AVOIDANCE TRAJECTORIES FOR DRIVER ASSISTANCE SYSTEMS VIA SOLVERS FOR OPTIMAL CONTROL PROBLEMS*

I. XAUSA[†], R. BAIER[‡], M. GERDTS[§], M. GONTER[†], AND C. WEGWERTH[†].

Abstract. Avoidance trajectories for driver assistance systems is an important and active field of research in car industry. Assistance systems with active braking maneuvers rely on car models, e.g. the single-track model, which are modeled as control problems. The formulation of suitable objective functions serves as a tool to realize collision detection and avoidance. In two scenarios for overtaking maneuvers, an optimal trajectory is computed via fixing a secure target state or by computing reachable sets from the initial starting point. First numerical experiments show approximations to optimal trajectories, controls and reachable sets. The sensitivity analysis in both, the optimal trajectory and the reachable set, reveal parameters that significantly influence the solution.

Key words. sensitivity analysis; direct discretization methods; optimal control; reachable sets

AMS subject classifications. 90C31, 49K40, 49M37

1. Scenarios for Avoidance Trajectories. Over the years many passive and active safety systems have been developed for modern passenger cars with the aim to reduce the number of casualties in traffic accidents. In contrast to passive safety systems (chassis, airbags, seat belts), semi-active safety systems and driver assistance systems become active in critical situations before an accident occurs and intend to prevent accidents. Next to the required technical devices, intelligent software systems and algorithms play a crucial role. Future developments in active steering are one of the motivations for the study here presented. In this view the main tasks are to reliably indicate future collisions and – if possible – to provide escape trajectories if such exist. Such investigations are modeled as optimal control problems where the chosen car model and the scenario play a central role.

A simplifying assumption in this paper made to derive the single-track car model (a detailed presentation is provided in [4]) is that the rolling and pitching behavior of the car body can be neglected, that is, the roll angle and the pitch angle are small. These assumptions justify the replacement of the two wheels on the front and rear axle by a virtual wheel located in the center of the respective axle. Furthermore, due to the simplifying assumptions it can be presumed that the car's center of gravity is located on the roadway and therefore, it is sufficient to consider the motion of the car solely in the horizontal plane. The car model includes the two **control variables** $u := (w_\delta, F_B)^\top$ with control bounds

$$\left\{ \begin{array}{lllll} w_{\delta,min} & \leq & w_\delta & \leq & w_{\delta,max} \quad \text{(steering velocity)}, \\ F_{B,min} & \leq & F_B & \leq & F_{B,max} \quad \text{(braking force)}. \end{array} \right. \tag{1.1}$$

Note that negative values of F_B stand for acceleration and positive values specify braking. The vector z of the **state variables** consists of the car's center of gravity (x, y), yaw angle ψ, yaw angle rate w_ψ, velocities $x' = v_x$ and $y' = v_y$ in x- and y-direction, respectively, steering angle δ, that is $z = (x, y, \psi, v_x, v_y, w_\psi, \delta)^\top$. The **state**

*This work has been supported by the European Union Seventh Framework Programme [FP7-PEOPLE-2010-ITN] under grant agreement 264735-SADCO.

[†]Volkswagen AG, Group Research: Integrated Safety and Light, Wolfsburg, Germany.

[‡]Chair of Applied Mathematics, University of Bayreuth, Germany.

[§]Institut für Mathematik und Rechneranwendung, Universität der Bundeswehr München, Germany.

dynamics $z'(t) = f(z(t), u(t))$ for a.e. $t \in [0, t_f]$ is given by the following system of differential equations

$$\text{(1.2)} \qquad \begin{cases} x'' &= (F_x \cos(\psi) - F_y \sin(\psi))/m, \\ y'' &= (F_x \sin(\psi) + F_y \cos(\psi))/m, \\ \psi'' &= (\ell_f F_{sf} \cos(\delta) - \ell_r F_{sr} + \ell_f F_{\ell f} \sin(\delta))/I_{zz}, \\ \delta' &= w_\delta. \end{cases}$$

Herein, m, I_{zz}, ℓ_f, ℓ_r are constants and F_x, F_y, F_{sf}, F_{sr}, $F_{\ell f}$, $F_{\ell r}$ are nonlinear functions of the state z, which are subject to the state constraints

$$\text{(1.3)} \qquad \|(F_{sf}, F_{\ell f})\| \le F_{max,f}, \ \|(F_{sr}, F_{\ell r})\| \le F_{max,r} \quad \text{(Kamm's circle)}.$$

As a first step we look at two model scenarios which are easier than other ones, like cross traffic scenarios, that will be investigated in the future. For both scenarios a secure final state is defined by the following boundary constraints ensuring that the avoiding car at t_f moves parallel to the x-direction of the road:

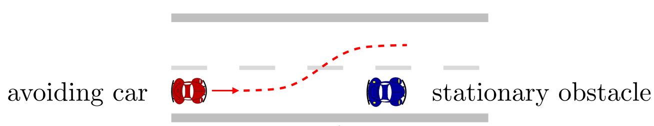
$$\text{(1.4)} \qquad v_y(t_f) = 0, \qquad \psi(t_f) = 0.$$

Moreover, we need to impose a state constraint so that the car stays on the road:

$$\text{(1.5)} \qquad road_{down} + \frac{width_{car}}{2} \le y(t) \le road_{up} - \frac{width_{car}}{2}.$$

Herein, $road_{down}$, $road_{up}$, and $width_{car}$ are given constants.

Scenario 1: A stationary obstacle located at the position $(x_{obstacle}, y_{obstacle})$ on the road relative to the avoiding car, which drives at a prescribed speed, has to be avoided:

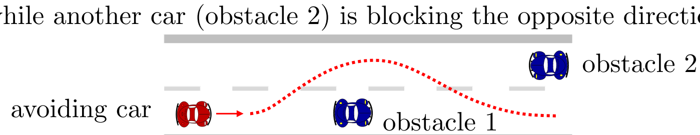


Within this scenario a secure region for the y-coordinate at the car's terminal position is given by

$$\text{(1.6)} \qquad y(t_f) \ge y_{target} := y_{obstacle} + \frac{width_{obstacle}}{2} + \frac{width_{car}}{2} + 0.3,$$

where 0.3 is a safety margin and $y_{obstacle}$ and $width_{obstacle}$ are constants. The avoidance maneuver terminates if $x(t_f) = d$, where d is either fixed or an optimizable parameter indicating the initial distance between avoiding car and obstacle.

Scenario 2: An overtaking maneuver on a rural road is considered. The avoiding car has initiated the overtaking maneuver for obstacle 1 next to it with some relative velocity, while another car (obstacle 2) is blocking the opposite direction.



Here a safety region for the avoiding car is characterized by the following state and boundary constraints:

$$\text{(1.7)} \qquad \|(x(t), y(t)) - (x_{obstacle1}, y_{obstacle1})\| \ge \frac{length_{obstacle2}}{2} + \frac{length_{car}}{2} + 0.3,$$

$$\text{(1.8)} \qquad y(t_f) \le y_{target} := y_{obstacle2} - \frac{width_{obstacle2}}{2} - \frac{width_{car}}{2} - 0.3.$$

Herein, $x_{obstacle1}$, $y_{obstacle1}$, $length_{obstacle2}$, $length_{car}$, $y_{obstacle2}$, $width_{obstacle2}$ are given constants. The avoidance maneuver terminates if $x(t_f) = d_2$, where d_2 is either fixed or an optimizable parameter indicating the initial distance between avoiding car and obstacle 2. Likewise d_1 denotes the fixed or optimizable initial distance of the avoiding car to obstacle 1.

Let q denote d (Scenario 1) or (d_1, d_2) (Scenario 2), respectively, in case these parameters are not fixed but optimizable. The resulting optimal control problems (OCPs), which model the two scenarios, have the following structure (the objective function will be specified in Section 2):

$$\begin{array}{ll} \min & \varphi(z(t_f), t_f, q) + \int_0^{t_f} f_0(z(t), u(t))dt \\ s.t. & \text{(1.1)-(1.5), given initial state } z(0) = z_0, \\ & \text{(1.6) and } x(t_f) = d \text{ for Scenario 1 resp.} \\ & \text{(1.7)-(1.8) and } x(t_f) = d_2 \text{ for Scenario 2.} \end{array} \tag{1.9}$$

2. Computational Approaches for Avoidance Trajectories. Once an obstacle has been detected by suitable sensors (e.g. radar, lidar), the following approaches can be used to decide in the model whether a collision is going to happen or not.

Approach 1: Compute an (optimal) trajectory to a secure final state. The objective function is a linear combination of the final time t_f, the steering effort, and q, that is

$$\varphi(z(t_f), t_f, q) + \int_0^{t_f} f_0(z(t), u(t))dt = c_1 t_f + c_2^\top q + c_3 \int_0^{t_f} w_\delta(t)^2 dt$$

with appropriate constants $c_1, c_2, c_3 \geq 0$.

Since sometimes it is not clear whether a collision can be avoided at all, a constraint violation minimization technique is employed, for instance the function in equation (1.7) is maximized instead of being a hard state constraint.

The role of the steering effort term in the objective function is to calculate a driver friendly trajectory. This becomes clear whenever we look at the controls in Figure 2.1, where the task was to minimize the steering effort with fixed distances, i.e. $c_1 = c_2 = 0$, $c_3 = 1$. The controls appear to be more regular than in Figure 2.2, where the task was to minimize initial distances, i.e. $c_1 = c_3 = 0$, $c_2 = 1$.

The minimization of the initial distances represents the last point where the avoiding car can still avoid the collision. Figure 2.2 shows that the trajectory is shorter than that trajectory where the initial distance is not minimized, see Fig. 2.1.

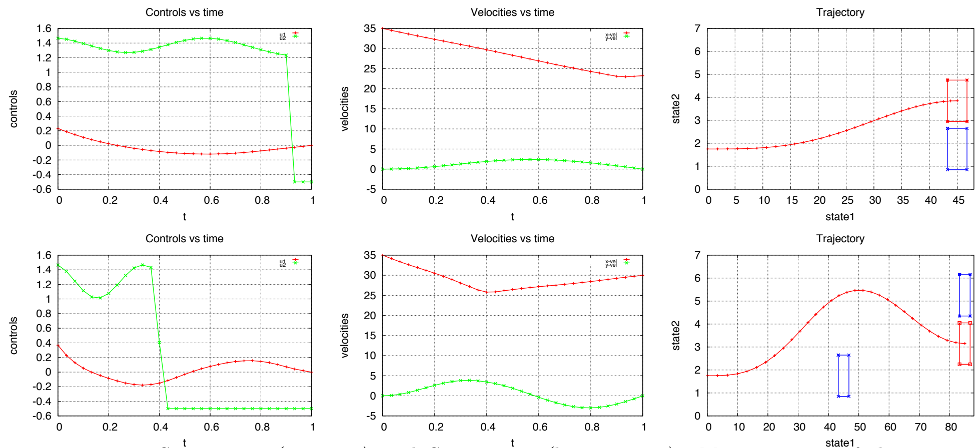


Fig. 2.1. *Scenario 1 (top row) and Scenario 2 (bottom row): Minimization of the steering effort. The frames in the pictures on the right indicate the avoiding car and the obstacles. Note the different scales in x- and y-direction.*

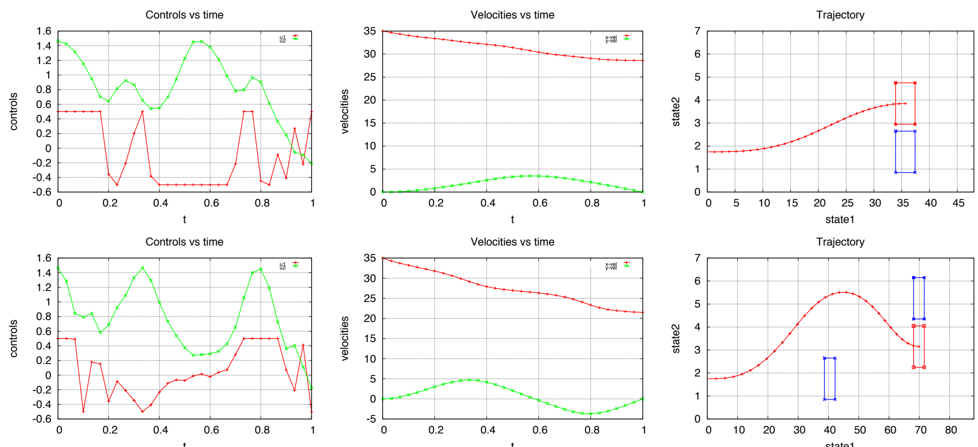


Fig. 2.2. *Scenario 1 (top row) and Scenario 2 (bottom row): Minimization of the initial distances between avoiding car and obstacles. The frames in the pictures on the right indicate the avoiding car and the obstacles. Note the different scales in x- and y-direction.*

Approach 2: Compute (projected) reachable set from initial position. The reachable set is characterized by distance functions of certain grid points g_h in state space. For each grid point g_h in some bounding box containing the reachable set, an OCP is solved via the minimization of the distance of the endpoint $z(t_f)$ of a trajectory to g_h plus a regularization term involving the steering effort.

Let $\hat{z}(t_f; g_h)$ denote the endpoint of the optimal trajectory which is close to some grid point g_h and has initial value $z(0) = z_0$. An approximation of the reachable set is then given by the union of all grid points g_h sufficiently close to $\hat{z}(t_f; g_h)$,

$$R_h(t_f) \approx \bigcup_{g_h : \|\hat{z}(t_f;g_h) - g_h\| \le Ch} \{g_h\}, \ C > 0 \text{ suitable,} \tag{2.1}$$

i.e. those belonging to an $O(h)$-neighborhood of g_h, see [1, 2].

The reachable sets and the trajectory funnels are calculated for an initial velocity of $v = 35m/s$ in Figure 2.3. The dotted points of the reachable set correspond to different free final times. Due to the initial speed and the end conditions (1.4), no other grid points from the dashed bounding box can be reached by the avoiding car.

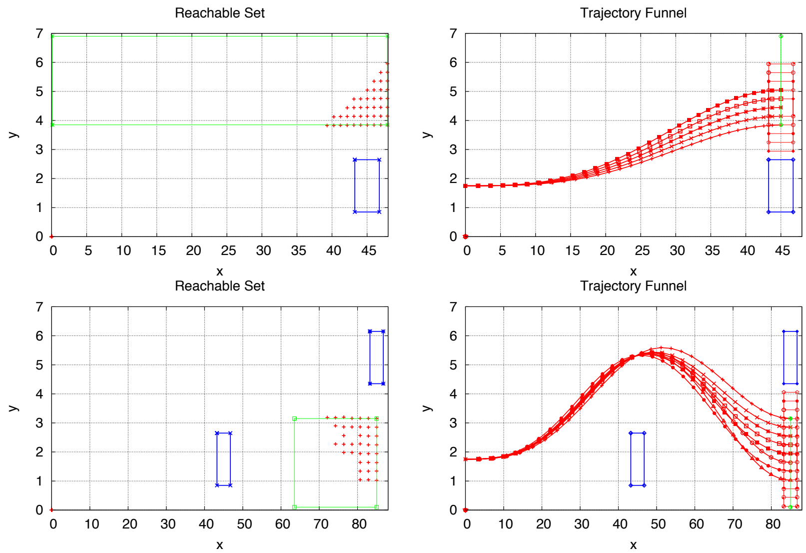


FIG. 2.3. *Figures in the first row concern Scenario 1, figures in the second row refer to Scenario 2.*

3. Sensitivity Analysis. In case of errors in the initial state owing to sensor perturbations we intend to perform a sensitivity analysis for this specific problem to study the influence of parameters on the solution (trajectory and controls) and on the reachable set of the perturbed OCP.

3.1. Sensitivity in Optimal Trajectory. With regard to the optimal trajectory we study problem (1.9) subject to the differential equation $z'(t) = f(z(t), u(t))$ and the perturbed initial value given by

$$
\begin{aligned}
z(0) &= (x(0), y(0), \psi(0), v_x(0), v_y(0), w_\psi(0), \delta(0)) \\
&=: (x_0(p), y_0(p), \psi_0(p), v_{x,0}(p), v_{y,0}(p), w_{\psi,0}(p), \delta_0(p)) \\
&=: (p_1, p_2, p_3, p_4 \cos(p_3), p_4 \sin(p_3), 0, 0, 0) =: z_0(p).
\end{aligned} \tag{3.1}
$$

The perturbation parameter $p = (p_1, \ldots, p_4)^\top$ models sensor perturbations that enter the mathematical model owing to measurement errors in the initial values. The vector p is used for sensitivity analysis and will denote perturbation parameters that enter the problem, but are not optimized.

Let $L^\infty([0, t_f], \mathbb{R}^{n_u}) \times \mathbb{R}^{n_p} \ni (u, p) \mapsto z(u, p)(\cdot) \in W^{1,\infty}([0, t_f], \mathbb{R}^{n_z})$ denote the control and parameter to state mapping, which maps a given control u and a given parameter p to the corresponding state trajectory $z(u, p)$. The aim is to investigate the dependence of the solution with respect to p with two different approaches:

Approach 1: Fiacco-Sensitivity. The first approach called *Fiacco-Sensitivity* is based on a parametric sensitivity analysis of the optimal solution of the optimal control problem (1.9) with respect to p, compare [3, Section 3.2 and 4.2]. To this end let $\hat{u} = \hat{u}(\hat{p})$ and $\hat{z} := z(\hat{u}(\hat{p}), \hat{p})$ denote the optimal solution of the optimal control problem (1.9) with (3.1) for a nominal parameter $\hat{p}$. Then, the Fiacco-Sensitivities of the state and the control are defined as

$$
\frac{dz}{dp}(\hat{u}, \hat{p}) = \frac{\partial z}{\partial u}(\hat{u}, \hat{p}) \frac{d\hat{u}}{dp}(\hat{p}) + \frac{\partial z}{\partial p}(\hat{u}, \hat{p}) \quad \text{and} \quad \frac{d\hat{u}}{dp}(\hat{p}). \tag{3.2}
$$

These sensitivities can be computed using the linearized necessary Karush-Kuhn-Tucker conditions in an optimal solution $(\hat{z}, \hat{u})$. An approximation to the optimal perturbed trajectory is given by

$$z(\hat{u}(p), p)(\cdot) \approx \hat{z}(\cdot) + \frac{dz}{dp}(\hat{u}, \hat{p})(\cdot)(p - \hat{p}). \tag{3.3}$$

An example of Fiacco-perturbed trajectories according to (3.3) with respect to parameter $p_i, i = 1, \ldots, 4$, is presented in Figure 3.1.

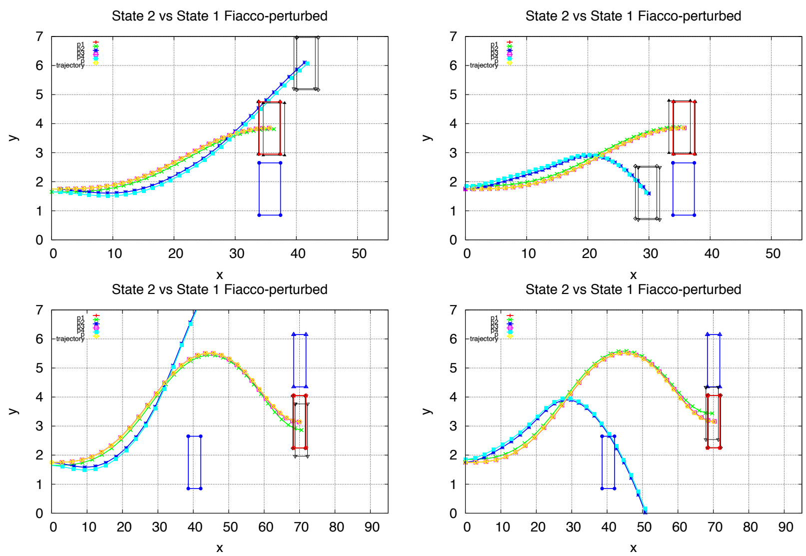


Fig. 3.1. *The six trajectories depicted in each picture show the nominal trajectory and its perturbation with respect to* $p_1, \ldots, p_4$ *and w.r.t. all parameters combined. We see that a perturbation* p_3 *in the yaw angle has the largest influence on the trajectory. Perturbations of the initial position and velocity do not have significant influence on the trajectory. The two pictures in the first row show Scenario 1 with a perturbation of* -0.1 *and* 0.1*, respectively. The two pictures in the second row show Scenario 2 with a perturbation of* -0.1 *and* 0.1*, respectively.*

Approach 2: ODE-Sensitivity. The second approach called *ODE-Sensitivity* investigates the dependence of the solution of the initial value problem on p for a fixed (optimal) control $\hat{u}$ and the nominal parameter $\hat{p}$. To this end let $\hat{u} = \hat{u}(\hat{p})$ be given and let $\hat{z} := z(\hat{u}, \hat{p})$ denote the corresponding solution of the initial value problem

$$z'(t) = f(z(t), \hat{u}(t)), \quad z(0) = z_0(\hat{p}).$$

Then, the ODE-Sensitivity of the state is defined as

$$S(\cdot) := \frac{\partial z}{\partial p}(\hat{u}, \hat{p})(\cdot).$$

Note that this is just the partial derivative of the state mapping w.r.t. p for a fixed control and not the total derivative as in (3.2). An approximation to the perturbed trajectory is obtained similar as in (3.3). The ODE-Sensitivity is given by solving the sensitivity differential equation

$$S'(t) = f'_z(\hat{z}(t), \hat{u}(t))S(t), \qquad S(0) = \frac{dz_0}{dp}(p).$$

An example of ODE-perturbed trajectories with respect to each parameter $p_i, i = 1, \ldots, 4$, is illustrated in Figure 3.2.

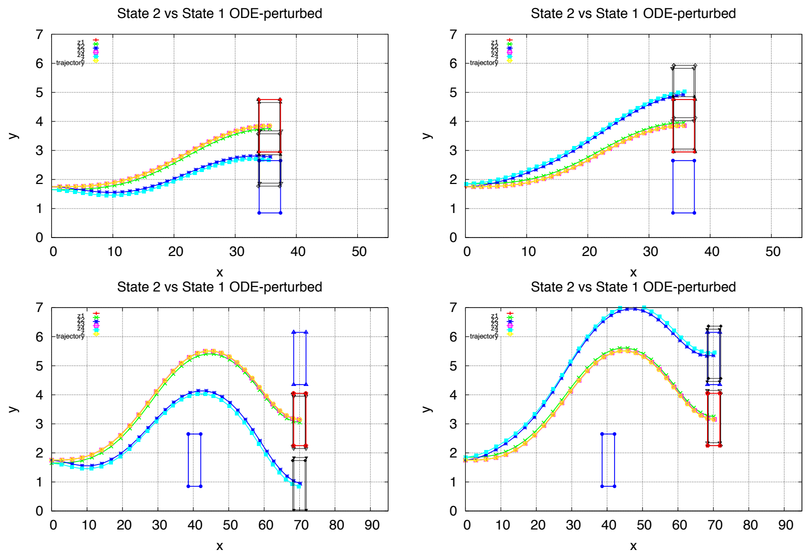


Fig. 3.2. *The six trajectories depicted in each picture show the nominal trajectory and its perturbation with respect to $p_1, \ldots, p_4$ and w.r.t. all parameters combined. We see that a perturbation p_3 in the yaw angle has the largest influence on the trajectory. Perturbations of the initial position and velocity do not have significant influence on the trajectory. The two pictures in the first row show Scenario 1 with a perturbation of -0.1 and 0.1, respectively. The two pictures in the second row show Scenario 2 with a perturbation of -0.1 and 0.1, respectively. In Scenario 1 a crash occurs whenever we have negative perturbation, in this specific case in parameter p_3. In Scenario 2 for negative or positive perturbation of parameter p_3 the car violates the state constraint (1.5).*

3.2. Sensitivity in Trajectory Funnels and Reachable Sets. A method to investigate the dependence of the reachable set on p uses the optimal control approach for reachable sets shown in Approach 2. We can perform a sensitivity analysis of the corresponding optimal control problem w.r.t. the perturbation parameter p at the nominal parameter $\hat{p}$ for each of the optimal solutions $\hat{z}(t_f; g_h, \hat{p})$ with Fiacco- or ODE-Sensitivities. Hence, an approximation of the reachable set for p in (2.1) can be obtained by linearization during the calculation of the reachable set for the nominal parameter:

$$R_h(t_f, p) \approx \bigcup_{g_h : \|\hat{z}(t_f; g_h, \hat{p}) + \hat{z}'_p(t_f; g_h, \hat{p})(p - \hat{p}) - g_h\| \leq Ch} \{g_h\}, \ C > 0 \text{ suitable},$$

where $\hat{z}'_p$ denotes one of the previously discussed Fiacco- or ODE-Sensitivities.

Figures 3.3–3.4 show trajectory funnels, i.e. several solutions reaching different endpoints, and reachable set approximations for the two scenarios.

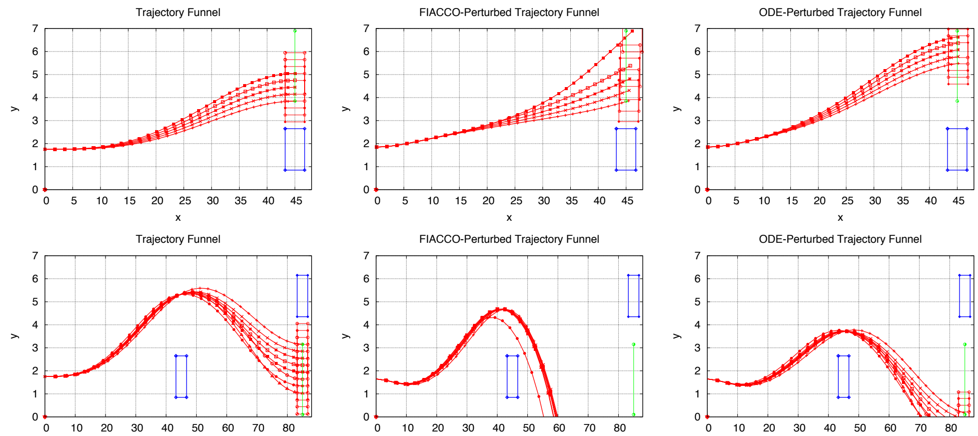


FIG. 3.3. *Trajectory funnels for Scenario 1 (first row) and Scenario 2 (second row): Left pictures show the non-perturbed trajectories (nominal parameters), then the approximations by Fiacco- and ODE-Sensitivity with a positive perturbation of* 0.1 *w.r.t. all combined parameters (Scenario 1) and a negative perturbation of* -0.1 *w.r.t. all combined parameters (Scenario 2). The ODE-sensitivity leads to bigger perturbations of the trajectory funnel for both scenarios than the Fiacco one. In the latter scenario even infeasible trajectories are created.*

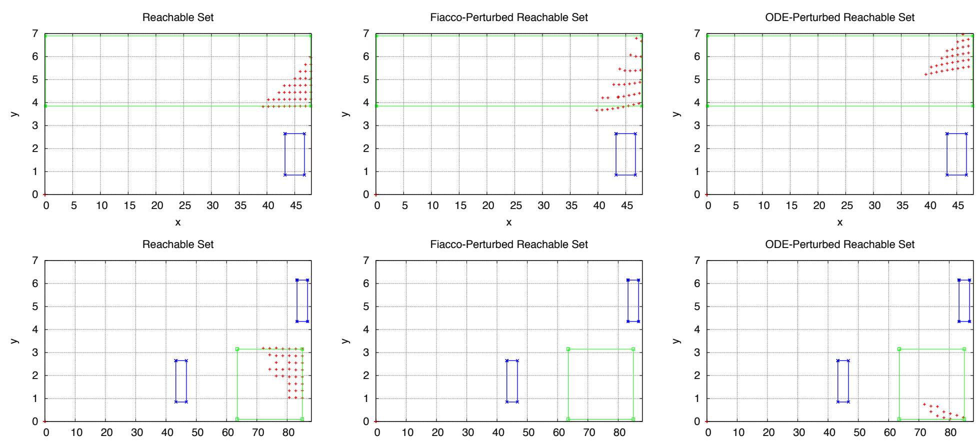


FIG. 3.4. *Reachable sets for Scenario 1 (first row) and Scenario 2 (second row): Left pictures show the non-perturbed trajectories (nominal parameters), then the approximations by Fiacco- and ODE-Sensitivity with a positive perturbation of* 0.1 *w.r.t. all combined parameters (Scenario 1) and a negative perturbation of* -0.1 *w.r.t. all combined parameters (Scenario 2). The ODE-sensitivity leads to bigger perturbations of the trajectory funnel for both scenarios than the Fiacco one. In the latter scenario some points from the reachable sets are infeasible.*

REFERENCES

[1] R. BAIER, M. GERDTS, *A computational method for non-convex reachable sets using optimal control*, Proceedings of the European Control Conference (ECC) 2009, Budapest (Hungary), August 23–26, 2009, EUCA, Budapest, 97–102.

[2] R. BAIER, M. GERDTS, I. XAUSA, *Approximation of reachable sets using optimal control algorithms*, 36 pages, submitted in October 2011.

[3] A. V. FIACCO, *Introduction to Sensitivity Analysis in Nonlinear Programming*, Academic Press Inc., 1983.

[4] M. GERDTS, *A variable time transformation method for mixed-integer optimal control problems*, Optimal Control Appl. Methods 27(3) (2006), pp. 169–182.

[5] M. GERDTS, I. XAUSA, *Collision avoidance using reachable sets and parametric sensitivity analysis*, 10 pages, submitted in January 2012.