Computation of Local ISS Lyapunov Functions Via Linear Programming*

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***Abstract*— In this paper, we present a numerical algorithm for computing a local ISS Lyapunov function for systems which are locally input-to-state stable (ISS) on compact subsets of the state space. The algorithm relies on a linear programming problem and computes a continuous, piecewise affine ISS Lyapunov function on a simplicial grid covering the given compact set excluding a small neighborhood of the origin. We show that the ISS Lyapunov function delivered by the algorithm is a viscosity subsolution of a partial differential equation.**

***Index Terms*— Nonlinear systems, Local input-to-state stability, Local ISS Lyapunov function, Linear programming, Viscosity subsolution**

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I. Introduction

The concept of input-to-state stability (ISS) was first introduced by Sontag [23] in the later 1980s. Basic results about ISS can be found in [23], [24], [25]. In [27], different equivalent formulations of ISS are given. In particular, it is shown that ISS is equivalent to the existence of an ISS Lyapunov function. The ISS notion is very useful in stability analysis of large scale systems. If all subsystems are ISS, then the stability of large scale systems can be analyzed by ISS small gain theorems [6], [7], [8], [9], [10], [17]. Motivated by these results, in this paper we study how to compute local ISS Lyapunov functions for low dimensional systems, as the knowledge of ISS Lyapunov functions leads to the knowledge of ISS gains which may be used in a small gain based stability analysis.

Based on [26, Lemma 2.10-2.14], it is shown in [4] that ISS Lyapunov functions in implication form may be calculated for the individual subsystems using a Zubov approach. An alternative Zubov type approach was developed in [20]. In these two papers, ISS Lyapunov functions can be obtained by computing robust Lyapunov functions for suitably designed auxiliary systems. This robust Lyapunov function can be characterized by the Zubov equation, a Hamilton-Jacobi-Bellman partial differential equation, which can be solved numerically [3]. This approach, however, only yields a numerical approximation of an ISS Lyapunov function but not a true ISS Lyapunov function. For discrete time systems, following the same auxiliary system approach, true Lyapunov functions can be computed by a set oriented approach [13]. This numerical approach, however, does not carry over to the continuous time setting. Moreover, the detour via the auxiliary system introduces conservatism, since the resulting Lyapunov function and ISS gains strongly depend on the way the auxiliary system is constructed.

In this paper, we propose a linear programming based algorithm for computing true ISS Lyapunov functions. The linear programming based algorithm for computing continuous, piecewise affine Lyapunov functions was first presented in [21]. In [14] it was proved that for exponentially stable equilibria the approach proposed in [21] always works. This method was subsequently extended to asymptotically stable systems [15], to asymptotically stable, arbitrary switched, non-autonomous systems [16] and to asymptotically stable differential inclusions [1]. The approaches proposed in these papers yield true Lyapunov functions except possible on a small neighborhood of the origin. Mainly inspired by [1], in this paper we will propose an analogous linear programming based algorithm for computing true ISS Lyapunov functions for local ISS systems. In this paper we focus on the formulation of the linear program, the proof that a solution of this program is an ISS Lyapunov function and the characterization of this solution in terms of viscosity solution theory. The existence of a solution to the proposed linear program will be addressed in the forthcoming paper [19].

The paper is organized as follows. In the ensuing Section II, we present the notation and preliminaries. In Section III we present our algorithm along with a couple of auxiliary results needed in order to formulate the constraints in the resulting linear program. Section IV contains the main results of the paper: we prove that upon successful termination the algorithm yields an ISS Lyapunov function outside a small neighborhood of the equilibrium, and the ISS Lyapunov function computed by the algorithm is a viscosity subsolution of a partial differential equation. In Section V, we illustrate our algorithm by a numerical example.

II. Notations and preliminaries

Let $\mathbb{R}_+ := [0, +\infty)$. For a vector $x \in \mathbb{R}^n$ we denote its transpose by $x^\top$. The standard inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$. We use the standard *norms* $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$ and $\|x\|_\infty := \max_{i\in\{1,\dots,n\}} |x_i|$ and let $B_p(z, r) := \{x \in \mathbb{R}^n \mid \|x-z\|_p < r\}$ denote the open

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ball of radius r around z in the norm $\|\cdot\|_p$. The *induced matrix norm* is defined by $\|A\|_p := \max_{\|x\|_p=1} \|Ax\|_p$. By $\|u\|_{\infty,p} = \operatorname{ess\,sup}_{t\geq 0} \|u(t)\|_p$ we denote the *essential supremum norm* of a measurable function $u : \mathbb{R}_+ \to \mathbb{R}^m$.

The *convex hull* of vectors $x_0, x_1, \ldots, x_m \in \mathbb{R}^n$ is given by

$$\operatorname{co}\{x_0, \ldots, x_m\} := \left\{ \sum_{i=0}^{m} \lambda_i x_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^{m} \lambda_i = 1 \right\}.$$

A set of vectors $x_0, x_1, \ldots, x_m \in \mathbb{R}^n$ is called *affine independent* if $\sum_{i=1}^{m} \lambda_i (x_i - x_0) = 0$ implies $\lambda_i = 0$ for all $i = 1, \ldots, m$. This definition is independent of the numbering of the x_i, that is, of the choice of the reference point x_0.

In this paper we consider nonlinear perturbed systems described by the ordinary differential equation

$$\dot{x} = f(x, u) \tag{1}$$

with vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $f(0,0) = 0$, state $x \in \mathbb{R}^n$ and perturbation input $u \in \mathbb{R}^m$. The *admissible input values* are given by $U_R := \operatorname{cl} B_1(0, R) \subset \mathbb{R}^m$ for a constant $R > 0$ and the *admissible input functions* by $u \in \mathcal{U}_R := \{u : \mathbb{R} \to \mathbb{R}^m \text{ measurable} \,|\, \|u\|_{\infty,1} \leq R\}$.

We assume

(H1) The vector field f is twice continuously differentiable with respect to x and u.

The following definition specifies the stability property we are considering in this paper and uses $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions (see e.g. [18], [28]).

Definition 1: System (1) is called *locally input-to-state stable (ISS)*, if there exist $\rho^x > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for all $\|x(0)\|_2 \leq \rho^x$ and $\|u\|_\infty \leq \rho^u$

$$\|x(t, x(0), u)\|_2 \leq \beta(\|x(0)\|_2, t) + \gamma(\|u\|_\infty) \;\forall\, t \in \mathbb{R}_+. \tag{2}$$

If $\rho^x = \rho^u = \infty$, then system (1) is called *input-to-state stable (ISS)*.

Observe that ISS implies that the origin is an equilibrium of (1) which is locally asymptotically stable for $u \equiv 0$.

It is known that the ISS property of (1) is equivalent to the existence of an ISS Lyapunov function for (1), see [26]. While this Lyapunov function is even guaranteed to be smooth, in what follows we will work with nonsmooth Lyapunov functions. In order to define these functions, we need a generalized notion of the gradient and for our purpose Clarke's subdifferential turns out to be useful. Since we are exclusively dealing with Lipschitz functions, we can use the following definition, cf. [5, Theorem 2.5.1].

Definition 2: For a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, *Clarke's subdifferential* is given by

$$\partial_{Cl} V(x) := \operatorname{co}\big\{ \lim_{i\to\infty} \nabla V(x_i) \,|\, \lim_{i\to\infty} x_i = x, \; \nabla V(x_i) \text{ and } \lim_{i\to\infty} \nabla V(x_i) \text{ exist} \big\}. \tag{3}$$

The elements $\xi \in \partial_{Cl} V(x)$ are called *generalized gradients*. Now we can state the definition of a nonsmooth ISS Lyapunov function.

Definition 3: Let $\mathcal{G} \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} \mathcal{G}$. A Lipschitz continuous function $V : \mathcal{G} \to \mathbb{R}_+$ is said to be a *(local) nonsmooth ISS Lyapunov function* for system (1) on $\mathcal{G}$ if there exist $\mathcal{K}_\infty$ functions ψ_1, ψ_2, α and β such that

$$\psi_1(\|x\|_2) \leq V(x) \leq \psi_2(\|x\|_2), \tag{4}$$

$$\langle \xi, f(x,u) \rangle \leq -\alpha(\|x\|_2) + \beta(\|u\|_1) \tag{5}$$

hold for all $x \in \mathcal{G}$, $u \in U_R$ and $\xi \in \partial_{Cl} V(x)$. If $\mathcal{G} = \mathbb{R}^n$ and $R = \infty$, then V is called a *global nonsmooth ISS Lyapunov function*.

In order to state the relationship between the ISS Lyapunov function delivered by the algorithm which we will propose and a viscosity subsolution of a partial differential equation, we now recall the definition of viscosity solutions. For more details of this theory we refer to [2, Sec. II.1 and III.2]. Here, C^1-test functions are used to avoid the gradient of the solution at points of non-differentiability in the domain.

Definition 4 ([2, Chap. II, Def. 1.1]):
Given an open subset Ω of $\mathbb{R}^n$ and a continuous function $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, we consider the partial differential equation

$$H(x, W, DW) = 0 \qquad \forall\, x \in \Omega \tag{6}$$

for a continuous function $W : \mathbb{R}^n \to \mathbb{R}$. We say that a continuous function $W : \mathbb{R}^n \to \mathbb{R}$ is a *viscosity subsolution* (resp. *supersolution*) of the equation if for all test functions $\phi \in C^1(\Omega)$ and $x \in \arg\max_\Omega (W - \phi)$ (resp. $x \in \arg\min_\Omega (W - \phi)$) we have

$$H(x, W(x), D\phi(x)) \leq 0 \tag{7}$$

$$\text{resp. } H(x, W(x), D\phi(x)) \geq 0. \tag{8}$$

A continuous function $W : \Omega \to \mathbb{R}$ is said to be a *viscosity solution* of (6) if W is a viscosity supersolution and a viscosity subsolution of (6).

Remark 1: A Lipschitz continuous viscosity solution satisfies almost everywhere the partial differential equation (6) due to [2, Chap. II, Proposition 1.9].

Based on [2, Chap. II, Lemma 1.7], the set of derivatives $D\phi(x)$ for $x \in \arg\min_\Omega (W - \phi)$ coincides with the set

$$D^- W(x) := \{p \in \mathbb{R}^n : \liminf_{y\to x, y\in\Omega} \frac{W(y) - W(x) - \langle p, y - x\rangle}{\|x - y\|_2} \geq 0\} \tag{9}$$

and the set of derivatives $D\phi(x)$ for $x \in \arg\max_\Omega (W - \phi)$ equals the following set:

$$D^+ W(x) := \{p \in \mathbb{R}^n : \limsup_{y\to x, y\in\Omega} \frac{W(y) - W(x) - \langle p, y - x\rangle}{\|x - y\|_2} \leq 0\} \tag{10}$$

Therefore, one can equivalently define viscosity solution by the sets $D^- W(x)$ and $D^+ W(x)$ which are called *sub-* and *superdifferentials*, respectively, i.e.

$$H(x, W(x), p) \leq 0 \qquad \forall\, p \in D^+ W(x), \tag{11}$$

$$\text{resp. } H(x, W(x), p) \geq 0 \qquad \forall\, p \in D^- W(x). \tag{12}$$

Remark 2: For any locally Lipschitz continuous function $W : \Omega \to \mathbb{R}$ the sub- and superdifferentials satisfy (cf. [2, Chap. II, (4.6)])

$$D^- W(x) \cup D^+ W(x) \subseteq \partial_{Cl} W(x) \qquad \forall\, x \in \Omega. \quad (13)$$

III. The algorithm

In this section we are going to introduce the linear programming based algorithm to compute an ISS Lyapunov function on a compact set $\mathcal{G} \subset \mathbb{R}^n$ with $0 \in \operatorname{int} \mathcal{G}$ and perturbation inputs from the set $U_R \subset \mathbb{R}^m$. The algorithm uses linear programming and the representation of the function on a simplicial grid in order to obtain a numerical representation as a continuous, piecewise affine function. By taking into account interpolation errors, the algorithm yields a true ISS Lyapunov function, not only an approximative one.

A. Definitions

We recall the following basic definitions: A simplex in $\mathbb{R}^n$ is a set of the form $\Sigma = \operatorname{co}\{x_0, x_1, \dots, x_j\}$, where $x_0, x_1, \dots, x_j$ are affine independent. The faces of Σ are given by $\operatorname{co}\{x_{i_0}, \dots, x_{i_k}\}$, where $\{x_{i_0}, \dots, x_{i_k}\}$ ranges over the subsets of $\{x_0, x_1, \dots, x_j\}$. An n-simplex is generated by a set of $n+1$ affine independent vertices. A collection $\mathcal{S}$ of simplices in $\mathbb{R}^n$ is called a simplicial complex, if

(i) for every $\Sigma \in \mathcal{S}$, all faces of Σ are in $\mathcal{S}$,
(ii) for all $\Sigma_1, \Sigma_2 \in \mathcal{S}$ the intersection $\Sigma_1 \cap \Sigma_2$ is a face of both Σ_1 and Σ_2 (or empty).

Some authors consider the empty simplex to be a face of every Σ, so that the last statement in (ii) is superfluous, but this will have no relevance in the present paper. The *diameter* of a simplex Σ is defined as $\operatorname{diam}(\Sigma) := \max_{x,y \in \Sigma} \|x - y\|_2$.

We now return to our problem. We assume that $\mathcal{G} \subset \mathbb{R}^n$ may be partitioned into finitely many n-simplices $\mathcal{T} = \{\Gamma_\nu \mid \nu = 1, \dots, N\}$, so that $\mathcal{T}$ defines a simplicial complex. Here we require that $0 \in \mathcal{T}_\nu$ if only if 0 is a vertex of $\mathcal{T}_\nu$. By assumption, we may also partition U_R into m-simplices $\mathcal{T}_u = \{\Gamma^u_\kappa \mid \kappa = 1, \dots, N_u\}$ defining a simplicial complex. We briefly write $h_{x,\nu} = \operatorname{diam}(\Gamma_\nu)$, $h_{u,\kappa} = \operatorname{diam}(\Gamma^u_\kappa)$ and $h_x = \max_{\nu=1,\dots,N} h_{x,\nu}$, $h_u = \max_{\kappa=1,\dots,N_u} h_{u,\kappa}$. For each $x \in \mathcal{G}$ we define the *active index set* $I_{\mathcal{T}}(x) := \{\nu \in \{1, \dots, N\} \mid x \in \Gamma_\nu\}$. For the simplices $\mathcal{T}_u$, we additionally assume that

$$\text{for each simplex } \Gamma^u_\kappa \in \mathcal{T}_u, \text{ the vertices of } \Gamma^u_\kappa \text{ are in the same closed orthant.} \quad (14)$$

Let $PL(\mathcal{T})$ denote the space of continuous functions $V : \mathcal{G} \to \mathbb{R}$ which are affine on each simplex, i.e., there are $a_\nu \in \mathbb{R}$, $w_\nu \in \mathbb{R}^n$, $\nu = 1, \dots, N$, such that

$$V|_{\Gamma_\nu}(x) = \langle w_\nu, x \rangle + a_\nu \qquad \forall\, x \in \Gamma_\nu,\ \Gamma_\nu \in \mathcal{T} \quad (15)$$

$$\nabla V_\nu := \nabla V|_{\operatorname{int} \Gamma_\nu} \equiv const = w_\nu \quad \forall\, \Gamma_\nu \in \mathcal{T}. \quad (16)$$

With $\nabla V_{\nu,k}$ we denote the k-th component of ∇V_ν.

Likewise, we define $PL^u(\mathcal{T}^u)$. Observe that our assumption on the vertices of the Γ^u_κ implies that the map $u \mapsto \|u\|_1$ is contained in $PL^u(\mathcal{T}^u)$.

Remark 3: In order to ensure that a given function $V \in PL(\mathcal{T})$ is an ISS Lyapunov function, in the algorithm we will in particular have to ensure the inequality

$$\langle \xi, f(x,u) \rangle \le -\|x\|_2 + r\|u\|_1 \quad \forall\, \xi \in \partial_{Cl} V(x). \quad (17)$$

To this end, observe that from the definition it follows that for any function $V \in PL(\mathcal{T})$ Clarke's subdifferential is given by

$$\partial_{Cl} V(x) = \operatorname{co}\{\nabla V_\nu \mid \nu \in I_{\mathcal{T}}(x)\}. \quad (18)$$

Hence, for fixed x and u inequality (17) becomes

$$\langle \xi, f(x,u) \rangle \le -\|x\|_2 + r\|u\|_1 \quad \forall\, \xi \in \operatorname{co}\{\nabla V_\nu \mid \nu \in I_{\mathcal{T}}(x)\}.$$

Linearity of the scalar product in its first argument implies the equivalence to

$$\langle \nabla V_\nu, f(x,u) \rangle \le -\|x\|_2 + r\|u\|_1 \quad \forall\, \nu \in I_{\mathcal{T}}(x). \quad (19)$$

An inequality of this type will be used for ensuring (17) in the algorithm.

B. Interpolation errors

As in [1], [16], the key idea for the numerical computation of a true Lyapunov function lies in incorporating estimates for the interpolation errors on $\mathcal{T}$ — and in this paper also on $\mathcal{T}^u$ — into the constraints of a linear program. In this section we analyze the error terms we need for this purpose.

Let $x \in \Gamma_\nu = \operatorname{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$, $x = \sum_{i=0}^n \lambda_i x_i$, $0 \le \lambda_i \le 1$, $\sum_{i=0}^n \lambda_i = 1$ and $u \in \Gamma^u_\kappa = \operatorname{co}\{u_0, u_1, \dots, u_m\} \in \mathcal{T}_u$, $u = \sum_{j=0}^m \mu_j u_j$, $0 \le \mu_j \le 1$, $\sum_{j=0}^m \mu_j = 1$.

The basic idea of the algorithm is to impose conditions on V in the vertices x_i of the simplices $\Gamma_\nu \in \mathcal{T}$ which ensure that the function V satisfies the inequalities (4) and (19) on the whole set $\mathcal{G}$.

In order to ensure (4), we impose the condition

$$V(x_i) \ge \|x_i\|_2 \quad (20)$$

for every vertex $x_i \in \Gamma_\nu$ and $V(0) = 0$. Note that this implies (4) for all $x \in \mathcal{G} \setminus B_2(0, \epsilon)$.

In order to make sure that $V(x)$ satisfies (19) for all $x \in \Gamma_\nu \in \mathcal{G}$, $u \in \Gamma^u_\kappa \in \mathcal{G}_u$ via imposing inequalities in the node values $V(x_i)$, we need to incorporate an estimate of the interpolation error into the inequalities. To this end, we demand that

$$\langle \nabla V_\nu, f(x_i, u_j) \rangle - r\|u_j\|_1 + \|\nabla V_\nu\|_1 A_{\nu,\kappa} \le -\|x_i\|_2, \quad (21)$$

for all $i = 0, 1, 2, \dots, n$, $j = 0, 1, \dots, m$. Here $A_{\nu,\kappa} \ge 0$ is a bound for the interpolation error of f in the points (x, u) with $x \in \Gamma_\nu \in \mathcal{G}$, $u \in \Gamma^u_\kappa \in \mathcal{G}_u$, $x \ne x_i$, $u \ne u_j$.

Remark 4: Close to the origin the positive term $\|\nabla V_\nu\|_1 A_{\nu,\kappa}$ may become predominant on the left hand side of (21), thus rendering (21) infeasible. This is the reason for excluding a small ball $B_2(0, \epsilon)$ in the construction of V. Under certain conditions on f this problem can be circumvented by choosing suitably shaped simplices near the origin. In order to keep the presentation in this paper concise we do not go into details here and refer to [11], instead.

Since (21) will be incorporated as an inequality constraint in the linear optimization problem to be formulated, below, we need to derive an estimate for $A_{\nu,\kappa}$ before we can formulate the algorithm. For this purpose we introduce the following Proposition 1. Here, for a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ which is twice differentiable w.r.t. to its first argument, we denote the Hessian of $g(x,u)$ w.r.t. x at z by

$$H_g(z,u) = \begin{bmatrix} \frac{\partial^2 g(x,u)}{\partial x_1^2}\Big|_{x=z} & \cdots & \frac{\partial^2 g(x,u)}{\partial x_1 \partial x_n}\Big|_{x=z} \\ & \cdots & \\ \frac{\partial^2 g(x,u)}{\partial x_n \partial x_1}\Big|_{x=z} & \cdots & \frac{\partial^2 g(x,u)}{\partial x_n^2}\Big|_{x=z} \end{bmatrix}.$$

For the first argument $x \in \Gamma_\nu$, let

$$H_x(u) := \max_{z \in \Gamma_\nu} \|H_g(z,u)\|_2, \tag{22}$$

and let $\overline{K}_x$ denote a positive constant satisfying

$$\max_{\substack{z \in \Gamma_\nu \\ r,s=1,2,\ldots,n}} \left| \frac{\partial^2 g(z,u)}{\partial x_r \partial x_s} \right| \le \overline{K}_x \quad (u \in U_R). \tag{23}$$

In the next proposition which is proved in a similar way to [1, Proposition 4.1, Lemma 4.2 and Corollary 4.3], we state properties of scalar functions $g : \mathcal{G} \times U_R \to \mathbb{R}$ or vector functions $g : \mathcal{G} \times U_R \to \mathbb{R}^n$ with respect to their first argument. Analogous properties hold with respect to the second argument.

Proposition 1: Consider a convex combination $x = \sum_{i=0}^n \lambda_i x_i \in \Gamma_\nu$, $\Gamma_\nu = \mathrm{co}\{x_0, x_1, \ldots, x_n\}$, $\sum_{i=0}^n \lambda_i = 1$, $1 \ge \lambda_i \ge 0$, $u \in U_R$ and a function $g : \mathcal{G} \times U_R \to \mathbb{R}^p$ with components $g(x,u) = (g_1(x,u), g_2(x,u), \ldots, g_p(x,u))$. If $g_j(x,u)$ is twice continuously differentiable with respect to x with the bound $H_x(u)$ from (22) on its second derivative for some $j = 1, \ldots, p$, then

$$\left| g_j\left(\sum_{i=0}^n \lambda_i x_i, u\right) - \sum_{i=0}^n \lambda_i g_j(x_i,u) \right| \le$$
$$\frac{1}{2}\sum_{i=0}^n \lambda_i H_x(u) \|x_i - x_0\|_2$$
$$\left(\max_{z \in \Gamma_\nu} \|z - x_0\|_2 + \|x_i - x_0\|_2\right) \le H_x(u) h_{x,\nu}^2.$$

Under the same differentiability assumption for all $j = 1, \ldots, p$, the estimate

$$\left\| g\left(\sum_{i=0}^n \lambda_i x_i, u\right) - \sum_{i=0}^n \lambda_i g(x_i,u) \right\|_\infty \le n\overline{K}_x h_{x,\nu}^2 \tag{24}$$

holds for all $u \in U_R$ by assuming the bounds from (23).

C. The algorithm

Now we have collected all the preliminaries to formulate the linear programming algorithm for computing an ISS Lyapunov function V for system (1). In this algorithm, we introduce the values $V(x_i)$ as optimization variables. Since it is desirable to obtain an ISS Lyapunov function in which the influence of the perturbation is as small as possible, the objective of the linear program will be to minimize r in (17).

As explained in Remark 4, we only consider x satisfying $x \in \mathcal{G} \setminus B_2(0,\epsilon)$ for a small $\epsilon > 0$. To this end we define the subsets

$$\mathcal{T}^\epsilon := \{\Gamma_\nu \,|\, \Gamma_\nu \cap B_2(0,\epsilon) = \emptyset\} \subset \mathcal{T}. \tag{25}$$

$$\mathcal{G}^\epsilon := \bigcup_{\Gamma_\nu \in \mathcal{T}^\epsilon} \Gamma_\nu \tag{26}$$

In the following algorithm, we will only impose the conditions (20) in those nodes $x_i \in \mathcal{G}$ and (21) in those nodes x_i which belong to simplices $\Gamma \in \mathcal{T}^\epsilon$. Moreover, the interpolation errors $A_{\nu,\kappa}$ will be replaced by the values derived in the previous section.

Algorithm

We solve the following linear optimization problem.

Inputs:
$$\begin{cases} \epsilon > 0, \\ x_i,\ \|x_i\|_2 \text{ for all vertices } x_i \text{ of each simplex } \Gamma_\nu \in \mathcal{T}, \\ u_j,\ \|u_j\|_1 \text{ for all vertices } u_j \text{ of each simplex } \Gamma_\kappa^u \in \mathcal{T}_u, \\ h_{x,\nu} \text{ for each simplex } \Gamma_\nu \in \mathcal{T}^\epsilon, \\ h_{u,\kappa} \text{ for each simplex } \Gamma_\kappa^u \in \mathcal{T}_u\ , \\ \overline{K}_x, \overline{K}_u \text{ from (24) with respect to } x,\ u, \text{ for } g(x,u) = f(x,u) \text{ from (1).} \end{cases} \tag{27}$$

Optimization variables:
$$\begin{cases} V_{x_i} \text{ for all vertices } x_i \text{ of all simplices } \Gamma_\nu \in \mathcal{T}, \\ C_{\nu,k} \text{ for } k = 1,2,\ldots,n \text{ and every } \Gamma_\nu \in \mathcal{T}^\epsilon, \\ r \in \mathbb{R}_+. \end{cases} \tag{28}$$

Optimization problem: (29)

minimize r

subject to

$$(A1):\ V_{x_i} \ge \|x_i\|_2 \text{ for all vertices } x_i \text{ of each simplex } \Gamma_\nu \in \mathcal{T},$$
$$(A2):\ |\nabla V_{\nu,k}| \le C_{\nu,k} \text{ for each simplex } \Gamma_\nu \in \mathcal{T}^\epsilon,\ k = 1,2,\ldots,n,$$
$$(A3):\ \langle \nabla V_\nu, f(x_i,u_j)\rangle - r\|u_j\|_1 + (n\overline{K}_x h_{x,\nu}^2 + m\overline{K}_u h_{u,\kappa}^2) \sum_{k=1}^n C_{\nu,k} \le -\|x_i\|_2,$$
for all vertices x_i of each simplex $\Gamma_\nu \in \mathcal{T}^\epsilon$, all vertices u_j of each simplex $\Gamma_\kappa^u \in \mathcal{T}_u$.

Remark 5: (i) The condition $(A1)$ makes sure $V(x) \ge \|x\|_2$ for $x \in \mathcal{G}$.

(ii) The condition $(A2)$ defines linear constraints on the optimization variables V_{x_i}, $C_{\nu,k}$.

Remark 6: If the above linear optimization problem has a feasible solution, then the values $V(x_i) = V_{x_i}$ from this feasible solution at all vertices x_i of all simplices $\Gamma_\nu \in \mathcal{T}$ and

the condition $V(x) \in PL(\mathcal{T})$ uniquely define a continuous, piecewise affine function

$$V : \mathcal{G} \to \mathbb{R}. \tag{30}$$

IV. Main result

In this section we formulate and prove our two main results. We show that any feasible solution $V(x)$ of our algorithm defines an ISS Lyapunov function on $\mathcal{G}^\epsilon$ and prove that this ISS Lyapunov function is a viscosity subsolution of a Hamilton-Jacobi-Bellman partial differential equation. We start with the former.

Theorem 1: If the assumption (H1) holds, and the linear optimization problem (29) has a feasible solution, then the function V from (30) is an ISS Lyapunov function on $\mathcal{G}^\epsilon$, i.e., it satisfies (4) and (5) for all $x \in \mathcal{G}^\epsilon$.

Proof: Consider convex combinations $x = \sum_{i=0}^{n} \lambda_i x_i \in \Gamma_\nu$, $\Gamma_\nu = \operatorname{co}\{x_0, x_1, \ldots, x_n\} \in \mathcal{T}^\epsilon$, $\sum_{i=0}^{n} \lambda_i = 1$, $0 \le \lambda_i \le 1$, and $u = \sum_{j=0}^{m} \mu_j u_j \in \Gamma_\kappa^u$, $\Gamma_\kappa^u = \operatorname{co}\{u_0, u_1, \ldots, u_m\} \in \mathcal{T}_u$, $\sum_{j=0}^{m} \mu_j = 1$, $0 \le \mu_j \le 1$.

As $V(x)$ is affine on Γ_ν and the constraint from $(A1)$ and the convexity of the norm $\|\cdot\|_2$ hold, we obtain

$$\begin{aligned} V(x) &= V\left(\sum_{i=0}^{n} \lambda_i x_i\right) = \sum_{i=0}^{n} \lambda_i V(x_i) \ge \sum_{i=0}^{n} \lambda_i \|x_i\|_2 \\ &\ge \left\|\sum_{i=0}^{n} \lambda_i x_i\right\|_2 = \|x\|_2 > 0 \end{aligned} \tag{31}$$

for all $x \in \mathcal{G}^\epsilon$. Thus $V(x)$ is positive on $\mathcal{G}^\epsilon$ and the existence of ψ_1 and ψ_2 satisfying (4) follows.

In order to prove inequality (17) we compute

$$\begin{aligned} &\langle \nabla V_\nu, f(x,u)\rangle = \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle \\ &+ \langle \nabla V_\nu, f(\sum_{i=0}^{n} \lambda_i x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle \\ &- \sum_{i=0}^{n} \lambda_i \langle \nabla V_\nu, f(x_i, \sum_{j=0}^{m} \mu_j u_j)\rangle \\ \le &\sum_{i=0}^{n} \lambda_i \sum_{j=0}^{m} \mu_j \langle \nabla V_\nu, f(x_i, u_j)\rangle \\ &+ \|\nabla V_\nu\|_1 \left\| f(\sum_{i=0}^{n} \lambda_i x_i, u) - \sum_{i=0}^{n} \lambda_i f(x_i, u)\right\|_\infty \\ &+ \sum_{i=0}^{n} \lambda_i \|\nabla V_\nu\|_1 \left\| f(x_i, u) - \sum_{j=0}^{m} \mu_j f(x_i, u_j)\right\|_\infty . \end{aligned}$$

According to Proposition 1, the constraint $(A3)$ ensures that V satisfies

$$\langle \nabla V_\nu, f(x,u)\rangle \le -\|x\|_2 + r\|u\|_1 \tag{32}$$

for all $x \in \mathcal{G}^\epsilon$ and all $u \in U_R$. ∎

Remark 7: We point out that an ISS-Lyapunov function defined on $\mathcal{G}^\epsilon$ will only imply ISS if the maximum of V on $\mathcal{G} \setminus \mathcal{G}^\epsilon$ is smaller than the minimum of V on $\partial\mathcal{G}$. This condition is checked a posteriori for the solution provided by our algorithm. We remark that this condition is always satisfied if the system is ISS on $\mathcal{G}$.

Now we turn to the second objective of this section, i.e., to prove that the ISS Lyapunov function delivered by the algorithm is a viscosity subsolution of a partial differential equation.

Theorem 2: We assume that (H1) holds and the linear optimization problem (29) has a feasible solution, then the function V from (30) is a viscosity subsolution of the partial differential equation (33) on $\mathcal{G}^\epsilon$

$$H(x, V(x), DV(x)) = 0 \tag{33}$$

with the Hamiltonian

$$H(x, V, p) = \sup_{u \in U_R} \{\langle p, f(x,u)\rangle + \|x\|_2 - r\|u\|_1\} \tag{34}$$

defined for $x, p \in \mathbb{R}^n$, $V \in \mathbb{R}$.

Proof: It follows from the proof of Theorem 1

$$\langle \nabla V_\nu, f(x,u)\rangle \le -\|x\|_2 + r\|u\|_1 \tag{35}$$

for all $x \in \mathcal{G}^\epsilon$ and all $u \in U_R$.

According to Remark 3, we obtain

$$\langle \xi, f(x,u)\rangle \le -\|x\|_2 + r\|u\|_1 \qquad \forall\, \xi \in \partial_{Cl} V(x) \tag{36}$$

for all $x \in \mathcal{G}^\epsilon$ and all $u \in U_R$.

Thus the following inequality (37) holds for $V(\cdot)$ and all $x \in \mathcal{G}^\epsilon$ based on Remark 2:

$$\sup_{u \in U_R} \{\langle p, f(x,u)\rangle + \|x\|_2 - r\|u\|_1\} \le 0 \quad \forall\, p \in D^+ V(x) \tag{37}$$

Therefore the function V from (30) is a viscosity subsolution of the partial differential equation (33) on $\mathcal{G}^\epsilon$. ∎

The partial differential equation (33) can be transformed to a Hamilton-Jacobi-Bellman equation (38) which is studied e.g. in [12, Sec. 3.5].

Corollary 1: Under the assumptions of Theorem 2, the function V from (30) is a viscosity subsolution of the partial differential equation (38) on $\mathcal{G}^\epsilon$:

$$\sup_{u \in U_R:\ 2r\|u\|_1 \le \|x\|_2} \{\langle DV(x), f(x,u)\rangle + \frac{1}{2}\|x\|_2\} = 0 \tag{38}$$

Proof: The proof is similar to Theorem 2, but we use

$$\langle \xi, f(x,u)\rangle \le -\frac{1}{2}\|x\|_2 \qquad \forall\, \xi \in \partial_{Cl} V(x) \tag{39}$$

for all $x \in \mathcal{G}^\epsilon$ satisfying $\|x\|_2 \ge 2r\|u\|_1$, $u \in U_R$. ∎

Remark 8: Based on Theorem 2 and Corollary 1, it is concluded that a viscosity subsolution of a certain Hamilton-Jacobi-Bellman equation such as (33), (38) can be obtained by solving the linear optimization problem (29).

V. Example

In this section we illustrate the algorithm by a numerical example.

We consider the following system which is adapted from [22]

$$\begin{aligned} \dot{x}_1 &= -x_1[0.5 - (x_1^2 + x_2^2)] + 0.1u_1^2, \\ \dot{x}_2 &= -x_2[0.5 - (x_1^2 + x_2^2)] - 0.1u_2^2, \end{aligned} \tag{40}$$

where $x \in \mathbb{R}^2$ with $\|x\|_2 \leq 0.3$ and $u \in \mathbb{R}^2$, $\|u\|_2 \leq 0.3$. The algorithm calculates the ISS Lyapunov function (see Figure 1) on the domain excluding a ball with radius $\varepsilon = 0.012$ around the origin. The minimization of r delivers the optimal value $r = 0.213014$.

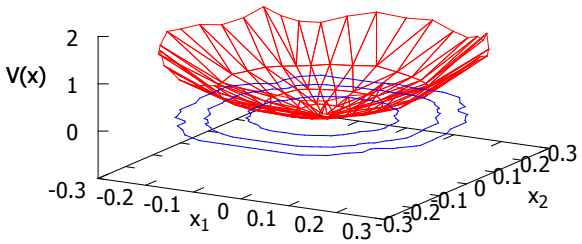

Fig. 1. Numerical ISS Lyapunov function $V(x)$ delivered by the algorithm for system (40) and its level sets, $\epsilon = 0.012$, $r = 0.213014$.

VI. Conclusion and Outlook

In this paper, a numerical algorithm for computing true ISS Lyapunov functions has been presented for systems which are locally ISS on compact subsets of the state space. The corresponding ISS gain is also computed by this algorithm. The results delivered by the algorithm make it is easier to analyze stability of interconnected systems by ISS small gain theorems. Furthermore, it has been proved that this ISS Lyapunov function computed by the algorithm is a viscosity subsolution of a Hamilton-Jacobi-Bellman partial differential equation.

In our future paper [19] we will prove that if the system is locally ISS, then the inequality (17) can be satisfied on a given compact subset of the state space excluding a ball around the origin. We will also state conditions under which the algorithm proposed in this paper has a feasible solution. Moreover, in future research we intend to study whether the proposed algorithm can be used to compute viscosity subsolutions also for other types of Hamilton-Jacobi-Bellman equations, e.g., for equations characterizing optimal control problems [2, Chap. III and IV].

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