraints and possibly nonlinear dynamics, MPC has become one of the most popular modern control method in industry [reference to MPC in Practice]. While in the literature various variants of this basic scheme are discussed, here we restrict ourselves to this most widely used basic MPC scheme.

When analyzing an MPC scheme, three properties are important:

- Recursive Feasibility, i.e., the property that the constraints (4) can be satisfied in Step (ii) in each sampling instant
- Stability, i.e., in particular convergence of the closed loop solutions $x_{\mu_N}(k)$ to a desired equilibrium x_* as $k \to \infty$
- Performance, i.e., appropriate quantitative properties of $x_{\mu_N}(k)$

Here we discuss these three issues for two widely used MPC variants:

(a) MPC with terminal constraints and costs

(b) MPC with neither terminal constraints nor costs

In (a), F and $\mathbb{X}_0$ in (3)–(4) are specifically designed in order to guarantee proper performance of the closed loop. In (b), we set $F \equiv 0$ and $\mathbb{X}_0 = \mathbb{X}$. Thus, the choice of ℓ and N in (3) is the most important part of the design procedure.

3. RECURSIVE FEASIBILITY

Since the ability to handle constraints is one of the key features of MPC, it is important to ensure that the constraints $x_{\mu_N}(k) \in \mathbb{X}$ and $\mu_N(x_{\mu_N}(k)) \in \mathbb{U}$ are satisfied for all $k \geq 0$. However, beyond constraint satisfaction, the stronger property $x_{\mu_N}(k) \in \mathbb{X}_N$ is required, where $\mathbb{X}_N$ denotes the *feasible set* for horizon N,

$$\mathbb{X}_N := \{x \in \mathbb{X} \,|\, \text{there exists } \mathbf{u} \text{ such that (4) holds}\}.$$

The property $x \in \mathbb{X}_N$ is called *feasibility* of x. Feasibility of $x = x_{\mu_N}(k)$ is a prerequisite for the MPC feedback μ_N being well defined, because the nonexistence of such an admissible control sequence $\mathbf{u}$ would imply that solving (2) under the constraints (4) in Step (ii) of the MPC iteration is impossible.

Since for $k \geq 0$ the state $x_{\mu_N}(k+1) = f(x_{\mu_N}(k), u^\star(0))$ is determined by the solution of the previous optimal control problem, the usual way to address this problem is via the notion of *recursive feasibility*. This property demands the existence of a set $A \subseteq \mathbb{X}$ such that

- for each $x_0 \in A$ the problem (2)–(4) is feasible
- for each $x_0 \in A$ and the optimal control $u^\star$ from (2)–(4) the relation $f(x_0, u^\star(0)) \in A$ holds.

It is not too difficult to see that this property implies $x_{\mu_N}(k) \in A$ for all $k \geq 1$ if $x_{\mu_N}(0) \in A$.

For terminal constrained problems, recursive feasibility is usually established by demanding that the terminal constraint set $\mathbb{X}_0$ is *viable* or *controlled forward invariant*. This means that for each $x \in \mathbb{X}_0$ there exists $u \in \mathbb{U}$ with $f(x,u) \in \mathbb{X}_0$. Under this assumption it is quite straightforward to prove that the feasible set $A = \mathbb{X}_N$ is also recursively feasible (Grüne and Pannek, 2011, Lemma 5.11). Note that viability of $\mathbb{X}_0$ is immediate if $\mathbb{X}_0 = \{x_*\}$ and $x_* \in \mathbb{X}$ is an equilibrium, i.e., a point for which there exists $u_* \in \mathbb{U}$ with $f(x_*, u_*) = x_*$. This setting is referred to as *equilibrium terminal constraint*.

For MPC without terminal constraints, the most straightforward way to ensure recursive feasibility is to assume that the state constraint set $\mathbb{X}$ is viable (Grüne and Pannek, 2011, Theorem 3.5). However, checking viability and even more constructing a viable state constraint set is in general a very difficult task. Hence, other methods for establishing recursive feasibility are needed. One method is to assume that the sequence of feasible sets $\mathbb{X}_N$, $N \in \mathbb{N}$ becomes *stationary* for some N_0, i.e., that $\mathbb{X}_{N+1} = \mathbb{X}_N$ holds for all

$N \geq N_0$. Under this assumption, recursive feasibility of $\mathbb{X}_{N_0}$ follows (Kerrigan, 2000, Theorem 5.3). However, like viability, stationarity is difficult to verify.

For this reason, a conceptually different approach to ensure recursive feasibility was presented in (Grüne and Pannek, 2011, Theorem 8.20); a similar approach for linear systems can be found in Primbs and Nevistić (2000). The approach is suitable for stabilizing MPC problems in which the stage cost ℓ penalizes the distance to a desired equilibrium x_* (cf. Section 4). Assuming the existence — but not the knowledge — of a viable neighborhood $\mathcal{N}$ of x_*, one can show that any initial point x_0 for which the corresponding open loop optimal solution satisfies $x_{\mathbf{u}^\star}(j) \in \mathcal{N}$ for some $j \leq N$ is contained in a recursively feasible set. The fact that ℓ penalizes the distance to x_* then implies $x_{\mathbf{u}^\star}(j) \in \mathcal{N}$ for suitable initial values. Together, these properties yield the existence of recursively feasible sets A_N which become arbitrarily large as N increases.

4. STABILITY

Stability in the sense of this article refers to the fact that a prespecified equilibrium $x_* \in \mathbb{X}$ — typically a desired operating point — is asymptotically stable for the MPC closed loop for all initial values in some set $\mathcal{S}$. This means that the solutions $x_{\mu_N}(k)$ starting in $\mathcal{S}$ converge to x_* as $k \to \infty$ and that solutions starting close to x_* remain close to x_* for all $k \geq 0$. Note that this setting can be extending to time varying reference solutions, see [reference to Tracking MPC].

In order to enforce this property, we assume that the stage cost ℓ penalizes the distance to the equilibrium x_* in the following sense: ℓ satisfies

$$\ell(x_*, u_*) = 0 \quad \text{and} \quad \alpha_1(|x|) \leq \ell(x,u) \tag{6}$$

for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Here α_1 is a $\mathcal{K}_\infty$ function, i.e., a continuous function $\alpha_1 : [0,\infty) \to [0,\infty)$ which is strictly increasing, unbounded and satisfies $\alpha_1(0) = 0$. With $|x|$ we denote the norm on X. In this article we exclusively discuss stage costs ℓ satisfying (6). More general settings using appropriate detectability conditions are discussed, e.g., in (Rawlings and Mayne, 2009, Section 2.7) or Grimm et al. (2005) in the context of stabilizing MPC. Even more general ℓ are allowed in the context of economic MPC, see [reference to the economic MPC article].

In case of terminal constraints and terminal costs, a compatibility condition between ℓ and F is needed on $\mathbb{X}_0$ in order to ensure stability. More precisely, we demand that for each $x \in \mathbb{X}_0$ there exists a control value $u \in \mathbb{U}$ such that $f(x,u) \in \mathbb{X}_0$ and

$$F(f(x,u)) - F(x) \leq -\ell(x,u) \tag{7}$$

holds. Observe that the condition $f(x,u) \in \mathbb{X}_0$ is again the viability condition which we already imposed for ensuring recursive feasibility. Note that (7) is trivially satisfied for $F \equiv 0$ in case of $\mathbb{X}_0 = \{x_*\}$ by choosing $u = u_*$.

Stability is now concluded by using the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ s.t. } (4)} J_N(x_0, \mathbf{u})$$

as a Lyapunov function. This will yield stability on $\mathcal{S} = \mathbb{X}_N$, as $\mathbb{X}_N$ is exactly the set on which V_N is defined. In order to prove that V_N is a Lyapunov function, we need to check that V_N is bounded from below and above by $\mathcal{K}_\infty$ functions α_1 and α_2 and that V_N is strictly decaying along the closed loop solution.

The first amounts to checking

$$\alpha_1(|x|) \leq V_N(x) \leq \alpha_2(|x|) \tag{8}$$

for all $x \in \mathbb{X}_N$. The lower bound follows immediately from (6) (with the same α_1), the upper bound can be ensured by conditions on the problem data, see, e.g., (Rawlings and Mayne, 2009, Section 2.4.5) or (Grüne and Pannek, 2011, Section 5.3).

For ensuring that V_N is strictly decreasing along the closed loop solutions we need to prove

$$V_N(f(x,\mu_N(x))) \leq V_N(x) - \ell(x,\mu_N(x)). \tag{9}$$

In order to prove this inequality, one uses on the one hand the dynamic programming principle stating that

$$V_{N-1}(f(x,\mu_N(x))) = V_N(x) - \ell(x,\mu_N(x)). \tag{10}$$

On the other hand, one shows that (7) implies

$$V_{N-1}(x) \geq V_N(x) \tag{11}$$

for all $x \in \mathbb{X}_N$. Inserting (11) with $f(x,\mu_N(x))$ in place of x into (10) then immediately yields (9). Details of this proof can be found, e.g., in Mayne et al. (2000), Rawlings and Mayne (2009) or Grüne and Pannek (2011). The survey Mayne et al. (2000) is probably the first paper which develops the conditions needed for this proof in a systematic way, a continuous time version of these results can be found in Fontes (2001).

Summarizing, for MPC with terminal constraints and costs, under the conditions (6)–(8) we obtain asymptotic stability of x_* on $\mathcal{S} = \mathbb{X}_N$.

For MPC without terminal constraints and costs, i.e., with $\mathbb{X}_0 = \mathbb{X}$ and $F \equiv 0$, these conditions can never be satisfied, as (7) will immediately imply $\ell(x,u) = 0$ for all $x \in \mathbb{X}$, contradicting (6). Moreover, without terminal constraints and costs one cannot expect (9) to be true. This is because without terminal constraints the inequality $V_{N-1}(x) \leq V_N(x)$ holds, which together with the dynamic programming principle implies that if (9) holds then it holds with equality. This, however, would imply that μ_N is the infinite horizon optimal feedback law, which — though not impossible — is very unlikely to hold.

Thus, we need to relax (9). In order to do so, instead of (9) we assume the relaxed inequality

$$V_N(f(x,\mu_N(x))) \le V_N(x) - \alpha\ell(x,\mu_N(x)) \qquad (12)$$

for some $\alpha > 0$ and all $x \in \mathbb{X}$, which is still enough to conclude asymptotic stability of x_* if (6) and (8) holds. The existence of such an α can be concluded from bounds on the optimal value function V_N. Assuming the existence of constants $\gamma_K \ge 0$ such that the inequality

$$V_K(x) \le \gamma_K \min_{u\in\mathbb{U}} \ell(x,u) \qquad (13)$$

holds for all $K = 1,\ldots,N$ and $x \in \mathbb{X}$, there are various ways to compute α from $\gamma_1,\ldots,\gamma_N$ (Grüne, 2012, Section 3). The best possible estimate for α, whose derivation is explained in detail in (Grüne and Pannek, 2011, Chapter 6), yields

$$\alpha = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^N(\gamma_i - 1)}{\prod_{i=2}^N \gamma_i - \prod_{i=2}^N(\gamma_i - 1)}. \qquad (14)$$

Though not immediately obvious, a closer look at this term reveals $\alpha \to 1$ as $N \to \infty$ if the γ_K are bounded. Hence, $\alpha > 0$ for sufficiently large N.

Summarizing the second part of this section, for MPC without terminal constraints and costs, under the conditions (6), (8) and (13) asymptotic stability follows on $\mathscr{S} = \mathbb{X}$ for all optimization horizons N for which $\alpha > 0$ holds in (14). Note that the condition (13) implicitly depends on the choice of ℓ. A judicious choice of ℓ can considerably reduce the size of the horizon N for which $\alpha > 0$ holds (Grüne and Pannek, 2011, Section 6.6) and thus the computational effort for solving (2)–(4).

5. PERFORMANCE

Performance of MPC controllers can be measured in many different ways. As the MPC controller is derived from successive solutions of (2), a natural quantitative way to measure its performance is to evaluate the infinite horizon functional corresponding to (3) along the closed loop, i.e.,

$$J_\infty^{cl}(x_0,\mu_N) := \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k),\mu_N(x_{\mu_N}(k)))$$

with $x_{\mu_N}(0) = x_0$. This value can then be compared with the optimal infinite horizon value

$$V_\infty(x_0) := \inf_{\mathbf{u}:\,u(k)\in\mathbb{U},x_{\mathbf{u}}(k)\in\mathbb{X}} J_\infty(x_0,\mathbf{u})$$

where

$$J_\infty(x_0,\mathbf{u}) := \sum_{k=0}^{\infty} \ell(x_{\mathbf{u}}(k),u(k)).$$

To this end, for MPC with terminal constraints and costs, by induction over (9) and using nonnegativity of ℓ it is fairly easy to conclude the inequality

$$J_\infty^{cl}(x_0,\mu_N) \le V_N(x_0)$$

for all $x \in \mathbb{X}_N$. However, due to the conditions on the terminal cost in (7), V_N may be considerably larger than V_∞ and an estimate relating these two functions is in general not easy to derive (Grüne and Pannek, 2011, Examples 5.18 and 5.19). However, it is possible to show that under the same assumptions guaranteeing stability the convergence

$$V_N(x) \to V_\infty(x)$$

holds for $N \to \infty$ (Grüne and Pannek, 2011, Theorem 5.21). Hence, we recover approximately optimal infinite horizon performance for sufficiently large horizon N.

For MPC without terminal constraints and costs, the inequality $V_N(x_0) \le V_\infty(x_0)$ is immediate, however, (9) will typically not hold. As a remedy, we can use (12) in order to derive an estimate. Using induction over (12) we arrive at the estimate

$$J_\infty^{cl}(x_0,\mu_N) \le V_N(x_0)/\alpha \le V_\infty(x_0)/\alpha.$$

Since $\alpha \to 1$ as $N \to \infty$, also in this case we obtain approximately optimal infinite horizon performance for sufficiently large horizon N.

6. SUMMARY AND FUTURE DIRECTIONS

MPC is a controller design method which uses the iterative solution of open loop optimal control problems in order to synthesize a sampled data feedback controller μ_N. The advantages of MPC are its ability to handle constraints, the rigorously provable stability properties of the closed loop and its approximate optimality properties. Assumptions needed in order to rigorously ensure these properties together with the corresponding mathematical arguments have been outlined in this article, both for MPC with terminal constraints and costs and without. Among the disadvantages of MPC are the computational effort and the fact that the resulting feedback is a full state feedback, thus necessitating the use of a state estimator to reconstruct the state from output data [reference to Moving Horizon Estimation].

Future directions include the application of MPC to more general problems than set point stabilization or tracking, the development of efficient algorithms for large scale problems including those originating from discretized infinite dimensional control problems and the understanding of the opportunities and limitations of MPC in increasingly complex environments, see also [reference to distributed MPC].

7. CROSS REFERENCES

→ Tracking MPC
→ Robust MPC
→ Stochastic MPC
→ Distributed MPC
→ Economic MPC

RECOMMENDED READING

MPC in the form known today was first described in Propoĭ (1963) and is now covered in several monographs, two recent ones being Rawlings and Mayne (2009) and Grüne and Pannek (2011). More information on continuous time MPC can be found in the survey by Findeisen and Allgöwer (2002). The nowadays standard framework for stability and feasibility of MPC with stabilizing terminal constraints is presented in Mayne et al. (2000), for a continuous time version see Fontes (2001). Stability of MPC without terminal constraints was proved in Grimm et al. (2005) under very general conditions, for a comparison of various such results see Grüne (2012). Feasibility without terminal constraints is discussed in Kerrigan (2000) and Primbs and Nevistić (2000).

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