A homotopy argument and its applications to the transformation rule for bi-Lipschitz mappings, the Brouwer fixed point theorem and the Brouwer degree

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Introduction

The transformation rule for multiple integrals is on the one hand a constitutive part of the lectures on analysis. On the other hand all proofs we find in literature are more or less cumbersome often technically difficult, depending on the level of preliminary knowledge of the students. It is therefore no surprise that we find until quite recently proposals for new, more simpler proofs. Here I'll like to mention especially the papers by P. D. Lax ([La1], [La2]) and the references given there.

In spring 1973 at the occasion of my "Habilitation" (in Germany: formal admission as an academic teacher) in my (officially) first public lecture ("Antrittsvorlesung") I presented a proof based on a simple homotopy argument whose naive idea can be described easily.

For simplicity let $G \subset \mathbb{R}^n$ be a domain (i. e. G is open and connected). Then the (classical) transformation rule reads as follows:

$$\text{(A)} \qquad \begin{cases} \text{Suppose that} \\ u \in C^1(G;\mathbb{R}^n),\ G^* := u(G) \\ \text{and that } u : G \to G^* \text{ is bijective,} \\ \det_n u'(x) \neq 0 \quad \forall x \in G. \end{cases}$$

Then for $f \in C^0(G^*) \cap L^1(G^*)$,
$(f \circ u) \cdot |\det_n u'| \in L^1(G)$ and

$$\text{(0.1)} \qquad \int_{G^*} f(y)dy = \int_G f(u(x))|\det_n u'(x)|dx$$

The basic observation for the homotopy argument is the following:

Let $I := [0,1]$ and suppose that

$$u \in C^0(\bar{G} \times I; \mathbb{R}^n) \cap C^1(G \times I; \mathbb{R}^n) \tag{0.2}$$

is a homotopic family such that

1^0 $u(x) := u(x,0)\ \forall x \in \bar{G}$,
$u(x,1) = x\ \forall x \in \bar{G}$

2^0 $\det_n u'(x,t) \neq 0\ \forall x \in G \times I$

3^0 let $G_t^* := \{u(x,t) | x \in G\}$ for $t \in [0,1]$ and assume that $u(.,t) : G \to G_t^*$ is bijective

4^0 there is an open set $U \subset \mathbb{R}^n$ such that $u(\partial G \times I) \subset U$.

Because of (0.2), 2^0 and the connectedness of G, the sign of $\det_n u'(x,t)$ is constant in $G \times I$ and because of 1_2^0:

$$\operatorname{sgn} \det_n u'(x,t) = 1. \tag{0.3}$$

Suppose now that $f \in C_c^0(\mathbb{R}^n)$ satisfies $f\big|_U = 0$. Then, for each $t \in [0,1]$ we may apply (0.1) to $u(.,t)$ and in (0.1) we may replace G^* by $\mathbb{R}^n$. From (0.1) we finally get (observe (0.3))

$$h(t) := \int_G f(u(x,t)) \det_n u'(x,t) dt = \int_{\mathbb{R}^n} f(y) dy, \tag{0.4}$$

is constant for $t \in [0,1]$.

Last observation was the starting point of my consideration: I was seeking for conditions on a family of mappings $u(.,t)$ such that the function $h : I \to \mathbb{R}$ defined by the first identity in (0.4) is *constant* on I. The most simple idea is to make assumptions on u and f allowing to differentiate the expression defining h "under the integral sign" and to prove that $h'(t) = 0$ for $t \in I$. Following this idea (Lemma 2.2) one is quasi automatically lead to the use of the differential equation (1.9) for the cofactors of a differentiable map. This identity was decisively used by E. Heinz [He] in his elementary analytic theory of the degree of mapping (compare e. g. [Dei], Proposition 2.2). By mollifying procedures the regularity assumption of our Lemma 2.2 can be decisively weakened (Theorem 2.5) so that the transformation rule can be finally proved for injective, locally bi-Lipschitz mappings (Theorem 5.3).

The idea of proof of (0.1) for $f \in C_c^0(\mathbb{R}^n)$ and linear maps $u(x) := Ax$, where $A \in M(n)$ denotes a $n \times n$-matrix with $\det_n A \neq 0$ is very simple. By elementary transformations such a matrix can be deformed in a diagonal matrix D such that $\det_n A = \det_n D$. This procedure may be regarded as a series of homotopies (proof of Theorem 3.2). Finally for the diagonal matrix D the change of variables formula (1.11) is then an easy consequence of iterated integration and the one-dimensional formula. This procedure is similar to that of T. M. Apostol [Ap]. The case of a differentiable map $u : G \to \mathbb{R}^n$ with $\det_n u'(x) \neq 0$ $(x \in G)$ is then reduced locally by regarding in a sufficiently small ball a homotopy between the map u and its affine-linear approximation (Lemma 4.3). The general case (section 5) is based on partition of unity arguments and some further approximations of $f \in L^1(u(G))$.

It was H. Leinfelder [Le/Si] who observed that our homotopy Lemma allows as well an elementary proof of Brouwer's fixed point theorem (Theorem 6.1) as a proof of the homotopy invariance of the Brouwer degree (Lemma 6.4 and Theorem 6.9). The definition of the degree is based on a simple calculation (Theorem 6.2, compare [Le/Si], p. 354).

The purpose of the underlying paper is to give a detailed elementary proof of the transformation rule for injective, locally bi-Lipschitz mappings. The necessary preliminaries I listed in the first section, where in addition some properties of locally (bi-)Lipschitz mappings are studied (Theorems 1.4, 1.6, Lemma 1.7). Our considerations are based on a famous theorem by H. Rademacher [Ra] guaranteeing that a Lipschitz mapping $u : G \to \mathbb{R}$, $(G \subset \mathbb{R}^n$ open) is totally differentiable almost everywhere. We note this (Theorem 1.5) without giving a proof. In Lemma 4.4 we present a simple proof for the fact observed by H. Rademacher [Ra], Satz IV, p. 354) that in a domain $G \subset \mathbb{R}^n$ a bi-Lipschitz mapping $u : G \to \mathbb{R}^n$ has the additional property that $\det_n u'(x)$ exists a. e., is different from zero and has constant sign in G. Further we prove as a special case of L. E. J. Brouwer's open mapping theorem that a locally bi-Lipschitz mapping is open (Theorem 4.5). For locally bi-Lipschitz mappings there is a complete analogue to the classical theorem concerning local diffeomorphisms (Theorem 4.6). In addition within the framework of Lipschitz mappings we give a version of the implicit function theorem (Theorem 4.8).

I have included all details of proofs one should present in lectures for students of the second year. It is obvious how the proofs can be simplified if only continuously differentiable maps $u : G \to \mathbb{R}^n$ with $\det_n u'(x) \neq 0$ are regarded.

1 Preliminaries

By $M(n)$ we denote the linear space of real $n \times n$ - matrices $A = (a_{ik})$. If $n \geq 2$ for $A \in M(n)$ we write $(i, j \in \{1, \dots, n\})$

$$
A_{ij} := \begin{array}{cc}
\begin{pmatrix}
a_{11} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \dots & a_{ij} & \dots & a_{in} \\
\vdots & & \vdots & & \vdots \\
a_{n1} & \dots & a_{nj} & \dots & a_{nn}
\end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow i \\ \\ \\ \end{array} \\
\begin{array}{ccccc} \phantom{a_{11}} & & \uparrow & & \phantom{a_{1n}} \\ & & j & & \end{array} &
\end{array}
\tag{1.1}
$$

for the matrix $A_{ij} \in M(n-1)$, where we deleted the i-th row and the j-th column of A. The matrix $\tilde{A}$ *complementary to* A is defined by

$$
\tilde{A} := (b_{ij}), \quad b_{ij} := (-1)^{i+j} \det\nolimits_{n-1} A_{ji} \text{ for } i, j = 1, \dots, n \tag{1.2}
$$

Further there are the expansions by column and by row

$$
\begin{cases}
\sum\limits_{j=1}^{n} (-1)^{i+j} a_{ij} \det\nolimits_{n-1} A_{kj} = \delta_{ik} \det\nolimits_n A & \text{for } i, k \in \{1, \dots, n\} \\
\sum\limits_{i=1}^{n} (-1)^{i+j} a_{ij} \det\nolimits_{n-1} A_{ik} = \delta_{jk} \det\nolimits_n A & \text{for } j, k \in \{1, \dots, n\}.
\end{cases} \tag{1.3}
$$

If we regard $\det_n A$ as a function of the n^2 variables a_{ik} $(i, k \in \{1, \dots, n\})$ then we get from (1.3_1)

$$
\frac{\partial \det_n A}{\partial a_{ik}} = (-1)^{i+k} \det\nolimits_{n-1} A_{ik}. \tag{1.4}
$$

Let $G \subset \mathbb{R}^m$ be open and let be given differentiable functions

$$
a_{ik} : G \to \mathbb{R}, \qquad i, k = 1, \dots, n.
$$

We consider

$$
A(x) := (a_{ik}(x)), \qquad A : G \to M(n).
$$

Then $\det_n A : G \to \mathbb{R}$ is differentiable and by the chain rule and (1.4)

$$
\frac{\partial}{\partial x_j} \det\nolimits_n A(x) = \sum_{i,k=1}^{n} (-1)^{i+k} \det\nolimits_{n-1} A_{ik}(x) \frac{\partial a_{ik}(x)}{\partial x_j} \tag{1.5}
$$

We need decisively the forthcoming identity (1.9). For the proof we follow Ch. B. Morrey jr. [Mo1], Lemma 1.1, p. 9. As a preparation we prove

Lemma 1.1 *Let $G \subset \mathbb{R}^n (n \geq 2)$ be open and let $g_r \in C^2(G)$ for $r = 1, \ldots, n-1$. For $j = 1, \ldots, n$ we set for $x \in G$*

$$B_j(x) := \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_j} & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_{n-1}}{\partial x_1}(x) & \cdots & \frac{\partial g_{n-1}}{\partial x_j} & \cdots & \frac{\partial g_{n-1}(x)}{\partial x_n} \end{pmatrix} \in M(n-1) \tag{1.6}$$

Then

$$\sum_{j=1}^{n} (-1)^j \frac{\partial}{\partial x_j} \det\nolimits_{n-1} B_j(x) = 0 \text{ for } x \in G. \tag{1.7}$$

Proof. We perform induction on $n \geq 2$. Let $n = 2$. Then

$$\det\nolimits_1 B_1 = \frac{\partial g_1}{\partial x_2}, \qquad \det\nolimits_1 B_2 = \frac{\partial g_1}{\partial x_1}$$
$$(-1)^1 \frac{\partial}{\partial x_1} \det\nolimits_1 B_1 + (-1)^2 \frac{\partial}{\partial x_2} \det\nolimits_1 B_2 =$$
$$= -\frac{\partial^2 g_1}{\partial x_1 \partial x_2} + \frac{\partial^2 g_1}{\partial x_2 \partial x_1} = 0$$

Assume that (1.7) is true for some $n \geq 2$. Let $G \subset \mathbb{R}^{n+1}$ be open and let $g_r \in C^2(G)$ for $r = 1, \ldots, n$. For $j, k = 1, \ldots, n+1$, $j \neq k$, and for $i = 1, \ldots, n$ let

$$A_{jk}^{(i)} := \begin{array}{cc} i \rightarrow & \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_j} & \cdots & \frac{\partial g_1}{\partial x_k} & \cdots & \frac{\partial g_1}{\partial x_{n+1}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial g_i}{\partial x_1} & \cdots & \frac{\partial g_i}{\partial x_j} & \cdots & \frac{\partial g_i}{\partial x_k} & \cdots & \frac{\partial g_i}{\partial x_{n+1}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_j} & \cdots & \frac{\partial g_n}{\partial x_k} & \cdots & \frac{\partial g_n}{\partial x_{n+1}} \end{pmatrix} \in M(n-1) \\ & \begin{matrix} \qquad\qquad\quad \uparrow \qquad\qquad \uparrow \\ \qquad\qquad\quad j \qquad\qquad\; k \end{matrix} \end{array}$$

where we deleted the j-th and k-th column and the i-th row.

For $j = 1, \ldots, n+1$ let B_j be defined by (1.6) with n replaced by $n+1$. For $j = 1, \ldots, n$ we expand $det_n B_j$ with respect to the $(n+1)$-th column:

$$\det_n B_j = \sum_{i=1}^{n} (-1)^{i+n} \frac{\partial g_i}{\partial x_{n+1}} \det_{n-1} A^{(i)}_{j(n+1)}$$

Then

$$\begin{aligned}
&\sum_{j=1}^{n+1} (-1)^j \frac{\partial}{\partial x_j} \det_n B_j = \\
&= \sum_{j=1}^{n} (-1)^j \frac{\partial}{\partial x_j} \left[\sum_{i=1}^{n} (-1)^{i+n} \frac{\partial g_i}{\partial x_{n+1}} \det_{n-1} A^{(i)}_{j(n+1)} \right] \\
&\quad + (-1)^{n+1} \frac{\partial}{\partial x_{n+1}} \det_n B_{n+1} \\
&= \sum_{j=1}^{n} (-1)^j \sum_{i=1}^{n} (-1)^{i+n} \frac{\partial^2 g_i}{\partial x_j \partial x_{n+1}} \det_{n-1} A^{(i)}_{j(n+1)} \\
&\quad + \sum_{j=1}^{n} (-1)^j \sum_{i=1}^{n} (-1)^{i+n} \frac{\partial g_i}{\partial x_{n+1}} \frac{\partial}{\partial x_j} \det_{n-1} A^{(i)}_{j(n+1)} \\
&\quad + (-1)^{n+1} \frac{\partial}{\partial x_{n+1}} \det_n B_{n+1} =: I_1 + I_2 + I_3
\end{aligned}$$

Observe that

$$I_2 = \sum_{i=1}^{n} (-1)^{i+n} \frac{\partial g_i}{\partial x_{n+1}} \left[\sum_{j=1}^{n} (-1)^j \frac{\partial}{\partial x_j} \det_{n-1} A^{(i)}_{j(n+1)} \right] \tag{1.8}$$

For $x_{n+1} \in \mathbb{R}$ fixed let

$$G_{x_{n+1}} := \{x' \in \mathbb{R}^n : (x', x_{n+1}) \in G\}.$$

Then $G_{x_{n+1}}$ is open and in case that it is not empty we regard

$$g_r^*(x') := g_r(x', x_{n+1}) \quad \text{for } x' \in G_{x_{n+1}}, \quad r = 1, \ldots, n, \quad r \neq i.$$

Then $g_r^* \in C^2(G_{x_{n+1}})$ and by induction hypothesis in (1.8) the expression in brackets vanishes and so does I_2. By (1.5)

$$\frac{\partial}{\partial x_{n+1}} \det_n B_{n+1} = \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} \frac{\partial^2 g_i}{\partial x_j \partial x_{n+1}} \det_{n-1} A^{(i)}_{j(n+1)}$$

Therefore

$$\begin{aligned}
I_1 + I_3 &= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j+n} \frac{\partial^2 g_i}{\partial x_j \partial x_{n+1}} \det_{n-1} A^{(i)}_{j(n+1)} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j+n+1} \frac{\partial^2 g_i}{\partial x_j \partial x_{n+1}} \det_{n-1} A^{(i)}_{j(n+1)} = 0
\end{aligned}$$

■

Theorem 1.2 *Let $G \subset \mathbb{R}^n (n \geq 2)$ be open and let $f_r \in C^2(G)(r = 1, \dots, n)$. For $x \in G$ let*

$$A(x) = (a_{ij}(x)) \in M(n),$$
$$a_{ij}(x) := \frac{\partial f_i(x)}{\partial x_j}, \quad i, j = 1, \dots, n$$

With $A_{ij}(x)$ defined by (1.1),

$$\sum_{j=1}^{n} (-1)^j \frac{\partial}{\partial x_j} \det_{n-1} A_{ij}(x) = 0 \qquad \forall x \in G, \qquad i = 1, \dots, n. \tag{1.9}$$

Proof. Let $i \in \{1, \dots, n\}$. For $r \in \{1, \dots, n\}, r \neq i$, consider $g_r := f_r$. Then (1.9) follows from Lemma 1.1. ■

If $f \in C_c^0(\mathbb{R}^n)(n \geq 2)$, the function

$$\varphi(x') := \int_{-\infty}^{+\infty} f(x', x_n) dx', \quad x' \in \mathbb{R}^{n-1}$$

satisfies $\varphi \in C_c^0(\mathbb{R}^{n-1})$. With an elementary induction argument with respect to $n \in \mathbb{N}$ it is readily seen that the "elementary form of the Fubini-theorem" holds true:

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} f(x_1, \dots, x_{n-1}, x_n) dx_n \right) dx_{n-1} \dots \right] dx_1 \tag{1.10}$$

If $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in M(n)$ is diagonal with $\det_n D \neq 0$ and if $b \in \mathbb{R}$, then using (1.10) and n-fold application of the one-dimensional change of variables formula one readily derives

$$\int_{\mathbb{R}^n} f(y) dy = |\det_n D| \int_{\mathbb{R}^n} f(Dx + b) dx. \tag{1.11}$$

Similarly for $f \in C_c^1(\mathbb{R}^n)$ using the fundamental theorem of calculus, by (1.10) for $i = 1, \dots, n$

$$\int_{\mathbb{R}^n} \partial_i f(x) dx = 0 \tag{1.12}$$

(where we use the notation $\partial_i := \frac{\partial}{\partial x_i}$).

For approximating functions by smooth functions we use mollifiers. Let

$$
(1.13) \qquad \begin{cases} \omega \in C_c^\infty(\mathbb{R}^n),\ \omega \geq 0,\ \omega(x) = 0 \text{ for } |x| \geq 1 \\ \int\limits_{\mathbb{R}^n} \omega(x)dx = 1. \end{cases}
$$

Then for $\varrho > 0$ put

$$
\omega_\varrho(x) := \varrho^{-n}\omega\left(\frac{x}{\varrho}\right).
$$

By (1.11)

$$
(1.14) \qquad \begin{cases} \int\limits_{\mathbb{R}^n} \omega_\varrho(x)dx = 1 \quad \forall \varrho > 0, \\ \operatorname{supp}\omega_\varrho \subset B_\varrho. \end{cases}
$$

Let $G \subset \mathbb{R}^n$ be open and let

$$
\begin{aligned}
\mathcal{L}^1_{\text{loc}}(G) := \Big\{ & f : G \to \mathbb{R} \mid f \text{ measurable and} \\
& \int\limits_K |f|dx < \infty \text{ for every compact } K \subset G \Big\}.
\end{aligned}
$$

For $f, g \in \mathcal{L}^1_{\text{loc}}(G)$ an equivalence is defined by

$$
(1.15) \qquad f \sim g \underset{Def.}{\Leftrightarrow} f = g \quad \text{a. e. in } G
$$

We write for $f \in \mathcal{L}^1_{\text{loc}}(G)$

$$
\begin{aligned}
[f] &= \{g \in \mathcal{L}^1_{\text{loc}}(G) \mid g \sim f\} \\
L^1_{\text{loc}}(G) &:= \big\{[f] \mid f \in \mathcal{L}^1_{\text{loc}}(G)\big\}.
\end{aligned}
$$

Similarly

$$
\mathcal{L}^1(G) := \left\{ f \in \mathcal{L}^1_{\text{loc}}(G) \mid \int\limits_G |f|dx < \infty \right\}
$$

and

$$
L^1(G) := \big\{[f] \mid f \in \mathcal{L}^1(G)\big\}
$$

If $f \in \mathcal{L}^1_{\text{loc}}(G)$ and $x \in G$, $0 < \varrho < \operatorname{dist}(x, \partial G)$ the ball

$$B_\varrho(x) := \{y \in \mathbb{R}^n \mid \|y-x\| < \varrho\}$$

satisfies $\overline{B_\varrho(x)} \subset G$. Then

$$f_\varrho(x) := \int_G \omega_\varrho(x-y)f(y)dy \tag{1.16}$$

is well defined. It is well known that for any $G' \subset\subset G$, $0 < \varrho < \operatorname{dist}(G', \partial G)$

$$\begin{cases} f_\varrho \in C^\infty(G') \cap \mathcal{L}^1(G') \\ \int_{G'} |f(x) - f_\varrho(x)|dx \to 0 \text{ as } \varrho \to 0 \end{cases} \tag{1.17}$$

(see e.g. [Fr], part 1, Theorem 6.2, [Gi/Tr], section 7.2).

If $f \in L^1(G)$ then f_ϱ is well defined by (1.15) for all $x \in \mathbb{R}^n$ and all $\varrho > 0$. Moreover (1.17) holds true with G' replaced by G. If $f \in C_c^0(\mathbb{R}^n)$, then $f_\varrho \in C_c^\infty(\mathbb{R}^n)$ and

$$f_\varrho \to f \quad (\varrho \to 0) \text{ uniformly in } \mathbb{R}^n \tag{1.18}$$

Moreover if f is continuous in G, $G' \subset\subset G$ and $0 < \varrho < \operatorname{dist}(G', \partial G)$. Then $f_\varrho(x)$ is well defined and $f_\varrho \mid_{G'} \to f \mid_{G'}$ $(\varrho \to 0)$ even uniformly.

We say that a function $f \in \mathcal{L}^1_{\text{loc}}(G)$ has a generalized D_i-derivative in $\mathcal{L}^1_{\text{loc}}(G)$ if there is $g_i \in \mathcal{L}^1_{\text{loc}}(G)$ such that $(D_i = \frac{\partial}{\partial xi} = \partial_i)$.

$$\int_G f D_i \phi dx = -\int_G g_i \phi dx \qquad \forall \phi \in C_c^\infty(G). \tag{1.19}$$

If with $\tilde{g}_i \in \mathcal{L}^1_{\text{loc}}(G)$ the identity (1.19) holds true too, then $\tilde{g}_i = g_i$ a. e. in G. Therefore $\tilde{g}_i \in [g_i]$, and we write

$$D_i f := [g_i]$$

and call any representative of $[g_i]$ a weak D_i-derivative of f. As usual, in a sloppy way we don't distinguish typographically between classes $[f]$ and their representatives f. Let $f \in L^1_{\text{loc}}(G)$ have a weak derivative $D_i f \in L^1_{\text{loc}}(G)$. Let $G' \subset\subset G$ and let $0 < \varrho < \operatorname{dist}(G', \partial G)$. If $x \in G'$, then $\overline{B_\varrho(x)} \subset G$ and therefore $\omega_\varrho(x - .) \in C_c^\infty(G)$. By (1.19)

$$\begin{aligned} \int_G \frac{\partial}{\partial x_i}\omega_\varrho(x-y)f(y)dy &= -\int_G \frac{\partial}{\partial y_i}\omega_\varrho(x-y)f(y)dy \\ &= \int_G \omega_\varrho(x-y)D_i f(y)dy \end{aligned}$$

By (1.16) this implies

$$D_i\left(f_\varrho(x)\right) = (D_i f)_\varrho(x) \qquad \forall x \in G', \forall 0 < \varrho < \operatorname{dist}(G', \partial G) \tag{1.20}$$

Because of (1.17_2) (applied to $D_i f$ instead of f)

$$\int\limits_{G'} |D_i f - D_i f_\varrho| dx \to 0 \text{ as } \varrho \to 0,\ 0 < \varrho < \operatorname{dist}(G', \partial G). \tag{1.21}$$

Let us emphasize that even in case f, $D_i f \in L^1(G)$ the identity (1.20) is true in general only for $x \in G' \subset\subset G$ and $0 < \varrho < \operatorname{dist}(G', \partial G)$. Similarly (1.21) applies only to $G' \subset\subset G$.

In addition, if $f \in C^1(G)$, then f, $D_i f \in L^1_{loc}(G)$ and by (1.12) the identity (1.19) is valid with $g_i = D_i f$. That means that the classical derivative of f is a weak derivative of f too.

In the sequel, we consider Lipschitz maps.

Definition 1.3 *Let $G \subset \mathbb{R}^n$ be an open set.*

1. *A mapping $u : G \to \mathbb{R}^m (m, n \in \mathbb{N})$ is called* Lipschitz continuous *(or briefly: Lipschitz)* in G *if there is a constant $L \geq 0$ such that*

$$\|u(x) - u(x')\| \leq L\|x - x'\| \qquad \forall x, x' \in G$$

2. *A mapping $u : G \to \mathbb{R}$ is called* locally Lipschitz continuous *(locally Lipschitz)* in G *if for every $x \in G$ there exists $\varepsilon_x = \varepsilon(x, G, u) > 0$ and a constant $L_x \geq 0$ such that*

   i) $B_{\varepsilon_x}(x) \subset G$

   ii) $\|u(y) - u(y')\| \leq L_x\|y - y'\| \qquad \forall y, y' \in B_{\varepsilon_x}(x)$

3. *A mapping $u : G \to \mathbb{R}^n$ is called* bi-Lipschitz in G, *if there are constants $0 < L_1 \leq L_2$ such that*

$$L_1\|x - x'\| \leq \|u(x) - u(x')\| \leq L_2\|x - x'\| \qquad \forall x, x' \in G.$$

4. *A mapping $u : G \to \mathbb{R}^n$ is called* locally bi-Lipschitz in G, *if for every $x \in G$ there exists $\varepsilon_x = \varepsilon(x, G, u) > 0$ and constants $L_i = L_i(x) > 0,\ i = 1, 2,$ $0 < L_1 \leq L_2$, such that*

$$\begin{aligned} &i)\ B_{\varepsilon_x}(x) \subset G \\ &ii)\ L_1\|y - y'\| \leq \|u(y) - u(y')\| \leq L_2\|y - y'\| \quad \forall y, y' \in B_{\varepsilon_x}(x) \end{aligned} \tag{1.22}$$

Theorem 1.4 *Let $G \subset \mathbb{R}^n$ be an open set and let $u : G \to \mathbb{R}^m (n, m \in \mathbb{N})$.*

a) It is equivalent

i) u is locally Lipschitz in G.

ii) For every compact $K \subset G$ there exists $L_K \geq 0$ such that

$$\|u(x) - u(x')\| \leq L_K \|x - x'\| \text{ for all } x, x' \in K$$

b) It is equivalent

i) u is locally bi-Lipschitz in G and $u : G \to u(G)$ is injective

ii) For every compact subset $K \subset G$ there are constants $0 < L'_K \leq L_K$ such that

$$L'_K \|x - x'\| \leq \|u(x) - u(x')\| \leq L_K \|x - x'\| \quad \forall x, x' \in K.$$

Proof.

1. Suppose that a.ii) resp. b.ii) hold true. Let $x \in G$. Since G is open there exists $\delta > 0$ such that $B_\delta(x) \subset G$. Let $\varepsilon := \delta/2$ and $K = \overline{B_\varepsilon(x)} \subset G$. Then in case a.ii) there exists $L_K \geq 0$ such that

$$\|u(y) - u(y')\| \leq L_K \|y - y'\| \quad \forall y, y' \in \overline{B_\varepsilon(x)}.$$

2. Similarly in case of condition b.ii) estimate (1.22) is valid even for $y, y' \in \overline{B_\varepsilon(x)}$. Let now $x, x' \in G$, $x \neq x'$. The set $K := \{x, x'\}$ is compact and by b.ii) there is $L'_K > 0$ such that

$$0 < L'_K \|x - x'\| \leq \|u(x) - u(x')\|.$$

Then $u(x) \neq u(x')$ and u is injective.

3. Suppose now conversely that a.i) resp. b.i) holds true. Let $K \subset G$ be a compact set. For $x \in K$ let $\varepsilon_x > 0$ such that conditions 2 resp. 4 of Definition 1.3 are satisfied. Then

$$\left\{ B_{\varepsilon_{x/2}}(x) \mid x \in K \right\}$$

forms an open covering of K. Hence there are $x_i \in K (i = 1, \ldots, N;\ \varepsilon_i := \varepsilon_{x_i})$

$$K \subset \bigcup_{i=1}^{N} B_{\varepsilon_{i/2}}(x_i) \subset \bigcup_{i=1}^{N} B_{\varepsilon_i}(x_i)$$

Let

$$\varepsilon_0 := \frac{1}{4}\min\{\varepsilon_i \mid i = 1, \ldots, N\}.$$

Let $y, y' \in K$, $\|y - y'\| < \varepsilon_0$. Then there is $i_0 \in \{1, \ldots, N\}$ such that $y \in B_{\varepsilon_{i_0/2}}(x_{i_0})$. Further,

$$\|y' - x_{i_0}\| \leq \|y - x_{i_0}\| + \|y - y'\| < \frac{1}{2}\varepsilon_{x_{i_0}} + \varepsilon_0 < \varepsilon_{x_{i_0}}$$

Therefore, $y, y' \in B_{\varepsilon_i}(x_i)$ and by assumption there is $L_{i_0} = L_{i_0}(x_{i_0}) > 0$ such that $\|u(y) - u(y')\| \leq L_{i_0}\|y - y'\|$. Let $\tilde{L}_K := \max(L_1, \ldots, L_N)$. Then

$$\|u(y) - u(y')\| \leq \tilde{L}_K\|y - y'\| \quad \forall y, y' \in K \text{ with } \|y - y'\| < \varepsilon_0. \tag{1.23}$$

By continuity of u,

$$M := \max_{x \in K}\|u(x)\| < \infty.$$

For $y, y' \in K$ with $\|y - y'\| \geq \varepsilon_0$,

$$\frac{\|u(y) - u(y')\|}{\|y - y'\|} \leq \frac{2M}{\varepsilon_0}$$

Define $L_K := \max\left(\tilde{L}_K, \frac{2M}{\varepsilon_0}\right)$. Then

$$\|u(y) - u(y')\| \leq L_K\|y - y'\| \qquad \forall y, y' \in K.$$

In case that assumption b.i) holds true we see in addition that there are $0 < L_i' \leq L_i''$ such that

$$L_i'\|y - y'\| \leq \|u(y) - u(y')\| \leq L_i\|y - y_i\| \quad \forall y, y' \in B_{\varepsilon_i}(x_i).$$

Then the number L_K constructed above is positive. In addition, let

$$\tilde{L}_K' := \min(L_1', \ldots, L_N') > 0.$$

Then we see similarly for $y, y' \in K$, $\|y - y'\| < \varepsilon_0$ that

$$\tilde{L}_K'\|y' - y'\| \leq \|u(y) - u(y')\|.$$

The set

$$C := \left\{(x, y) \in \mathbb{R}^{2n} \middle| \|x - y\| \geq \varepsilon_0\right\}$$

is closed and therefore

$$\tilde{K} := (K \times K) \cap C \subset \mathbb{R}^{2n}$$

is compact. Suppose $\tilde{K} \neq \phi$. Then the continuous function

$$f(x,y) := \|u(x) - u(y)\|$$

attains its minimum d in a point $(x_0, y_0) \in \tilde{K}$. Suppose that $d = 0$. Then $u(x_0) = u(y_0)$ and by injectivity of u finally $x_0 = y_0$, contradicting $\|x_o - y_0\| \geq \varepsilon_o$. Let

$$D := \operatorname{diam} \tilde{K} = \sup \left\{ \|x - y\| \mid x, y \in \tilde{K} \right\} > 0$$

For $(x,y) \in \tilde{K}$

$$\frac{d}{D}\|x - y\| \leq d = \|u(x_0) - u(y_0)\| \leq \|u(x) - u(y)\|$$

We set

$$L'_K := \begin{cases} \tilde{L}'_K & \text{if } \tilde{K} = \phi \\ \min\left(\frac{d}{D}, \tilde{L}'_K\right) & \text{if } \tilde{K} \neq \phi. \end{cases}$$

Then $L'_K > 0$ and

$$L'_K \|x - y\| \leq \|u(x) - u(y)\| \qquad \forall x, y \in K.$$

■

Examples and Remarks. Suppose that $G \subset \mathbb{R}^n$ is open.

i) Let $u \in C^1(G)$. If $x_0 \in G$, there is $\varepsilon > 0$ such that $B_{2\varepsilon}(x_0) \subset G$. Then $\overline{B_\varepsilon(x_0)} \subset B_{2\varepsilon}(x_0)$ and

$$\max \left\{ \|\nabla u(x)\| \mid x \in \overline{B_\varepsilon(x_0)} \right\} =: C_\varepsilon.$$

If $x, y \in B_\varepsilon(x_0)$, then $tx + (1-t)y \in B_\varepsilon(x_0)$ for all $t \in [0,1]$. Hence

$$(1.24) \qquad u(x) - u(y) = \int_0^1 \frac{d}{dt}[u(tx + (1-t)y)]dt =$$

$$= \int_0^1 \sum_{k=1}^n (D_k u)(tx + (1-t)y)(x_k - y_k)dt$$

By Schwarz' inequality

$$\left|\sum_{k=1}^{n}(D_k u)(tx+(1-t)y)(x_k-y_k)\right| \leq \|(\nabla u)(tx+(1-t)y)\|\|x-y\| \leq C_\varepsilon\|x-y\|$$

Therefore $|u(x)-u(y)| \leq C_\varepsilon\|x-y\|$ $\forall x,y \in B_\varepsilon(x_0)$ and u is *locally* Lipschitz continuous.

ii) It is easy to construct even *bounded domains* $G \subset \mathbb{R}^n$ and functions $u \in C^1(G)$ such that

$$|\nabla u(x)| \leq C \qquad \forall x \in G$$

but u *is not Lipschitz in* G, even not uniformly continuous.

iii) Let $u \in C^1(G, \mathbb{R}^n)$ and define

$$u'(x) := (D_i u_k(x)) \in M(n) \text{ for } i,k = 1,\dots,n$$

Assume that

$$\det\nolimits_n u'(x) \neq 0 \qquad \forall x \in G.$$

If $x_0 \in G$, there exists $L_1 = L(x_0) > 0$ such that

$$\|u'(x_0)\xi\| \geq L_1\|\xi\| \qquad \forall \xi \in \mathbb{R}^n.$$

We choose now $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset G$ and $|D_k u_i(x) - D_k u_i(z)| \leq \frac{L_1}{2n}$ for all $x,z \in B_\varepsilon(x_0)$ and for all $i,k \in \{1,\dots,n\}$. If we apply (1.24) to each component of $u = (u_1,\dots,u_n)$ we see

$$\begin{aligned}
u_i(x) - u_i(y) &= \int_0^1 \sum_{k=1}^{n}(D_k u_i)(tx-(1-t)y)(x_k-y_k)dt = \\
&= \sum_{k=1}^{n}(D_k u_i)(x_0)(x_k-y_k) + \\
&\quad + \int_0^1 \sum_{k=1}^{n}\left[(D_k u_i)(tx+(1-t)y) - D_k u_i(x_0)\right](x_k-y_k)dt \\
&=: a_i + b_i
\end{aligned}$$

Then $|b_i| \leq \frac{L_1}{2} n^{-\frac{1}{2}}\|x-y\|$. Therefore

$$
\begin{aligned}
\|u(x)-u(y)\| &\geq \|a\|-\|b\| \geq \|u'(x_0)(x-y)\| - \frac{L_1}{2}\|x-y\| \\
&\geq \frac{L_1}{2}\|x-y\| \qquad \forall x,y \in B_\varepsilon(x_0).
\end{aligned}
$$

Therefore, $u \in C^1(G,\mathbb{R}^n)$ with $\det_n' u(x) \neq 0$ for all $x \in G$ is locally bi-Lipschitz.

In a similar way in case $m > n$ and $u \in C^1(G,\mathbb{R}^m)$ it is easily proved that u is locally bi-Lipschitz, if we assume for the $m \times n$-matrix

$$
u'(x) = (D_i u_k(x)) \qquad k = 1,\dots,m,\ i = 1,\dots,n
$$

rank $u'(x) = n$.

There is a celebrated theorem by Rademacher [Ra] (compare e.g. [Mo2], p. 65) on the total differentiability of Lipschitz functions.

Theorem 1.5 *(Rademacher) Let $G \subset \mathbb{R}^n$ be an open set and let $u : G \to \mathbb{R}$ be Lipschitz on G,*

$$
\|u(x)-u(y)\| \leq L\|x-y\| \qquad \forall x,y \in G.
$$

Then there exists a subset $N \subset G$ of measure zero such that:

i) For each $x \in G \setminus N$ the function u is totally differentiable, e.g. for each $x \in G \setminus N$ there exists a neighborhood $B_\varepsilon(x) \subset G$, a vector $w(x) = (w_1(x),\dots,w_n(x))$ and a function $\varphi(x,.) : B_\varepsilon(0) \to \mathbb{R}$ such that

$$
u(x+\xi) = u(x) + \sum_{j=1}^{n} w_j(x)\xi_j + \varphi(x,\xi) \qquad \forall \xi \in B_\varepsilon(0) \tag{1.25}
$$

$$
\textit{and} \quad \lim_{\substack{|\xi|\to 0 \\ \xi \in B_\varepsilon(0)\setminus\{0\}}} \frac{\varphi(x,\xi)}{\|\xi\|} = 0 \tag{1.26}
$$

ii) The map $\omega : G \setminus N \to \mathbb{R}^n$ is measurable and

$$
|w(x)| \leq L \qquad \forall x \in G \setminus N \tag{1.27}
$$

iii) If we define for $i = 1,\dots,n$

$$
D_i u(x) := \begin{cases} w_i(x) & \textit{for } x \in G \setminus N \\ 0 & \textit{for } x \in N \end{cases}
$$

then $D_i u \in L^1_{\mathrm{loc}}(G) \cap L^\infty(G)$ is a weak derivative of u.

Clearly for $x \in G \setminus N$ $\omega_i(x)$ is the partial derivative $\partial_i u(x)$. An easy consequence is

Theorem 1.6 *Let $G \subset \mathbb{R}^n$ and let $u : G \to \mathbb{R}^m (m, n \in \mathbb{N})$ be bi-Lipschitz,*

$$(1.28) \quad L_1\|x-y\|_n \le \|u(x)-u(y)\|_m \le L_2\|x-y\|_n \qquad \forall x, y \in G \quad 0 < L_1 \le L_2$$

Then there is a subset $N \subset G$, $|N| = 0$, such that u is totally differentiable at each $x \in G \setminus N$. For $x \in G \setminus N$ for the total derivative

$$u'(x) = (D_i u_k(x)) \in M(m \times n)$$

we have the estimate

$$(1.29) \qquad L_1\|\eta\|_n \le \|u'(x)\eta\|_m \le L_2\|\eta\|_n, \qquad \forall x \in G \setminus N \qquad \forall \eta \in \mathbb{R}^n.$$

Therefore $m \ge n$ and

$$\textit{rank}\ u'(x) = n \qquad \forall x \in G \setminus N$$

Proof. We apply Theorem 1.5 to each component of $u = (u_1, \ldots, u_m)$ and we set for $x \in G \setminus N$

$$\varphi(x, \xi) := (\varphi_1(x, \xi), \ldots, \varphi_m(x, \xi)) \ \text{ for } \xi \in B_\varepsilon(0),$$

where $\varepsilon := \min(\varepsilon_1, \ldots, \varepsilon_n)$. Let $\xi \in \mathbb{R}^n$, $|\xi| = 1$. By (1.25), (1.26) for $0 < \varrho < \varepsilon$

$$L_1\|\varrho\xi\| \le \|u(x+\varrho\xi) - u(x)\| = \|u'(x)\varrho\xi + \varphi(x, \varrho\xi)\| \le L_2\|\varrho\xi\|.$$

Therefore

$$L_1\|\xi\| \le \|u'(x)\xi\| + \frac{\|\varphi(x, \varrho\xi)\|}{\|\varrho\xi\|} \to \|u'(x)\xi\| \text{ as } \varrho \to 0$$

Similarly $\|u'(x)\xi\| \le L_2\|\xi\|$. If $0 \ne \eta \in \mathbb{R}^n$ is arbitrary, we consider $\xi := \frac{\eta}{\|\eta\|}$ and derive (1.29). By (1.29) the kernel of the linear map

$$A : \mathbb{R}^n \to \mathbb{R}^m, \qquad A_\eta = u'(x)\eta \text{ for } \eta \in \mathbb{R}^n$$

has dimension zero. Since

$$\begin{aligned} n &= \dim\ \text{kernel}\ A + \dim\ \text{image}\ A = \\ &= \dim\ \text{image of}\ A = \ \text{rank of}\ A, \end{aligned}$$

we see in addition $m \ge n$. ■

Let us consider Theorem 1.6 in case $m = n$. If we *assume* that $G^* = u(G)$ is open, we could apply Theorem 1.6 to the map $v := u^{-1} : G^* \to G$ and find $N^* \subset G^*$, $|N^*| = 0$, such that v is differentiable in $G^* \setminus N^*$. But later (Theorem 4.5) we will *prove* that bi-Lipschitz maps are open, e.g. they map open subsets $U \subset G$ on open subsets of $\mathbb{R}^n$.

Let us recall that $N \subset \mathbb{R}^n$ has measure zero if and only if for each $\varepsilon > 0$ there is a sequence of balls $B_i = B_{r_i}(x_i) \subset \mathbb{R}^n$ such that

$$N \subset \bigcup_{i=1}^{\infty} B_i \text{ and } \sum_{i=1}^{\infty} |B_i| < \varepsilon. \tag{1.30}$$

Clearly it is equivalent to demand that

$$\sum_{i=1}^{\infty} r_i^n < \varepsilon.$$

Lemma 1.7 *Let $G \subset \mathbb{R}^n$ be open and let $u : G \to \mathbb{R}^m$ $(m \geq n)$ be locally Lipschitz. Let $N \subset G$ be a set of n-dimensional measure zero: $|N|_n = 0$. Then, $N^* := u(N)$ has m-dimensional measure zero: $|N^*|_m = 0$.*

Proof.

i) Let $G' \subset\subset G$ and choose $G' \subset\subset G'' \subset\subset G$. Suppose that $L = L(G'') \geq 0$,

$$\|u(x) - u(x')\| \leq L\|x - x'\| \qquad \forall x, x' \in G''.$$

Let $N' := N \cap G'$, then $|N'|_n \leq |N|_n = 0$. If $\varepsilon > 0$ is given, let

$$0 < \varepsilon_0 < \min\left(\varepsilon, 1, \left[2^{-1} \operatorname{dist}(G', \partial G'')\right]^{\frac{1}{n}}\right).$$

Then there exist $B_i = B_{r_i}(x_i) \subset \mathbb{R}^n$ such that

$$N' \subset \bigcup_{i=1}^{\infty} B_i \text{ and } \sum_{i=1}^{\infty} r_i^n < \frac{\varepsilon_0}{L^m} \tag{1.31}$$

Without loss of generality we may assume $N' \cap B_i \neq \phi$ for all $i \in \mathbb{N}$. Then $B_i \subset G''$. Let $y_i = u(x_i)$. Then

$$u(B_i) \subset \tilde{B}_i := \{y \in \mathbb{R}^m : \|y - y_i\| < Lr_i\}$$

Because $r_i < 1$ and $m \geq n$, $r_i^{m-n} \leq 1$ and

$$\sum_{i=1}^{\infty} (Lr_i)^m = L^m \sum_{i=1}^{\infty} r_i^n r_i^{m-n} \leq L^m \sum_{i=1}^{\infty} r_i^n < \varepsilon_0 \leq \varepsilon$$

Since $u(N') \subset u\left(\bigcup\limits_{i=1}^{\infty} B_i\right) \subset \bigcup\limits_{i=1}^{\infty} u(B_i) \subset \bigcup\limits_{i=1}^{\infty} \tilde{B}_i$

we see $|u(N')|_m = 0$.

ii) Let now (G_k) be an exhausting sequence for G,

$$G_k \subset\subset G_{k+1} \subset\subset G \qquad \forall k \in \mathbb{N}, \qquad \bigcup_{k=1}^{\infty} G_k = G.$$

Then $$N = \bigcup_{k=1}^{\infty} (N \cap G_k)$$ and

$$N^* = u(N) \subset \bigcup_{k=1}^{\infty} u(N \cap G_k).$$

By part i) of proof, $|u(N \cap G_k)|_m = 0$ and therefore $|N^*|_m = 0$. ■

2 The Homotopy Theorems

Throughout this section let

$$I := [0, 1] \tag{2.1}$$

Lemma 2.1 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^0(\bar{G} \times I, \mathbb{R}^n)$. Let $U \subset \mathbb{R}^n$ be an open set such that*

$$\bigcup_{t \in I} u(\partial G, t) \subset U.$$

Then

i) $\bigcup_{t \in I} u(\partial G, t) = u(\partial G \times I)$ *is compact.*

ii) $\operatorname{dist}(u(\partial G \times I), \partial U) > 0$

iii) There exists a compact set $K \subset G$ such that

$$x \in \bar{G} \setminus K \Rightarrow u(x, t) \in U \qquad \forall t \in [0, 1]$$

(i. e. $u\big((\bar{G} \setminus K) \times I\big) \subset U$)

Proof.

a)

$$y_0 \in \bigcup_{t \in I} u(\partial G, t) \Leftrightarrow \exists t_0 \in I : y_0 \in u(\partial G, t_0) \Leftrightarrow \exists x_0 \in \partial G : y_0 = u(x_0, t_0) \Leftrightarrow$$
$$\Leftrightarrow y_0 \in u(\partial G \times I).$$

Since $\partial G \times I$ is compact, by continuity of u, $u(\partial G \times I)$ is compact too.

b) If $U = \mathbb{R}^n$ we set $K := \phi$. If $\phi \neq U \neq \mathbb{R}^n$, then $\partial U \neq \phi$ and it is closed. Because of compactness of $u(\partial G \times I)$

$$0 < d := \operatorname{dist}\left(u(\partial G \times I), \partial U\right)$$

If $y_0 \in u(\partial G \times I)$, then $B_{\frac{d}{2}}(y_0) \subset U$. By uniform continuity of $u : \bar{G} \times I \to \mathbb{R}^n$ there is $\delta > 0$ such that

$$(2.2)\quad \|u(x_1,t) - u(x_2,t)\| < \frac{d}{2} \qquad \forall x_1, x_2 \in \bar{G} \quad \text{with } \|x_1 - x_2\| < \delta, \quad \forall t \in I$$

We define

$$K := \{x \in G \mid \operatorname{dist}(x, \partial G) \geq \delta\}$$

Then K is compact. If $x \in \bar{G} \setminus K$, then $\operatorname{dist}(x, \partial G) < \delta$. By compactness of ∂G there is $x_0 \in \partial G$ such that

$$\|x - x_0\| = \operatorname{dist}(x, \partial G) < \delta.$$

For $t \in I$, $(x_0, t) \in u(\partial G \times I)$ and by (2.2) $\|u(x,t) - u(x_0,t)\| < \frac{d}{2}$, that is

$$u(x,t) \in B_{\frac{d}{2}}\left(u(x_0,t)\right) \subset U.$$

Since $t \in I$ was arbitrary, $u(x,t) \in U \quad \forall t \in I$. ■

Lemma 2.2 *Let $G \subset \mathbb{R}^n$ be a bounded open set.*

Suppose that

(i) $u \in C^2(G \times I, \mathbb{R}^n) \cap C^0(\bar{G} \times I, \mathbb{R}^n)$

(ii) there is an open set $U \subset \mathbb{R}^n$ such that $u(\partial G \times I) \subset U$

(iii) $f \in C_c^1(\mathbb{R}^n)$ and $f|_U = 0$

Then

$$(2.3)\qquad h(t) := \int\limits_G f(u(x,t)) \det\nolimits_n u'(x,t)\,dx$$

is constant in I $\left(\text{here } u'(x,t) \text{ means the } n \times n\text{-matrix } \left(\left(\frac{\partial u_i}{\partial x_k}\right)(x,t)\right) \in M(n)\right)$.

Proof.

a) According Lemma 2.1 we choose a compact $K \subset G$ such that

$$u(x,t) \in U \qquad \forall x \in \bar{G} \setminus K, \qquad \forall t \in I.$$

Then

$$f\left(u(x,t)\right) = 0 \qquad \forall x \in \bar{G} \setminus K, \qquad \forall t \in I$$

that is, $\operatorname{supp} f(u(.,t)) \subset K \subset G$ $\forall t \in I$. Clearly h is differentiable in I and

$$\begin{aligned} h'(t) = &\int_G \sum_{j=1}^n \left(\frac{\partial f}{\partial y_j}\right)(u(x,t)) \frac{\partial u_j(x,t)}{\partial t} \det_n u'(x,t) dx \\ &+ \int_G f\left(u(x,t)\right) \frac{\partial}{\partial t} \det_n u'(x,t) dx =: I_1(t) + I_2(t). \end{aligned}$$

We write again

$$A(x,t) = (a_{ij}(x,t)) \text{ where } a_{ij}(x,t) := \frac{\partial u_i(x,t)}{\partial x_j}$$

According (1.5) (we use the notation (1.1))

$$\frac{\partial}{\partial t} \det_n u'(x,t) = \sum_{i=1}^n \sum_{k=1}^n (-1)^{i+k} \frac{\partial^2 u_i(x,t)}{\partial t \partial x_k} \det_{n-1} A_{ik}(x,t)$$

We consider $I_2(t)$ and integrate by parts (using (1.12)):

$$\begin{aligned} I_2(t) &= -\int_G \sum_{i,k=1}^n (-1)^{i+k} \frac{\partial u_i(x,t)}{\partial t} \frac{\partial}{\partial x_k} \left[f(u(x,t)) \det_{n-1} A_{ik}(x,t)\right] dx \\ &= -\int_G \sum_{i,k=1}^n (-1)^{i+k} \frac{\partial u_i(x,t)}{\partial t} \sum_{j=1}^n \left(\frac{\partial f}{\partial y_j}\right)(u(x,t)) \frac{\partial u_j(x,t)}{\partial x_k} \cdot \det_{n-1} A_{ik}(x,t) dx \\ &\quad - \int_G f\left(u(x,t)\right) \sum_{i=1}^n (-1)^i \frac{\partial u_i(x,t)}{\partial t} \left[\sum_{k=1}^n (-1)^k \frac{\partial}{\partial x_k} \det_{n-1} A_{ik}(x,t)\right] dx \end{aligned}$$

By Theorem 1.2 the expression in brackets [..] vanishes. For the first integral we observe by (1.3)

$$\sum_{k=1}^n (-1)^{i+k} \frac{\partial u_j(x,t)}{\partial x_k} \det_{n-1} A_{ik}(x,t) = \delta_{ji} \det_n A(x,t) = \delta_{ji} \det_n u'(x,t).$$

Therefore

$$I_2(t) = -\int\limits_G \sum_{j=1}^{n} \left(\frac{\partial f}{\partial yj}\right)(u(x,t))\,\frac{\partial u_j(x,t)}{\partial t}\,\det_n u'(x,t)dx = -I_1(t).$$

and $h'(t) = 0$ for $t \in I$. ■

We needed the high differentiability assumptions on u and f only for our *proof* of Lemma 2.2. But the expression at the right side of (2.3) is well defined if e.g. $f \in C_c^0(\mathbb{R}^n)$, $u \in C^0(\bar{G} \times I)$ and u has merely bounded *weak* derivatives. Therefore there is hope to diminish the assumptions by some mollification arguments, what we'll do in the sequel.

Lemma 2.3 *Let $G \subset \mathbb{R}^n$ be a bounded open set. Assume that hypotheses (i) and (ii) of Lemma 2.2 hold and*

(iii) $f \in C_c^0(\mathbb{R}^n)$ and $f\,|_U = 0$.

Then h defined by (2.3) is constant in I.

Proof.

a) If $U = \mathbb{R}^n$ then $h(t) = 0\ \forall t \in I$. We may assume $U \neq \mathbb{R}^n$, i. e. $\partial U \neq \phi$. Then

$$0 < d := \operatorname{dist}\left(u(\partial G \times I), \partial U\right) < \infty$$

Let

$$U' := \left\{ y \in U : \operatorname{dist}(y, u(\partial G \times I)) < \frac{d}{2} \right\} \tag{2.4}$$

U' is open, $U' \subset\subset U$ and $u(\partial G \times I) \subset U'$. Let $K' \subset G$ be the compact set constructed according Lemma 2.1 with respect to U' with $u\left((\bar{G} \setminus K') \times I\right) \subset U'$.

b) For $y \in \mathbb{R}^n$ consider the mollified functions

$$f_\varrho(y) := \int\limits_{\mathbb{R}^n} \omega_\varrho(y - z) f(z) dz \text{ for } 0 < \varrho < \frac{d}{4}$$

If $y \in U'$, then $B_\varrho(y) \subset U$ and since

$$f_\varrho(y) = \int\limits_{B_\varrho(y)} \omega_\varrho(y - z) f(z) dz = 0$$

we see

$$(2.5) \qquad f_\varrho\,|_{U'} = 0 \qquad \forall 0 < \varrho < \frac{d}{4}, f_\varrho \in C_c^\infty(\mathbb{R}^n)$$

Let $h^{(\varrho)} : I \to \mathbb{R}$ be defined by (2.3) with respect to f_ϱ. By Lemma 2.2 $h^{(\varrho)}$ is constant on I.

c) Since $u\left((\bar{G} \setminus K') \times I\right) \subset U'$, by (2.4) we see $\operatorname{supp} f_\varrho(u(.,t)) \subset K'$ $\forall t \in I$, $\forall 0 \le \varrho < \frac{d}{4}$ (where $f_0 := f$ for $\varrho = 0$). By continuity of $\det_n u'(.)$ on $K' \times I$,

$$L := \max\{|\det_n u'(x,t)| : (x,t) \in K' \times I\} < \infty$$

exists. By (1.18) we see $f_\varrho \to f$ uniformly in $\mathbb{R}^n$, therefore

$$|f\,(u(x,t)) - f_\varrho\,(u(x,t))| \le \max_{z \in \mathbb{R}^n} |f_\varrho(z) - f(z)| \to 0 (\varrho \to 0)$$

Then

$$\left| \int_G [f\,(u(x,t)) - f_\varrho\,(u(x,t))] \det_n u'(x,t) dx \right| \le$$
$$\le \max_{z \in \mathbb{R}^n} |f(z) - f_\varrho(z)| \cdot L|G| \to 0 (\varrho \to 0)$$
$$\forall t \in I.$$

■

Theorem 2.4 *Let $G \subset \mathbb{R}^n$ be a bounded open set. Suppose that*

(i) $u \in C^0(\bar{G} \times I, \mathbb{R}^n)$,
for each fixed $t \in I$ $u(.,t) \in C^1(G; \mathbb{R}^n)$ *and* $D_i u_k \in C^0(G \times I)$, $i, k = 1, \ldots, n$

(ii) there is an open set $U \subset \mathbb{R}^n$ *such that*

$$u(\partial G \times I) \subset U.$$

(iii) $f \in C_c^0(\mathbb{R}^n)$ *and* $f\,|_U = 0$

Then

$$h(t) := \int_G f\,(u(x,t)) \det_n u'(x,t) dx$$

is constant in I.

Proof.

a) Without loss of generality we may assume that there is $0 < \delta \leq \frac{1}{3}$ such that u is constant with respect to t for $0 \leq t \leq \delta$ and $1 - \delta \leq t \leq 1$. Otherwise, let $\varphi_\delta : \mathbb{R} \to I$

$$\varphi_\delta(t) := \begin{cases} 0 & \text{for } t < \delta \\ \frac{1}{1-2\delta}(t-\delta) & \text{for } \delta \leq t \leq 1-\delta \\ 1 & \text{for } 1-\delta < t \end{cases}$$

and consider

$$\begin{gathered} u^{(\delta)} : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^n, \\ u^{(\delta)}(x,t) := u(x, \varphi_\delta(t)) \text{ for } (x,t) \in \bar{\Omega} \times \mathbb{R} \end{gathered}$$

Then $u^{(\delta)}$ satisfies (i) and (ii) too and

$$u^{(\delta)}(x,t) = \begin{cases} u(x,0) & \text{for } t \leq \delta \\ u(x, \frac{1}{1-2\delta}(t-\delta) & \text{for } \delta \leq t \leq 1-\delta \\ u(x,1) & \text{for } t \geq 1-\delta \end{cases}$$

Let $h^{(\delta)} : \mathbb{R} \to \mathbb{R}$ be defined by (2.3) with respect to $u^{(\delta)}$. Then for $t \in I$

$$h(t) = h^{(\delta)}(\delta + t(1-2\delta))$$

and the Theorem is proved for h if it is proved for $h^{(\delta)}$.

b) In case $U = \mathbb{R}^n$ the assertion is trivial. We assume $\phi \neq U \neq \mathbb{R}^n$. Let $U' \subset\subset U$ and K' be constructed as in part a) of proof of Lemma 2.3 such that

$$u\big((\bar{G} \setminus K') \times I\big) \subset U' \tag{2.6}$$

We choose now an open G' such that $K' \subset G' \subset\subset G$. Then $K' \cap \partial G' = \phi$ and if $x \in \partial G'$ then $x \in \bar{G}' \setminus K'$ and therefore

$$\begin{cases} u(\partial G' \times I) \subset U' \\ d_1 := \operatorname{dist}(u(\partial G' \times I), \partial U') > 0 \end{cases} \tag{2.7}$$

Let $D := \operatorname{dist}(\bar{G}', \partial G) > 0$ and $\varrho_0 := \frac{1}{2}(\delta, D)$. Let ω denote an $(n+1)$-dimensional mollifier kernel and consider the mollified mapping

$$u_\varrho : \bar{G}' \times I \to \mathbb{R}^n, \qquad 0 < \varrho < \varrho_0.$$

We observe that by part a) of proof u_ϱ is constant with respect to t for $0 \le t \le \frac{\delta}{2}$ and for $1 - \frac{\delta}{2} \le t \le 1$. Then $u_\varrho \in C^\infty(\bar{G}' \times I)$

$$u_\varrho \to u \ (\varrho \to 0) \text{ uniformly on } \bar{G}' \times I.$$

Because of

$$\frac{\partial}{\partial x_i} (u_k)_\varrho (x,t) = \left(\frac{\partial u_k}{\partial x_i} \right)_\varrho (x,t) \text{ for } (x,t) \in \bar{G}' \times I$$

and the continuity of $D_i u_k$ on $G \times I$

$$\frac{\partial}{\partial x_i} (u_k)_\varrho \big|_{\bar{G}' \times I} \to \frac{\partial}{\partial x_i} u_k \big|_{\bar{G}' \times I}$$

uniformly. Then there is $0 < \varrho_1 < \varrho_0$ such that

$$\| u_\varrho(x,t) - u(x,t) \| < \frac{d_1}{2} \qquad \forall (x,t) \in \bar{G}' \times I, \qquad \forall 0 < \varrho < \varrho_1$$

If $z \in \partial U$, then

$$\begin{aligned} \| u_\varrho(x,t) - z \| &\ge \| u(x,t) - z \| - \| u(x,t) - u_\varrho(x,t) \| \\ &\ge d_1 - \frac{d_1}{2} = \frac{d_1}{2} > 0 \qquad \forall (x,t) \in \bar{G}' \times I, \qquad \forall 0 < \varrho < \varrho_1 \end{aligned}$$

and therefore

$$u_\varrho(\partial G' \times I) \subset U' \qquad \forall 0 < \varrho < \varrho_1$$

All assumptions of Lemma 2.3 are satisfied with G replaced by G' and for u_ϱ. Therefore there is $c_\varrho \in \mathbb{R}$ such that

$$c_\varrho = \int\limits_{G'} f(u_\varrho(x,t)) \det\nolimits_n u'_\varrho(x,t) dx \qquad \forall t \in I$$

Since $f\left(u_\varrho(.,t)\right)$ has compact support in G' for $0 \le \varrho < \varrho_1$ and all functions are uniformly bounded, we may pass to the limit $\varrho \to 0$ and get finally

$$c = \int\limits_{G'} f\left(u(x,t)\right) \det\nolimits_n u'(x,t) dt = \int\limits_{G} f\left(u(x,t)\right) \det\nolimits_n u'(x,t) dt \qquad \forall t \in I.$$

■

For most applications Theorem 2.4 is sufficient. But if we want to deal with Lipschitz transforms we have to extend it further.

Theorem 2.5 *Let $G \subset \mathbb{R}^n$ be a bounded open set. Suppose that*

(i) $u \in C^0\left(\bar{G} \times I, \mathbb{R}^n\right)$

(ii) There exists $N \subset G$, $|N| = 0$ such that for each fixed $t \in I$ the mapping $u(.,t) : G \setminus N \to \mathbb{R}^n$, is (totally) differentiable, $D_i u_k(.,t) : G \setminus N \to \mathbb{R}$ are measurable and that there is $L > 0$ such that

$$|D_i u_k(x,t)| \leq L \qquad \forall x \in G \setminus N, \quad \forall t \in [0,1] \tag{2.8}$$

Further we assume, that for each $x \in G \setminus N$ the functions $D_i u_k(x,.) : I \to \mathbb{R}$ are continuous $\forall i, k = 1, \ldots, n$.

(iii) there is an open set $U \subset \mathbb{R}^n$ such that $u(\partial G \times I) \subset U$

(iv) $f \in C_c^0(\mathbb{R}^n)$ and $f\,|_U = 0$. Then

$$h(t) := \int\limits_G f\left(u(x,t)\right) \det\nolimits_n u'(x,t) dx$$

is constant in I.

Proof.

a) As in part b) of proof of Theorem 2.4 we construct U' and $K' \subset G' \subset\subset G$ such that (2.6) and (2.7) hold true. Now we consider an n-dimensional mollifier kernel ω and we set for $k = 1, \ldots, n$

$$u_{k\varrho}(x,t) := \int\limits_G \omega_\varrho(x-y) u_k(y,t) dy, \quad (x,t) \in \bar{G}' \times I \quad 0 < \varrho < d_1$$

Then $u_{k\varrho}(.,t) \in C^\infty(\bar{G}')$ $\forall t \in I$ and by (1.20)

$$\frac{\partial}{\partial x_i} u_{k\varrho}(x,t) = \int\limits_G \omega_\varrho(x-y) D_i u_k(y,t) dy \quad \forall t \in I \tag{2.9}$$

By (2.8)

$$|D_i u_{k\varrho}(x,t)| \leq L. \tag{2.10}$$

Let

$$(x_\nu, t_\nu) \in \bar{G}' \times I \quad \forall \nu \in \mathbb{N}, (x_\nu, t_\nu) \to (x_0, t_0) \in \bar{G}' \times I.$$

For $0 < \varrho < d_1$ fixed

$$
\begin{aligned}
|D_i u_{k\varrho}(x_\nu, t_\nu) - D_i u_k(x_0, t_0)| = \\
&\leq \left| \int_G \left[\omega_\varrho(x_\nu - y) - \omega_\varrho(x_0 - y)\right] D_i u_k(y, t_\nu) dy \right| \\
&+ \left| \int_G \omega_\varrho(x_0 - y) \left[D_i u_k(y, t_\nu) - D_i u_k(y, t_0)\right] dy \right| \\
&\leq L \int_G |\omega_\varrho(x_\nu - y) - \omega_\varrho(x_0 - y)| dy \\
&\quad + \int_G \omega_\varrho(x_0 - y) \left|D_i u_k(y, t_\nu) - D_i u_k(y, t_0)\right| dy
\end{aligned}
$$

For $y \in G \setminus N$

$$
D_i u_k(y, t_\nu) \to D_i u_k(y, t_0) \qquad (\nu \to \infty)
$$

and

$$
|D_i u_k(y, t_\nu) - D_i u_k(y, t_0)| \leq 2L \qquad \forall y \in G \setminus N, \ \forall \nu \in \mathbb{N}.
$$

Since $\omega_\varrho(x_0 - .)$ is bounded, the second integral tends to zero for $\nu \to \infty$ by Lebesgue's theorem. Since $\omega_\varrho : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous, $\omega_\varrho(x_\nu - y) \to \omega_\varrho(x - y)$ uniformly in $\mathbb{R}^n$ and since $|G| < \infty$, $\int_G |\omega_\varrho(x_\nu - y) - \omega_\varrho(x_0 - y)| dy \to 0$. Therefore, $D_i u_{k\varrho} \in C^0(\bar{G}' \times I)$. Because of uniform boundedness and uniform continuity of u in $\bar{G} \times I$ we see similarly $u_\varrho \in C^0(\bar{G}' \times I)$. Further for $(x, t) \in \bar{G}' \times I$ and $0 < \varrho < d_1$

$$
\int_G \omega_\varrho(x - y) dy = \int_{\mathbb{R}^n} \omega_\varrho(x - y) dy = 1
$$

(by (1.14)). This gives

$$
\begin{aligned}
\|u(x, t) - u_\varrho(x, t)\| &= \left\| \int_G \omega_\varrho(x - y) \left(u(y, t) - u(x, t)\right) dx \right\| \\
&\leq \max \left\{ \|u(x, t) - u(y, t)\| \, \middle| \, x, y \in \bar{G}, \|x - y\| \leq \varrho, \ t \in I \right\} \to 0 \ (\varrho \to 0)
\end{aligned}
$$

Therefore there is $0 < \varrho_1 < d_1$ such that

$$\|u_\varrho(x,t) - u(x,t)\| < \frac{d_1}{2} \qquad \forall (x,t) \in \bar{G}' \times I, \qquad \forall 0 < \varrho < \varrho_1.$$

As in part b) of proof of Theorem 2.4 this guarantees

$$u_\varrho(\partial G' \times I) \subset U' \qquad \forall 0 < \varrho < \varrho_1.$$

Then all assumptions of Theorem 2.4 are fulfilled for u_ϱ with respect to G'. Then there is $c_\varrho \in \mathbb{R}$ for $0 < \varrho < \varrho_1$ such that

$$c_\varrho = \int\limits_{G'} f\left(u_\varrho(x,t)\right) \det_n u'_\varrho(x,t) dx \qquad \forall t \in I$$

We observe that $\operatorname{supp} f\left(u_\varrho(.,t)\right) \subset K'$ $\forall t \in I$, $\forall 0 \leq \varrho < \varrho_1$ and $|f(z)| \leq C$ $\forall z \in \mathbb{R}^n$. Therefore all integrals in the next consideration can be taken over K'.

$$\begin{aligned}
I(\varrho) &:= \left| \int\limits_{K'} f\left(u_\varrho(x,t)\right) \det_n u'_\varrho(x,t) dx - \int\limits_{K'} f\left(u(x,t)\right) \det_n u'(x,t) dx \right| \\
&\leq \int\limits_{K'} \left| f\left(u_\varrho(x,t)\right) - f\left(u(x,t)\right)\right| \left|\det_n u'_\varrho(x,t)\right| dx \\
&+ \int\limits_{K'} \left| f\left(u(x,t)\right)\right| \left|\det_n u'_\varrho(x,t) - \det_n u'(x,t)\right| dx = \\
&=: I_1(\varrho) + I_2(\varrho)
\end{aligned}$$

If $A = (a_{ik}) \in M(n)$, $|a_{ik}| \leq L \;\; \forall i,k = 1,\dots,n$ then with the help of (1.3) it is readily seen by induction that

$$|\det_n A| \leq n! L^n. \tag{2.11}$$

Since $|\partial_i u_{k\varrho}(x)| \leq L$ for $x \in K'$ and $f(u_\varrho(.)) \to f(u(.))$ uniformly on $K' \times I$,

$$I_1(\varrho) \leq n! L^n \int\limits_{K'} \left| f\left(u_\varrho(x,t)\right) - f\left(u(x,t)\right)\right| dx \to 0 \qquad (\varrho \to 0).$$

For $j = 1,\dots,n$ we abbreviate the j-th column of u' by

$$D_j\underline{u} := \begin{pmatrix} D_j u_1 \\ \vdots \\ D_j u_n \end{pmatrix}$$

and write

$$A := (D_1\underline{u}, \ldots, D_n\underline{u}); \qquad A_\varrho := (D_1\underline{u}_\varrho, \ldots, D_n\underline{u}_\varrho)$$

Then $A = u', A_\varrho = u'_\varrho$. For $1 \leq k \leq n-1$ let

$$A^{(k)} := \Big(D_1\underline{u_\varrho}, \ldots, D_k\underline{u_\varrho}, D_{k+1}\underline{u}, \ldots, D_n\underline{u}\Big)$$

$A^{(o)} := A$, $A^{(n)} := A_\varrho$. Then

$$\det_n u'_\varrho - \det_n u' = \det_n A_\varrho - \det_n A = \sum_{k=1}^{n} \left(\det_n A^{(k)} - \det_n A^{(k-1)}\right)$$

The matrices $A^{(k)}$ and $A^{(k-1)}$ differ only in the k-th column. Therefore (see (1.1))

$$\det_{n-1} A^{(k)}_{ik} = \det_{n-1} A^{(k-1)}_{ik}, \quad i = 1, \ldots, n$$

Expanding $\det_n A^{(k)}$ and $\det_n A^{(k-1)}$ with respect to the k-th column (see (1.3_2)) gives

$$\det_n A^{(k)} - \det_n A^{(k-1)} = \sum_{i=1}^{n} (-1)^{i+k} \det_{n-1} A^{(k)}_{ik} \left(D_k u_{i\varrho} - D_k u_i\right).$$

By (2.11)

$$\left|\det_n A^{(k)} - \det_n A^{(k-1)}\right| \leq (n-1)! L^{n-1} \sum_{i=1}^{n} |D_k u_{i\varrho} - D_k u_i|$$

Finally we see by (1.21)

$$I_2(\varrho) \leq C(n-1)! L^{n-1} \int\limits_{K'} \sum_{k,i=1}^{n} |D_k u_{i\varrho}(x,t) - D_k u_i(x,t)| dx \to 0 (\varrho \to 0) \quad \forall t \in I.$$

■

3 The change of variables formula for linear maps and $f \in C_c^0(\mathbb{R}^n)$.

For $n \in \mathbb{N}$, $n \geq 2$, $c \in \mathbb{R}$ we consider for $j \neq k$ the matrices

$$
(3.1) \qquad M_{jk}(c) := \begin{array}{rc} & \begin{array}{cccccc} & & & & k & \\ & & & & \downarrow & \end{array} \\ j \rightarrow & \begin{pmatrix} 1 & & 0 & \cdots & 0 & 0 \\ 0 & & 1 & & & \cdot \\ & \cdot & & 1 & c & \cdot \\ & \cdot & & & 0 & \cdot \\ & \cdot & & & 1 & \cdot \\ & \cdot & & & & \cdot \\ & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix} \end{array} \in M(n)
$$

having the number c in the i-th row and the j-th column and being identical with the unit matrix I otherwise. For $j = k$ we set

$$
(3.2) \qquad M_{jj}(c) := I
$$

Then for $B \in M(n)$

$$
(3.3) \qquad M_{jk}(c)B = \begin{array}{cl} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{j1} + cb_{k1} & \cdots & b_{jn} + cb_{kn} \\ \vdots & & \vdots \\ b_{k1} & \cdots & b_{kn} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow j \\ \\ \leftarrow k \\ \\ \\ \end{array} \end{array}
$$

Similarly

$$
(3.4) \qquad BM_{kj}(c) = \begin{array}{c} \begin{pmatrix} b_{11} & \cdots & b_{1j} + cb_{1k} & \cdots & b_{1k} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} + cb_{nk} & \cdots & b_{nk} & \cdots & b_{nn} \end{pmatrix} \\ \begin{array}{ccccccc} \phantom{b_{n1}} & & \uparrow & & \uparrow & & \phantom{b_{nn}} \\ & & j & & k & & \end{array} \end{array}
$$

We observe $M_{kj}(0) = I$,

$$
(3.5) \qquad \det_n M_{jk}(c) = 1 \qquad \forall j, k \in \{1, \ldots, n\}, \qquad \forall c \in \mathbb{R}
$$

Lemma 3.1 *Let $B \in M(n)$, $\det B \neq 0$. If $f \in C_c^0(\mathbb{R}^n)$, $c \in \mathbb{R}$, $j, k = 1, \ldots, n$, then*

$$
(3.6) \qquad \left\{ \begin{array}{l} f(B.), f(MB.), f(BM.) \in C_c^0(\mathbb{R}^n) \text{ and} \\ \int\limits_{\mathbb{R}^n} f\left(M_{jk}(c)Bx\right) dx = \int\limits_{\mathbb{R}^n} f(Bx) dx = \int\limits_{\mathbb{R}^n} f\left(BM_{kj}(c)x\right) dx \end{array} \right.
$$

Proof.

a) For $t \in I$ and $x \in \mathbb{R}^n$ let

$$u(x,t) := M_{jk}(ct)Bx.$$

Then

$$\begin{aligned}\det_n u'(x,t) &= \det_n \left(M_{jk}(ct)B\right) = \\ &= \det_n M_{jk}(ct) \det_n B = \det_n B \neq 0\end{aligned}$$

Let $S_n := \left\{\xi \in \mathbb{R}^n \middle| \|\xi\| = 1\right\}$. The continuous function

$$\|u(\xi,t)\| = \|M_{jk}(ct)B\xi\|, \qquad (\xi,t) \in S_n \times I$$

attains its minimum $m \geq 0$ at a point $(\xi_0, t_0) \in S_n \times I$. Since $\|\xi_0\| = 1$ and $\det_n \left(M_{jk}(ct_0)B\right) \neq 0$, we see $m > 0$. If $0 \neq x \in \mathbb{R}^n$, then $\xi := \frac{x}{\|x\|} \in S_n$ and therefore by homogeneity

$$\|u(x,t)\| = \|M_{jk}(ct)Bx\| \geq m\|x\| \qquad \forall x \in \mathbb{R}^n \times I \tag{3.7}$$

b) Let $R > 0$ be chosen such that $\operatorname{supp} f \subset B_R(0)$. Because of (3.7) $f(Bx)$ and $f\left(M_{jk}(c)Bx\right)$ vanish for $\|x\| \geq \frac{R}{m}$. We set

$$\begin{aligned}G &:= \left\{x \in \mathbb{R}^n \middle| \|x\| < \frac{2R}{m}\right\} \\ U &:= \left\{y \in \mathbb{R}^n \middle| \|y\| > R\right\}.\end{aligned}$$

By (3.7)

$$\|u(x,t)\| \geq m \cdot \frac{2R}{m} = 2R \qquad \forall (x,t) \in \partial G \times I,$$

hence $u(\partial G \times I) \subset U$. By Lemma 2.3

$$\begin{aligned}\int\limits_{\mathbb{R}^n} f\left(M_{jk}(c)Bx\right) \det_n B dx &= \int\limits_{\mathbb{R}^n} f\left(u(x,1)\right) \det_n u'(x,1) dx \\ = \int\limits_{\mathbb{R}^n} f\left(u(x,0)\right) \det_n u'(x,0) dx &= \int\limits_{\mathbb{R}^n} f(Bx) \det_n B dx.\end{aligned}$$

If we devide by $\det B \neq 0$ we see the first identity of (3.6). The second is proved analogously. ∎

Theorem 3.2 *Let $A \in M(n)$, $\det A \neq 0$ and let $b \in \mathbb{R}^n$. Then for $f \in C_c^0(\mathbb{R}^n)$*

$$\int\limits_{\mathbb{R}^n} f(y)dy = \int\limits_{\mathbb{R}^n} f(Ax+b)|\det A|dx. \tag{3.8}$$

Proof.

a) Let $g(x) = f(Ax)$. Then by (3.6) $g \in C_c^0(\mathbb{R}^n)$ and by (1.11)

$$\int\limits_{\mathbb{R}^n} f(Ax)dx = \int\limits_{\mathbb{R}^n} g(x)dx = \int\limits_{\mathbb{R}^n} g(x+b)dx = \int\limits_{\mathbb{R}^n} f(Ax+b)dx \tag{3.9}$$

b) By iterated application of Lemma 3.1 (if $n \geq 2$) we construct in the sequel a diagonal matrix $D \in M(n)$ with

$$\begin{cases} \det_n D = \det_n A \\ \int\limits_{\mathbb{R}^n} f(Dx)dx = \int\limits_{\mathbb{R}^n} f(Ax)dx. \end{cases} \tag{3.10}$$

Consider the last column of A. If $a_{nn} = 0$, there is $k \in \{1, \ldots, n\}$ such that $a_{kn} \neq 0$. Consider

$$A^{(1)} := M_{nk}(1)A, \qquad A^{(1)} = \left(a_{ij}^{(1)}\right)$$

Then $a_{nn}^{(1)} = a_{kn} \neq 0$ and by Lemma 3.1

$$\int\limits_{\mathbb{R}^n} f\left(A^{(1)}x\right)dx = \int\limits_{\mathbb{R}^n} f(Ax)dx.$$
$$\det A^{(1)} = \det A$$

Without loss of generality we may therefore assume $a_{nn} \neq 0$. Then, for $j = 1, \ldots, n-1$ we consider the matrices

$$M_{nj}\left(-\frac{a_{jn}}{a_{nn}}\right) \cdot A$$

Applying successively Lemma 3.1 for $j = 1, \ldots, n-1$, we get a matrix

$$B = (b_{ij}) \in M_n, \qquad \det B = \det A,$$
$$b_{jn} = 0 \text{ for } j = 1, \ldots, n-1, \quad b_{nn} \neq 0$$
$$\int\limits_{\mathbb{R}^n} f(Bx)dx = \int f(Ax)dx.$$

We consider now the matrices

$$B \cdot M_{nk}\left(-\frac{b_{nk}}{b_{nn}}\right), \qquad k = 1, \dots, n-1$$

and apply each time Lemma 3.1. After at most $n-1$ steps we get a matrix

$$\begin{gathered} C = (c_{ij}) \in M(n), \quad \det C = \det A \\ c_{jn} = c_{nj} = 0, \quad j = 1, \dots, n-1, \quad c_{nn} \neq 0 \\ \int\limits_{\mathbb{R}^n} f(Cx)dx = \int\limits_{\mathbb{R}^n} f(Ax)dx \end{gathered}$$

This procedure we apply step by step to all further columns and rows. After at most n^2 steps we find the desired diagonal matrix D with (3.10). By (1.11)

$$\begin{aligned} \int\limits_{\mathbb{R}^n} f(y)dy = |\det{}_n D| \int\limits_{\mathbb{R}^n} f(Dx)dx = \\ = |\det{}_n A| \int\limits_{\mathbb{R}^n} f(Ax)dx. \end{aligned}$$

Combining last identity with (3.9) yields (3.8). ∎

4 Some properties of bi-Lipschitz mappings

Throughout this chapter, let $G \subset \mathbb{R}^n$ be an open set and, if not otherwise stated, let $u : G \to \mathbb{R}^n$ be a *bi-Lipschitz* mapping. Let

$$N \subset G, \qquad |N| = 0$$

such that u is differentiable on $G \setminus N$ (compare Theorem 1.5).

Lemma 4.1 *Let $x_0 \in G \setminus N$. Then there is $\delta = \delta(x_0, G, N, u) > 0$ such that $B_\delta(u(x_0)) \subset u(G)$.*

Proof.

a) By Theorem 1.5 there is $\varepsilon' > 0$ such that $B_{\varepsilon'}(x_0) \subset G$ and there is

$$\varphi : B_{\varepsilon'}(0) \to \mathbb{R}^n, \qquad \lim_{\substack{B_{\varepsilon'}(0) \ni \xi \to 0 \\ \xi \neq 0}} \frac{\varphi(\xi)}{\|\xi\|} = 0$$

such that

$$(4.1)\qquad u(x) = u(x_0) + u'(x_0)(x - x_0) + \varphi(x - x_0) \qquad \forall x \in B_\varepsilon(x_0).$$

We choose $0 < \varepsilon < \varepsilon'$ such that

$$(4.2)\qquad \|\varphi(\xi)\| \leq \frac{L_1}{4}\|\xi\| \qquad \forall \xi \in \overline{B_\varepsilon}(0)$$

(where $0 < L_1 \leq L_2$ by Theorem 1.6). We set $\delta := \frac{L_1}{4}\varepsilon > 0$ and for $y \in B_\delta(u(x_0))$

(4.3)
$$u(x,t) := tu(x) + (1-t)\left[u(x_0) + u'(x_0)(x - x_0)\right] - y \text{ for } (x,t) \in \overline{B_\varepsilon(x_0)} \times I.$$

It is immediately seen that hypotheses (i) and (ii) of Theorem 2.5 hold (with G replaced by $B_\varepsilon(x_0)$). For $\|x - x_0\| = \varepsilon$, $t \in [0,1]$ by (4.1)

$$u(x,t) = u'(x_0)(x - x_0) + t\varphi(x - x_0) + u(x_0) - y$$

By (1.29) and (4.2)

$$(4.4)\qquad \begin{cases} \|u(x,t)\| & \geq \|u'(x_0)(x - x_0)\| - \|\varphi(x - x_0)\| - \|u(x_0) - y\| \\ & \geq L_1 \cdot \varepsilon - \frac{L_1}{4}\varepsilon - \frac{L_1}{4}\varepsilon = \frac{L_1}{2}\varepsilon = 2\delta > 0 \\ & \qquad\qquad \forall \|x - x_0\| = \varepsilon, \quad \forall t \in I \end{cases}$$

Let $U := \{y \in \mathbb{R}^n : \|y\| > \delta\}$. Then $u(x,t) \in U$ for $(x,t) \in \partial B_\varepsilon(x_0) \times I$.
By Theorem 2.5 (applied to $B_\varepsilon(x_0)$) for $f \in C_c^0(\mathbb{R}^n)$ with $f\big|_U = 0$

$$(4.5)\qquad \int\limits_{B_\varepsilon(x_0)} f(u(x) - y)\det_n u'(x)dx =$$
$$= \int\limits_{B_\varepsilon(x_0)} f\left(u(x_0) + u'(x_0)(x - x_0) - y\right) \cdot \det_n u'(x_0)dx$$

b) Assume now that there is no $x \in \overline{B_\varepsilon(x_0)}$ such that $u(x) = y$. Then

$$\min\left\{\|u(x) - y\| \big| x \in \overline{B_\varepsilon(x_0)}\right\} =: \sigma > 0.$$

For $x = x_0$ we see because of $y \in B_\delta(u(x_0))$

$$(4.6)\qquad \sigma \leq \|u(x_0) - y\| < \delta.$$

Let

$$\varphi(t) := \begin{cases} \left(\frac{\sigma}{2} - t\right) & \text{for } 0 \le t \le \frac{\sigma}{2} \\ 0 & \text{for } t \ge \frac{\sigma}{2} \end{cases} \tag{4.7}$$

and $f(z) := \varphi(\|z\|)$ for $z \in \mathbb{R}^n$. $f \in C_c^0(\mathbb{R}^n)$ and because of (4.6) it vanishes on U. Further, $f\,(u(x) - y) = 0$ for $x \in B_\varepsilon(x_0)$ and therefore the integral at the left of (4.5) is zero.

On the other hand, $\det_n u'(x_0) \neq 0$, and there exists a unique $z \in \mathbb{R}^n$ such that $y - u(x_0) = u'(x_0) \cdot z$. By (1.29)

$$\frac{L_1 \varepsilon}{4} = \delta > \|y - u(x_0)\| = \|u'(x_0) \cdot z\| \ge L_1 \|z\|,$$

hence $\|z\| < \frac{\varepsilon}{4}$ and

$$\begin{aligned} x_1 &:= x_0 + z \in B_{\varepsilon/4}(x_0), \\ f\,(u(x_0) + u'(x_0)(x_1 - x_0) - y) &= f(0) = \frac{\sigma}{2}. \end{aligned}$$

By continuity there is $\tau > 0$ such that $B_\tau(x_1) \subset B_\varepsilon(x_0)$ and

$$f\,(u(x_0) + u'(x_0)(x - x_0) - y) \ge \frac{\sigma}{4} \qquad \forall x \in B_\tau(x_1).$$

Then

$$\begin{aligned} |\det_n u'(x_0)| \int_{B_\varepsilon(x_0)} f\,(u(x_0) + u'(x_0)(x - x_0) - y)\,dx \ge \\ \ge \frac{\sigma}{4} |B_\tau(x_1)|\, |\det_n u'(x_0)| > 0, \end{aligned}$$

a contradiction. ■

Corollary 4.2 *Let $N^* := u(N) \subset u(G)$, $|N^*| = 0$ (Lemma 1.7). Then there are open sets V, $V^* \subset \mathbb{R}^n$ such that*

$$\begin{gathered} u(V) = V^* \\ G \setminus N \subset V \subset G, \quad u(G) \setminus N^* \subset V^* \subset u(G). \end{gathered}$$

Proof. For $x \in G \setminus N$ let $B_{\delta_x}(x)$ be according Lemma 4.1 and set

$$V^* := \bigcup_{x \in G \setminus N} B_{\delta_x}(x)$$

Then $V^* \subset \mathbb{R}^n$ is open, $V^* \subset u(G)$ and $V := u^{-1}(V^*) \subset G$ is open too. Clearly by construction, $G \setminus N \subset V$ and because u is injective, $u(G) \setminus N^* \subset V^*$.

Lemma 4.3 *For each $x_0 \in G \setminus N$ there is $\delta = \delta(x_0, G, N, u) > 0$ such that*

i) $B_\delta(u(x_0)) \subset u(G)$

ii) $\operatorname{sgn} \det_n u'(x) = \operatorname{sgn} \det_n u'(x_0)$ *for all* $x \in u^{-1}\left(B_\delta(u(x_0)) \setminus N\right)$

iii) for every $f \in C_c^0(\mathbb{R}^n)$ *with* $\operatorname{supp} f \subset B_\delta(u(x_0))$

$$\int_{u(G)} f(y)dy = \int_G f(u(x))|\det_n u'(x)|dx \tag{4.8}$$

Proof.

a) We proceed similarly as in the proof of Lemma 4.1. We choose $\varepsilon > 0$ and $\delta = \frac{L_1 \varepsilon}{4}$ as there. Now we consider

$$u(x,t) := tu(x) + (1-t)\left[u(x_0) + u'(x_0)(x - x_0)\right] \qquad \forall (x,t) \in \overline{B_\varepsilon(x_0)} \times I.$$

Again hypotheses i) and ii) of Theorem 2.5 hold with G replaced by $B_\varepsilon(x_0)$. By (4.1)

$$u(x,t) = u(x_0) + u'(x_0)(x - x_0) + t\varphi(x - x_0).$$

For $\|x - x_0\| = \varepsilon$, $t \in I$, by (1.29) and (4.2)

$$\|u(x,t) - u(x_0)\| \geq L_1\|x - x_0\| - \frac{L_1}{4}\|x - x_0\| = \frac{3}{4}L_1 \cdot \varepsilon = 3\delta \tag{4.9}$$

Let now

$$U' := \left\{y \in \mathbb{R}^n \middle| \|y - u(x_0)\| > \delta\right\}$$

By (4.9)

$$u(x,t) \in U' \text{ for } (x,t) \in \partial B_\varepsilon(x_0) \times I$$

and by Theorem 2.5 for all $f \in C_c^0(\mathbb{R}^n)$ with $\operatorname{supp} f \in B_\delta(u(x_0))$

(4.10)
$$\int\limits_{B_\varepsilon(x_0)} f(u(x)) \det_n u'(x) dx = \int\limits_{B_\varepsilon(x_0)} f(u(x_0) + u'(x_0)(x - x_0)) \det_n u'(x_0) dx$$

b) Let for $x \in B_\varepsilon(x_0) \setminus N$

$$\sigma := \operatorname{sgn} \det_n u'(x_0)$$

By Theorem 3.2 and (4.9), (4.10) for $f \in C_c^0 \left(B_\delta \left(u(x_0)\right)\right)$

$$\begin{aligned}
\int\limits_{B_\delta(u(x_0))} f(y) dy &= \int\limits_{\mathbb{R}^n} f(y) dy = \int\limits_{\mathbb{R}^n} f(u(x_0) + u'(x_0)(x - x_0)) |\det_n u'(x_0)| dx \\
&= \sigma \int\limits_{B_\varepsilon(x_0)} f(u(x_0) + u'(x_0)(x - x_0)) \det_n u'(x_0) dx \qquad (4.11) \\
&= \sigma \int\limits_{B_\varepsilon(x_0)} f(u(x)) \det_n u'(x) dx
\end{aligned}$$

c) Let $\Omega := u^{-1} \left(B_\delta \left(u(x_0)\right)\right)$. Ω is open, $\Omega \subset B_\varepsilon(x_0)$. Let $h \in C_c^0(\Omega)$. Then $u(\operatorname{supp} h) \subset B_\delta(u(x_0))$ is compact and with

$$f(y) := h(u^{-1}(y))$$

we see $\operatorname{supp} f = u(\operatorname{supp} h)$ and therefore $f \in C_c^0(B_\delta(u(x_0))$ and it is admissible for (4.11). Suppose in addition, that $h \geq 0$, then $f \geq 0$ too. We consider the case $\sigma = 1$. Then by (4.11)

$$0 \leq \int\limits_{B_\varepsilon(x_0)} f(u(x)) \det_n u'(x) dx = \int\limits_{\Omega} h(x) \det_n u'(x) dx. \qquad (4.12)$$

This is true for all $h \in C_c^0(\Omega)$, $h \geq 0$. Let

$$M := \left\{x \in \Omega \setminus N : \det_n u'(x) \leq 0\right\}.$$

Then M is measurable, $|M| \leq |B_\varepsilon(x_0)| < \infty$. Hence the characteristic function χ_M of M is integrable. Then there is a sequence $(h_k) \subset C_c^0(\Omega)$ such that $\int\limits_\Omega |\chi_M - h_k| dx \to 0$. Clearly $|h_k| \in C_c^0(\Omega)$ and

$$\int\limits_\Omega |\chi_M - |h_k|| dx = \int\limits_\Omega ||\chi_M| - |h_k|| dx \leq \int\limits_\Omega |\chi_M - h_k| dx \to 0.$$

Since $|\det u'(x)| \le C \ \forall x \in \Omega$,

$$\int_\Omega |\chi_M(x) - |h_k(x)|| \, |\det_n u'(x)| dx \le$$

$$\le C \int_\Omega |\chi_M(x) - |h_k(x)|| \, dx.$$

By (4.12)

$$\int_{\Omega \setminus N} \chi_M(x) \det_n u'(x) dx = \lim_{k\to\infty} \int_{\Omega\setminus N} |h_k(x)| \det_n u'(x) dx \ge 0$$

On the other hand $\chi_M(x) \det_n u'(x) \le 0$, for $x \in \Omega \setminus N$. Therefore $|M| = 0$ and $\det_n u'(x) > 0$ a. e. in $\Omega \setminus N$, $\sigma \cdot \det_n u'(x) = |\det_n u'(x)|$ and (4.7) follows from (4.11). The case $\sigma = -1$ is treated analogously. ■

Lemma 4.4 *Let $G \subset \mathbb{R}^n$ be in addition connected. Then either $\det_n u'(x) > 0$ or $\det_n u'(x) < 0 \ \forall x \in G \setminus N$.*

Proof. Let $x_i \in G \setminus N (i = 0, 1)$ $x_0 \ne x_1$. Then there is a continuous curve $\gamma : I \to G$ such that $\gamma(i) = x_i (i = 0, 1)$. Let

$$\bar\gamma := \left\{\gamma(t) \middle| t \in [0, 1]\right\}.$$

Then $\bar\gamma \subset G$ is compact, hence $d := \operatorname{dist}(\bar\gamma, \partial G) > 0$ (if $\partial G \ne \phi$; set $d := 1$ if $G = \mathbb{R}^n$). We choose $\delta_i > 0$ according Lemma 4.3 such that $B_{\delta_i}(x_i) \subset u(G)$ $(i = 0, 1)$. Let

$$\delta := \frac{1}{2} \min\left(\delta_1, \delta_2, d, \frac{\|x_0 - x_1\|}{2}\right) > 0$$

Then

$$G' := \left\{x \in G \middle| \operatorname{dist}(x, \bar\gamma) < \delta\right\}$$

is a bounded open set subset of G, $G' \subset\subset G$. We set

$$u(x, t) := u(x) - u(\gamma(t)) \text{ for } (x, t) \in \bar{G}' \times I.$$

Clearly, $\partial G' \subset \left\{x \in G \middle| \operatorname{dist}(x, \bar\gamma) = \delta\right\}$. By compactness of $\bar\gamma$ for every $x \in \partial G'$ there exists $t_0 \in I$ such that

$$\|x - \gamma(t_0)\| = \delta = \inf\left\{\|x - \gamma(t)\| \middle| t \in I\right\}.$$

Then for $t \in I$, $x \in \partial G'$

$$\|u(x, t)\| = \|u(x) - u(\gamma(t))\| \ge L_1 \|x - \gamma(t)\| \ge L_1 \|x - \gamma(t_0)\| = L_1 \cdot \delta \tag{4.13}$$

Let

$$U := \left\{ y \in \mathbb{R}^n \middle| \operatorname{dist}(y, u(\bar{\gamma})) > L_1 \frac{\delta}{2} \right\}$$

By (4.13) $u(\partial G' \times I) \subset U$. Let now $\varphi \in C_c^0(\mathbb{R}^n)$, $\operatorname{supp} \varphi \subset B_\delta(0)$ and

$$\int\limits_{B_\delta(0)} \varphi(y) dy = 1.$$

By Theorem 2.5 (applied to G')

$$\int\limits_{G'} \varphi(u(x,0)) \det_n u'(x,0) dx = \int\limits_{G'} \varphi(u(x,1)) \det_n u'(x,1) dx$$

that is

$$\int\limits_{G'} \varphi(u(x) - u(x_0)) \det_n u'(x) dx = \int\limits_{G'} \varphi(u(x) - u(x_1)) \det_n u'(x) dx$$

Since $\varphi(u(x) - u(x_i))$ vanishes for $x \notin u^{-1}(B_\delta(u(x_i))$, the domain of integration may be replaced by these sets respectively. According Lemma 4.3 ii)

$$\operatorname{sgn} \det_n u'(x) = \operatorname{sgn} \det_n u'(x_i) =: \sigma_i \qquad \forall x \in u^{-1}(B_\delta u(x_i)) \setminus N,\ i = 1, 2$$

Therefore

$$\sigma_0 \int\limits_{u^{-1}(B_\delta(u(x_o)))} \varphi(u(x) - u(x_0)) |\det_n u'(x)| dx =$$

$$= \sigma_1 \int\limits_{u^{-1}(B_\delta(u(x_1)))} \varphi(u(x) - u(x_1)) |\det_n u'(x)| dx$$

We apply Lemma 4.3 to each of this integrals and to the maps $v_i(x) := u(x) - u(x_i)$ and we get

$$\sigma_0 \int\limits_{B_\delta(0)} \varphi(y) dy = \sigma_1 \int\limits_{B_\delta(0)} \varphi(y) dy$$

and therefore $\sigma_0 = \sigma_1$. Since $x_i \in G \setminus N$ $(i = 0, 1)$ had been arbitrary $(x_0 \neq x_1)$, the claim is proved. ■

The next result is a special case of a famous theorem by L. E. J. Brouwer. This theorem guarantees that a continuous, locally injective mapping $f : G \to \mathbb{R}^n$, where $G \subset \mathbb{R}^n$ is open, is an *open mapping*, i. g. it maps open subsets $U \subset G$ onto open subsets $f(U) \subset \mathbb{R}^n$. Especially, $f(G)$ is open. Here, locally injective means that to every $x \in G$ there exists a neighborhood $U_x \subset G$ such that $f\big|_{U_x} : U_x \to f(U_x)$ is injective (see e.g. [Dei], Theorem 4.3).

Clearly, locally bi-Lipschitz mappings are continuous and locally injective.

Theorem 4.5 *Let $G \subset \mathbb{R}^n$ be open and let $u : G \to \mathbb{R}^n$ be a locally bi-Lipschitz mapping. Then u is open.*

Proof.

a) We prove first that for each $x \in G$ there is $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset G$ and there is $\delta_x > 0$ such that $B_{\delta_x}(u(x)) \subset u(B_{\varepsilon_x}(x))$. Without loss of generality let $x_0 = 0 \in G$ and $u(0) = 0$. Otherwise consider

$$\tilde{u}(x) := u(x - x_0) - u(x_0)$$

for $x \in \tilde{G} := \left\{ y + x_0 \middle| y \in G \right\}$. Then there is $\varepsilon > 0$ such that $B_\varepsilon(0) \subset G$ and $u\big|_{B_\varepsilon(0)}$ is bi-Lipschitz (compare Definition 1.3, part 4). Clearly, by continuity, there is a unique bi-Lipschitz extension to $\overline{B_\varepsilon(0)} : 0 < L_1 \le L_2$

$$(4.14) \qquad L_1 \|x - x'\| \le \|u(x) - u(x')\| \le L_2 \|x - x'\| \qquad \forall x, x' \in \overline{B_\varepsilon(0)}$$

Consider $y \in B_\delta(0)$, where $\delta := \frac{L_1 \varepsilon}{2} > 0$ and consider

$$u(x, t) := u(x) - ty \text{ for } (x, t) \in \overline{B_\varepsilon(0)} \times I.$$

Then for $(x, t) \in \partial B_\varepsilon(0) \times I$ by (4.14) and $u(0) = 0$

$$(4.15) \quad \|u(x, t)\| \ge \|u(x)\| - \|y\| \ge L_1 \|x\| - \|y\| > L_1 \cdot \varepsilon - \frac{L_1 \varepsilon}{2} = \frac{L_1 \varepsilon}{2} = \delta$$

Suppose that $y \notin u\left(\overline{B_\varepsilon(0)}\right)$. Then

$$\sigma := \min \left\{ \|u(x) - y\| \middle| x \in \overline{B_\varepsilon(0)} \right\} > 0.$$

Clearly,

$$\sigma \le \|u(0) - y\| = \|y\| < \delta = \frac{L_1 \varepsilon}{2}.$$

Consider again φ defined by (4.7) and $f(z) := \varphi(\|z\|)$ for $z \in \mathbb{R}^n$. Let

$$U := \left\{ y \in \mathbb{R}^n \middle| \|y\| > \delta \right\}.$$

By (4.15) $u(\partial B_\varepsilon(0) \times I) \subset U$, $f \in C_c^0(\mathbb{R}^n)$ and $f\big|_U = 0$. By Theorem 2.5 we see

$$(4.16) \qquad \int\limits_{B_\varepsilon(0)} \varphi(\|u(x)-y\|)\det u'(x) = \int\limits_{B_\varepsilon(0)} \varphi(\|u(x)\|)\det u'(x)dx$$

Then $\varphi(\|u(x)-y\|) = 0\ \forall x \in B_\varepsilon(0)$ and the left integral vanishes. On the other hand $u(0) = 0$, therefore $\varphi(\|u(0)\|) = \frac{\sigma}{2}$. Then there is $0 < \varepsilon' < \varepsilon$ such that

$$\varphi(\|u(x)\|) \geq \frac{\sigma}{4} > 0 \qquad \forall x \in B_{\varepsilon'}(0).$$

Since $B_\varepsilon(0)$ is a domain, by Lemma 4.4 $\det_n u'(x)$ is of constant sign on $B_\varepsilon(0)\setminus N$. Without loss of generality let $\operatorname{sgn}\det_n u'(x) = 1$ for $x \in B_\varepsilon(0)\setminus N$. By (4.16)

$$(4.17) \qquad 0 = \int\limits_{B_\varepsilon(0)} \varphi(\|u(x)\|)\det{}_n u'(x)dx \geq \frac{\sigma}{4}\int\limits_{B_{\varepsilon'}(0)} \det{}_n u'(x)dx.$$

Because of the first inequality in (1.19), $\det_n u'(x) \neq 0\ \forall x \in B_{\varepsilon'}(0)\setminus N$. If we suppose $\int\limits_{B_{\varepsilon'}(0)} \det_n u'(x)dx = 0$, we would conclude (observe $\det_n u'(x) \geq 0$ in $B_{\varepsilon'}(0)$) $\det_n u'(x) = 0$ a. e. in $B_{\varepsilon'}(0)$, a contradiction. Therefore $\int\limits_{B_{\varepsilon'}(0)} \det_n u'(x)dx > 0$ and we get by (4.17) a contradiction. Since $y \in \overline{B_\delta(0)}$ was arbitrary we see $B_\delta(0) \subset \overline{B_\delta(0)} \subset u(B_\varepsilon(x))$.

b) By part a) of proof, for every $x \in G$ there is $\delta_x > 0$ such that $B_{\delta_x}(u(x)) \subset u(G)$. Then $\bigcup\limits_{x\in G} B_{\delta_x}(u(x))$ is open in $\mathbb{R}^n$ and clearly

$$u(G) \subset \bigcup_{x\in G} B_{\delta_x}(u(x)) \subset u(G).$$

Therefore $u(G)$ is open in $\mathbb{R}^n$.

c) If $G' \subset G$ is any open set, then we apply parts a) and b) of proof to G' in place of G and we see by b) that $u(G')$ is open in $\mathbb{R}^n$. ■

Theorem 4.6 *Let $G \subset \mathbb{R}^n$ be open and let $u : G \to \mathbb{R}^n$ be a locally bi-Lipschitz mapping. Then*

i) $G^* := u(G)$ is open

ii) For every $x_0 \in G$ there are open neighborhoods $U_{x_0} \subset G$ and $V_{y_0} \subset G^* (y_0 := u(x_0))$ such that $u\big|_{U_{x_0}} : U_{x_0} \to V_{y_0}$ is bijective and there is a set $M \subset V_{y_0}$, $|M| = 0$, such that

$$v : V_{y_0} \to U_{x_0}, \qquad v := \left(u\big|_{U_{x_0}}\right)^{-1}$$

is differentiable on $V_{y_0}\setminus M$.

Proof. Due to Theorem 4.5, G^* is open. There is a neighborhood U_{x_0} such that $u\big|_{U_{x_0}} : U_{x_0} \to \mathbb{R}^n$ is a bi-Lipschitz mapping, hence $u : U_{x_0} \to V_{y_0} := u(U_{x_0})$ is bijective and V_{y_0} is open by Theorem 4.5. Due to Rademacher's theorem (Theorem 1.4) there is $M \subset V_{y_0}$, $|M| = 0$ such that $v : V_{y_0} \setminus M \to U_{y_0}$ is differentiable. ■

Theorem 4.6 is a generalization of the classical theorem on local diffeomorphisms.

Theorem 4.7 *Let $G \subset \mathbb{R}^n$ be open and let $u : G \to \mathbb{R}^n$ be an injective, locally bi-Lipschitz mapping. Then*

i) $G^* := u(G)$ is open,
$u : G \to G^*$ is bijective and differentiable on $G \setminus N$, $|N| = 0$.

ii) There is a set $M \subset G^*$, $|M| = 0$ such that $v := u^{-1}$ is differentiable on $G^* \setminus M$.

Proof. By Theorem 4.5, G^* is open and because of global injectivity, $u : G \to G^*$ is bijective. Let $(G_k^*) \subset G^*$ be an exhausting sequence for G^*:

$$G_k^* \subset\subset G_{k+1}^* \subset\subset G^* \quad \forall k \in \mathbb{N}, \quad \bigcup_{k=1}^{\infty} G_k^* = G^*$$

Due to Theorem 4.6 for each $y_0 \in \overline{G_k^*}$ there is an open V_{y_0} and a set M_{y_0}, $|M_{y_0}| = 0$ such that $v\big|_{V_{y_0} \setminus M}$ is differentiable. By compactness of $\overline{G_k^*}$ there are $y_i^{(k)} \in \overline{G_k^*}$ ($i = 1, \ldots, m_k$) such that $\overline{G_k^*} \subset \bigcup_{i=1}^{m_k} V_{y_i^{(k)}}$, $M_i^{(k)} \subset V_{y_i}^{(k)}$, $|M_i^{(k)}| = 0$ and $v\big|_{V_{y_i}^{(k)} \setminus M_i^{(k)}}$ is differentiable. Then $v\big|_{G_k^* \setminus M_k}$ is differentiable, where $M_k := \bigcup_{i=1}^{m_k} M_i^{(k)}$, $|M_k| = 0$. Let $M := \bigcup_{k=1}^{\infty} M_k \subset G^*$, $|M| = 0$. Then v is differentiable on $G^* \setminus M$. ■

As a further corollary we derive

Theorem 4.8 *(implicit function theorem) Let $m, n \in \mathbb{N}$ and let $U_m \subset \mathbb{R}^m$, $U_n \subset \mathbb{R}^n$ be open sets, $U := U_m \times U_n \subset \mathbb{R}^{n+m}$. Let*

$$f : U \to \mathbb{R}^n$$

and suppose that

i) there is $a \in U_m$, $b \in U_n$ such that

$$f(a, b) = 0$$

ii) there exists $L > 0$ such that

$$\|f(x, y) - f(x', y')\|_n \leq L(\|x - x'\|_m + \|y - y'\|_n) \qquad \forall x, x' \in U_m, \qquad \forall y, y' \in U_n$$

iii) there exists $K > 0$ such that

$$\|f(x,y) - f(x,y')\|_n \geq K\|y - y'\|_n \qquad \forall y, y' \in U_n \quad \forall x \in U_m.$$

Then there exists an open neighborhood $V_m \subset U_m$ of a and a Lipschitz mapping

$$g : V_m \to \mathbb{R}^n$$

such that

1^0 $(x, g(x)) \in U \quad \forall x \in V_m$

2^0 *there is a subset $N_m \subset V_m$, $|N_m| = 0$ and g is differentiable on $V_m \setminus N_m$*

3^0 $g(a) = b$ *and* $f(x, g(x)) = 0 \quad \forall x \in V_m$

4^0 $\{(x,y) \in V_m \times U_n \big| f(x,y) = 0\} = \{(x, g(x)) \big| x \in V_m\}$

Proof.

a) Let $h : U \to \mathbb{R}^{m+n}$ be defined by

$$\begin{aligned} h_i(x,y) &:= \alpha x_i \text{ for } i = 1, \ldots, m \\ & \qquad\qquad\qquad\qquad (x,y) \in U_m \times U_n = U \\ h_{m+k}(x,y) &:= f_k(x,y) \text{ for } k = 1, \ldots, n \end{aligned}$$

where $\alpha = \left[\frac{1}{2}K^2 + L^2\right]^{\frac{1}{2}} > 0$.
For (x,y), $(x',y') \in U$

$$\|f(x,y) - f(x',y')\|_n \geq \big|\|f(x,y) - f(x,y')\|_n - \|f(x,y') - f(x',y')\|_n\big|.$$

Then by ii) and iii) (for $\varepsilon > 0$: $2|a \cdot b| \leq \varepsilon a^2 + \varepsilon^{-1}b^2$; choose $\varepsilon := \frac{1}{2}$)

$$\begin{aligned} \|f(x,y) - f(x',y')\|_n^2 &\geq (\|f(x,y) - f(x,y')\|_n - \|f(x,y') - f(x,y')\|_n)^2 \\ &\geq \frac{1}{2}\|f(x,y) - f(x,y')\|_n^2 - \|f(x,y') - f(x,y')\|_n^2 \\ &\geq \frac{1}{2}K^2\|y - y'\|_n^2 - L^2\|x - x'\|_m^2 \end{aligned}$$

Hence

$$\begin{aligned} \|h(x,y) - h(x',y')\|_{n+m}^2 &= \alpha^2\|x - x'\|_m^2 + \|f(x,y) - f(x',y')\|_n^2 \\ &\geq (\alpha^2 - L^2)\|x - x'\|_m^2 + \frac{1}{2}K^2\|y - y'\|_n^2 \\ &\geq \frac{1}{2}K^2\|(x,y) - (x',y')\|_{n+m}^2 \end{aligned}$$

and therefore

(4.18)

$$\|h(x,y)-h(x',y')\|_{n+m} \geq L_1\|(x,y)-(x',y')\|_{n+m}, \quad \forall (x,y),(x',y') \in U$$

$$L_1 := \frac{1}{2}K \cdot \sqrt{2} > 0.$$

(4.19)

$$\begin{aligned}\|h(x,y)-h(x',y')\|_{n+m} &\leq \alpha\|x-x'\|_m + L(\|x-x'\|_m + \|y-y'\|_m)\\ &\leq L_2\|(x,y)-(x',y')\|_{n+m} \quad \forall (x,y),(x',y') \in U\end{aligned}$$

where $L_2 := \alpha + 2L > 0$.

Because of (4.18), (4.19) $h : U \to \mathbb{R}^n$ is a bi-Lipschitz mapping and by Theorem 4.7 $U^* := h(U)$ is open. Let

$$v := h^{-1} : U^* \to U$$

and by (4.18), (4.19) for $z \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$, $(z,\omega) \in U^*$

(4.20)

$$\begin{aligned}L_2^{-1}\|(z,\omega)-(z',\omega')\|_{n+m} &\leq \|v(z,\omega)-v(z',\omega')\|_{n+m}\\ &\leq L_1^{-1}\|(z,\omega)-(z',\omega')\|_{n+m} \quad \forall (z,\omega),(z',\omega') \in U^*.\end{aligned}$$

b) Since $h(a,b) = (\alpha a, 0) \in U^*$ and U^* is open, there is $\delta > 0$ such that

$$B_\delta(\alpha a, 0) = \left\{(z,\omega) \in \mathbb{R}^{n+m} \big| \|(z,\omega)-(\alpha a,0)\|_{n+m} < \delta\right\} \subset U^*.$$

Let

$$B_m := \left\{z \in \mathbb{R}^m \Big| \|z-\alpha a\|_m < \frac{\delta}{\sqrt{2}}\right\}$$

$$B_n := \left\{\omega \in \mathbb{R}^n \Big| \|\omega\|_n < \frac{\delta}{2}\right\}$$

Obviously $(\alpha a, 0) \in B_m \times B_n \subset U^*$. Let

$$V_m := \left\{x \in \mathbb{R}^m \Big| \|x-a\|_m < \frac{\delta}{\alpha\sqrt{2}}\right\}$$

Observe that $x \in V_m$ if and only if $\alpha x \in B_m$. We write $v = \left(\tilde{v}, \tilde{\tilde{v}}\right)$, where

$$\tilde{v} := (v_1, \ldots, v_m), \qquad \tilde{\tilde{v}} := (v_{m+1}, \ldots, v_{m+n}).$$

$(z, \omega) \in U^*$ if and only if there is a unique $(x, y) \in U$ such that

$$(z, \omega) = h(x, y) = (\alpha x, f(x, y)).$$

Therefore $z = \alpha x$, $\omega = f(x, y)$, hence

$$\tilde{v}(z, \omega) = x, \qquad \tilde{\tilde{v}}(z, \omega) = y$$

Since $h(a, b) = (\alpha a, f(a, b)) = (\alpha a, 0)$,

$$\tilde{\tilde{v}}(\alpha a, 0) = b.$$

If $(x, \omega) \in V_m \times B_n$, then $(\alpha x, \omega) \in B_m \times B_n$ and

$$(\alpha x, \omega) = h(v(\alpha x, \omega)) = (\alpha x, f(x, \tilde{\tilde{v}}(\alpha x, \omega)),$$

hence

$$\omega = f(x, \tilde{\tilde{v}}(\alpha x, \omega))$$

If $\omega = 0$, $0 \in B_n$, $x \in V_m$, then

$$\big(x, \tilde{\tilde{v}}(\alpha x, 0)\big) \in U.$$

We set

$$g(x) := \tilde{\tilde{v}}(\alpha x, 0) \text{ for } x \in V_m \tag{4.21}$$

g is continuous, $(x, g(x)) \in U$ and

$$0 = f(x, g(x)) \qquad \forall x \in V_m.$$

Further, $g(a) = \tilde{\tilde{v}}(\alpha a, 0) = b$. If $(x, y) \in V_m \times U_n \subset U$ and $f(x, y) = 0$, then $h(x, y) = (\alpha x, 0) \in U^*$, therefore $v(\alpha x, 0) = (x, \tilde{\tilde{v}}(\alpha x, 0)) = (x, g(x))$. Then trivially 4^0 is true. By (4.20) for $x, x' \in V_m$

$$\begin{aligned}\|g(x) - g(x')\|_n &= \|\tilde{\tilde{v}}(\alpha x, 0) - \tilde{\tilde{v}}(\alpha x', 0)\|_n \leq \\ &\leq \|v(\alpha x, 0) - v(\alpha x', 0)\|_{n+m} \leq L_1^{-1} \|\alpha(x - x')\|_m = \\ &= L_1^{-1} \alpha \|x - x'\|_m\end{aligned}$$

By Rademacher's theorem there is $N_m \subset V_m$, $|N_m|_m = 0$, such that g is differentiable on $V_m \setminus N_m$. ■

Corollary 4.9 *Let the hypotheses of Theorem 4.8 hold true. Then there is* $N \subset \mathbb{R}^{n+m}$, $|N|_{n+m} = 0$ *such that* f *is differentiable on* $U \setminus N$. *For* $(x, y) \in U \setminus N$ *let*

$$(D_y f)(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(x,y) & \cdots & \frac{\partial f_1}{\partial y_n}(x,y) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1}(x,y) & \cdots & \frac{\partial f_n}{\partial y_n}(x,y) \end{pmatrix}$$

Then

$$K\|\eta\|_n \le \|(D_y f)(x,y)\eta\|_n \le L\|\eta\|_n \qquad \forall \eta \in \mathbb{R}^n,\ \forall (x,y) \in U \setminus N \tag{4.22}$$

and

$$\text{rank } (D_y f)(x,y) = n \qquad \forall (x,y) \in U \setminus N.$$

Proof. The proof is completely analogous to the proof of Theorem 1.5 ■

Clearly f and g are a.e. differentiable and $f(x, g(x)) = 0\ \forall x \in V_m$ too. At points $x \in V_m \setminus N_m$ such that $(x, g(x)) \notin N \subset \mathbb{R}^{n+m}$ (N denoting the set where f is not differentiable), then the chain rule may be applied to $f(x, g(x))$. But it may happen that $(x, g(x)) \in N\ \forall x \in V_m$. A simple example:
Let $U := \left\{(x,y) \in \mathbb{R}^2 \middle| |x| < \frac{1}{2}, |y| < 1\right\}$

$$f(x,y) := \begin{cases} (1+x)y & \text{for } y \ge 0 \\ \\ (1+x)2y & \text{for } y < 0 \end{cases} \qquad\qquad (x,y) \in U$$

Then $f(0,0) = 0$,

$$\begin{aligned} &\|f(x,y) - f(x',y')\|_1 \le 3(\|x-x'\|_1 + \|y-y'\|_1) \\ &\|f(x,y) - f(x,y')\|_1 \ge \frac{1}{2}\|y-y'\|_1 \quad \forall |x| < \frac{1}{2} \quad \forall |y| < 1. \end{aligned}$$

All hypotheses of Theorem 4.8 hold true. Let $V_m := \left\{x \in \mathbb{R} \middle| |x| < \frac{1}{2}\right\}$, $g : V_m \to \mathbb{R}$, $g(x) := 0$. Then $f(x, g(x)) = 0\ \forall x \in V_m$. The set where f is not differentiable is

$$N := \left\{(x,y) \in U \middle| y = 0\right\} \subset \mathbb{R}^2, \quad |N|_2 = 0$$

and $\left\{(x, g(x)) \middle| x \in V_m\right\} = N$. The chain rule is *not* applicable.

5 The change of variables formula for (locally) bi-Lipschitz mappings.

We assume firstly

(A.1) Let $G \subset \mathbb{R}^n$ be an open set

(A.2) Let $u : G \to \mathbb{R}^n$ be a bi-Lipschitz mapping. Let $N \subset G$, $|N| = 0$, be the set where u is not differentiable

For $x \in G$ we write again

$$u'(x) := \begin{cases} (D_i u_k(x))_{i,k=1,\dots,n} & \text{if } x \in G \setminus N \\ 0 & \text{if } x \in N \end{cases}$$

Lemma 5.1 *Assume (A.1) and (A.2). Then $u(G)$ is open and for every $x_0 \in G \setminus N$ there is $\delta = \delta(x_0, G, N) > 0$ such that*

i) $B_\delta(u(x_0)) \subset u(G)$

ii) for every $f \in L^1(\mathbb{R}^n)$ such that $f(y) = 0$ for $y \in \mathbb{R}^n \setminus B_\delta(u(x_0))$ $f(y(.))|\det_n u'(.)| \in L^1(G)$ and

$$\int\limits_{u(G)} f(y)dy = \int\limits_G f(y(x))|\det_n u'(x)|dx \tag{5.1}$$

Proof.

i) By Theorem 4.5 $u(G)$ is open and therefore there exists $\delta > 0$ sucht that i) holds true. Assume in addition $f \geq 0$ and $f \in L^\infty(\mathbb{R}^n)$. For $k \in \mathbb{N}$, $k \geq 2$ let

$$\delta_k := \left(1 - \frac{1}{k}\right)\delta$$

and let χ_k denote the characteristic function of $B_{\delta_k}(u(x_0))$, let $f_k := f \cdot \chi_k$. For $k \in \mathbb{N}$ fixed, let (ϱ_ν) be a sequence with $0 < \varrho_\nu < \frac{1}{2}(\delta - \delta_k)$, $\varrho_\nu \to 0$, and consider the mollified functions $(f_k)_{\varrho_\nu} \in C_c^0(\mathbb{R}^n)$, $\mathrm{supp}(f_k)_{\varrho_\nu} \subset B_\delta(u(x_0))$

$$\int\limits_{\mathbb{R}^n} |f_k - (f_k)_{\varrho_\nu}|\, dy \to 0 \quad (u \to \infty).$$

There is a subsequence (again denoted by ϱ_ν) such that $(f_k)_{\varrho_\nu} \to f_k$ a. e. in $\mathbb{R}^n$. Therefore there is a set $M^* \subset B_\delta(u(x_0))$, $|M^*|_n = 0$ and $(f_k)_{\varrho_\nu}(y) \to f_k(y)$ for $y \in B_\delta(u(x_0)) \setminus M^*$. Let $v := u^{-1} : u(G) \to G$. By Lemma 1.7 with

$$M := v(M^*) \subset u^{-1}(B_\delta(u(x_0)), \quad |M|_n = 0.$$

For $x \in u^{-1}\left(B_\delta(u(x_0))\right) \setminus M$

$$(f_k)_{\varrho_\nu}(u(x)) \to (f_k)(u(x)) \quad (\nu \to \infty) \tag{5.2}$$

By Lemma 4.3 for all $\nu \in \mathbb{N}$

$$(5.3) \qquad \int\limits_{u(G)} (f_k)_{\varrho_\nu}(y)dy = \int\limits_G (f_k)_{\varrho_\nu}(u(x))|\det_n u'(x)|dx$$

Since $|(f_k)_{\varrho_\nu}(y)| \leq \|f\|_{L^\infty(\mathbb{R}^n)}$, by Lebesgue's theorem the left hand side of (5.3) tends to $\int\limits_{u(G)} f_k(y)dy$. By (5.2) $f_k \circ u$ is measurable and because of

$$|\det_n u'(x)| \leq n! L_2^n$$

again by Lebesgue's theorem the right hand side of (5.3) converges too. Therefore (5.1) holds with f replaced by f_k. Now $\int\limits_G f_k(y)dy \to \int\limits_G f(y)dy$ by Levi's theorem. Since $f_k(u(x)) \to f(u(x))\ \forall x \in u^{-1}(B_\delta(u(x_0)))$, $(k \to \infty)$ we see the measurability of $f \circ u$ and again by Levi's theorem

$$\int\limits_G f_k(u(x))|\det_n u'(x)|dx \to \int\limits_G f(u(x))|\det_n u'(x)|dx.$$

ii) If $0 \leq f \in L^1(B_\delta(u(x_0)))$, then we consider for $j \in \mathbb{N}$

$$f^{(j)}(y) := \begin{cases} f(y) & \text{if } f(y) \leq j \\ j & \text{if } f(y) > j \end{cases}$$

By part i) of proof the assertion is true for all $f^{(j)}$. Again using at both sides of (5.1) Levi's theorem, we finally prove the Lemma for $0 \leq f \in L^1(B_\delta(x_0))$.

iii) If $f \in L^1(B_\delta(u(x_0)))$, apply ii) to $0 \leq f_+, f_- \in L^1(B_\delta(u(x_0)))$, $f = f_+ - f_-$. ■

Theorem 5.2 *Assume (A.1) and (A.2). Then $u(G)$ is open and for each $f \in L^1(u(G))$*

$$(5.4) \qquad \begin{aligned} &1^0 \quad (f \circ u)|\det_n u'(.)| \in L^1(G) \\ &2^0 \int\limits_{u(G)} f(y)dy = \int\limits_G f(u(x))|\det_n u'(x)|dx. \end{aligned}$$

Proof.

i) $G^* := u(G) \subset \mathbb{R}^n$ is open by Theorem 4.5 and by Lemma 1.7 $N^* := u(N)$ has n-dimensional measure zero. Therefore, for each $k \in \mathbb{N}$ there exists an open set $O_k \subset \mathbb{R}^n$ such that $N^* \subset O_k$ and $|O_k|_n \leq \frac{1}{k}$. Let

$$U_k^* := \bigcap_{j=1}^k O_j \cap G^*, \quad |U_k^*|_n \leq |O_k|_n \leq \frac{1}{k}$$

Then U_k^* is open, $N^* \subset U_k^* \subset G^*$. Finally let $U^* := \bigcap\limits_{k=1}^{\infty} U_k^*$. Then

$$N^* \subset U^* \subset G^*, \qquad |U^*|_n = 0.$$

Let χ_k resp. λ denote the characteristic function for $G^* \setminus U_k^*$ resp. $G^* \setminus U^*$. Then $\lambda_k(y) \to \lambda(y)\ \forall y \in G^*$ $(k \to \infty)$. Let $U := u^{-1}(U^*) = v(U^*)$, if $v := u^{-1}$ denotes the inverse mapping. Since v is Lipschitz, $|U|_n = 0$.

ii) Let $G_1^* \subset\subset G^*$ and let $0 \le f \in L^1(G_1^*)$. We extend f by zero to $\mathbb{R}^n$. Since U_k^* is open, the set

$$K_k := \overline{G_1^*} \cap (\mathbb{R}^n \setminus U_k^*) \subset G^* \setminus N^*$$

is compact and $f\chi_k$ vanishes on $\mathbb{R}^n \setminus K_k$. If $y \in K_k$, there is $x \in G \setminus N$ such that $u(x) = y$. According Lemma 5.1 we choose $\delta_x > 0$ such that $B_{\delta_x}(u(x)) \subset u(G)$ and (5.1) is true for $f \in L^1\left(B_{\delta_x}(u(x))\right)$. Then

$$\left\{ B_{\delta_x}(u(x)) \middle| x \in u^{-1}(K_k) \right\}$$

forms an open covering of the compact set K_k. Then there is $m_k \in \mathbb{N}$ such that

$$K_k \subset \bigcup_{i=1}^{m_k} B_i, \qquad B_i = B_{\delta_{x_i}}(u(x_i)),\ x_i \in u^{-1}(K_k).$$

Let χ_i denote the characteristic function of B_i.
Then for $y \in K_k : \sum\limits_{j=1}^{m_k} \chi_j(y) \ge 1$. We set

$$\varphi_i(y) := \frac{\chi_i(y)}{\sum\limits_{j=1}^{m_k} \chi_j(y)}$$

Then $\varphi_i(y) = 0$ for $y \notin B_i$, $\sum\limits_{i=1}^{m_k} \varphi_i(y) = 1\ \forall y \in K_k$. Further, $f\chi_{G_1^*}\chi_k\varphi_i \in L^1(\mathbb{R}^n)$ and it vanishes on $\mathbb{R}^n \setminus B_i$. By Lemma 5.1

$$f\chi_{G_1^*}\chi_k\varphi_i \circ u |\det{}_n' u(.)| \in L^1(G)$$

and (5.4) is true for all $i = 1, \ldots, m_k$ with f replaced by $f\chi_k\varphi_i$. If we observe that

$$\sum_{i=1}^{m_k} f\chi_k\varphi_i = f\chi_k$$

after summation we get

$$(5.5) \quad \int_{u(G)} f(y)\chi_{G_1^*} \cdot \chi_k(y)dy = \int_G f(u(x))\chi_{G_1^*}(u(x))\chi_k(u(x))|\det{}_n u'(x)|dx.$$

The sequence $f\chi_{G_1^*}\chi_k$ is monotonically increasing and it converges pointwise to

$$f\chi_{G_1^*}\chi = f\chi_{G_1^*\setminus U^*} \leq f \in L^1(G^*).$$

By Levi's theorem the left hand side of (5.5) tends to $\int\limits_{u(G)} f(y)\chi_{G_1^*\setminus U^*}(y)dy$. For $x \in u^{-1}(G_1^* \setminus U^*)$ we see

$$f(u(x))\chi_{G_1^*}(u(x))\chi_k(u(x)) \to f(u(x))\chi_{G_1^*\setminus U^*}(u(x)).$$

monotonically for $k \to \infty$. If $x \notin u^{-1}(G_1^* \setminus U^*)$ all these numbers are zero. Again by Levi's theorem the integrals at the right hand side converge and finally by (5.4)

$$(5.6) \quad \int_{u(G)} f(y)\chi_{G_1^*\setminus U^*}(y)dy = \int_{u(G)} f(u(x))\chi_{G_1^*\setminus U^*}(u(x))|\det_n u'(x)|dx.$$

Since $|U^*|_n = 0$, $\chi_{G_1^*\setminus U^*}(y) = \chi_{G_1^*}(y)$ for almost all $y \in \mathbb{R}^n$, therefore

$$\int_{u(G)} f(y)\chi_{G_1^*\setminus U^*}(y)dy = \int_{u(G)} f(y)\chi_{G_1^*}(y)dy.$$

Let $G_1 := u^{-1}(G_1^*)$, $U := u^{-1}(U^*)$. Then

$$\chi_{G_1^*\setminus U^*}(u(x)) = \chi_{G_1\setminus U_1}(x) \quad \text{and} \quad \chi_{U^*}(u(x)) = \chi_U(x).$$

Since $\chi_{G_1\setminus U_1} = \chi_{G_1}$ a. e. in G,

$$\int_G f(u(x))\chi_{G_1^*\setminus U^*}(u(x))|\det{}_n u'(x)|dx = \int_G f(u(x))\chi_{G_1^*}(u(x)|\det{}_n u'(x)|dx$$

Finally we get for $G_1^* \subset\subset G^*$ and $0 \leq f \in L^1(\mathbb{R}^n)$, $f(y) = 0$ for $y \notin G_1^*$ identity (5.4) from (5.6).

iii) Let now $0 \le f \in L^1(G^*)$. Let (G^*_j) be an exhausting sequence for G^*:

$$G^*_j \subset\subset G^*_{j+1} \subset\subset G \quad \forall j \in \mathbb{N}, \qquad \bigcup_{j=1}^{\infty} G^*_j = G^*.$$

Let χ_j denote the characteristic functions of G^*_j. By part ii) of proof

$$[(f\chi_j) \circ u] \, |\det_n u'(.)| \in L^1(G)$$

and formula (5.4) is valid for $f\chi_j$ in place of f. Since $(f\chi_j)$ is monotonically increasing on $u(G)$, $(f\chi_j)(y) \to f(y)$ $\forall y \in u(G)$,

$$0 \le \int\limits_{u(G)} f(y)\chi_j(y)dy \le \int\limits_{u(G)} f(y)dy \qquad \forall j \in \mathbb{N},$$
$$\int\limits_{u(G)} f(y)\chi_j(y)dy \to \int\limits_{u(G)} f(y)dy.$$

On the other hand

$$f(u(x))\chi_j(u(x)) \to f(u(x)) \quad (j \to \infty) \quad \forall x \in G.$$

Again by Levi's theorem and (5.4) for $f\chi_j$ we see i) and ii) in case of $0 \le f \in L^1(u(G))$.

iv) If $f \in L^1(u(G))$ is arbitrary, we decompose $f = f_1 - f_2$ with $0 \le f_i \in L^1(u(G))$ and we apply part iii) of proof separately to f_i. ∎

Theorem 5.3 *Let $G \subset \mathbb{R}^n$ be an open set and let $u : G \to \mathbb{R}^n$ be injective and locally bi-Lipschitz. Then $u(G)$ is open and for each $f \in L^1(G)$*

1^0 $(f \circ u)|\det_n u'(.)| \in L^1(G)$

2^0 $\int\limits_{u(G)} f(y)dy = \int\limits_{G} f(u(x))|\det_n u'(x)|dx$

Proof.

i) By Theorem 4.5 the mapping u is open. Let (G_j) be an exhausting sequence for G,

$$G_j \subset\subset G_{j+1} \subset\subset G \quad \forall j \in \mathbb{N}, \quad \bigcup_{j=1}^{\infty} G_j = G.$$

Let $G^*_j := u(G_j)$ for $j \in \mathbb{N}$, $G^* = u(G)$. Then G^*_j is open, $G^*_j \subset\subset G^*_{j+1} \subset\subset G^*$, $\bigcup\limits_{j=1}^{\infty} G^*_j = G^*$. Let χ_j denote the characteristic function of G^*_j. Clearly, $\chi_j \circ u$ is the characteristic function of G_j.

ii) By Theorem 1.4 b) $u\big|_{\overline{G_j}} : \overline{G_j} \to \mathbb{R}^n$ is a bi-Lipschitz mapping. Let now $f \in L^1(u(G))$ and $f \geq 0$. By Theorem 5.2 (applied to G_j)

$$\int_{G_j^*} f(y)dy = \int_{G_j} f(u(x))|\det_n u'(x)|dx$$

Using the characteristic functions we rewrite this identity

$$\int_{G^*} f(y)\chi_j(y)dy = \int_{G} f(u(x))\chi_j(u(x)|\det_n u'(x)|dx. \tag{5.7}$$

The sequence $f\chi_j$ is monotonically increasing and tends pointwise to f.

$$0 \leq \int_{G^*} f(y)\chi_j(y)dy \leq \int_{G^*} f(y)dy \tag{5.8}$$

By Levi's theorem,

$$\lim_{j\to\infty} \int_{G^*} f(y)\chi_j(y)dy = \int_{G^*} f(y)dy. \tag{5.9}$$

Similarly the sequence $(f \circ u)(\chi_j \circ u)|\det_n u'(.)|$ is monotonically increasing and it converges to $(f \circ u)|\det_n u'(.)|$ in $G \setminus N$, where $N \subset G$ denotes the set such that $|N|_n = 0$ and u is differentiable in $G \setminus N$. Because of (5.7), (5.8) the sequence of the integrals extended over $G \setminus N$ remains bounded. Again by Levi's theorem, we see assertion 1^0. Further,

$$\lim_{j\to\infty} \int_{G\setminus N} f(u(x))\chi_j(u(x))|\det_n u'(x)dx = \int_{G\setminus N} f(u(x))|\det_n u'(x)|dx =$$

$$= \int_{G} f(u(x))|\det_n u'(x)|$$

and finally because of (5.7) and (5.9) we see 2^0.

iii) If $f \in L^1(u(G))$ is arbitrary, we split

$$f = f_+ - f_-, \qquad 0 \leq f_+, f_- \in L^1(u(G))$$

and apply part ii) of proof to f_+ and f_-. By additivity of the integral, 1^0 and 2^0 follows for f. ■

6 Some further applications of the homotopy theorem

6.1 Brouwer's fixed point theorem

Let

$$B := B_1(0) = \left\{x \in \mathbb{R}^n \middle| \|x\| < 1\right\}.$$

Theorem 6.1 *Every continuous mapping $u : \bar{B} \to \bar{B}$ has at least one fixed point.*

Proof.

i) First we prove that it suffices to consider those continuous $u : \bar{B} \to \bar{B}$ with the additional property $u \in C^\infty(B; \bar{B})$. Let $\bar{u} : \mathbb{R}^n \to \bar{B}$ be defined by

$$\bar{u}(x) := \begin{cases} u(x) & \text{for } x \in \bar{B} \\ u\left(\frac{x}{\|x\|}\right) & \text{for } x \in \mathbb{R}^n \setminus \bar{B} \end{cases}$$

Obviously $\bar{u}(x) \in \bar{B}$ for all $x \in \mathbb{R}^n$ and $\bar{u}$ is continuous in $B \cup (\mathbb{R}^n \setminus \bar{B})$. If $(x_k) \subset \mathbb{R}^n$ is a sequence such that $x_k \to x_0 \in \partial B$, then we may assume $x_k \neq 0$ for all k. Let $\xi_k := \frac{x_k}{\|x_k\|}$. Then $\xi_k \to \frac{x_0}{\|x_0\|} = x_0$,

$$\bar{u}(x_0) = u(x_0) = \lim_{\nu \to \infty} u(\xi_0) = \lim_{\nu \to \infty} \bar{u}(x_\nu)$$

proving continuity of $\bar{u}$ for $x_0 \in \partial B$. For $\varrho > 0$ consider the mollified functions

$$\bar{u}_{i\varrho}(x) = \int\limits_{\mathbb{R}^n} \omega_\varrho(x-y)\bar{u}_i(y)dy, \quad i = 1, \ldots, n,\ x \in \mathbb{R}^n.$$

$\bar{u}_\varrho := (\bar{u}_{1\varrho}, \ldots, \bar{u}_{n\varrho}) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Clearly, $\bar{u}_\varrho \to \bar{u} = u$ uniformly in $\bar{B}$. By Schwarz' inequality

$$\begin{aligned} |\bar{u}_{i\varrho}(x)| &\leq \left| \int\limits_{\mathbb{R}^n} \omega_\varrho(x-y)^{\frac{1}{2}} \omega_\varrho(x-y)^{\frac{1}{2}} \bar{u}_i(y)dy \right| \\ &\leq \left(\int\limits_{\mathbb{R}^n} \omega_\varrho(x-y)dy \right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^n} \omega_\varrho(x-y) \cdot \bar{u}_i(y)^2 dy \right)^{\frac{1}{2}} \end{aligned}$$

Since $\int\limits_{\mathbb{R}^n} \omega_\varrho(x-y)dy = 1$ and $\|\bar{u}(y)\| \leq 1$,

$$\|\bar{u}_\varrho(x)\|^2 = \sum_{i=1}^n |\bar{u}_{i\varrho}(x)|^2 \le \int\limits_{\mathbb{R}^n} \omega_\varrho(x-y) \sum_{i=1}^n \bar{u}_i(y)^2 dy$$
$$\le \int\limits_{\mathbb{R}^n} \omega_\varrho(x-y) dy = 1,$$

hence $\bar{u}_\varrho \in C^\infty(\mathbb{R}^n; \bar{B})$. Let (ϱ_k) be a sequence with $\varrho_k > 0$ for all k and $\varrho_k \to 0$. Let

$$\bar{u}^{(k)} := \bar{u}_{\varrho_k}.$$

Suppose that for each $k \in \mathbb{N}$ there is $x_k \in \bar{B}$ such that $\bar{u}^{(k)}(x_k) = x_k$. Then there is a subsequence (again denoted by (x_k)) and $x_0 \in \bar{B}$ such that $x_k \to x_0$. Since $\bar{u}^{(k)} \to \bar{u} = u$ uniformly,

$$\|u(x_k) - \bar{u}^{(k)}(x_k)\| \to 0 \quad (k \to \infty),$$

hence

$$u(x_0) = u(x_0) - u(x_k) + u(x_k) - \bar{u}^{(k)}(x_k) + \bar{u}^{(k)}(x_k) - x_k + x_k \to x_0 (k \to \infty),$$

$u(x_0) = x_0$.

ii) Assume now $u \in C^2(B; \bar{B}) \cap C^0(\bar{B}, \bar{B})$ and assume that $u(x) \neq x$ for all $x \in \bar{B}$. For $(x,t) \in \bar{B} \times I$ $(I := [0,1])$ we set

$$u(x,t) := x - tu(x). \tag{6.1}$$

Then $u \in C^2(B \times I; \mathbb{R}^n) \cap C^0(\bar{B} \times I, \mathbb{R}^n)$. By assumption $u(x) \neq x$ for $x \in \bar{B}$

$$\|u(x,1)\| = \|x - u(x)\| > 0 \qquad \forall x \in \bar{B} \tag{6.2}$$
$$\|u(x,t)\| \ge \|x\| - t\|u(x)\| \ge 1 - t > 0 \qquad \forall x \in \partial B,\ \forall 0 \le t < 1. \tag{6.3}$$

By (6.2), (6.3)

$$\|u(x,t)\| > 0 \qquad \forall (x,t) \in \partial B \times I$$

Therefore

$$\delta_1 := \min\left\{\|u(x,t)\| \big| (x,t) \in \partial B \times I\right\} > 0.$$

Because of (6.2)

$$\delta_2 := \min\left\{\|u(x,1)\|\big|x \in \bar{B}\right\} > 0.$$

Let $\delta := \frac{1}{2}\min(\delta_1, \delta_2) > 0$. Then

$$\|u(x,t)\| > \delta \qquad \forall (x,t) \in \partial B \times I \tag{6.4}$$

$$\|u(x,1)\| > \delta \qquad \forall x \in \bar{B}. \tag{6.5}$$

Let now $G := B$, $U := \left\{y \in \mathbb{R}^n \big| \|y\| > \delta\right\}$. By (6.4) $u(\partial B \times I) \subset U$. Let

$$f(y) := \omega_{\delta/2}(y)$$

where $\omega_{\delta/2}$ denotes the mollifier kernel, $\omega_{\delta/2} \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\omega_{\delta/2} \subset B_{\delta/2}(0)$,

$$1 = \int\limits_{B_{\delta/2}} \omega_{\delta/2}(x)dx = \int\limits_{B} \omega_{\delta/2}(x)dx. \tag{6.6}$$

By Lemma 2.2 we conclude

$$\int\limits_{B} \omega_{\delta/2}(u(x,0)) \det\nolimits_n u'(x,0)dx = \int\limits_{B} \omega_{\delta/2}(u(x,1) \det\nolimits_n u'(x,1)dx,$$

hence by (6.1)

$$\int\limits_{B} \omega_{\delta/2}(x)dx = \int_{B} \omega_{\delta/2}(u(x,1)) \det\nolimits_n u'(x,1)dx. \tag{6.7}$$

Because of (6.5) we see $\omega_{\delta/2}(u(x,1)) = 0\ \forall x \in B$ and by (6.7) $\int\limits_B \omega_{\delta/2}(x)dx = 0$, contradicting (6.6). ■

6.2 A preparation for the definition of the Brouwer-degree

Theorem 6.2 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let*

i) $u \in C^0(\bar{G}, \mathbb{R}^n) \cap C^2(G; \mathbb{R}^n)$.

ii) *Let* $z \in \mathbb{R}^n$, $z \notin u(\partial G)$, $d := \operatorname{dist}(z, u(\partial G)) > 0$.

iii) Let $f_i \in C_c^0(B_d(0))$ with

$$\int\limits_{B_d(0)} f_i(y)dy = 1 \quad (i = 1, 2)$$

Then $f_i(u(x) - z) \det_n u'(.) \in C_c^0(G)$ and

$$\int\limits_G f_1(u(x) - z) \det_n u'(x)dx = \int\limits_G f_2(u(x) - z) \det_n u'(x)dx$$

Proof.

i) Since $\operatorname{supp} f_i \subset B_d(0)$, the functions $f_i(u(.) - z)$ vanish in a neighborhood of ∂G. Let $h := f_1 - f_2$. Then $h \in C_c^0(B_d(0))$ and

$$\int\limits_{B_d(0)} h(y)dy = 0 \tag{6.8}$$

Let $0 < \varrho_0 := \operatorname{dist}(\operatorname{supp} h, \partial B_d(0))$. For $0 < \varrho < \frac{1}{2}\varrho_0$, $i = 1, \dots, n$ and $y \in B_d(0)$ let (with ω_ϱ by (1.14))

$$\phi_i^{(\varrho)}(y) := -\int\limits_0^1 \left[\int\limits_{B_d(0)} \omega_\varrho(y - ts)s_i h(s)ds \right] dt$$

Then $\phi_i^{(\varrho)}(y) = 0$ if $\operatorname{dist}(y, \partial B_d(0)) < \varrho$, hence $\phi^{(\varrho)} := \left(\phi_1^{(\varrho)}, \dots, \phi_n^{(\varrho)}\right) \in C_c^\infty(B_d(0); \mathbb{R}^n)$. Furthermore

$$\operatorname{div} \phi^{(\varrho)}(y) = -\int\limits_0^1 \int\limits_{B_d(0)} \sum_{i=1}^n (\partial_i \omega_\varrho)(y - ts)s_i h(s)dsdt$$

Since

$$\frac{d}{dt}\omega_\varrho(y - ts) = -\sum_{i=1}^n (\partial_i \omega_\varrho)(y - ts)s_i$$

we see because of (6.8)

$$\operatorname{div}\phi^{(\varrho)}(y) = \int\limits_{B_d(0)} \left[\int\limits_0^1 \frac{d}{dt}\omega_\varrho(y-ts)dt\right] h(s)ds =$$
$$= \int\limits_{B_d(0)} \omega_\varrho(y-s)h(s)ds - \omega_\varrho(y)\int\limits_{B_d(0)} h(s)ds$$
$$= \int\limits_{B_d(0)} \omega_\varrho(y-s)h(s)ds = h_\varrho(y).$$

ii) Let $A(x) := u'(x)$, let $\tilde{A}(x)$ be the matrix complementary to $A(x)$ (see (1.1), (1.2)) and let

(6.9)

$$w^{(\varrho)}(x) := \tilde{A}(x)\phi^{(\varrho)}(u(x)-z),$$
$$w_i^{(\varrho)}(x) = \sum_{j=1}^n (-1)^{i+j}\det{}_{n-1} A_{ji}(x)\phi_j^{(\varrho)}(u(x)-z), \quad i=1,\dots,n, \quad x\in G$$

Then $\omega_i^{(\varrho)} \in C^1(G)$ and

$$\partial_i\omega_i^{(\varrho)}(x) = \sum_{j=1}^n (-1)^{i+j}\partial_i\left[\det{}_{n-1} A_{ji}(x)\right]\phi_j^{(\varrho)}(u(x)-z)+$$
$$+\sum_{j=1}^n (-1)^{i+j}\det{}_{n-1} A_{ji}(x)\sum_{k=1}^n (\partial_k\phi_j^{(\varrho)})(u(x)-z)\partial_i u_k(x)$$

Hence by the fundamental identity (1.9) and by (1.3_1)

$$\operatorname{div}\omega^{(\varrho)}(x) = \sum_{j=1}^n (-1)^j \underbrace{\left[\sum_{i=1}^n (-1)^i\partial_i\det{}_{n-1} A_{ji}(x)\right]}_{=0 \text{ by } (1.9)}\phi_j^{(\varrho)}(u(x)-z)+$$
$$+\sum_{j=1}^n\sum_{k=1}^n \left(\partial_k\phi_j^{(\varrho)}\right)(u(x)-z)\underbrace{\left[\sum_{i=1}^n (-1)^{i+j}\det{}_{n-1} A_{ji}(x)\partial_i u_k(x)\right]}_{=\delta_{jk}\det_n u'(x) \text{ by } (1.3_1)} =$$
$$= \sum_{j=1}^n \left(\partial_j\phi_j^{(\varrho)}\right)(u(x)-z)\cdot\det{}_n u'(x) =$$
$$= h_\varrho(u(x)-z)\det{}_n u'(x).$$

Since $\operatorname{supp}\phi^{(\varrho)} \subset B_d(0)$, because of hypothesis ii) $\omega^{(\varrho)} \in C_c^1(G)$, hence by the trivial form of the Gaussian theorem (1.12)

$$0 = \int_G \operatorname{div} \omega^{(\varrho)}(x)dx = \int_G h_\varrho(u(x)-z)\det\nolimits_n u'(x)dx =$$
$$= \int_G f_{1\varrho}(u(x)-z)\det\nolimits_n u'(x)dx - \int_G f_{2\varrho}(u(x)-z)\det\nolimits_n u'(x)dx.$$

Because of $f_{i\varrho} \to f_i (\varrho \to 0, i = 1,2)$ uniformly in $B_d(0)$ the claim is proved. ■

6.3 The Brouwer degree

Now we are in the position to define the degree of mapping for C^1-maps (following E. Heinz [He]).

Definition 6.3 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let*

i) $u \in C^2(G;\mathbb{R}^n) \cap C^0(\bar{G};\mathbb{R}^n)$

ii) Let $z \in \mathbb{R}^n$ such that

$$d := \min_{x\in\partial G} \|u(x) - z\| > 0$$

iii) Let $f \in C_c^0(B_d(0))$ satisfy

$$\int_{B_d(0)} f(y)dy = 1 \tag{6.10}$$

Then the Brouwer degree $d[u,G,z]$ is defined by

$$d[u,G,z] := \int_G f(u(x)-z)\det\nolimits_n u'(x)dx. \tag{6.11}$$

Because of Theorem 6.2 this definition is independent of the choice of $f \in C_c^0(B_d(0))$ enjoying property (6.10). We admit here slightly more general functions f than E. Heinz [He] did in his famous paper. He considered radially depending functions $f(y) := \phi(\|y\|)$ which in addition vanish in a neighborhood of zero.

If instead of u we consider $v : \bar{G} \to \mathbb{R}^n$, $v(x) := u(x) - z$, then $d = \min\{\|v(x)\| \big| x \in \partial G\} > 0$ and we see immediately

$$d[u,G,z] = d[u-z,G,0]. \tag{6.12}$$

One of the most important properties of the Brouwer degree is its *homotopy invariance*. A first version we derive from our homotopy theorem (Theorem 2.4).

Lemma 6.4 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let*

i) $u_i \in C^2(G; \mathbb{R}^n) \cap C^0(\bar{G}; \mathbb{R}^n)$ $(i = 1, 2)$

ii) $z \in \mathbb{R}^n$,

$$\min\{\|u_2(x) - z\| : x \in \partial G\} =: 2d > 0 \tag{6.13}$$

and let

$$\|u_1(x) - u_2(x)\| < d \text{ for } x \in \partial G \tag{6.14}$$

Then

$$\|u_1(x) - z\| > d \text{ for } x \in \partial G$$

and

$$d[u_1, G, z] = d[u_2, G, z] \tag{6.15}$$

Proof. For $t \in I := [0, 1]$ let

$$u(x, t) := tu_1(x) + (1 - t)u_2(x) - z \text{ for } x \in \bar{G}.$$

Then $u \in C^2(G \times I; \mathbb{R}^n)$ and the derivatives with respect to x depend continuously on t. Further, for $(x, t) \in \partial G \times I$

$$\begin{aligned} \|u(x, t)\| &= \|u_2(x) - z + t(u_1(x) - u_2(x)\| \geq \\ &\geq \|u_2(x) - z\| - \|u_1(x) - u_2(x)\| > d \end{aligned}$$

Therefore, $\|u_1(x) - z\| = \|u(x, 1)\| > d$. Let $U := \{y \in \mathbb{R}^n \big| \|y\| > d\}$. Then U is open and $u(\partial G \times I) \subset U$. Let $f \in C_c^0(B_d(0))$ satisfy (6.10) and let f be extended by zero to $\mathbb{R}^n$. By Theorem 2.4

$$\begin{aligned} d[u_1, G, z] &= \int_G f(u(x, 1)) \det_n u'(x, 1) dx = \\ &= \int_G f(u(x, 0)) \det_n u'(x, 0) dx = d[u_2, G, z]. \end{aligned}$$

■

Lemma 6.5 *(Tietze's extension theorem). Let $C \subset \mathbb{R}^n$ be a compact set and let $v : C \to \mathbb{R}$ be continuous. Then there exists a continuous $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) = v(x)$ for $x \in C$.*

A method of proof, going back to M. Nagumo [Na] is presented in Heinz' paper ([He], Lemma 3, p. 236). Another very elegant proof can be found in Dunford-Schwartz ([Du/Sch], p. 15/16), compare also Rudin's textbook [Ru], Theorem 20.4.

Lemma 6.6 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^0(\bar{G};\mathbb{R}^n)$. Then there exists a sequence $(u_k) \subset C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that*

$$u_k\big|_{\bar{G}} \to u \text{ uniformly } (k \to \infty).$$

Proof. For each component u_i of $u = (u_1, \ldots, u_n)$ we consider the continuous extension $U_i : \mathbb{R}^n \to \mathbb{R}$ according Lemma 6.5. Let $(\varrho_k) \subset \mathbb{R}$ be a sequence such that $\varrho_k > 0$, $\varrho_k \to 0$. Then the mollifications

$$u_{ik} := U_{i\varrho_k}, \qquad u_k = (u_{1k}, \ldots, u_{nk})$$

satisfy $u_k \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $u_k\big|_{\bar{G}} \to u(k \to \infty)$ uniformly. ∎

Let now $G \subset \mathbb{R}^n$ be open and bounded, let $u \in C^0(\bar{G} \times \mathbb{R}^n)$ and let $z \in \mathbb{R}^n$ satisfy $u(x) \neq z$ for $x \in \partial G$. Then

$$3d := \min_{x \in \partial G} \|u(x) - z\| > 0.$$

Let $(u_k) \subset C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be chosen such that $u_k\big|_{\bar{G}} \to u$ $(k \to \infty)$ uniformly (according Lemma 6.6 such a sequence exists). Then there exists $k_0 \in \mathbb{N}$ such that

$$\|u(x) - u_k(x)\| < d, \qquad \|u_k(x) - u_j(x)\| < d, \quad \forall k, j \geq k_0 \quad \forall x \in \bar{G}$$

Hence

$$\|u_k(x) - z\| \geq 2d \qquad \forall x \in \partial G, \qquad \forall k \geq k_0 \tag{6.16}$$

For $k \geq k_0$ the degree $d[u_k, G, z]$ is well defined and by Lemma 6.4

$$d[u_k, G, z] = d[u_j, G, z] \qquad \forall k, j \geq k_0.$$

This consideration justifies.

Definition 6.7 *Let $G \subset \mathbb{R}^n$ be a bounded open set, let $u \in C^0(\bar{G};\mathbb{R}^n)$ and let $z \in \mathbb{R}^n$ such that $u(x) \neq z$ $\forall x \in \partial G$. Let $(u_k) \subset C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be any sequence such that*

$$u_k\big|_{\bar{G}} \to u \ (k \to \infty) \text{ uniformly in } \bar{G}.$$

Let $k_0 \in \mathbb{N}$ be such that for $k \geq k_0$

$$\min_{x \in \partial G} \|u_k(x) - z\| \geq \frac{2}{3} \min_{x \in \partial G} \|u(x) - z\| > 0$$

Then the Brouwer degree $d[u, G, z]$ is uniquely defined by

$$d[u, G, z] = \lim_{\substack{k \to \infty \\ k \geq k_0}} d[u_k, G, z]. \tag{6.17}$$

We study now some properties of this degree.

Theorem 6.8 *(domain additivity) Let $G_i \subset \mathbb{R}^n$ $(i = 1, 2)$ be bounded open sets such that $G_1 \cap G_2 = \phi$. Let $u \in C^0(\bar{G}_1 \cup \bar{G}_2; \mathbb{R}^n)$ and let*

$$z \in \mathbb{R}^n, \qquad u(x) \neq z \qquad \forall x \in \partial G_1 \cup \partial G_2.$$

Then

$$d[u, G_1 \cup G_2, z] = d[u, G_1, z] + d[u, G_2, z].$$

Proof. Let $(u_k) \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be any approximating sequence for u in the sense of Definition 6.7, then for $k \geq k_0$ by (6.11)

$$\begin{aligned} d[u_k, G_1 \cup G_2, z] &= \int\limits_{G_1 \cup G_2} f(u_k(x) - z) \det\nolimits_n u_k'(x) dx = \\ &= \int\limits_{G_1} f(u_k(x) - z) \det\nolimits_n u_k'(x) dx + \int\limits_{G_2} f(u_k(x) - z) \det\nolimits_n u_k'(x) dx \\ &= d[u_k, G_1, z] + d[u_k, G_2, z] \end{aligned}$$

and the claim follows by Definition 6.7. ■

Theorem 6.9 *(homotopy invariance). Let $G \subset \mathbb{R}^n$ be a bounded open set, let $I := [0, 1]$ and let $u \in C^0(\bar{G} \times I; \mathbb{R}^n)$. Assume that $z \in \mathbb{R}^n$ and $u(x, t) \neq z$ $\forall (x, t) \in \partial G \times I$. Then*

$$d[u(., t), G, z] = d[u(., 0), G; z] \qquad \forall t \in I.$$

Proof. Let

$$3d := \min \big\{ \|u(x, t) - z\| \big| (x, t) \in \partial G \times I \big\} > 0.$$

u being uniformly continuous in $\bar{G} \times I$, there is $\delta > 0$ such that

$$\|u(x, t) - u(x, t')\| < \frac{d}{2} \qquad \forall x \in \bar{G}, \quad \forall t, t' \in I \quad \text{such that } |t - t'| \leq \delta.$$

Let now $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \delta$. We choose sequences $(u_{ik})_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that $u_{ik}\big|_{\bar{G}} \to u(., t_i)$ $(k \to \infty)$, $i = 1, 2$ uniformly. Then there is $k_0 \in \mathbb{N}$ such that

$$\|u_{ik}(x) - u(x, t_i)\| < \frac{d}{4} \quad \forall x \in \bar{G}, i = 1, 2.$$

Then

$$\begin{aligned} \|u_{ik}(x) - z\| &\geq 2d \qquad \forall x \in \partial G,\ i = 1, 2,\ \forall k \geq k_0 \\ \|u_{1k}(x) - u_{2k}(x)\| &< d \qquad \forall x \in \partial G,\ \forall k \in \mathbb{N} \end{aligned}$$

By Lemma 6.4

$$d[u_{1k}, G, z] = d[u_{2k}, G, z].$$

Passing to the limit $k \to \infty$ yields

$$d[u(., t_1), G, z] = d[u(., t_2), G, z]$$

Since $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \delta$ had been arbitrary, the assertion follows. ∎

As a corollary we derive

Theorem 6.10 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^0(\bar{G}; \mathbb{R}^n)$. Let*

$$z_0 \in \mathbb{R}^n \quad \text{and} \quad u(x) \neq z_0 \qquad \forall x \in \partial G$$

Then there exists an $\varepsilon > 0$ such that

$$u(x) \neq z \quad \forall z \in B_\varepsilon(z_0), \quad \forall x \in \partial G$$

and

$$d[u; G, z] = d[u, G, z_0] \qquad \forall z \in B_\varepsilon(z_0). \tag{6.18}$$

Proof. Let $\varepsilon := \frac{1}{2} \min_{x \in \partial G} \|u(x) - z_0\| > 0$. For $t \in [0, 1]$, $z \in B_\varepsilon(z_0)$ and $x \in \bar{G}$ we set $u(x, t) := u(x) - t(z - z_0)$. Then for $x \in \partial G$, $t \in [0, 1]$ $\|u(x, t) - z_0\| = \|u(x) - z_0 - t(z - z_0)\| \geq \varepsilon$. By Theorem 6.9

$$d[u, G, z] = d[u(., 1), G, z_0] = d[u(., 0), G, z_0] = d[u, G, z_0].$$

∎

Theorem 6.11 *Let $G \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^0(\bar{G}; \mathbb{R}^n)$. Suppose that $z \in \mathbb{R}^n$, $u(x) \neq z$ $\forall x \in \partial G$ and $d[u, G, z] \neq 0$. Then there exists $x_0 \in G$ such that $u(x_0) = z$.*

Proof. Let $(u_k) \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be an approximating sequence for u in the sense of Definition 6.7. If we assume $u(x) \neq z$ for all $x \in \bar{G}$, then

$$2d := \min_{x \in \bar{G}} \|u(x) - z\| > 0.$$

There is $k_0 \in \mathbb{N}$ such that

$$\|u_k(x) - z\| \geq d \qquad \forall x \in \bar{G}, \quad \forall k \geq k_0$$

Let now $f \in C_c^0(B_d(0))$. Then

$$f(u_k(x) - z) = 0 \qquad \forall x \in \bar{G}, \quad \forall k \geq k_0$$

and

$$d[u, G, z] = \lim_{\substack{k \geq k_0 \\ k \to \infty}} d[u_k, G, z] = \lim_{\substack{k \to \infty \\ k \geq k_0}} \int_G f(u_k(x) - z) \det_n u_k'(x) dx = 0,$$

a contradiction. ■

For all further properties of the degree we refer to [He].

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