Regularity of Set-Valued Maps and Their Selections through Set Differences. Part 2: One-Sided Lipschitz Properties

Robert Baier Elza Farkhi

Dedicated to the 65th birthday of Asen L. Dontchev
and to the 60th birthday of Vladimir M. Veliov.

Abstract

We introduce one-sided Lipschitz (OSL) conditions of set-valued maps with respect to given set differences. The existence of selections of such maps that pass through any point of their graphs and inherit uniformly their OSL constants is studied. We show that the OSL property of a convex-valued set-valued map with respect to the Demyanov difference with a given constant is characterized by the same property of the generalized Steiner selections. We prove that an univariate OSL map with compact images in $\mathbb{R}^1$ has OSL selections with the same OSL constant. For such a multifunction which is OSL with respect to the metric difference, one-sided Lipschitz metric selections exist through every point of its graph with the same OSL constant.

1 Introduction

The One-Sided Lipschitz (OSL) property of a multifunction F, introduced in [12], generalizes the Lipschitz condition with respect to the Hausdorff metric, as well as the monotonicity of $I - F$ (here I is the identity operator). This condition may be satisfied without any requirement of single-valuedness [12], in contrast to the monotonicity condition [30, 20, 21, 22]. This property implies weak exponential stability of the solution mapping of differential inclusions [13], and, moreover, weak asymptotic stability when the OSL constant is negative. The existence of one-sided Lipschitz selections of such maps may have impact on analysis and numerics of dynamical systems, control problems and other fields related to set-valued analysis (see e.g. [3, Chap. 9], [2, 1, 28, 6]).

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Following the general framework of [5], we define different OSL notions of set-valued maps with respect to various differences of two sets. Our approach is based on the observation that some known OSL-type conditions, as the uniform OSL (UOSL) condition which appears in [17, 24] can be represented by the algebraic difference of sets. We extend this approach to other set differences, as the Demyanov difference, the geometric difference of Hadwiger-Pontryagin and the metric difference [19].

The OSL condition of the map F with respect to the set difference $\ominus$ is defined as

$$\delta^*(x-y, F(x) \ominus F(y)) \le L\|x-y\|^2.$$

In this paper we focus our attention mainly on one-sided Lipschitz condition with respect to the metric and the Demyanov set differences. The analogous Lipschitz notions based on set differences are studied in [5]. An important advantage of this approach is that the inclusion hierarchy between set differences implies directly the hierarchy of the corresponding OSL (Lipschitz, monotonicity) conditions.

We first review known one-sided Lipschitz conditions [14, 12, 16, 27, 26] and present them with known set differences. We define here also new OSL and monotonicity conditions based on the Demyanov, the metric and the geometric set difference.

Special attention is given to one-sided Lipschitz conditions with respect to the metric difference of compacts or the Demyanov difference of convex compacts in $\mathbb{R}^n$. It is shown that the OSL condition for set-valued functions with respect to the Demyanov difference of sets is equivalent to the same OSL condition satisfied uniformly by the generalized Steiner (GS) selections in [7, 4]. Also, we prove that a univariate OSL multifunction with one-dimensional compact values has OSL selections with the same OSL constant, passing through any point of its graph. If this multifunction is OSL with respect to the metric difference, it has OSL metric selections. Therefore, if such a multifunction is monotone with respect to the Demyanov (or the metric) difference, it has monotone generalized Steiner (resp. metric) selections.

The paper is organized as follows. In the next section we recall some notions of set differences and recall some basic properties of such differences derived in [5]. In Section 3 various OSL conditions with respect to given set differences are introduced and studied. Special cases and properties, arithmetic operations, as well as the hierarchy of these notions are studied. The representation of OSL mappings with respect to the Demyanov (resp. the metric) difference by OSL generalized Steiner (resp. metric) selections is presented in Section 4. A result on the existence of OSL metric selections through any point of the graph of OSL maps is also derived. In the last section a collection of examples illustrating the hierarchy of different OSL notions is presented.

2 Set Differences and Their Properties

We refer the reader to [5] for all definitions and notation. We recall shortly only some notation. We denote by $\mathcal{K}(\mathbb{R}^n)$ the set of nonempty compact subsets of $\mathbb{R}^n$ and by $\mathcal{C}(\mathbb{R}^n)$ the set of nonempty convex compact subsets of $\mathbb{R}^n$. By $\|\cdot\|$ we denote some vector norm in $\mathbb{R}^n$ and with $\|\cdot\|_2$ the Euclidean norm, with $\|M\|_2$ the spectral norm of a matrix $M \in \mathbb{R}^{n\times n}$ and for a set $A \in \mathcal{K}(\mathbb{R}^n)$, $\|A\| := \sup\{\|a\| \,:\, a \in A\}$. The notation $\|\cdot\|_\infty$ is used for the maximum norm in $\mathbb{R}^n$. The convex hull of the set A is denoted by $\operatorname{co}(A)$, $\overline{\operatorname{co}}(A)$ is the closed convex hull of A. S_{n-1} denotes the unit sphere in $\mathbb{R}^n$.

The *support function* resp. the *supporting face* for a set $C \in \mathcal{C}(\mathbb{R}^n)$ is defined for $l \in \mathbb{R}^n$ as

$$\delta^*(l,C) := \max_{c\in C} \langle l,c\rangle\,, \quad Y(l,C) := \{c \in C \,:\, \langle l,c\rangle = \delta^*(l,C)\}\,.$$

A *supporting point* (an element of $Y(l,C)$) is denoted by $y(l,C)$.

Recall that the *Hausdorff distance* between two compact, nonempty sets A, B is denoted by $\mathrm{d_H}(A,B)$. The *Demyanov distance* is formed by subtracting unique supporting points of common directions:

$$\mathrm{d_D}(A,B) := \sup\{\|y(l,A) - y(l,B)\| \,:\, l \in T_A \cap T_B\} \tag{1}$$

The notations λA, $-A$ and $A + B$ are used for the *multiplication with a nonnegative scalar* resp. with -1 and for the *Minkowski sum* (see [5]).

We now recall the notation of some known differences of nonempty, compact subsets A, B of $\mathbb{R}^n$:

- $A \ominus_A B$: *algebraic difference*
- $A \ominus_G B$: *geometric difference*
- $A \ominus_D B$: *Demyanov difference*
- $A \ominus_M B$: *metric difference*

We will use the following inclusions and estimates for the above differences which are derived in [5].

$$A \ominus_M B \subseteq A \ominus_A B\,, \tag{2}$$

$$A \ominus_G B \subseteq A \ominus_D B \subseteq A \ominus_A B\,, \tag{3}$$

$$\delta^*(l, A \ominus_G B) \le \delta^*(l,A) - \delta^*(l,B) \le \delta^*(l, A \ominus_D B) \quad (l \in S_{n-1})\,, \tag{4}$$

$$\delta^*(l, A \ominus_G B) \le \delta^*(l,A) - \delta^*(l,B) \le \delta^*(l, A \ominus_M B) \quad (l \in S_{n-1}) \tag{5}$$

The metric and the Demyanov difference determine metrics in $\mathcal{K}(\mathbb{R}^n)$ resp. $\mathcal{C}(\mathbb{R}^n)$ (see [5]) via

$$\mathrm{d_H}(A,B) = \|A \ominus_M B\| \quad (A, B \in \mathcal{K}(\mathbb{R}^n))\,, \tag{6}$$

$$\mathrm{d_D}(C,D) = \|C \ominus_D D\| \quad (C, D \in \mathcal{C}(\mathbb{R}^n))\,. \tag{7}$$

We also remind the known fact that $\mathrm{d_H}(A,B) \le \mathrm{d_D}(A,B)$.

For the set difference "$\ominus_\Delta$" we list the following stronger forms of the axioms in [5] for a general set difference of compact sets $A, B, C, A_i, B_i \in \mathcal{K}(\mathbb{R}^n)$, $i = 1, 2$.

$$\begin{array}{ll}
\text{(A2')} & B \ominus_\Delta A = -(A \ominus_\Delta B)\,, \\
\text{(A3')} & A \ominus_\Delta B \subset (A \ominus_\Delta C) \; + \; (C \ominus_\Delta B)\,, \\
\text{(A4')} & (\alpha A) \ominus_\Delta (\alpha B) = \alpha(A \ominus_\Delta B) \quad (\alpha \geq 0)\,, \\
\text{(A5')} & (\alpha A) \ominus_\Delta (\beta A) = (\alpha - \beta)A \qquad (\alpha \geq \beta \geq 0)\,, \\
\text{(A6')} & (A_1 + A_2) \ominus_\Delta (B_1 + B_2) \subset (A_1 \ominus_\Delta B_1) + (A_2 \ominus_\Delta B_2)
\end{array}$$

We also need the following weakened form of (A6'):

$$(A_1 + a_2) \ominus_\Delta (B_1 + b_2) = (A_1 \ominus_\Delta B_1) + (a_2 - b_2) \tag{8}$$

It is known that the Demyanov difference satisfies (A2')–(A6') on $\mathcal{K}(\mathbb{R}^n)$, (A2') and (A4') hold for the metric difference, whereas the algebraic difference fulfills (A2')–(A4') and (A6') and the geometric difference satisfies (A4') and (8), (A5') only holds in $\mathcal{C}(\mathbb{R}^n)$.

3 OSL Multimaps through Set Differences

In this section we recall some known one-sided Lipschitz conditions and introduce some new OSL notions with respect to set differences.

3.1 One-Sided Lipschitz Conditions

Throughout the paper we consider a closed set $X \subset \mathbb{R}^n$ and set-valued maps $F : X \rightrightarrows \mathbb{R}^n$ with images in the metric space $\mathcal{X} = \mathcal{K}(\mathbb{R}^n)$ or $\mathcal{X} = \mathcal{C}(\mathbb{R}^n)$ supplied with the metric

$$\mathrm{d}_\Delta(A, B) := \|A \ominus_\Delta B\| \quad (A, B \in \mathcal{X},\ \Delta \in \{M, D\})\,.$$

We focus in this subsection on the One-Sided Lipschitz (OSL) property, which also extends monotonicity-type behavior of set-valued maps. We will define it with respect to a given set difference. Recall first some known variants of the OSL property.

Definition 3.1 *A set-valued map $F : X \to \mathcal{C}(\mathbb{R}^n)$ is called*

(i) UOSL *(*uniformly OSL *in [17]* [1]*) with constant $\mu \in \mathbb{R}$ if for all $x, y \in X$ and for all $u \in F(x), v \in F(y)$*

$$\langle x - y, u - v \rangle \leq \mu \|x - y\|^2\,, \tag{9}$$

[1] In [25] this property is called just one-sided Lipschitz, but this term is also used in the later appearing articles [12, 13].

(ii) S-UOSL *(*strengthened UOSL *in [24, 26]* [2]*) with constant* $\mu \in \mathbb{R}$*, if for all* $x, y \in X$*, all* $u \in F(x), v \in F(y)$ *and all indices* $i = 1, \ldots, n$

$$\langle e^i, x - y \rangle \cdot \langle e^i, u - v \rangle \le \mu \|x - y\|^2$$

holds, where e^i *is the* i*-th standard unit vector* [3]*,*

(iii) OSL *(*one-sided Lipschitz *in [14, 12]) with constant* $\mu \in \mathbb{R}$*, if for all* $x, y \in X$ *and all* $u \in F(x)$ *there exists* $v \in F(y)$ *such that*

$$\langle x - y, u - v \rangle \le \mu \|x - y\|^2 \,,$$

(iv) S-OSL *(*strengthened OSL *in [27]* [4]*) with constant* $\mu \in \mathbb{R}$*, if the condition "for all* $v \in F(y)$*" in the definition of S-UOSL is replaced by "there exists* $v \in F(y)$*" (as in the spirit of the OSL property).*

These conditions can be regarded as generalized monotonicity-type conditions, since the UOSL property of the negation of $F(\cdot)$, $-F(\cdot)$ with a constant 0 means monotonicity of F.

We now express the first two notions by set differences. We can define the OSL property with respect to a given set difference.

Definition 3.2 *Let* $F : X \to \mathcal{C}(\mathbb{R}^n)$*. F is called* OSL with respect to the set difference "$\ominus_\Delta$" *(*Δ-OSL*) with constant* $\mu \in \mathbb{R}$*, if for all* $x, y \in X$

$$\delta^*(x - y, F(x) \ominus_\Delta F(y)) \le \mu \|x - y\|^2 \,. \tag{10}$$

The map F is called monotone with respect to the set difference *"*$\ominus_\Delta$*" (or shortly,* Δ-monotone*), if* $-F$ *is* Δ*-OSL with a constant zero.*

The following claim follows directly from Definitions 3.1 and 3.2.

Proposition 3.3 *A set-valued map* $F : X \to \mathcal{C}(\mathbb{R}^n)$ *is*

(i) UOSL iff it is OSL with respect to the algebraic difference, namely

$$\delta^*(x - y, F(x) \ominus_A F(y)) \le \mu \|x - y\|^2 \quad (x, y \in X) \,, \tag{11}$$

(ii) S-UOSL iff for every $i = 1, 2, ..., n$*,*

$$\langle e^i, x - y \rangle \delta^*(e^i, F(x) \ominus_A F(y)) \le \mu \|x - y\|^2 \quad (x, y \in X) \,, \tag{12}$$

[2]In [24, 25, 13] this property is called strengthened OSL (S-OSL) and in [26] it is called strongly OSL. Since it involves *all* choices $v \in F(y)$ as in the UOSL property we add the term "uniform" as in [12].

[3]In [26] a generalization to an arbitrary orthonormal basis of $\mathbb{R}^n$ is given.

[4]Do not mix this with the property called this way in the publications [24, 25, 13], see also the footnote for the S-UOSL property.

(iii) OSL iff [12]

$$\delta^*(x-y, F(x)) - \delta^*(x-y, F(y)) \le \mu\|x-y\|^2 \quad (x, y \in X)\,, \tag{13}$$

(iv) S-OSL iff for every $i = 1, 2, ..., n$,

$$\langle e^i, x-y\rangle(\delta^*(e^i, F(x)) - \delta^*(e^i, F(y))) \le \mu\|x-y\|^2 \quad (x, y \in X)\,. \tag{14}$$

Remark 3.4 *The expression of the OSL and the S-OSL property by some set difference is more complicated. Such a difference can be defined in an abstract vector space, in which the convex subsets of* $\mathbb{R}^n$ *are embedded, as the space of differences of support functions. Such a construction can be found e.g. in [15, Example 1], but it is beyond the scope of this paper.*

If we replace in (10) the algebraic difference by another difference, the Demyanov difference for instance, then we get a new OSL property.

Definition 3.5 *Let* $F : X \to \mathcal{C}(\mathbb{R}^n)$. *F is called* D-OSL *(* OSL with respect to the Demyanov difference*) with constant* $\mu \in \mathbb{R}$, *if for all* $x, y \in X$

$$\delta^*(x-y, F(x) \ominus_D F(y)) \le \mu\|x-y\|_2^2\,. \tag{15}$$

Similarly we define M-OSL *as* OSL with respect to the metric difference*, and the* G-OSL property with respect to the geometric difference. *For the latter to be well-defined we use the convention that the support function of the empty set is equal to* $-\infty$.

Let us note that since all differences coincide for singletons, the corresponding OSL properties with respect to any of these differences coincide for single-valued $F(\cdot)$.

Proposition 3.6 (single-valued case) *Let* $f : X \to \mathbb{R}^n$ *and set* $F(x) := \{f(x)\}$. *Then:*

(i) The properties G-OSL, OSL, D-OSL, M-OSL and UOSL are identical and are equivalent to the OSL condition of $f(\cdot)$.

(ii) The properties S-OSL and S-UOSL are equivalent.

Proof: The assertions follow directly from the definitions. ■

Example 5.7 presents an example of a single-valued OSL function not being S-OSL.

In the 1d case several notions coincide (as it is proved for UOSL and S-UOSL in [25, 3.1 Lemma]).

Proposition 3.7 (1d case) *Let* $F : I \Rightarrow \mathbb{R}$ *with images in* $\mathcal{C}(\mathbb{R})$ *and a closed set* $I \subset \mathbb{R}$ *be given. Then:*

(i) If $F(\cdot)$ is G-OSL and $F(s) \ominus_G F(t) \neq \emptyset$ for all $s, t \in I$, then $F(\cdot)$ is OSL.

(ii) If $F(t) = [a(t), b(t)]$, then $F(\cdot)$ being OSL is equivalent to $a(\cdot), b(\cdot)$ being both OSL.

(iii) The properties D-OSL, OSL and S-OSL coincide.

(iv) The properties UOSL and S-UOSL are equivalent.

Proof:
(i) As in the proof of [5, Proposition 3.9] the geometric difference coincides with the Demyanov one so that (4) assures (13).

(ii) In the 1d case there are essentially two cases to prove (13) for $s, t \in I$ (for $s - t = 0$ there is nothing to prove): if $s - t$ is positive, (13) is equivalent to the OSL condition for $b(\cdot)$ and if $s - t$ is negative, (13) is equivalent to the OSL condition for $a(\cdot)$.

(iii) We can argue similarly to (i) to derive the D-OSL condition from the OSL one. Due to the 1d case, OSL coincides with S-OSL.

(iv) Obviously, UOSL and S-UOSL coincide by definition for $n = 1$. ∎

The assumption in (i) that the geometric difference is never empty is quite restrictive and requires that the diameter of $F(\cdot)$ is constant. Example 5.1 illustrates such a case.

Remark 3.8 *The 1d case $F : [a, b] \Rightarrow \mathbb{R}$ and the single-valued case $F(x) = \{f(x)\}$ with an OSL function $f : X \to \mathbb{R}^n$ are cases for OSL maps in which all GS-selections are uniformly OSL via Propositions 3.6 and 3.7. Clearly, this follows from the equivalence of OSL to D-OSL in these two cases.*

A class of examples for OSL maps are polytopes whose vertices are single-valued OSL functions. The next result generalizes parts of Propositions 3.6 and 3.7 for OSL maps.

Lemma 3.9 *Consider $F(x) = \operatorname{co}\{f_j(x) \,:\, j = 1, \dots, M\}$ with $f_j : X \to \mathbb{R}^n$ OSL for $j = 1, \dots, M$.*
Then, $F(\cdot)$ is OSL with constant $\mu = \max\limits_{1 \le j \le M} \mu_j$.

Proof: Let $x, y \in X$ and let $\xi = \sum_{j=1}^M \alpha_j f_j(x) \in F(x)$ with a convex combination $(\alpha_j)_{j=1,\dots,M}$. We choose $\eta = \sum_{j=1}^M \alpha_j f_j(y) \in F(y)$ such that

$$\begin{aligned} \langle x - y, \xi - \eta \rangle &= \sum_{j=1}^M \alpha_j \langle x - y, f_j(x) - f_j(y) \rangle \le \sum_{j=1}^M \alpha_j \mu_j \|x - y\|^2 \\ &\le \mu \|x - y\|^2 \end{aligned}$$

which proves the OSL condition for $F(\cdot)$. ■

A class of D-OSL maps are scalar Lipschitz functions multiplied with a constant set.

Lemma 3.10 *Consider a convex, compact, nonempty set $U \subset \mathbb{R}^n$ and a Lipschitz function $r : X \to [0, \infty)$ with constant L.*
If the set difference "$\ominus_\Delta$" fulfills (A2') and (A5') then $F(x) := r(x)U$ for $x \in X$ is Δ-OSL (especially D-OSL) and S-OSL.

Proof: Let us first consider the case "$r(x) \geq r(y)$" in which we use (A5'):

$$\begin{aligned}&\delta^*(x - y, (r(x)U) \ominus_\Delta (r(y)U)) = \delta^*(x - y, (r(x) - r(y))U)\\ &= (r(x) - r(y))\delta^*(x - y, U) \leq L \cdot \|U\| \cdot \|x - y\|^2\end{aligned}$$

If $r(x) < r(y)$, then (A2') yields

$$\begin{aligned}&\delta^*(x - y, (r(x)U) \ominus_\Delta (r(y)U)) = \delta^*(y - x, -((r(x)U) \ominus_\Delta (r(y)U)))\\ &= \delta^*(y - x, (r(y)U) \ominus_\Delta (r(x)U)) \leq L \cdot \|U\| \cdot \|y - x\|^2 \,.\end{aligned}$$

Since the Demyanov difference satisfies both required equations, we can show the D-OSL property.
For S-OSL this follows immediately from $\delta^*(e^i, F(x)) = r(x)\delta^*(e^i, U)$. ■

Setting $r(x) = 1$ we get that constant set-valued maps are also D-OSL.

3.2 Properties and Relations Between the Different Notions

The next proposition is well-known for Lipschitz, OSL, UOSL and S-OSL maps (see e.g. [25, 3.4 Lemma]).

Proposition 3.11 *Let $F_1, F_2 : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{K}(\mathbb{R}^n)$ be OSL with constants $\mu_1, \mu_2 \in \mathbb{R}$ with respect to the set difference "$\ominus_\Delta$" and $\alpha \geq 0$. We set $F(\cdot) = \alpha F_1(\cdot)$ and $G(\cdot) = F_1(\cdot) + F_2(\cdot)$.*
(i) If (A4') holds for the set difference "$\ominus_\Delta$", then $F(\cdot)$ is OSL with respect to the set difference "$\ominus_\Delta$" and constant $\alpha\mu_1$.
(ii) If (A6') holds for "$\ominus_\Delta$", then $G(\cdot)$ is OSL with respect to the set difference "$\ominus_\Delta$" and constant $\mu_1 + \mu_2$.
(iii) Both properties (i)–(ii) hold for S-OSL and S-UOSL maps.

Proof: (i) Since $\alpha \geq 0$ and (A4') holds,

$$\begin{aligned}&\delta^*(x - y, F(x) \ominus_\Delta F(y)) = \delta^*(x - y, \alpha(F_1(x) \ominus_\Delta F_1(y)))\\ &= \alpha \cdot \delta^*(x - y, F_1(x) \ominus_\Delta F_1(y)) \leq \alpha \cdot \mu \cdot \|x - y\|^2 \,.\end{aligned}$$

(ii) The result for the sum follows from (A6') and

$$\begin{aligned}
&\delta^*(x-y, G(x) \ominus_\Delta G(y)) \\
&\le \delta^*\Big(x-y, (F_1(x) \ominus_\Delta F_1(y)) + (F_2(x) \ominus_\Delta F_2(y))\Big) \\
&= \delta^*(x-y, F_1(x) \ominus_\Delta F_1(y)) + \delta^*(x-y, F_2(x) \ominus_\Delta F_2(y)) \\
&\le \mu_1 \cdot \|x-y\|^2 + \mu_2 \cdot \|x-y\|^2 .
\end{aligned}$$

(iii) Since (A4') and (A6') are fulfilled for the algebraic difference and the properties S-OSL and S-UOSL are based on each coordinate, both properties also hold for these two classes. ■

The condition (A4') holds for the geometric, the metric, the Demyanov and the algebraic difference, hence for G-OSL, M-OSL, D-OSL and UOSL maps. Since the condition (A6') holds only for the Demyanov and algebraic difference, but not for the metric or the geometric difference, the sum property holds for D-OSL and UOSL maps and cannot be expected for G-OSL and M-OSL maps in general. It is easy to see that (i) cannot be true for $\alpha < 0$.

The following corollary is easy to derive (compare [12, Example 2.3] for an OSL example).

Corollary 3.12 *Let $G : X \Rightarrow \mathbb{R}^n$ have images in $\mathcal{K}(\mathbb{R}^n)$ and $h : X \to \mathbb{R}^n$ be OSL with constant μ_h. We consider the set-valued map $F : X \Rightarrow \mathbb{R}^n$ with $F(x) = G(x) + h(x)$.*
(i) If $G(\cdot)$ is Δ-OSL with constant μ_G for a set difference "$\ominus_\Delta$" which fulfills (8), then $F(\cdot)$ is Δ-OSL with constant $\mu_G + \mu_h$.
(ii) If $G(\cdot)$ is S-OSL resp. S-UOSL, then $F(\cdot)$ is also S-OSL resp. S-UOSL.

Proof: (i) By Proposition 3.6 the set-valued map $H(\cdot) = \{h(\cdot)\}$ is Δ-OSL. Similarly to the proof of Proposition 3.11, we use (8) instead of (A6').
(ii) The arguments are the same as in the proof of Proposition 3.11. ■

The following theorem is proved in [30, 20] for monotone maps. Here, we formulate it for UOSL maps.

Theorem 3.13 ([30, 20]) *Let $F : X \Rightarrow \mathbb{R}^n$ have nonempty images and be UOSL. Then, $F(\cdot)$ is almost everywhere single-valued.*

This theorem is easily reduced to the same result for monotone maps by considering the monotone map $\mu I - F$, where μ is the UOSL constant of F and I is the identity map. Let us summarize some of the known and established results.

Remark 3.14 *Let us recall some well-known facts (see e.g. [12]) together with new results. To compare with the one-sided Lipschitz condition, in the* *case of single-valued functions, the five properties G-OSL, OSL, M-OSL, D-OSL and UOSL coincide but still are weaker than the Lipschitz condition (LC) (see Proposition 3.6). Moreover, they do not imply continuity of the mapping (see e.g. Example 5.1).*

For set-valued functions, the OSL property is weaker than the six properties LC, UOSL, M-OSL, D-OSL, S-UOSL and S-OSL (see Theorems 3.16 and 3.17). If $n = 1$, UOSL is equivalent to S-UOSL, but still stronger than OSL and S-OSL (see Proposition 3.7 and Example 5.1). One can easily construct examples of maps which are UOSL but not LC (even not continuous as in Example 5.6), and also examples of LC (even constant) maps that are not UOSL (see Lemma 3.10 with $r(x) = 1$).

Let us remark that the OSL property of the supporting faces $Y(l, F(\cdot))$ uniformly in $l \in S_{n-1}$ is equivalent to the D-OSL condition. This uniform property is obviously a sufficient condition for OSL (compare in this respect the hierarchy of OSL notions stated in Theorem 3.17).

Proposition 3.15 *Let $F : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$ be given. Then, $F(\cdot)$ is D-OSL iff $Y(l, F(\cdot))$ is OSL uniformly in $l \in S_{n-1}$.*

Proof: Sufficiency for D-OSL: Motivated by the definition of the Demyanov difference, we consider $l \in T_{F(x)} \cap T_{F(y)}$. By the uniform OSL property of $Y(l, F(\cdot))$ and for the direction l of unique support, we get

$$\langle x - y, y(l, F(x)) - y(l, F(y)) \rangle \leq \mu \|x - y\|^2 . \tag{16}$$

Taking the union over all such directions l of unique support, we can estimate the support function of the Demyanov difference and obtain the D-OSL condition.
Necessity for D-OSL: The left-hand side in (16) for $l \in T_{F(x)} \cap T_{F(y)}$ can be bounded from above by the support function of $F(x) \ominus_D F(y)$ and the D-OSL condition. In a standard way using the Theorem of Straszewicz in [29, Theorem 1.4.7] (the convex set coincides with the closed convex hull of its exposed points that are actually points of unique support) we can extend the D-OSL estimate for any point in the supporting face $Y(l, F(x))$. ■

The following result shows that D-OSL is a generalization of D-Lipschitz as OSL is a generalization of Lipschitz maps (see [12, Remark 2.2]).

Theorem 3.16 *Let $F : X \Rightarrow \mathbb{R}^n$ be given with images in $\mathcal{K}(\mathbb{R}^n)$. If $F(\cdot)$ is Lipschitz with respect to the set difference "$\ominus_\Delta$", it is also Δ-OSL.*

Proof: The claim follows by the Cauchy-Schwarz inequality:

$$\delta^*(x - y, F(x) \ominus_\Delta F(y)) \leq \|x - y\|_2 \cdot \|F(x) \ominus_\Delta F(y)\|_2 \leq L\|x - y\|_2^2$$

■

If we take in (10) a set difference generating a smaller set than the algebraic difference, then we clearly will get a generalization of the UOSL property. Natural candidates for this are the Demyanov difference and the metric difference. More generally, we get the following

Theorem 3.17 (hierarchy for OSL conditions)
Let $F : X \Rightarrow \mathbb{R}^n$ be given with images in $\mathcal{C}(\mathbb{R}^n)$.
Then, the following implications hold:

(i) S-UOSL ⇒ UOSL ⇒ D-OSL ⇒ OSL ⇒ G-OSL,
(ii) UOSL ⇒ M-OSL ⇒ OSL,
(iii) S-UOSL ⇒ S-OSL ⇒ OSL

Proof: (i) All implications follow from the relations (2)–(4). The implication S-UOSL ⇒ UOSL is proved in [25, 3.3 Lemma].

(ii) By (2), the metric difference is a subset of the algebraic one, so that the implication "UOSL ⇒ M-OSL" is obvious.
By (5), the support function of the metric difference majorizes the difference of support functions so that the implication "M-OSL ⇒ OSL" is also clear.

(iii) In the third chain of implications the first implication is obvious from the definition and the second one can be proved along the lines of [25, 3.3 Lemma]. ■

The hierarchy of notions proved above easily follows from inclusions that hold for set differences resp. by estimates of support functions of the set differences.

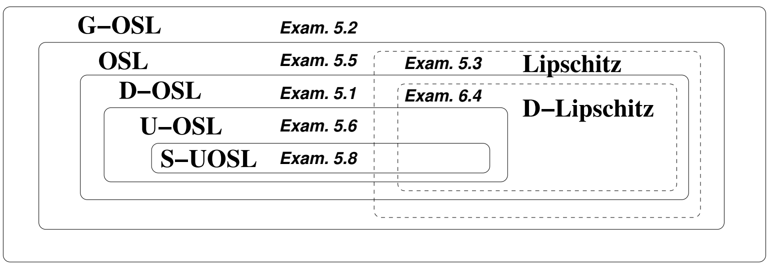

Figure 1: hierarchy of OSL and Lipschitz notions

Each Lipschitz map is also OSL as well as each D-Lipschitz map is also D-OSL. The diagram in Figure 1 illustrates the first chain of implications in Theorem 3.17 as inclusions of regularity classes for set-valued maps. It also shows that the inclusions are strict. The examples in the next section will demonstrate these facts, e.g. Example 5.5 presents a set-valued map which is OSL, but neither D-OSL nor Lipschitz. Example 6.4 in this figure refers to [5].

Remark 3.18 *[5, Example 2.4] indicates that we cannot hope for the implication M-OSL $\Rightarrow$ D-OSL or for the inverse one. Examples 5.3 and 5.4 present set-valued maps which are M-OSL but not D-OSL and vice versa.*

To illustrate the sharpness of inclusions in the second and third chain of inclusions: Example 5.1 lists a M-OSL map which is not UOSL, while Example 5.4 contains the definition of an OSL map which is not M-OSL.

For the third chain, Lemma 3.10 shows that the constant map $F(x) = B_1(0) \subset \mathbb{R}^n$ is S-OSL with constant 0, but not S-UOSL (since it is not UOSL by Theorem 3.13). Example 5.6 shows a map that is UOSL (and hence, OSL), but not S-OSL.

4 One-Sided Lipschitz Selections

Here, we present results on the existence of OSL selections passing through any point of the graph of set-valued maps which are one-sided Lipschitz of various kinds (D-OSL, M-OSL and OSL). For set-valued maps with convex, compact images we characterize the D-OSL property by the existence of uniformly OSL generalized Steiner selections. In the nonconvex case we can show the existence of uniformly OSL metric selections for M-OSL maps in the one-dimensional case. Similarly to the D-OSL case, we characterize OSL maps by the existence of a family of uniformly OSL selections passing through any point of their graphs.

4.1 OSL Generalized Steiner Selections

In this subsection we turn our attention to D-OSL mappings and prove that the GS-selections of a D-OSL mapping are uniformly OSL. In Subsection 5.2 we will see that the existence of OSL selections require more than the OSL condition for the set-valued map. This corresponds to [5] in which the existence of uniform Lipschitz selections requires more than Lipschitz regularity (namely Lipschitzness with respect to the Demyanov difference).

Let us recall the *generalized Steiner selections* for convex-valued set-valued maps as introduced in [7, 8, 9]. We refer to the notations in [5] and will denote by $\mathrm{St}_\alpha(C)$ the *generalized Steiner point* of a convex set $C \in \mathcal{C}(\mathbb{R}^n)$. In [8, Theorem 3.4], the *Castaing representation*

$$F(x) = \overline{\bigcup_{\alpha \in \mathcal{SM}} \{\mathrm{St}_\alpha(F(x))\}} \tag{17}$$

for a set-valued map $F : X \Rightarrow \mathbb{R}^n$ with convex images is studied, where $\mathcal{SM}$ is the set of smooth measures. In [4] the set $\mathcal{AM}$ of atomic measures $\alpha_{[l]}$ which is concentrated in a single point $l \in S_{n-1}$ is introduced as $\mathrm{St}_{\alpha_{[l]}}(F(x))$ via the *Steiner point* $\mathrm{St}(Y(l, F(x)))$. Following this we get the second Castaing representation

$$F(x) = \overline{\mathrm{co}} \left\{ \bigcup_{l \in S_{n-1}} \{\mathrm{St}_{\alpha_{[l]}}(F(x))\} \right\}. \tag{18}$$

As in [5], we denote by $\mathcal{M}_{\mathrm{sp}}$ either the measure class $\mathcal{AM}$ or $\mathcal{SM}$.

We next state a generalization of [4, Proposition 5.1] in which D-Lipschitzness is characterized by the uniform Lipschitz continuity of generalized Steiner selections.

Theorem 4.1 (characterization of D-OSL) *Let $F : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$ be given. Then, F is D-OSL with constant $\mu \in \mathbb{R}$ if and only if the family $(\mathrm{St}_\alpha(F(\cdot)))_{\alpha \in \mathcal{M}_{sp}}$ is uniformly OSL with constant μ.*

Proof: By [4, Theorem 4.6] the Demyanov difference can be represented as closed convex hull of differences of generalized Steiner selections, i.e.

$$F(x) \ominus_D F(y) = \overline{\mathrm{co}}\{ \bigcup_{l \in S_{n-1}} (\mathrm{St}_{\alpha_{[l]}}(F(x)) - \mathrm{St}_{\alpha_{[l]}}(F(y)))\}. \tag{19}$$

If $F(\cdot)$ is D-OSL, then

$$\langle x - y, \mathrm{St}_{\alpha_{[l]}}(F(x)) - \mathrm{St}_{\alpha_{[l]}}(F(y)) \rangle \le \delta^*(x - y, F(x) \ominus_D F(y)) \le \mu \|x - y\|^2$$

for all $l \in S_{n-1}$ holds, i.e. the uniform OSL property for the selections $\mathrm{St}_{\alpha_{[l]}}(F(\cdot))$.
On the other hand, the uniform OSL property

$$\langle x - y, \mathrm{St}_{\alpha_{[l]}}(F(x)) - \mathrm{St}_{\alpha_{[l]}}(F(y)) \rangle \le \mu \|x - y\|^2 \quad (l \in S_{n-1})$$

together with (19) yields

$$\begin{aligned} \delta^*(x - y, F(x) \ominus_D F(y)) &= \sup_{l \in S_{n-1}} \langle x - y, \mathrm{St}_{\alpha_{[l]}}(F(x)) - \mathrm{St}_{\alpha_{[l]}}(F(y)) \rangle \\ &\le \mu \|x - y\|^2, \end{aligned}$$

i.e. the D-OSL property holds. The same reasoning works for the smooth measures in $\mathcal{SM}$. ■

The following corollary provides a positive answer to the question in [6], in what cases there exist continuous and uniform OSL selections for a given set-valued OSL map.

Corollary 4.2 *Let $F : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$ be given. Then, F is D-OSL with constant $\mu \in \mathbb{R}$ and D-continuous if and only if the family $\left\{ \mathrm{St}_\alpha(F(\cdot)) \right\}_{\alpha \in \mathcal{M}_{sp}}$ is uniformly OSL with constant μ and uniformly continuous.*

In view of Proposition 3.7 we get

Corollary 4.3 *Let $F : I \Rightarrow \mathbb{R}$ with images in $\mathcal{C}(\mathbb{R})$ and $I \subset \mathbb{R}$ be given. Then, F is OSL if and only if the family $\big\{ \mathrm{St}_\alpha(F(\cdot)) \big\}_{\alpha \in \mathcal{M}_{sp}}$ is uniformly OSL with constant μ.*

From the Castaing representation (17) it follows that GS-points are sufficient to test the corresponding OSL condition for the set difference "$\ominus_\Delta$".

Proposition 4.4 *Let $F : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$ be given.*
Then, conditions

(i) $\langle x - y, \mathrm{St}_{\alpha_{[l]}}(F(x) \ominus_G F(y)) \rangle \le \mu \|x - y\|^2$ *for all $l \in S_{n-1}$ assuming that $F(x) \ominus_G F(y) \neq \emptyset$ for all $x, y \in X$,*

(ii) $\langle x - y, \mathrm{St}_{\alpha_{[l]}}(F(x)) - \mathrm{St}_{\alpha_{[l]}}(F(y)) \rangle \le \mu \|x - y\|^2$ *for all $l \in S_{n-1}$,*

(iii) $\langle x - y, \mathrm{St}_{\alpha_{[l]}}(F(x) \ominus_\Delta F(y)) \rangle \le \mu \|x - y\|^2$ *for all $l \in S_{n-1}$*

are equivalent respectively to G-OSL (i), to D-OSL (ii) and to Δ-OSL (iii).

4.2 One-Sided Lipschitz Metric Selections

In this subsection we turn our attention to M-OSL mappings and prove that the metric selections of a M-OSL map $F : [a, b] \to \mathcal{K}(\mathbb{R})$ are OSL with the same constant.

According to Definition 3.2, a map is *M-OSL* if

$$\delta^*(x - y, F(x) \ominus_M F(y)) \le \mu \|x - y\|^2 \quad (x, y \in [a, b]) \,. \tag{20}$$

Metric selections are useful for studying M-OSL maps. We use the following construction which is a slight modification of the construction from [18] and from [5, Definition 5.1]. Here, the x-coordinate of the given point in the graph is incorporated in the grid points and thus a non-uniform grid is defined, since the constructed selections automatically pass through the given point of the graph.

Definition 4.5 *Let $F : [a, b] \to \mathcal{K}(\mathbb{R})$ and $(\hat{x}, \hat{y}) \in Graph(F)$ be given. We take a uniform partition of $[a, b]$, $a = \tau_0 < \tau_1 < ... < \tau_N = b$ with $\tau_i = a + i(b - a)/N$, $i = 0, ..., N$, and add the given point $\hat{x}$ to the partition. We denote the new partition points by x_i, $i = 0, 1, ..., N + 1$, and $h_i = x_{i+1} - x_i$. Without loss of generality denote $\hat{x} = x_{i_0}, \hat{y} = y_{i_0}$. Starting from (x_{i_0}, y_{i_0}), we determine $y_{i_0+1} \in F(x_{i_0+1})$ by $\|y_{i_0+1} - y_{i_0}\| = \mathrm{dist}(y_{i_0}, F(x_{i_0+1}))$. Then, for every $i \ge i_0 + 1$, and for given x_i, x_{i+1}, y_i we find subsequently $y_{i+1} \in F(x_{i+1})$ satisfying $\|y_{i+1} - y_i\| = \mathrm{dist}(y_i, F(x_{i+1}))$.*

In a similar way we proceed backwards, to the left of x_{i_0}: for $i \le i_0$, we find $y_{i-1} \in F(x_{i-1})$ such that $\|y_i - y_{i-1}\| = \mathrm{dist}(y_i, F(x_{i-1}))$.

The obtained sequence $\{(x_i, y_i)\}_{i=0,...,N+1}$ *is called* metric chain. *The piecewise-linear interpolant* $y^N(x)$ *of such points* (x_i, y_i), $i = 0, ..., N+1$, *of a metric chain is called* metric piecewise-linear interpolant.

To prove that M-OSL mappings have uniformly OSL metric selections, we cannot use the Arzelà-Ascoli theorem, since M-OSL maps as well as their metric selections are not necessarily continuous. Instead, we use the following theorem (see e.g. [23, Chap. 10, Subsec. 36.5]) for a sequence of functions of bounded variation, in particular for monotone functions.

Theorem 4.6 *(Helly's selection principle) A uniformly bounded sequence of functions with uniformly bounded variation, defined on a compact interval, has a subsequence that converges at every point of this interval.*

Obviously, in the case of a sequence of monotone functions, the limit function in the above theorem is monotone. Let us recall that any monotone function has a countable number of discontinuity points [23, Chap. 9, Subsec. 31.1], thus the limit function is continuous almost everywhere except for at most countable number of points in the given 1d domain.

Now, we prove the theorem on the metric selections of M-OSL multivalued mappings.

Theorem 4.7 *Let* $F : [a, b] \to \mathcal{K}(\mathbb{R})$ *be bounded, with closed graph and M-OSL with constant* μ*. Then through any point* $(\hat{x}, \hat{y})$ *of the graph of* F *there is an OSL metric selection of* F *which is OSL with constant* μ*.*

Proof: We construct a metric chain as in Definition 4.5. For any natural N and step-size $h = (b-a)/N$ we take a uniform partition of $[a, b]$, $\tau_i = a + ih$, $i \in \{0, 1, ..., N\}$, and for the given point $(\hat{x}, \hat{y})$, $\hat{y} \in F(\hat{x})$ we add $\hat{x}$ to the partition, construct a metric chain $\{(x_i, y_i),\ y_i \in F(x_i)\}_{i=0,...,N+1}$ and consider the metric piecewise-linear interpolant y^N with knots (x_i, y_i), $i \in \{0, 1, ..., N+1\}$. By (20), we get for $i \in \{0, 1, ..., N\}$

$$(x_{i+1} - x_i)(y_{i+1} - y_i) \leq \mu h_i^2 \ \Leftrightarrow\ y_{i+1} - y_i \leq \mu h_i. \tag{21}$$

To complete the proof, we prove the following claims:
1. The piecewise linear interpolants y^N are OSL with constant μ.
2. The sequence y^N has a pointwise converging subsequence, and the limit function is OSL with constant μ.
3. The limit function is a selection of F, which passes through (x, y).

To prove the first claim, we show that for all $x' < x'' \in [a, b]$,

$$y^N(x'') - y^N(x') \leq \mu(x'' - x'). \tag{22}$$

We first show that (22) is true when x', x'' are both in the same subinterval $[x_i, x_{i+1}]$. By the construction in Definition 4.5, if $x'' - x' = \lambda(x_{i+1} - x_i)$, with $0 \le \lambda \le 1$, then $y^N(x'') - y^N(x') = \lambda(y_{i+1} - y_i)$. Then, by (21) we get

$$\begin{aligned} y^N(x'') - y^N(x') &= \lambda\,(y_{i+1} - y_i) \\ &\le \lambda\mu(x_{i+1} - x_i) = \mu(x'' - x'). \end{aligned}$$

Now, let x', x'' belong to different subintervals of the partition. Without loss of generality suppose $x' \in [x_0, x_1], x'' \in [x_i, x_{i+1}]$, $i \in \{0, \dots, N\}$, with $x_1 - x' = \lambda(x_1 - x_0)$, $\lambda \in [0,1]$, and $x'' - x_i = \nu(x_{i+1} - x_i)$, $\nu \in [0,1]$. Then $x'' - x' = \nu h_i + \sum_{j=2}^{i} h_{j-1} + \lambda h_0$. Using (21) and (22) in any subinterval of the partition, we get

$$\begin{aligned} & y^N(x'') - y^N(x') \\ &= y^N(x'') - y^N(x_i) + \sum_{j=2}^{i} \big(y^N(x_j) - y^N(x_{j-1})\big) + y^N(x_1) - y^N(x') \\ &= \nu\,\big(y^N(x_{i+1}) - y^N(x_i)\big) + \sum_{j=2}^{i} \big(y^N(x_j) - y^N(x_{j-1})\big) \\ &\quad + \lambda\,\big(y^N(x_1) - y^N(x_0)\big) \le \mu(\nu h_i + \sum_{j=2}^{i} h_{j-1} + \lambda h_0) = \mu(x'' - x'), \end{aligned}$$

which completes the proof of the first claim.

To prove the second claim, we observe that the functions $y^N(x) - \mu x$ are monotone decreasing, which follows trivially from (22), and also bounded on $[a,b]$, since F is bounded. Then by the Helly's selection principle, the sequence $y^N(\cdot)$ has a pointwise convergent subsequence on $[a,b]$. Denote the limit function of this subsequence by y^∞. Now, we show that y^∞ is OSL. Indeed, since the functions $y^N(x) - \mu x$ are monotone decreasing, the limit function $y^\infty(x) - \mu x$ is also monotone decreasing, which implies that y^∞ is OSL, and also that it has a countable set of discontinuity points.

To prove the third claim, we first note that the limit function passes through the given point $(\hat{x}, \hat{y})$, since all polygonal functions pass through this point. To show that the limit function is a selection of F, we take a subsequence of binary partitions, i.e. we take $N_k = 2^k$, $k = 1, 2, \dots$. Then a fixed binary point $x = a + \frac{i_0(b-a)}{2^{k_0}}$ appears in all refined partitions for $k > k_0$, and $y^{N_k}(x) \in F(x)$ for all $k > k_0$. Since the graph of F is closed, the limit of this sequence satisfies $y^\infty(x) \in F(x)$ for all binary points x.

Now, since every point $x \in [a,b]$ is a limit of a sequence of binary points, and since the graph of F is closed, it follows that $y^\infty(x) \in F(x)$ at every point $x \in [a,b]$. Hence, y^∞ is a selection of F. ■

4.3 One-Sided Lipschitz Selections of OSL Maps

In this subsection we prove that a OSL map $F : [a,b] \to \mathcal{K}(\mathbb{R})$ with a compact graph has OSL selections with the same constant passing through every point of its graph. We use the following construction in the proof.

Definition 4.8 *Let* $F : [a, b] \to \mathcal{K}(\mathbb{R})$ *and* $(\hat{x}, \hat{y}) \in \mathrm{Graph}(F)$. *We take a uniform partition of* $[a, b]$, $a = \tau_0 < \tau_1 < ... < \tau_N = b$ *with* $\tau_i = a + i(b-a)/N$, $i = 0, ..., N$, *and add the given point* $\hat{x}$ *to the partition. We denote the so-obtained new partition points by* x_i, $i = 0, 1, ..., N+1$, *and* $h_i = x_{i+1} - x_i$. *Let* $\hat{x} = x_{i_0}$, $\hat{y} = y_{i_0}$. *Starting from* (x_{i_0}, y_{i_0}), *we determine* $y_{i_0+1} \in F(x_{i_0+1})$ *by the OSL condition:*

$$(x_{i_0+1} - x_{i_0})(y_{i_0+1} - y_{i_0}) \le \mu |x_{i_0+1} - x_{i_0}|^2 \quad \Longleftrightarrow \quad y_{i_0+1} - y_{i_0} \le \mu h_{i_0}.$$

Then, increasing i *from* i_0+1 *to* N, *for given* x_i, x_{i+1}, y_i *we find subsequently* y_{i+1} *satisfying*

$$(x_{i+1} - x_i)(y_{i+1} - y_i) \le \mu |x_{i+1} - x_i|^2 \quad \Longleftrightarrow \quad y_{i+1} - y_i \le \mu h_i. \tag{23}$$

In a similar way we proceed backwards, to the left of x_{i_0}: *for* $i \le i_0 - 1$, *we find* $y_i \in F(x_i)$ *satisfying* (23).

We then construct a piecewise-linear interpolant $y^N(x)$ *of the points* (x_i, y_i), $i = 0, ..., N+1$.

Theorem 4.9 *Let* $F : [a, b] \to \mathcal{K}(\mathbb{R})$ *be bounded, with closed graph and OSL with constant* μ. *Then through any point* $(\hat{x}, \hat{y})$ *of the graph of* F *there is a selection of* F *which is OSL with constant* μ.

Proof: We construct the piecewise-linear functions passing through $(\hat{x}, \hat{y})$ as in Definition 4.8. Then the steps of the proof are the same as in that of Theorem 4.7. ■

It is easy to show that if there exists an OSL selection through any point of the graph of a multifunction $F(\cdot)$ and if all these selections have the same OSL constant, then $F(\cdot)$ is OSL. Thus we obtain

Corollary 4.10 *Let* $F : [a, b] \to \mathcal{K}(\mathbb{R})$ *be bounded with closed graph.* $F(\cdot)$ *is OSL with constant* μ *iff through any point* $(\hat{x}, \hat{y})$ *of the graph of* $F(\cdot)$ *there is a OSL selection with the same constant* μ.

Remark 4.11 *A direct consequence of Theorem 4.9 is that D-OSL and M-OSL maps have OSL selections through each point of their graphs, but these selections are not necessarily GS-selections or metric ones. In this sense, the stronger requirements of Theorems 4.1 and 4.7 lead to stronger conclusions — they provide existence of special OSL selections, namely generalized Steiner resp. metric selections.*

Remark 4.12 *We conjecture that the last theorem can be extended to maps with images in* $\mathbb{R}^n$ *which are S-OSL (in the sense of Lempio/Veliov [27]).*

5 Examples

The examples presented below demonstrate that the OSL notions in Section 3 are indeed different and that the inclusions in the diagram in Figure 1 are strict. Furthermore, some of them highlight the selection results in Section 4 and study generalized Steiner and metric selections.

5.1 Examples Illustrating Strict Inclusions in Theorem 3.17

The first 1d example illustrates Proposition 3.7. As in [12, Example 2.1], the OSL map in this example is even discontinuous, see Figure 2.

Example 5.1 *Set* $F : \mathbb{R} \to \mathbb{R}$ *as*

$$F(t) = [-\operatorname{sign}(t), -\operatorname{sign}(t) + 1] \quad (t \in \mathbb{R})\,.$$

Then, $F(\cdot)$ *is G-OSL, OSL, M-OSL and D-OSL, but neither UOSL nor Lipschitz.*

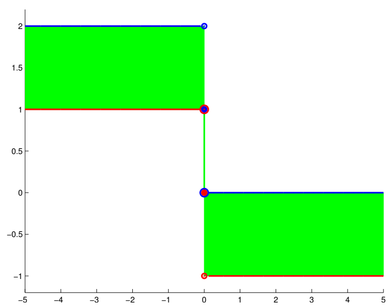

Figure 2: OSL set-valued map in Example 5.1

(i) The D-OSL property follows from Lemma 3.10 ($r(x) = 1$*) and Corollary 3.12 for* $F(t) = [0, 1] + (-\operatorname{sign}(t))$*. Although Theorem 3.17 holds, the G-OSL property with* $\mu = 0$ *follows directly from* $\operatorname{diam}(F(t)) = 1$*,*

$$F(s) \ominus_G F(t) = \{-\operatorname{sign}(s) + \operatorname{sign}(t)\}\,,$$
$$\delta^*(s - t, F(s) \ominus_G F(t)) = (s - t)(-\operatorname{sign}(s) + \operatorname{sign}(t)) \le 0$$

by the OSL condition of the negative signum function.

(ii) $F(\cdot)$ *is M-OSL*
To verify the M-OSL property with $\mu = 0$*, we check only two cases. The other cases for* $s, t \in \mathbb{R}$ *can be treated similarly.*
(a) "$s < 0 < t$*":*

$$F(s) \ominus_M F(t) = [1, 2] \ominus_M [-1, 0] = \bigcup_{a \in F(s)} \{a - 0\} \cup \bigcup_{b \in F(t)} \{1 - b\} = [1, 2]\,,$$

$$\delta^*(s - t, F(s) \ominus_M F(t)) = (s - t) < 0$$

(b) "$0 < t < s$":

$$\begin{aligned} F(s) \ominus_M F(t) &= [-1,0] \ominus_M [-1,0] = \{0\}\,, \\ \delta^*(s-t, F(s) \ominus_M F(t)) &= 0 \end{aligned}$$

(iii) Since the map is not almost everywhere single-valued, it cannot be UOSL by Theorem 3.13. The discontinuity at $t = 0$ prevents the Lipschitz property.

We next state an example of a G-OSL map which is not OSL which shows that G-OSL is a weaker assumption than OSL.

Example 5.2 *Set $X = [0,\infty) \times \mathbb{R}$ and $F : X \to \mathbb{R}^2$ as*

$$F(x) = \mathrm{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ \sqrt{x_1} \end{pmatrix} \right\} \quad (x \in X)\,.$$

Then, $F(\cdot)$ is G-OSL, but not OSL.

(i) We show that $F(\cdot)$ is not OSL by considering

$$h_m = \frac{1}{m}\,,\ x^m = \begin{pmatrix} 2h_m \\ h_m \end{pmatrix},\ y^m = \begin{pmatrix} h_m \\ 0 \end{pmatrix} \quad (m \in \mathbb{N})\,.$$

Then,

$$\delta^*(x^m - y^m, F(x^m)) - \delta^*(x^m - y^m, F(y^m)) = h_m^2 + (\sqrt{2}-1) h_m \sqrt{h_m}\,,$$

which, for large $m \in \mathbb{N}$, is bigger than

$$\mu \| x^m - y^m \|_2^2 = 2\mu h_m^2\,.$$

(ii) $F(\cdot)$ is G-OSL with constant $\mu = 0$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$:

$$\begin{aligned} F(x) \ominus_G F(y) &= \begin{cases} \emptyset & \text{for } x, y \text{ with } x_1 \neq y_1, \\ \{0\} & \text{for } x, y \text{ with } x_1 = y_1, \end{cases} \\ 0 \geq \delta^*(x - y, F(x) \ominus_G F(y)) &= \begin{cases} -\infty & \text{for } x, y \text{ with } x_1 \neq y_1, \\ 0 & \text{for } x, y \text{ with } x_1 = y_1 \end{cases} \end{aligned}$$

The next example is stated in [10, Example 3.1] to demonstrate that continuity with respect to the Hausdorff and the Demyanov distance differs ($F(\cdot)$ is proved to be Hausdorff continous, but not D-continuous). But it can also show that D-OSL is a stronger condition than OSL and that there exist Lipschitzian maps which are not D-OSL.

Example 5.3 ([10, Example 3.1]) *Set* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *as*

$$F(x) = \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(x_1) \\ \sin(x_1) \end{pmatrix} \right\} \quad (x \in \mathbb{R}^2)\,.$$

Then, $F(\cdot)$ *is obviously Lipschitz and hence, OSL. We show that it is also M-OSL, but neither D-Lipschitz nor D-OSL.*

(i) $F(\cdot)$ *is M-OSL, since it is Lipschitz and Theorem 3.16 applies for the metric difference.*

(ii) $F(\cdot)$ *is not D-OSL.*
The support function for $l = (l_1, l_2)^\top \in S_1$ *is*

$$\delta^*(l, F(x)) = \max\{0, l_1 \cos(x_1) + l_2 \sin(x_1)\}\,.$$

With $l := \binom{1}{0}$ *and*

$$h_m = \frac{1}{m}\,,\ x^m = \left(\frac{\pi}{2} - h_m\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad z^m = \left(\frac{\pi}{2} + h_m\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$y^{1,m} := y(l, F(x^m)) = \begin{pmatrix} \cos\left(\frac{\pi}{2} - h_m\right) \\ \sin\left(\frac{\pi}{2} - h_m\right) \end{pmatrix}, \quad y^{2,m} := y(l, F(z^m)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The left and right-hand side of the D-OSL condition is given by

$$\langle x^m - z^m, y^{1,m} - y^{2,m} \rangle = -2h_m \left(\cos\left(\frac{\pi}{2} - \frac{1}{m}\right) - \sin\left(\frac{\pi}{2} - \frac{1}{m}\right) \right), \quad (24)$$

$$\mu \| x^m - z^m \|_2^2 = \left\| -2h_m \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|_2^2 = 8h_m^2\,,$$

we get the inequality

$$-\cos\left(\frac{\pi}{2} - \frac{1}{m}\right) + \sin\left(\frac{\pi}{2} - \frac{1}{m}\right) \le 4h_m$$

which cannot hold for large $m \in \mathbb{N}$*, since the left-hand side tends to* 1.

(iii) $F(\cdot)$ *is not D-Lipschitz by Theorem 3.16.*

We now show a D-OSL map with nonconvex images which is not M-OSL.

Example 5.4
Consider $F : \mathbb{R} \rightrightarrows \mathbb{R}$ *with images in* $\mathcal{K}(\mathbb{R})$ *(see Figure 3) defined as*

$$F(t) = \begin{cases} [-2, -1] \cup [1, 2] & \text{for } t < 0, \\ [-2, 2] & \text{for } t \ge 0. \end{cases}$$

$F(\cdot)$ *is OSL and D-OSL with constant* 0*, but neither M-OSL nor Lipschitz.*

(i) $F(\cdot)$ is OSL, since $\operatorname{co}(F(t)) = [-2, 2]$ *for all* $t \in \mathbb{R}$.

(ii) $F(\cdot)$ is D-OSL by the same reasoning.

(iii) $F(\cdot)$ is not M-OSL, since if we assume this and consider $h_m = \frac{1}{m}$,

$$F(-h_m) \ominus_M F(0) = \bigcup_{b\in[-1,0]} \{-1-b\} \cup \bigcup_{b\in[0,1]} \{1-b\} = [-1,1]\,,$$
$$\delta^*(-h_m - 0, F(-h_m) \ominus_M F(0)) = \delta^*(-h_m, [-1,1]) = h_m$$

which cannot be less or equal than $\mu|-h_m - 0|^2 = \mathcal{O}(h_m^2)$ for large $m \in \mathbb{N}$.

(iv) $F(\cdot)$ is obviously not Lipschitz in $t = 0$.

The construction of the next example is based on Corollary 3.12. The set-valued map is a sum of a Lipschitz map and an OSL function, which is used in [13, Example 5.4]. Here, $F_1(\cdot)$ is Lipschitz, but not D-Lipschitz and $f_2(\cdot)$ is OSL, but not Lipschitz at the origin.

Example 5.5 *Consider $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ as in Example 5.3, $v = \binom{1}{-1}$ and set*

$$F(x) = F_1(x) + \left\{ f_2\left(x - \frac{\pi}{2}v\right)\right\} \quad (x \in \mathbb{R}^2)\,,$$
$$f_2(x) = \begin{pmatrix} -(x_1)^{\frac{1}{3}} + (x_2)^{\frac{2}{3}} \\ -(x_2)^{\frac{1}{3}} + (x_1)^{\frac{2}{3}} \end{pmatrix}.$$

Then, $F(\cdot)$ is OSL, but not Lipschitz or D-OSL.

(i) $F(\cdot)$ is OSL, since it is a sum of a Lipschitz set-valued map and a OSL function with constant $\mu = \frac{1}{2}$ by [13, Example 5.4].

(ii) $F(\cdot)$ is not D-OSL
Assume that $F(\cdot)$ is D-OSL and consider $h_m = \frac{1}{m}$, $x^m = \left(\frac{\pi}{2} - h_m\right)\binom{1}{-1}$, $z^m = \left(\frac{\pi}{2} + h_m\right)\binom{1}{-1}$ and $y^{1,m}, y^{2,m} \in \mathbb{R}^2$ as in Example 5.3. Consider the

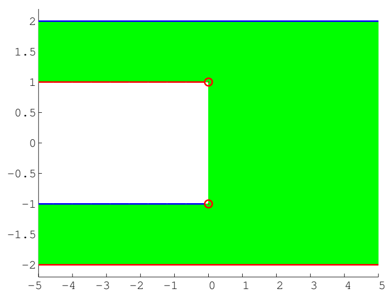

Figure 3: D-OSL set-valued map in Example 5.4

following scalar product:

$$\begin{aligned}
&\Big\langle x^m - z^m,\, f_2\left(x^m - \frac{\pi}{2}v\right) - f_2\left(z^m - \frac{\pi}{2}v\right)\Big\rangle \\
=&\; -2h_m \left\langle v, \begin{pmatrix} -(-h_m)^{\frac{1}{3}} + h_m^{\frac{2}{3}} \\ -h_m^{\frac{1}{3}} + (-h_m)^{\frac{2}{3}} \end{pmatrix} - \begin{pmatrix} -h_m^{\frac{1}{3}} + (-h_m)^{\frac{2}{3}} \\ -(-h_m)^{\frac{1}{3}} + h_m^{\frac{2}{3}} \end{pmatrix} \right\rangle \\
=&\; -2h_m \left(1 \cdot 2h_m^{\frac{1}{3}} + (-1)\cdot(-2h_m^{\frac{1}{3}})\right) = -8h_m^{\frac{4}{3}}
\end{aligned}$$

The right-hand side of the D-OSL condition is

$$\mu\|x^m - z^m\|_\infty^2 = \mu\| - 2h_m v\|_\infty^2 = 4\mu h_m^2 \,,$$

while the left-hand side (see also (24)) can be estimated for big $m \in \mathbb{N}$ as

$$\begin{aligned}
&\langle x^m - z^m, y(l, F(x^m)) - y(l, F(z^m))\rangle \\
=&\; \langle x^m - z^m, y^{1,m} - y^{2,m}\rangle + \Big\langle x^m - z^m, f_2\left(x^m - \frac{\pi}{2}v\right) - f_2\left(z^m - \frac{\pi}{2}v\right)\Big\rangle \\
=&\; 2h_m\left(\sin\left(\frac{\pi}{2} - \frac{1}{m}\right) - \cos\left(\frac{\pi}{2} - \frac{1}{m}\right)\right) - 8h_m^{\frac{4}{3}} \geq 2h_m\left(\frac{9}{10} - 4h_m^{\frac{1}{3}}\right) \geq h_m \,.
\end{aligned}$$

This leads to the contradiction $h_m \leq 4\mu h_m^2$ for large $m \in \mathbb{N}$. Hence, $F(\cdot)$ is not D-OSL.

(iii) $F(\cdot)$ is not Lipschitz
With a similar reasoning as in (ii), one can show after some calculations that $F(\cdot)$ is not Lipschitz in $x = \frac{\pi}{2}v$.

The next example shows that there are UOSL maps that are not S-UOSL.

Example 5.6 *Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and define $F : \mathbb{R}^n \to \mathcal{C}(\mathbb{R}^n)$ as $F(x) = -\partial f(x)$.*
If $f(x) = \|x\|_2$, then $F(\cdot)$ is UOSL but neither S-UOSL nor S-OSL (for $n \geq 2$) or Lipschitz.

(i) $F(\cdot)$ is UOSL.
It is well-known that the subdifferential is maximal monotone by [2, Sec. 3.4, Proposition 1], hence $F(\cdot)$ is UOSL with constant $\mu = 0$.

(ii) $F(\cdot)$ is neither S-UOSL nor S-OSL.
The subdifferential is given by

$$\partial f(x) = \begin{cases} \left\{ \frac{1}{\|x\|_2} x \right\} & \text{if } x \neq 0, \\ B_1(0) & \text{if } x = 0. \end{cases}$$

Let us consider the unit vectors $e^i \in \mathbb{R}^n$, $i = 1, \ldots, n$, and

$$h_m = \frac{1}{m}\,, \ x^m = h_m(3e^1 + 4e^2)\,, \qquad y^m = h_m(e^1 + e^2)\,,$$
$$\xi^m = -\frac{1}{\|x^m\|_2} \cdot x^m \in F(x^m)\,, \ \eta^m = -\frac{1}{\|y^m\|_2} \cdot y^m \in F(y^m)$$

for all $m \in \mathbb{N}$ which will lead to

$$\|x^m\|_2 = \sqrt{9h_m^2 + 16h_m^2} = 5h_m\,, \quad \|y^m\|_2 = \sqrt{h_m^2 + h_m^2} = \sqrt{2}h_m\,.$$

The following left-hand side of the S-UOSL condition

$$(x_1^m - y_1^m)\delta^*(e^1, F(x) \ominus_A F(x^m)) \geq (x_1^m - y_1^m)(\xi_1^m - \eta_1^m)$$
$$= 2h_m \left(-\frac{1}{5h_m} 3h_m + \frac{1}{\sqrt{2}h_m} h_m \right) = 2h_m \left(\frac{\sqrt{2}}{2} - \frac{3}{5} \right) > 0$$

and its right-hand side

$$\mu\|x^m - y^m\|_\infty^2 = \mu(3h_m)^2 = 9\mu h_m^2$$

yield a contraction for large $m \in \mathbb{N}$, since the estimate $0 < 2\left(\frac{\sqrt{2}}{2} - \frac{3}{5}\right) \leq 9\mu h_m$ cannot hold true for large $m \in \mathbb{N}$. Since $F(\cdot)$ is single-valued for $x \neq 0$, $F(\cdot)$ cannot be S-OSL as well.

The next example is a modification of the previous one and shows that single-valued functions may be OSL, but not S-OSL.

Example 5.7 *Let $F(x)$ be defined as in Example 5.6 and consider the function $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ and $F(x) = \{f(x)\}$ for $x \neq 0$.*
Then, $f(\cdot)$ is OSL, but not S-OSL.

The next example states a typical S-UOSL example which is not Lipschitz continuous (even discontinuous).

Example 5.8 *Let us consider the function $f(x) = \sum_{i=1}^n |x_i|$ in Example 5.6. The function $F(x) = -\partial f(x)$ is S-UOSL and discontinuous.*

(i) The set-valued map $F(\cdot)$ is S-UOSL.
For $n = 1$, $F(x)$ equals the set-valued signum function

$$\mathrm{Sign}(x) = \begin{cases} \{1\} & \text{for } x < 0, \\ [-1, 1] & \text{for } x = 0, \\ \{-1\} & \text{for } x > 0, \end{cases}$$

for higher dimensions $F(x)$ coincides with

$$F(x) = \sum_{i=1}^{n} \text{Sign}(x_i) e^i \,.$$

Since $\text{Sign}(x_i)$ *is the negative subdifferential of* $f_i(x_i) = |x_i|$*, [25, Lemma 3.6] applies showing the S-UOSL property with constant* $\mu = 0$.

(ii) $F(\cdot)$ *is obviously discontinuous, e.g. in* $x = 0$.

5.2 Examples of One-Sided Lipschitz Selections

We continue the discussion of Example 5.3 (see also the comments in [5, Example 6.3]), to show that the generalized Steiner selections for atomic measures are not Lipschitz, although the set-valued map is Lipschitz. Here, we show that these selections are not OSL either, although the map is even Lipschitz (and hence, OSL).

Example 5.9 *For* $F(\cdot)$ *defined in Example 5.3,* $\text{St}_{\alpha_{[l]}}(F(\cdot))$ *is not OSL for some direction* $l \in S_1$ *in* $x = \frac{\pi}{2}\binom{1}{-1}$*, although* $F(\cdot)$ *is Lipschitz.*

Indeed, we calculate

$$\begin{aligned} \text{St}_{\alpha_{[l]}}(F(x)) &= \text{St}(Y(l, F(x))) \\ &= \begin{cases} \binom{\cos(x_1)}{\sin(x_1)}\,, & \text{if } l_1\cos(x_1) + l_2\sin(x_1) > 0, \\ \text{St}(F(x)) = \frac{1}{2}\binom{\cos(x_1)}{\sin(x_1)}\,, & \text{if } l_1\cos(x_1) + l_2\sin(x_1) = 0, \\ \binom{0}{0}\,, & \text{else.} \end{cases} \end{aligned}$$

Let us consider $l = \binom{1}{0}$ *and the sequences* $(x^m)_m$*,* $(z^m)_m$ *as in Example 5.3. Since*

$$\text{St}_{\alpha_{[l]}}(F(z)) = \text{St}(Y(l, F(z))) = y(l, F(z))$$

for $z \in \{x^m, z^m \,:\, m \in \mathbb{N}\}$*, we can assume that* $\text{St}_{\alpha_{[l]}}(F(\cdot))$ *is OSL and arrive to the same contradiction as in Example 5.3.*
As the generalized Steiner selection is discontinuous, $F(\cdot)$ *cannot be D-OSL by Theorem 4.1.*

The next example studies the 1d case for which uniform OSL selections exist for OSL maps. The reason for this is the equivalence of OSL and D-OSL maps for the 1d case (see Proposition 3.7). It also demonstrates that not all GS and metric selections need to coincide.

Example 5.10 *For* $F(\cdot)$ *defined in Example 5.1, we prove that the GS-selections are uniformly OSL.*

(i) The GS selections are uniformly OSL with constant 0.
Since $Y(l, F(t))$ *is just a point for* $l = \pm 1$*, the generalized Steiner point* $\mathrm{St}_{\alpha_{[l]}}(F(t))$ *coincides with* $y(l, F(t))$*. Clearly, we have*

$$\begin{aligned} \mathrm{St}_{\alpha_{[1]}}(F(t)) &= y(1, F(t)) = -\operatorname{sign}(t) + 1 \quad && (t \in \mathbb{R}), \\ \mathrm{St}_{\alpha_{[-1]}}(F(t)) &= y(-1, F(t)) = -\operatorname{sign}(t) \quad && (t \in \mathbb{R}), \end{aligned}$$

i.e. the two GS-selections are discontinuous (see Figure 4), but (uniformly) OSL with constant 0.

(ii) The metric selections are uniformly OSL with constant 0.

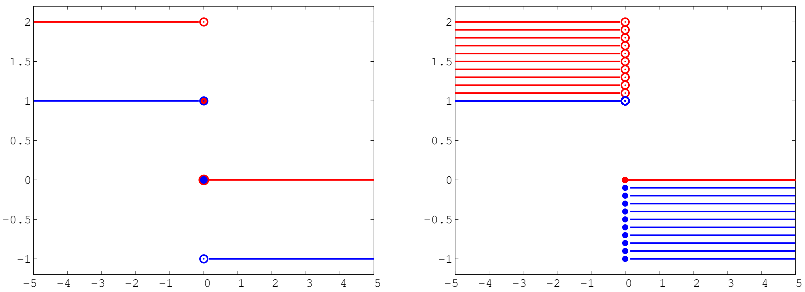

Figure 4: generalized Steiner (left) and metric selections (right) for Example 5.10

The metric selections passing through $(-1, \alpha)$ *resp.* $(1, \beta)$ *from the graph of* $F(\cdot)$ *(see Figure 4) can be parametrized as*

$$\eta_\alpha(t) = \begin{cases} \alpha & \text{for } t < 0, \\ 0 & \text{for } t \geq 0, \end{cases}$$

$$\eta^\beta(t) = \begin{cases} 1 & \text{for } t < 0, \\ \beta & \text{for } t \geq 0 \end{cases}$$

for $\alpha \in F(-1) = [1, 2]$ *and* $\beta \in F(1) = [-1, 0]$*. They are all OSL with constant 0. It turns out that the metric selection* $\eta^{-1}(\cdot)$ *and* $\eta_2(\cdot)$ *coincide with one of the two generalized Steiner selections in (i).*

Conclusions

In this paper we investigated the question raised in [6], whether there exist continuous and uniformly OSL selections for a set-valued OSL map and give a partial answer to it. In the subclass of the D-OSL mappings with convex images, the generalized Steiner selections are OSL with the same OSL constant as the set-valued map. If in addition the map is D-continuous, these selections are continuous with the same modulus of continuity as the given mapping. A sufficient condition for a map with compact, not necessarily convex images in $\mathbb{R}$, to have uniformly continuous and uniformly OSL selections passing through every point of its graph, is that this map is continuous (in the Hausdorff metric) and M-OSL. A direct consequence is that a M-monotone map (that is a map F with $-F$ being M-OSL with OSL constant zero) has monotone selections, and if in addition it is continuous, then these selections are also continuous. Similarly, a D-monotone map, that is a map F such that $-F$ is D-OSL with a constant zero, has monotone generalized Steiner selections, and if F is also D-continuous, then these selections are also continuous. If we are interested only in existence of uniformly OSL selections, then it is sufficient that the univariate map with images in $\mathbb{R}$ is OSL.

Generalized Steiner selections for the convex case give an interesting way to derive selection results for set-valued maps. They are closely related to the Demyanov difference of sets. As we saw in [5], uniformly Lipschitz GS selections provide a characterization of the class of D-Lipschitz set-valued maps and uniformly OSL generalized Steiner selections provide a characterization of the class of D-OSL mappings.

Results for the rather weak notions G-Lipschitz and G-OSL set-valued maps remain a future task. The collection of examples presented here illustrates the established hierarchies and hopefully provide more insight in the various regularity classes for set-valued maps.

To make a brief comparison of the OSL-type conditions, we remind that the S-UOSL condition is stronger than the UOSL condition by [25, 3.3 Lemma], but it is remarkable, since it implies $\mathcal{O}(h)$ convergence of the Euler method for differential inclusions with a unique solution, although it does not imply continuity [25, Theorem 2.4]. The S-OSL condition [27] is not so well-known. It is essentially weaker than S-UOSL, but still guarantees the convergence order $\mathcal{O}(h)$ for the Euler method applied to autonomous systems (see [27, Theorem 2.4 and Theorem 3.3]). It does not require continuity of the right-hand side of the differential inclusion. On the other hand, the OSL condition for autonomous right-hand sides ensures at least convergence order $\mathcal{O}(\sqrt{h})$ for the Euler's method (see [13, 27] resp. [11] for the non-autonomous case). The question that naturally arises is what order of convergence is achieved by D-OSL maps. Can we do better than $\mathcal{O}(\sqrt{h})$ then? Can the GS-selections help to find the sharp order?

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Robert Baier
Chair of Applied Mathematics
University of Bayreuth
95440 Bayreuth, Germany
e-mail: robert.baier@uni-bayreuth.de

Elza Farkhi
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv 69978, Israel
e-mail: elza@post.tau.ac.il