Enumeration of generalized polyominoes

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Abstract

As a generalization of polyominoes we consider edge-to-edge connected nonoverlapping unions of regular k-gons. For $n \leq 4$ we determine formulas for the number $a_k(n)$ of generalized polyominoes consisting of n regular k-gons. Additionally we give a table of the numbers $a_k(n)$ for small k and n obtained by computer enumeration. We finish with some open problems for k-polyominoes.

1 Introduction

A polyomino, in its original definition, is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge. In other words, it is an edge-connected union of cells in the planar square lattice. For the origin of polyominoes we quote Klarner [13]: "Polyominoes have a long

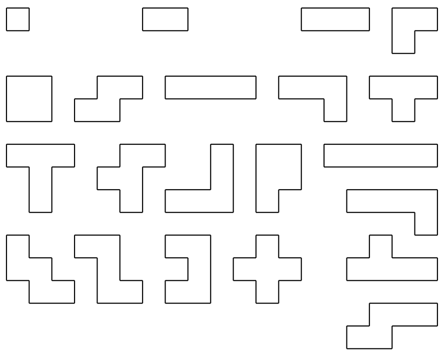

Figure 1: Polyominoes with at most 5 squares.

history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb i.e. [5, 6, 7], then by Martin Gardner in his *Scientific American* columns." At the present time they are widely known by mathematicians, physicists, chemists and have been considered in many different applications, i.e. in the *Ising Model* [2]. To give an illustration of polyominoes Figure 1 depicts the polyominoes consisting of at most 5 unit squares.

One of the first problems for polyominoes was the determination of there number. Although there has been some progress, a solution to this problem remains outstanding. In the literature one sometimes speaks also of the cell-growth problem and uses the term animal instead of polyomino.

Due to its wide area of applications polyominoes were soon generalized to the two other tessellations of the plane, to the eight Archimedean tessellations [4] and were also considered as unions of d-dimensional hypercubes instead of squares. For the known numbers we refer to the "Online Encyclopedia of Integer Sequences" [19].

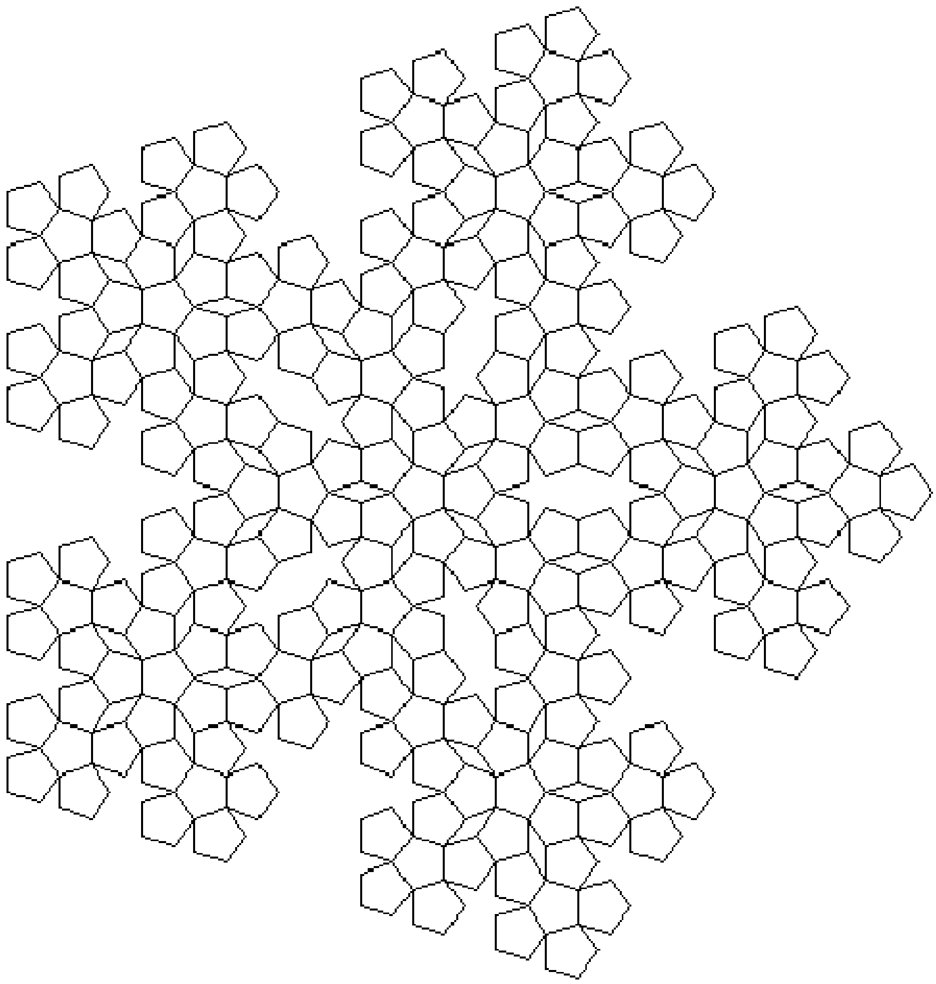

Figure 2: A nice 5-polyomino.

In this article we generalize concept of polyominoes to unions of regular nonoverlapping edge-to-edge connected k-gons. For short we call them k-polyominoes. An example of a 5-polyomino, which reminds somewhat to Penrose's famous non-periodic tiling of the plane, is depicted in Figure 2. In the next sections we determine exact formulas for the number $a_k(n)$ of nonisomorphic k-polyominoes with $k \leq 4$ and give some further values for small parameters k and n obtained by computer enumeration. So far edge-to-edge connected unions of regular k-gons were only enumerated if overlapping of the k-gons is permitted [9]. We finish with some open problems for k-polyominoes.

2 Formulas for the number of k-polyominoes

By $a_k(n)$ we denote the number of nonisomorphic k-polyominoes consisting of n regular k-gons as cells where $a_k(n) = 0$ for $k < 3$. For at most two cells we have $a_k(1) = a_k(2) = 1$. If $n \geq 3$ we characterize three edge-to-edge connected cells $\mathcal{C}_1$,

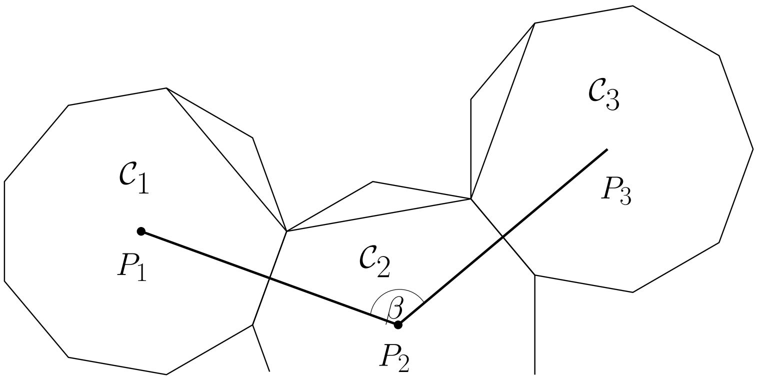

Figure 3: Angle $\beta = \angle(P_1, P_2, P_3)$ between three neighbored cells.

$\mathcal{C}_2$ and $\mathcal{C}_3$ of a k-polyomino, see Figure 3, by the angle $\beta = \angle(P_1, P_2, P_3)$ between the centers of the cells. Since these angles are multiples of $\frac{2\pi}{k}$ we call the minimum

$$\min\left(\angle(P_1, P_2, P_3)\frac{k}{2\pi}, \left(2\pi - \angle(P_1, P_2, P_3)\right)\frac{k}{2\pi}\right)$$

the discrete angle between $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ and denote it by $\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

Lemma 1 *Two k-gons $\mathcal{C}_1$ and $\mathcal{C}_3$ joined via an edge to a k-gon $\mathcal{C}_2$ are nonoverlapping if and only if $\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \geq \left\lfloor \frac{k+5}{6} \right\rfloor$. The three k-gons are neighbored pairwise if and only if $k \equiv 0 \,mod\, 6$.*

Proof. We consider Figure 3 and set $\beta = \delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)\frac{2\pi}{k}$. If the cells $\mathcal{C}_1$ and $\mathcal{C}_3$ are nonoverlapping we have $\overline{P_1P_3} \geq \overline{P_1P_2}$ because the lengths of the lines $\overline{P_1P_2}$ and $\overline{P_2P_3}$ are equal. Thus $\beta \geq \frac{2\pi}{6}$ and $\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \geq \left\lfloor \frac{k+5}{6} \right\rfloor$ is necessary. Now we consider the circumcircles of the cells $\mathcal{C}_1$ and $\mathcal{C}_3$, see Figure 4. Due to $\beta \geq \frac{2\pi}{6}$ only the circlesegments between points P_4, P_5 and P_6, P_7 may intersect. The last step is to check that the corresponding lines $\overline{P_4P_5}$ and $\overline{P_6P_7}$ do not intersect and they touch each other if and only if $k \equiv 0 \bmod 6$. $\square$

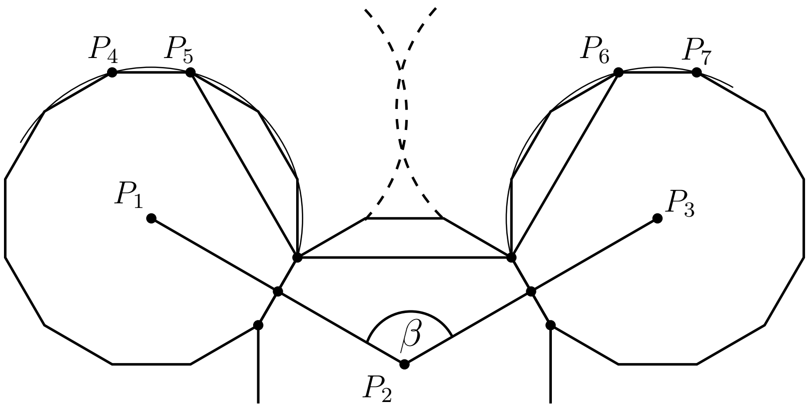

Figure 4: Nonoverlapping 12-gons.

Corollary 2 *The number of neighbors of a cell in a k-polyomino is at most*

$$\min\left(k, \frac{k}{\left\lfloor \frac{k+5}{6} \right\rfloor}\right) \leq 6\,.$$

With the aid of Lemma 1 we are able to determine the number $a_k(3)$ of k-polyominoes consisting of 3 cells.

Theorem 3

$$a_k(3) = \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k+5}{6} \right\rfloor + 1 \quad \textit{for } k \geq 3\,.$$

Proof. It suffices to determine the possible values for $\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. Due to Lemma 1 we have $\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \geq \left\lfloor \frac{k+5}{6} \right\rfloor$ and due to symmetry considerations we have $\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \leq \left\lfloor \frac{k}{2} \right\rfloor$. $\square$

In order to determine the number of k-polyominoes with more than 3 cells we describe the classes of k-polyominoes by graphs. We represent each k-gon by a vertex and join two vertices exactly if they are connected via an edge.

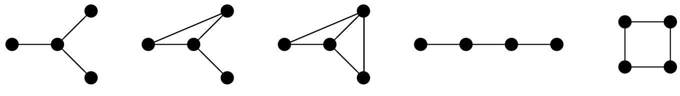

Figure 5: The possible graphs of k-polyominoes with 4 vertices.

Lemma 4 *The number of k-polyominoes with a graph isomorphic to one of the first three ones in Figure 5 is given by*

$$\left\lfloor \frac{\left(k-3\left\lfloor\frac{k+5}{6}\right\rfloor\right)^2+6\left(k-3\left\lfloor\frac{k+5}{6}\right\rfloor\right)+12}{12}\right\rfloor .$$

Proof. We denote the cell corresponding to the unique vertex of degree 3 in the graph by $\mathcal{C}_0$ and the three other cells by $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. With $\delta_1=\delta(\mathcal{C}_1,\mathcal{C}_0,\mathcal{C}_2)-\left\lfloor\frac{k+5}{6}\right\rfloor$, $\delta_2=\delta(\mathcal{C}_2,\mathcal{C}_0,\mathcal{C}_3)-\left\lfloor\frac{k-1}{6}\right\rfloor$, and $\delta_3=\delta(\mathcal{C}_3,\mathcal{C}_0,\mathcal{C}_1)-\left\lfloor\frac{k-1}{6}\right\rfloor$ we set $m=\delta_1+\delta_2+\delta_3=k-3\left\lfloor\frac{k+5}{6}\right\rfloor$. Because the k-polyominoes with a graph isomorphic to one of the first three ones in Figure 5 are uniquely described by $\delta_1,\delta_2,\delta_3$, due to Lemma 1 and due to symmetry their number equals the number of partitions of m into at most three parts. This number is the coefficient of x^m in the Taylor series of $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ in $x=0$ and can be expressed as $\left\lfloor\frac{m^2+6m+12}{12}\right\rfloor$. $\square$

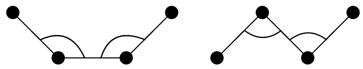

Figure 6: Paths of lengths 3 representing chains of four neighbored cells.

In Lemma 1 we have given a condition for a chain of three neighbored cells avoiding an overlapping. For a chain of four neighbored cells we have to consider the two cases of Figure 6. In the second case the two vertices of degree one are not able to overlap so we need a lemma in the spirit of Lemma 1 only for the first case.

Lemma 5 *Four k-gons $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ arranged as in the first case of Figure 6 are nonoverlapping if and only if Lemma 1 is fulfilled for the two subchains of length 3 and*

$$\delta(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+\delta(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)\geq\left\lfloor\frac{k+1}{2}\right\rfloor .$$

The chain is indeed a 4-cycle if and only if

$$\delta(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+\delta(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)=\frac{k}{2} .$$

Proof. We start with the second statement and consider the quadrangle of the centers of the 4 cells. Because the angle sum of a quadrangle is 2π we have

$$\delta(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+\delta(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)+d(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_1)+\delta(\mathcal{C}_4,\mathcal{C}_1,\mathcal{C}_2)=k .$$

Due to the fact that the side lengths of the quadrangle are equal we have

$$\delta(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+\delta(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)=\delta(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_1)+\delta(\mathcal{C}_4,\mathcal{C}_1,\mathcal{C}_2)$$

which is equivalent to the statement.

Thus $\delta(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+\delta(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)\geq\left\lfloor\frac{k+1}{2}\right\rfloor$ is a necessary condition. Similar to the proof of Lemma 1 we consider the circumcircles of the cells $\mathcal{C}_1$, $\mathcal{C}_4$ and check that the cells do not intersect. $\square$

Lemma 6 *For $k \geq 3$ the number of k-polyominoes with a graph isomorphic to one of the last two ones in Figure 5 is given by*

$$\begin{array}{llll}
\dfrac{5k^2+4k}{48} & \text{for } k \equiv 0 \bmod 12, & \dfrac{5k^2+6k-11}{48} & \text{for } k \equiv 1 \bmod 12,\\
\dfrac{5k^2+12k+4}{48} & \text{for } k \equiv 2 \bmod 12, & \dfrac{5k^2+14k+9}{48} & \text{for } k \equiv 3 \bmod 12,\\
\dfrac{5k^2+20k+32}{48} & \text{for } k \equiv 4 \bmod 12, & \dfrac{5k^2+22k+5}{48} & \text{for } k \equiv 5 \bmod 12,\\
\dfrac{5k^2+4k-12}{48} & \text{for } k \equiv 6 \bmod 12, & \dfrac{5k^2+6k+1}{48} & \text{for } k \equiv 7 \bmod 12,\\
\dfrac{5k^2+12k+16}{48} & \text{for } k \equiv 8 \bmod 12, & \dfrac{5k^2+14k-3}{48} & \text{for } k \equiv 9 \bmod 12,\\
\dfrac{5k^2+20k+20}{48} & \text{for } k \equiv 10 \bmod 12, & \dfrac{5k^2+22k+17}{48} & \text{for } k \equiv 11 \bmod 12\,.
\end{array}$$

Proof. Because each of the last two graphs in Figure 5 contains a path of length 3 as a subgraph we consider the two cases of Figure 6. We denote the two interesting discrete angles by δ_1 and δ_2. Due to symmetry we may assume $\delta_1 \leq \delta_2$ and because the graphs do not contain a triangle we have $\delta_2 \geq \delta_1 \geq \left\lfloor \frac{k+6}{6} \right\rfloor$ due to Lemma 1. From the definition of the discrete angle we have $\delta_1 \leq \delta_2 \leq \left\lfloor \frac{k}{2} \right\rfloor$. To avoid double counting we assume $\left\lfloor \frac{k+6}{6} \right\rfloor \leq \delta_1 \leq \delta_2 \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ in the second case, so that we get a number of

$$\binom{\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k+6}{6} \right\rfloor + 2}{2}$$

k-polyominoes. With Lemma 5 and a look at the possible symmetries the number of k-polyominoes in the first case is given by

$$\sum_{\delta_1=\left\lfloor \frac{k+6}{6} \right\rfloor}^{\left\lfloor \frac{k}{2} \right\rfloor} \; \sum_{\delta_2=\max\left(\delta_1,\left\lfloor \frac{k+1}{2} \right\rfloor-\delta_1\right)}^{\left\lfloor \frac{k}{2} \right\rfloor} 1\,.$$

A little calculation yields the proposed formulas. □

Theorem 7 *For $k \geq 3$ we have*

$$a_k(4) = \begin{cases}
\frac{3k^2+8k+24}{24} & \text{for } k \equiv 0 \bmod 12, \quad \frac{3k^2+4k-7}{24} \quad \text{for } k \equiv 1 \bmod 12,\\
\frac{3k^2+8k-4}{24} & \text{for } k \equiv 2 \bmod 12, \quad \frac{3k^2+10k+15}{24} \quad \text{for } k \equiv 3 \bmod 12,\\
\frac{3k^2+14k+16}{24} & \text{for } k \equiv 4 \bmod 12, \quad \frac{3k^2+16k+13}{24} \quad \text{for } k \equiv 5 \bmod 12,\\
\frac{3k^2+8k+12}{24} & \text{for } k \equiv 6 \bmod 12, \quad \frac{3k^2+4k-7}{24} \quad \text{for } k \equiv 7 \bmod 12,\\
\frac{3k^2+8k+8}{24} & \text{for } k \equiv 8 \bmod 12, \quad \frac{3k^2+10k+3}{24} \quad \text{for } k \equiv 9 \bmod 12,\\
\frac{3k^2+14k+16}{24} & \text{for } k \equiv 10 \bmod 12, \quad \frac{3k^2+16k+13}{24} \quad \text{for } k \equiv 11 \bmod 12.
\end{cases}$$

Proof. The list of graphs in Figure 5 is complete because the graphs have to be connected and the complete graph on 4 vertices is not a unit distance graph. Adding the formulas from Lemma 4 and Lemma 6 yields the theorem. □

3 Computer enumeration of k-polyominoes

For $n \geq 5$ we have constructed k-polyominoes with the aid of a computer and have obtained the following values of $a_k(n)$ given in Table 1 and Table 2.

$k \backslash n$	5	6	7	8	9	10	11	12	13
3	4	12	24	66	160	448	1186	3334	9235
4	12	35	108	369	1285	4655	17073	63600	238591
5	25	118	551	2812	14445	76092	403976	2167116	11698961
6	22	82	333	1448	6572	30490	143552	683101	3274826
7	25	118	558	2876	14982	80075	431889	2354991	12930257
8	50	269	1605	10102	65323	430302	2868320	19299334	130807068
9	82	585	4418	34838	280014	2285047	18838395	156644526	1311575691
10	127	985	8350	73675	664411	6078768	56198759	523924389	4918127659
11	186	1750	17501	181127	1908239	20376032	219770162	2390025622	
12	168	1438	13512	131801	1314914	13303523	136035511	1402844804	
13	187	1765	17775	185297	1968684	21208739	230877323	2534857846	
14	263	2718	30467	352375	4158216	49734303	601094660	7326566494	
15	362	4336	55264	725869	9707046	131517548	1800038803		
16	472	6040	83252	1180526	17054708	249598727	3690421289		
17	613	8814	134422	2104485	33522023	540742895	8810416620		
18	566	7678	112514	1694978	26019735	404616118			
19	615	8839	135175	2123088	33942901	549711709			
20	776	11876	195122	3291481	56537856	983715865			

Table 1: Number of k-polyominoes with n cells for small k and n.

Now we go into more detail how the computer enumeration was done. At first we have to represent k-polyominoes by a suitable data structure. As in Lemma 1 a k-polyomino can be described by the set of all discrete angles between three neighbored cells. By fixing one direction we can define the discrete angle between this direction and two neighbored cells and so describe a k-polyomino by an $n \times n$-matrix with integer entries. Due to Corollary 2 we can also describe it as a $6 \times n$-matrix by listing only the neighbors. To deal with symmetry we define a canonical form for these matrices.

Our general construction strategy is orderly generation [18], where we use a variant introduced in [15, 17]. Here a k-polyomino consisting of n cells is constructed by glueing two k-polyominoes consisting of $n-1$ cells having $n-2$ cells in common. There are two advantages of this approach. In a k-polyomino each two cells must be nonoverlapping. If we would add a cell in each generation step we would have to check $n-1$ pairs of cells whether they are nonoverlapping or not. By glueing two k-polyominoes we only need to perform one such check. To demonstrate the second advantage we compare in Table 3 the numbers $c_1(n,k)$ and $c_2(n,k)$ of candidates produced by the original version and the used variant via glueing of orderly generation.

To avoid numerical twists in the overlapping check we utilize Gröbner bases [1].

$k\backslash n$	5	6	7	8	$k\backslash n$	5	6	7
21	972	16410	294091	5402087	36	4575	130711	3943836
22	1179	20970	397852	7739008	37	4796	140434	4326289
23	1437	27720	566007	11832175	38	5380	163027	5204536
24	1347	24998	495773	10079003	39	6089	193587	6464267
25	1439	27787	568602	11917261	40	6760	221521	7634297
26	1711	34763	751172	16624712	41	7578	259396	9311913
27	2045	44687	1031920	24389611	42	7282	244564	8643473
28	2376	54133	1307384	32317393	43	7584	259838	9341040
29	2786	67601	1729686	45260884	44	8373	295558	10958872
30	2641	62252	1557663	39891448	45	9321	342841	13215115
31	2790	67777	1737915	45587429	46	10207	385546	15274792
32	3204	81066	2169846	59424885	47	11282	442543	18169170
33	3706	99420	2808616	81124890	48	10890	420154	17012270
34	4193	116465	3413064	102292464	49	11290	443178	18217475
35	4789	140075	4306774	135337752	50	12309	495988	20944951

Table 2: Number of k-polyominoes with n cells for small k and n.

n	4	5	6	7	8	9	10	11
$a_5(n)$	7	25	118	551	2812	14445	76092	403976
$c_1(n,5)$	21	74	242	1038	4476	21945	111232	580139
$c_2(n,5)$	19	62	192	816	3541	17297	87336	452215
$a_7(n)$	7	25	118	558	2876	14982	80075	431889
$c_1(n,7)$	31	107	356	1530	6682	33057	168881	889721
$c_2(n,7)$	19	62	196	821	3584	17778	91109	479814
$a_{13}(n)$	23	187	1765	17775	185297	1968684	21208739	230877323
$c_1(n,13)$	126	721	5059	43842	420958	4294445	45258582	485481211
$c_2(n,13)$	76	408	2697	23412	223789	2274489	23849241	254712159
$a_{17}(n)$	48	614	8814	134422	2104485	33522023	540742895	8810416620
$c_1(n,17)$	255	2039	22038	292887	4311681	66600525	1057440375	17043701525
$c_2(n,17)$	171	1261	12964	173839	2545538	39008006	614066925	9828917852

Table 3: Number of candidates $c_1(n,k)$ and $c_2(n,k)$ for k-polyominoes with n cells.

4 Open problems for k-polyominoes

For 4-polyominoes the maximum area of the convex hull was considered in [3]. If the area of a cell is normalized to 1 then the maximum area of a 4-polyomino consisting of n squares is given by $n+\frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$. The second author has proven an analogous result for the maximum content of the convex hull of a union of d-dimensional units hypercubes [14], which is given by

$$\sum_{I\subseteq\{1,\ldots,d\}} \frac{1}{|I|!}\prod_{i\in I}\left\lfloor\frac{n-2+i}{d}\right\rfloor$$

for n hypercubes. For other values of k the question for the maximum area of the convex hull of k-polyominoes is still open. Besides from [11] no results are known for the question of the minimum area of the convex hull, which is non trivial for $k \neq 3, 4$.

Another class of problems is the question for the minimum and the maximum number of edges of k-polyominoes. The following sharp inequalities for the number q of edges of k-polyominoes consisting of n cells were found in [8] and are also given in [10].

$$\begin{aligned}
k = 3: &\qquad n + \left\lceil \frac{1}{2}\left(n+\sqrt{6n}\right)\right\rceil \leq q \leq 2n+1 \\
k = 4: &\qquad 2n + \left\lceil 2\sqrt{n}\right\rceil \leq q \leq 3n+1 \\
k = 6: &\qquad 3n - \left\lceil \sqrt{12n-3}\right\rceil \leq q \leq 5n+1
\end{aligned}$$

In general the maximum number of edges is given by $(k-1)n+1$. The numbers of 4-polyominoes with a minimum number of edges were enumerated in [16].

Since for $k \neq 3, 4, 6$ regular k-gons do not tile the plane the question about the maximum density $\delta(k)$ of an edge-to-edge connected packing of regular k-gons arises. In [12]

$$\delta(5) = \frac{3\sqrt{5}-5}{2} \approx 0.8541$$

is conjectured.

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