Measure and Integration on Lipschitz-Manifolds

Joachim Naumann and Christian G. Simader

Abstract. The first part of this paper is concerned with various definitions of a k-dimensional Lipschitz manifold $\mathcal{M}^k$ and a discussion of the equivalence of these definitions. The second part is then devoted to the geometrically intrinsic construction of a σ-algebra $\mathcal{L}(\mathcal{M}^k)$ of subsets of $\mathcal{M}^k$ and a measure μ_k on $\mathcal{L}(\mathcal{M}^k)$.

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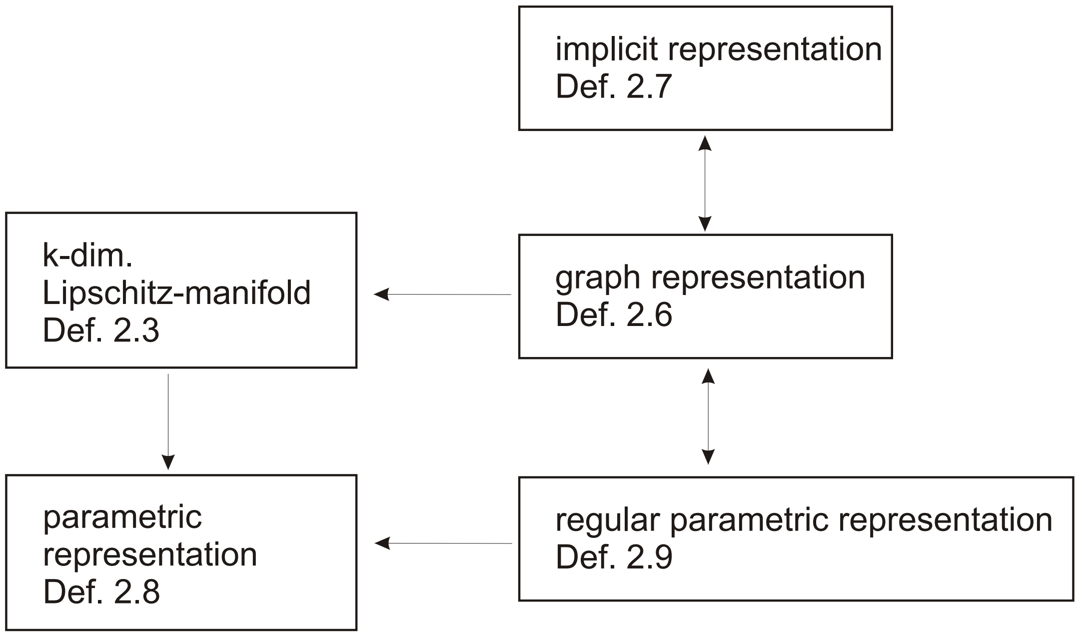
1 Introduction

In the case of k-dimensional manifolds of class C^m $(m \geq 1)$ in $\mathbb{R}^N$ there is a variety of equivalent definitions. If we replace the assumption of continuous differentiability by Lipschitz resp. bi-Lipschitz properties of certain maps we find several different possibilities to define k-dimensional Lipschitz manifolds (see Definitions 2.3, 2.6–2.9). The natural question arises if these definitions are equivalent. Here a certain hint is given by another consideration. If we consider a k-dimensional manifold $\mathcal{M}^k \subset \mathbb{R}^N$ of class C^m and if an open neighborhood $\mathcal{U}$ of $\mathcal{M}^k$ is mapped by a diffeomorphism ϕ of class $\mathcal{C}^m$ on an open set $\mathcal{U}^* \subset \mathbb{R}^N$, then $\phi(\mathcal{M}^k) \subset \mathcal{U}^* \subset \mathbb{R}^N$ is clearly again a k-dimensional manifold of class C^m in $\mathbb{R}^N$. As special $(N-1)$ - dimensional Lipschitz-manifolds Grisvard [4] considered boundaries of open subsets in $\mathbb{R}^N$. He gave two definitions. His Definition 1.2.1.1 (see [4, p.5]) coincides with our Definition 2.6 of a $(N-1)$-dimensional Lipschitz manifold in graph representation. The second definition of Grisvard (see [4, Definition 1.2.1.2, p. 6/7]) coincides with our Definition 2.3. Then Grisvard ([4, Lemma 1.2.1.3, p. 7]) pointed out that his Definition 1.2.1.2 is invariant under bi-Lipschitz homeomorphisms of a neighborhood of the manifold. But with the help of a very interesting counterexample (see [4, Lemma 1.2.1.4, p. 8/9]) he succeeded in proving that the graph representation needs not to be invariant under bi-Lipschitz homeomorphisms. We prove in Theorem 2.13 that our Definitions 2.3 and 2.8 are invariant under bi-Lipschitz homeomorphisms. In Theorem 2.11 we prove the equivalence of the definitions of a k-dimensional Lipschitz-manifold $\mathcal{M}_k$ in graph representation, in regular parametric representation and in implicit representation. Further, in Theorem 2.10 we prove that a k-dimensional Lipschitz-manifold in graph representation is a k-dimensional Lipschitz-manifold in the sense of Definition 2.3. Finally, Theorem 2.12 states that a k-dimensional Lipschitz-manifold in the sense of Definition 2.3 is a k-dimensional Lipschitz-manifold in parametric representation. We derive the following diagram:



For the definition of the measure space $(\mathcal{M}^k, \mathcal{L}(M^k), \mu_k)$ we use the parametric representation (Definition 2.8). As a justification for Definition 2.16 we prove in Theorem 2.15 that it suffices to consider parametric representations consisting in at most countable many charts. Contrary to the case of C^m-manifolds the Definition 2.6, 2.7 and 2.9 depend on the choice of a local Euclidean coordinate system (compare Definition 2.5). This fact is reflected by Example 1. Until now we not had been able to prove that the parametric representation implies Definition 2.3. In the case of "classical *continuous* differentiability" we replace property 1 of Definition 2.8 by

$$\psi \in C^1(0) \text{ and rank } \psi'(x) = k \text{ for } x \in \mathbb{N}$$

where

$$\psi'(t) = \begin{pmatrix} \partial_1\psi_1(t) & \cdots & \partial_N\psi_1(t) \\ \vdots & & \\ \partial_1\psi_k(t) & \cdots & \partial_N\psi_k(t) \end{pmatrix}.$$

Let $t_o := \psi^{-1}(x_o) \in \mathcal{O}$. After eventually renumbering coordinates in $\mathbb{R}^N$ for $\hat{\psi}(t) := (\psi_1(t), \ldots, \psi_k(t))$ we get $\det_k \hat{\psi}'(t_o) \neq 0$. Then there exists an open neighborhood $V \subset \mathcal{O}$ of t_o and an open neighborhood V' of $\hat{\psi}(t_o)$ such that $\hat{\psi}\mid_V : V \to V'$ is a homeomorphism with $\hat{\psi}^{-1} \in C^1(V')$. After choosing eventually a smaller neighborhood $W \subset V$ of t_o we see that $\hat{\psi}\mid_w : W \to \hat{\psi}(w)$ is bi-Lipschitz, hence we have a regular parametric representation in the sense of Definition 2.8. Therefore the decisive assumption in the classical procedure is the *continuity* of the derivatives. In Example 2 we construct a 1-dimensional Lipschitz manifold $\tilde{f}$ in $\mathbb{R}^2$, that is a bi-Lipschitz curve (see figure 2), which is never a graph or in regular parametric representation. But until now it is an open question if this map could be extended at least in a neigborhood of zero to a bi-Lipschitz map defined in a neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(0,0)$. Clearly, by a famous theorem of Kirszbraun [5], $\tilde{f}$ as well as $\tilde{f}^{-1}$ can be extended to $\mathbb{R}^2$ such that the Lipschitz constants are preserved. But the extension needs not to be bi-Lipschitz. Therefore the equivalence of Definitions 2.3 and 2.8 is an open problem.

In the third section we construct the measure space $\big(\mathcal{M}^k, \mathcal{L}(M^k), \mu_k\big)$. In section 3.1 the σ-algebra of measurable subsets of $\mathcal{M}^k$ is constructed and the measurability of a function $f : \mathcal{M}^k \to \bar{\mathbb{R}}$ is defined. Here one has to prove that both definitions are independent of the special parametric representation. After several preparations in section 3.2 the measure μ_k can be defined on the σ-Algebra $\mathcal{L}(M^k)$ (Definition 3.6 and Theorem 3.7). Finally, equivalent characterizations of sets of measure zero (Theorem 3.8) are given. Once a measure space is constructed, the integral is defined at least for non-negative measurable functions. In section 3.3 we prove some elementary properties of this integral. First, a relation between this integral and integrals using the parametric representation is studied (Theorem 3.9 and Corollary 3.10). For the remaining part of section 3.3 it is assumed that $\mathcal{M}^k$ has a finite parametric representation. Then estimates for the integral of nonnegative integrable functions are derived (Theorem 3.11 and Corollary 3.12) and a formula for the calculation of the integral with the help of a partition of unity is derived (Theorem 3.13). Finally, in section 3.4 we introduce the space $L^p(\mathcal{M}^k, \mathcal{L}(M^k), \mu_k)$.

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2 k-dimensional Lipschitz manifolds

2.1 Definitions. Equivalent characterizations

Definition 2.1 *Let $G \subset \mathbb{R}^n$ be an open set. A mapping $u : G \to \mathbb{R}^m$ $(m, n \in \mathbb{N})$ is called bi-Lipschitz in G if there are constants $0 < L_1 \le L_2$ such that*

$$L_1\|x - x'\|_n \le \|u(x) - u(x')\|_m \le L_2\|x - x'\|_n \quad \forall x, x' \in G \tag{2.1}$$

We summarize the following properties of bi-Lipschitz mappings.

Theorem 2.2 *Let $G \subset \mathbb{R}^n$ be open and let $u : G \to \mathbb{R}^m$ $(m, n \in \mathbb{N})$ satisfy (2.1).*

1. *There is a subset $N \subset G$, $|N| = 0$, such that u is totally differentiable at each $x \in G \setminus N$. For the total derivative*

$$u'(x) = (D_i u_k(x)) \in M(m \times n), \qquad x \in G \setminus N$$

we have the estimate

$$L_1\|\eta\|_n \le \|u'(x)\eta\|_m \le L_2\|\eta\|_n \qquad \forall x \in G \setminus N, \forall \eta \in \mathbb{R}^n \tag{2.2}$$

Therefore $m \ge n$ and rank $u'(x) = n$ $\forall x \in G \setminus N$.

2. *Let $m = n$. Then u is open, i.e. for every open $V \subset G$ the image $u(V)$ is open too.*

For the proof we refer e.g. to [7, Theorems 1.6 and 4.5]. In the sequel, let $N, k \in \mathbb{N}$, $N \geq 2$ and let $1 \leq k \leq N-1$.

Definition 2.3 *A subset $\mathcal{M}^k \subset \mathbb{R}^N$ is called a k-dimensional Lipschitz-manifold if for every $x_0 \in \mathcal{M}^k$ there exists an open set $\mathcal{U} \subset \mathbb{R}^N$ and a bi-Lipschitz mapping $\phi : \mathcal{U} \to \phi(\mathcal{U}) \subset \mathbb{R}^N$ such that $x_o \in \mathcal{U}$ and $\phi(\mathcal{M}^k \cap \mathcal{U}) = \mathbb{R}^{N-k}_o \cap \phi(\mathcal{U})$ where $\mathbb{R}^{N-k}_o = \{x \in \mathbb{R}^N : x_{k+1} = \ldots = x_N = 0\}$.*

Sometimes the following equivalent characterization is more convenient.

Theorem 2.4 *For $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ we write $x' := (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' := (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-k}$, $x = (x', x'')$. A subset $\mathcal{M}^k \subset \mathbb{R}^N$ is a k-dimensional Lipschitz-manifold if and only if for every $x_o \in \mathcal{M}^k$, $x_o = (x'_o, x''_o)$, there exist open subsets $V' \subset R^k$ and $V'' \subset \mathbb{R}^{N-k}$ such that with $V := V' \times V'' \subset \mathbb{R}^{N-k}$ holds true:*

1. *$x'_o \in V'$, $x''_o \in V''$, $x_o = (x'_o, x''_o) \in V$.*
2. *There exists a bi-Lipschitz mapping $H : V \to H(V)$ such that*

$$H(\mathcal{M}^k \cap V) = \{(x', x'') \in H(V) : x'' = 0\}$$

Proof

1. Let $\mathcal{M}^k$ be a k-dimensional Lipschitz manifold. Let $x_o \in \mathcal{M}^k$ and $\mathcal{U} \subset \mathbb{R}^N$ and $\phi : U \to \phi(\mathcal{U})$ be according Definition 2.3. Since $\mathcal{U}$ is open and $x_o \in \mathcal{U}$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathcal{U}$. Let $V' := B'_{\frac{\varepsilon}{\sqrt{2}}}(x'_o) \subset \mathbb{R}^k$, $V'' := B''_{\frac{\varepsilon}{\sqrt{2}}}(x''_o) \subset \mathbb{R}^{N-k}$. Then $V := \mathcal{V}' \times \mathcal{V}'' \subset B_\varepsilon(x)$. Let $H := \phi\,|_V$.
2. Clearly the converse statement holds true with $\mathcal{U} := V$ and $\phi := H$. ■

Definition 2.5 *Let $e_i := (\delta_{1i}, \ldots, \delta_{Ni})$, $i = 1, \ldots, N$, denote the canonical basis in R^N and let $x_o \in \mathbb{R}^N$. We say that $[O_{x_o}, f_1, \ldots, f_N]$ is a local Euclidean coordinate system with origin at x_o if there exists an orthogonal matrix S such that $f_i = Se_i$, $i = 1, \ldots, N$. For $x = \sum\limits_{i=1}^{N} x_i e_i \in \mathbb{R}^N$ let*

$$y =: \mathrm{T}x = S(x - x_o) = S\left(\sum_{i=1}^{N}(x_i - x'_{oi})e_i\right) = \sum_{i=1}^{N}(x_i - x_{oi})f_i$$

and conversely

$$\begin{aligned}
\mathrm{T}^{-1}y &= S^t y + x_o = S^t\left(\sum_{i=1}^{N}(x_i - x_{oi})f_i\right) + x_o = \\
&= \sum_{i=1}^{N}(x_i - x_{oi})S^t f_i + x_o = \sum_{i=1}^{N}(x_i - x_{oi})e_i + x_o = x - x_o + x_o = x
\end{aligned}$$

Definition 2.6 *A subset* $\mathcal{M}^k \subset \mathbb{R}^N$ *is called a k-dimensional Lipschitz-manifold in* **graph representation** *if for every* $x_o \in \mathcal{M}^k$ *there exists a local Euclidean coordinate system* $[O_{x_o}, f_1, \dots, f_N]$ *with origin in* x_o *and*

1. *there are open subsets* $V' \in \mathbb{R}^k$ *and* $V'' \subset \mathbb{R}^{N-k}$, $V := V' \times V''$ *and* $O_{x_o} \in V$.

2. *there exists a Lipschitz mapping* $h : V' \to V''$ *with*

$$\mathcal{M}^k \cap V = \{(x', h(x')) : x' \in V'\}$$

Definition 2.7 *A subset* $\mathcal{M}^k \subset \mathbb{R}^N$ *is called a k-dimensional Lipschitz-manifold in* **implicit representation** *if for every* $x_o \in \mathcal{M}^k$ *there exists a local Euclidean coordinate system* $[O_{x_o}, f_1, \dots, f_N]$ *with origin* O_{x_o} *in* x_o *such that*

1. *there are open subsets* $V' \subset \mathbb{R}^k$, $V'' \subset \mathbb{R}^{N-k}$, $V := V' \times V''$ *and* $O_{x_o} \in V$

2. *there exists a mapping* $F : V \to \mathbb{R}^{N-k}$ *with* $F(O_{x_o}) = 0$ *and there are two constants* $L_F > 0$, $K_F > 0$ *such that*

$$(2.3) \quad \|F(x', x'') - F(y', y'')\|_{N-k} \leq L_F(\|x' - y'\|_k + \|x'' - y''\|_{N-k}) \quad \forall x = (x', x''),\ \forall y = (y', y'') \in V$$

   and

$$(2.4) \qquad \|F(x', y'') - F(x', z'')\|_{N-k} \geq K_F \|y'' - z''\|_{N-k} \quad \forall x' \in V', \ \ \forall y'', z'' \in V''.$$

3. $\mathcal{M}^k \cap V = \{x \in V : F(x) = 0\}$

Definition 2.8 *A subset* $\mathcal{M}^k \subset \mathbb{R}^N$ *is called a k-dimensional Lipschitz-manifold in* **parametric representation** *if for every* $x_o \in \mathcal{M}^k$ *there exists an open set* $\mathcal{U} \subset \mathbb{R}^N$ *and an open set* $\mathcal{O} \subset \mathbb{R}^k$ *and a mapping* $\psi : \mathcal{O} \to \mathbb{R}^N$ *such that*

1. $\psi : \mathcal{O} \to \psi(\mathcal{O})$ *is bi-Lipschitz*

2. $x_o \in \mathcal{U}$

3. $\psi(\mathcal{O}) = \mathcal{M}^k \cap U$

Definition 2.9 *A subset* $\mathcal{M}^k \subset \mathbb{R}^N$ *is called a k-dimensional Lipschitz-manifold in* **regular parametric representation** *if for every* $x_o \in \mathcal{M}^k$ *there exists a local Euclidean coordinate system* $[O_{x_o}, f_1, \dots, f_N]$ *with origin* O_{x_o} *in* x_o *such that*

1. *there is an open set* $\mathcal{U} \subset \mathbb{R}^N$ *with* $O_{x_o} \in \mathcal{U}$ *and an open set* $\mathcal{O} \subset \mathbb{R}^k$ *and a mapping* $\psi : \mathcal{O} \to \mathbb{R}^N$ *such that*

   (a) $\psi : \mathcal{O} \to \psi(\mathcal{O})$ *is bi-Lipschitz* *(b)* $\psi(\mathcal{O}) = \mathcal{M}^k \cap \mathcal{U}$

(c) with

$$\hat{\psi}(t) := (\psi_1(t), \ldots, \psi_k(t)) \qquad t \in \mathcal{O}$$

the mapping $\hat{\psi} : \mathcal{O} \to \hat{\psi}(\mathcal{O}) \subset \mathbb{R}^k$ *is bi-Lipschitz.*

Theorem 2.10 *A k-dimensional Lipschitz-manifold* $\mathcal{M}^k$ *in graph representation (Definition 2.6) is a k-dimensional Lipschitz-manifold in the sense of Definition 2.3.*

Proof Let $x_o \in \mathcal{M}^k$ and let $[O_{x_o}, f_1, \ldots, f_N]$ be a local Euclidean coordinate system with origin at x_o, $f_i = Se_i$, $i = 1, \ldots, N$ with an orthogonal matrix S. Let the points $y \in \mathbb{R}^N$ be described with respect to the $[O_{x_o}, f_1, \ldots, f_N]$ frame. Let $V' \subset \mathbb{R}^k$, $V'' \subset \mathbb{R}^{N-k}$ be open, $O_{x_o} \in V := V' \times V''$ and let $h : V' \to V''$ be a Lipschitz mapping with

$$\mathcal{M}^k \cap V = \{(y', h(y')) : y' \in V'\}\,.$$

Let $\mathcal{U} := \{\mathrm{T}^{-1}y = S^t y + x_o : y \in V\}$ (where T is defined according Definition 2.5). For $x \in \mathcal{U}$ we write $y := \mathrm{T}x = ((\mathrm{T}x)', (\mathrm{T}x)'') \in V' \times V''$. Let now $\phi : \mathcal{U} \to \mathbb{R}^n$ be defined by $\phi(x) = ((\mathrm{T}x)', h((\mathrm{T}x)') - (\mathrm{T}x)'')$. Then for $x \in \mathcal{U}$

$$\phi(x) \in \mathbb{R}^{N-k}_o \Leftrightarrow (\mathrm{T}x)'' = h\left((\mathrm{T}x)'\right) \Leftrightarrow \\ ((\mathrm{T}x)', h\left((\mathrm{T}x)'\right)) \in \{(y', h(y')) : y' \in V'\} = \mathcal{M}^k \cap V.$$

We prove that ϕ is bi-Lipschitz. Let $L_h > 0$ such that

$$\|h(y') - h(y'')\|_{N-k} \le L_h \|y' - y''\|_k.$$

Then, for $x, z \in \mathcal{U}$

$$\begin{aligned}\|\phi(x) - \phi(z)\|_N^2 &= \|(\mathrm{T}x)' - (\mathrm{T}z)'\|_k^2 + \|h\left((\mathrm{T}x)'\right) - h\left((\mathrm{T}z)'\right) + (\mathrm{T}z)'' - (\mathrm{T}x)''\|_{N-k}^2 \le \\ &\le \|(\mathrm{T}x)' - (\mathrm{T}z)'\|_k^2 + \left(L_h\|(\mathrm{T}x)' - (\mathrm{T}z)'\|_k + \|(\mathrm{T}z)'' - (\mathrm{T}x)''\|_{N-k}\right)^2 \le \\ &\le (1 + 2L_h^2)\|(\mathrm{T}x)' - (\mathrm{T}z)'\|_k^2 + 2\|(\mathrm{T}z)'' - (\mathrm{T}x)''\|_{N-k}^2\end{aligned}$$

With $C := (1 + 2\max(1, L_h^2))^{\frac{1}{2}} > 0$ we see

$$\begin{aligned}\|\phi(x) - \phi(z)\|_N^2 &\le C^2\left(\|(\mathrm{T}x)' - (\mathrm{T}z)'\|_k^2 + \|(\mathrm{T}x)'' - (\mathrm{T}z)''\|_{N-k}^2\right) = \\ &= C^2\|\mathrm{T}x - \mathrm{T}z\|_N^2 = C^2\|Sx - Sz\|_N^2 = C^2\|x - z\|_N^2\end{aligned}$$

Let now $x \in \mathcal{U}$ and $\phi(x) = z \in \phi(\mathcal{U})$. Then

$$(\mathrm{T}x)' = z' \quad \text{and} \quad h((\mathrm{T}x)') - (\mathrm{T}x)'' = z''$$

whence $h(z') - z'' = (\mathrm{T}x)''$. Therefore

$$\mathrm{T}x = ((\mathrm{T}x)', (\mathrm{T}x)'') = (z', h(z') - z'')$$

and

$$x = S^t\left((z', h(z') - z'')\right) + x_o = \phi^{-1}\left(z', h(z') - z''\right).$$

Then for $z, w \in \phi(\mathcal{U})$

$$\begin{aligned}
&\left\|\phi^{-1}\left(z', h(z') - z''\right) - \phi^{-1}\left((w', h(w') - w'')\right)\right\|_N^2 = \\
&= \|(z', h(z') - z'') - (w', h(w') - w'')\|_N^2 = \\
&= \|z' - w'\|_k^2 + \|h(z') - h(w') + w'' - z''\|_{N-k}^2
\end{aligned}$$

As above, we see

$$\left\|\phi^{-1}(z) - \phi^{-1}(w)\right\|_N \leq C\|w - z\|_N,$$

whence u for $x, z \in \mathcal{U}$

$$C^{-1}\|x - z\|_N \leq \|\phi(x) - \phi(z)\|_N \leq C\|x - z\|_N$$

■

Theorem 2.11 *Let $\mathcal{M}^k \subset \mathbb{R}^N$. Then there are equivalent: $\mathcal{M}^k$ is a k-dimensional Lipschitz-manifold*

1. *in graph representation (Definition 2.6)*
2. *in regular parametric representation (Definition 2.9)*
3. *in implicit representation (Definition 2.7)*

Proof Througout this proof let $x_o \in \mathcal{M}^k$ and let $[O_{x_o}, f_1, \ldots, f_N]$ be a local Euclidean coordinate system with origin in x_o such that the respective representations hold true.

1. "$1° \Rightarrow 2°$" : Let $\mathcal{O} := V' \subset \mathbb{R}^k$ and let $\psi : \mathcal{O} \to \mathbb{R}^N$ be defined by

   $$\begin{aligned}
   \psi_i(x') &:= x_i, \quad i = 1, \ldots, k \\
   \psi_i(x') &:= h_{i-k}(x'), \quad i = k+1, \ldots, N
   \end{aligned}$$

   $(x' \in \mathcal{O})$. Then for $x', y' \in \mathcal{O}$

   $$\|\psi(x') - \psi(y')\|_N^2 = \|x' - y'\|_k^2 + \|h(x') - h(y')\|_{N-k}^2 \leq (1 + L_h^2)\|x' - y'\|_k^2$$

   where L_h denotes the Lipschitz constant of h. Clearly

   $$\|\psi(x') - \psi(y')\|_N^2 \geq \sum_{i=1}^{k} \|\psi_i(x') - \psi_i(y')\|_k^2 = \|x' - y'\|_k^2$$

   and with $\hat{\psi}(x') := (\psi_1(x'), \ldots, \psi_k(x')) = x'$ we see that ψ is a regular parametric representation.

2. "$2^\circ \Rightarrow 3^\circ$": Assume

$$L_1\|x' - y'\|_k \le \|\hat{\psi}(x') - \hat{\psi}(y')\|_k \le L_2\|x' - y'\|_k \quad \forall\ x', y' \in \mathcal{O}, \tag{2.5}$$

where $0 < L_1 \le L_2$. Let

$$\hat{\hat{\psi}}(x') := (\psi_{k+1}(x'), \dots, \psi_N(x)) \text{ for } x' \in \mathcal{O}$$

Since $\psi : \mathcal{O} \to \mathbb{R}^N$ is Lipschitz, $\hat{\hat{\psi}} : \mathcal{O} \to \mathbb{R}^{N-k}$ is Lipschitz too, and there is $K > 0$ such that

$$\|\hat{\hat{\psi}}(x') - \hat{\hat{\psi}}(y')\|_{N-k} \le K\|x' - y'\|_k \qquad \forall x', y' \in \mathcal{O}. \tag{2.6}$$

Since $\hat{\psi} : \mathcal{O} \to \hat{\psi}(\mathcal{O}) \subset \mathbb{R}^k$ is bi-Lipschitz, $V' := \hat{\psi}(\mathcal{O}) \subset \mathbb{R}^k$ is open ([7, Theorem 4.7]). Let $V'' := \mathbb{R}^{N-k}$ and $V := V' \times V'' \subset \mathbb{R}^N$. We define $F : V \to \mathbb{R}^{N-k}$ by

$$F(z', z'') := \hat{\hat{\psi}}\left(\hat{\psi}^{-1}(z')\right) - z'', \qquad (z', z'') \in V' \times V''$$

$z = (z', z'') \in \mathcal{M}^k \cap \mathcal{U} \Leftrightarrow \exists_1 x' \in \mathcal{O}$ such that

$$\begin{aligned} \psi(x') = \left(\hat{\psi}(x'), \hat{\hat{\psi}}(x'')\right) = (z', z'') \Leftrightarrow \hat{\psi}(x') = z' \in V', \\ z'' = \hat{\hat{\psi}}(x'') = \hat{\hat{\psi}}\left(\hat{\psi}^{-1}(z_1)\right) \in \mathbb{R}^{N-k} \Leftrightarrow (z', z'') \in V \text{ and } F(z', z'') = 0 \end{aligned}$$

For $z = (z', z'')$, $w = (w', w'') \in V$ and $x' = \hat{\psi}^{-1}(z')$, $y' := \hat{\psi}^{-1}(w')$, by (2.5)

$$\|x' - y'\|_k \le L_1^{-1}\|z' - w'\|_k$$

and by (2.6)

$$\|\hat{\hat{\psi}}\left(\hat{\psi}^{-1}(z')\right) - \hat{\hat{\psi}}\left(\hat{\psi}^{-1}(w')\right)\|_{N-k} \le KL_1^{-1}\|z' - w'\|_k$$

Therefore

$$\begin{aligned} &\|F(z', z'') - F(w', w'')\|_{N-k} \le \\ &\quad \le \left\|\hat{\hat{\psi}}\left(\hat{\psi}^{-1}(z')\right) - \hat{\hat{\psi}}\left(\hat{\psi}^{-1}(w')\right)\right\|_{N-k} + \|z'' - w''\|_{N-k} \le \\ &\quad \le KL_1^{-1}\|z' - w'\|_k + \|z'' - w''\|_{N-k} \le L_F\left(\|z' - w'\|_k + \|z'' - w''\|_{N-k}\right) \end{aligned}$$

where $L_F := \max\left(1, KL_1^{-1}\right)$. Furthermore, for $t' \in V'$, $z'', w'' \in V''$

$$\|F(t', z'') - F(t', w'')\|_{N-k} = \|z'' - w''\|_{N-k}.$$

3. "$3^\circ \Rightarrow 1^\circ$": By the Lipschitz variant of the implicit function theorem (compare e.g. [7, Theorem 4.8, p. 41/42]) there exists an open set $W' \subset V' \subset \mathbb{R}^k$ and a Lipschitz map $g : W' \to \mathbb{R}^{N-k}$ such that $(O_{x_o})' \in W'$ and

   (a) $(x', g(x')) \in V \;\forall x' \in W'$

   (b) $F\,(x', g(x')) = 0 \;\forall x' \in W'$

   (c) $\{(x', x'') \in W' \times V'' : F(x', x'') = 0\} = \{(x', g(x')) : x' \in W'\}$

   Therefore with $W := W' \times V''$ we see

$$\mathcal{M}^k \cap W = \{x \in W : F(x) = 0\}\,.$$

■

Theorem 2.12 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold in the sense of Definition 2.3. Then it is a k-dimensional Lipschitz-manifold in parametric representation too.*

Proof Let $x_o \in \mathcal{M}^k$ and let $\mathcal{U} \subset \mathbb{R}^N$ be open, $\phi : \mathcal{U} \to \phi(\mathcal{U}) \subset \mathbb{R}^N$ be bi-Lipschitz such that $x_o \in \mathcal{U}$ and $\phi(\mathcal{M}^k \cap \mathcal{U}) = \mathbb{R}^{N-k}_o \cap \phi(\mathcal{U})$. By [7, Theorem 4.6], $\phi(\mathcal{U})$ is open. Let

$$\mathcal{O} := \left\{x' \in \mathbb{R}^k : (x', 0) \in \mathbb{R}^{N-k}_o \cap \phi(\mathcal{U}) = \phi(\mathcal{M}^k \cap \mathcal{U})\right\}.$$

We prove that $\mathcal{O}$ is open. Let $(x', 0) \in \mathbb{R}^{N-k}_o \cap \phi(\mathcal{U})$. Since $\phi(\mathcal{U}) \subset \mathbb{R}^N$ is open, there is $\varepsilon > 0$ such that $\left\{y \in \mathbb{R}^N : \|y - (x', 0)\|_N < \varepsilon\right\} \subset \phi(\mathcal{U})$. Let

$$B'_\varepsilon(x') := \left\{y' \in \mathbb{R}^k : \|y' - x'\|_k < \varepsilon\right\}.$$

For $y' \in B'_\varepsilon(x')$ we see $\|(y', 0) - (x', 0)\|_N = \|y' - x'\|_k < \varepsilon$ and therefore $(y,0) \in \phi(\mathcal{U}) \cap \mathbb{R}^{N-k}_o$, that is $B'_\varepsilon(x') \subset \mathcal{O}$. Let $\psi : \mathcal{O} \to \mathbb{R}^N$, $\psi(x') := \phi^{-1}\left((x', 0)\right)$, $x' \in \mathcal{O}$. Then $\psi(\mathcal{O}) = \mathcal{M}^k \cap \mathcal{U}$ and because ϕ is bi-Lipschitz, ψ is bi-Lipschitz too. ■

In the case of k-dimensional C^1-manifolds it is easy to see that the parametric map $\psi : \mathcal{O} \to \mathcal{M}^k \cap \mathcal{U}$ can be locally extended to a diffeomorphism of an open set $\tilde{\mathcal{O}} \subset \mathbb{R}^N$ to an open set $\tilde{\mathcal{U}} \subset \mathcal{U}$ such that $\tilde{\psi}\left((x', 0)\right) = \psi(x')$ for $x' \in \mathcal{O}$. In the underlying case we had not been able to prove that a k-dimensional Lipschitz manifold in parametric representation is a k dimensional Lipschitz manifold in the sense of Definition 2.3. Conversely until now we could not find a counterexample too.

Theorem 2.13 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz manifold in the sense of Definition 2.3 (resp. in parametric representation). Let $\mathcal{W} \subset \mathbb{R}^N$ be open and $f : \mathcal{W} \to \mathbb{R}^N$ be bi-Lipschitz. Let $\mathcal{M}^k \subset \mathcal{W}$. Then $\tilde{\mathcal{M}}^k := f(\mathcal{M}^k)$ is a k-dimensional Lipschitz manifold in the sense of Definition 2.3 (resp. in parametric representation).*

Proof

1. Let $\mathcal{M}^k$ be a k-dimensional Lipschitz manifold in the sense of Definition 2.3. Let $\tilde{x}_o \in \tilde{\mathcal{M}}^k$. Then there exists a unique $x_o \in \mathcal{M}^k$ such that $\tilde{x}_o = f(x_o)$. By Definition 2.3, there exists an open set $\mathcal{U} \subset \mathbb{R}^N$ and a bi-Lipschitz mapping $\phi : \mathcal{U} \to \phi(\mathcal{U}) \subset \mathbb{R}^N$

such that $x_o \in \mathcal{U}$ and $\phi(\mathcal{M}^k \cap \mathcal{U}) = \mathbb{R}_o^{N-k} \cap \phi(\mathcal{U})$. Let $\tilde{\mathcal{U}} := f(U)$. Then $\tilde{\mathcal{U}}$ is open (see Theorem 2.2). Further, $\tilde{\phi} := \phi \circ f^{-1} : \tilde{\mathcal{U}} \to \mathbb{R}^N$ is bi-Lipschitz and by injectivity of f

$$\begin{aligned}\tilde{\phi}(\tilde{\mathcal{M}}^k \cap \tilde{\mathcal{U}}) &= \phi(f^{-1}(\tilde{\mathcal{M}}^k \cap \tilde{\mathcal{U}})) = \phi\left(f^{-1}(f(\mathcal{M}^k) \cap f(\mathcal{U}))\right) = \\ &= \phi(\mathcal{M}^k \cap \mathcal{U}) = \mathbb{R}_o^{N-k} \cap \phi(\mathcal{U}) = \mathbb{R}_o^{N-k} \cap \phi(f^{-1}f(\mathcal{U})) = \\ &= \mathbb{R}_o^{N-k} \cap \phi(f^{-1}(\tilde{\mathcal{U}})) = \mathbb{R}_o^{N-k} \cap \tilde{\phi}(\tilde{\mathcal{U}}).\end{aligned}$$

2. Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold in parametric form. Let $\tilde{x}_o \in \tilde{\mathcal{M}}^k$. Then there is a unique $x_o \in \mathcal{M}^k$ such that $\tilde{x}_o = f(x_o)$. By Definition 2.8 there is an open set $\mathcal{U} \subset \mathbb{R}^N$, an open set $\mathcal{O} \subset \mathbb{R}^k$ and a bi-Lipschitz mapping $\psi : \mathcal{O} \to \psi(\mathcal{O}) \subset \mathbb{R}^N$ such that $x_o \in \mathcal{U}$ and $\psi(\mathcal{O}) = \mathcal{M}^k \cap \mathcal{U}$. Let $\tilde{\mathcal{U}} := f(\mathcal{U})$ and let $\hat{\psi} : \mathcal{O} \to \mathbb{R}^N$, $\hat{\psi} := f \circ \psi$. Then $\hat{\psi} : \mathcal{O} \to \hat{\psi}(\mathcal{O})$ is bi-Lipschitz and

$$\hat{\psi}(\mathcal{O}) = f(\psi(\mathcal{O})) = f(\mathcal{M}^k \cap \mathcal{U}) = f(\mathcal{M}^k) \cap f(\mathcal{U}) = \tilde{\mathcal{M}}^k \cap \tilde{\mathcal{U}}.$$

■

Obviously the definition of a k-dimensional Lipschitz manifold $\mathcal{M}^k$ in parametric representation is the most general one and it is invariant under bi-Lipschitz transforms of a neighborhood $\mathcal{W}$ of $\mathcal{M}^k$. So we use Definition 2.8 for the remaining part of the paper. For the sake of brevity, in the sequel we call $\mathcal{M}^k \subset \mathbb{R}^N$ a k-dimensional Lipschitz-manifold if Definition 2.8 applies to $\mathcal{M}^k$. As a preparation we need

Lemma 2.14 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold. Then there exists a sequence $(\mathcal{K}_n)_{n\in\mathbb{N}} \subset \mathcal{M}^k$ such that*

1. *$\mathcal{K}_n$ is compact* $\forall n \in \mathbb{N}$
2. *$\mathcal{K}_n \subset \mathcal{K}_{n+1}$* $\forall n \in \mathbb{N}$
3. *$\mathcal{M}^k = \bigcup_{n=1}^{\infty} \mathcal{K}_n$*

Proof

1. Let $x_o \in \mathbb{Q}^N$, $q \in \mathbb{Q}$ and

$$B_q(x_o) := \left\{x \in \mathbb{R}^N : \|x - x_o\|_N < q\right\}.$$

Then $\mathcal{B} := \left\{B_q(x_o) : q \in \mathbb{Q}, x_o \in \mathbb{Q}^N\right\}$ forms a countable basis of the topology of $\mathbb{R}^N$. We choose an arbitrary but fixed numeration of $\mathcal{B}$ and write $\mathcal{B} = \{\mathcal{U}_i : i \in \mathbb{N}\}$. Then $\mathcal{B}^k := \{V_i := \mathcal{M}_k \cap \mathcal{U}_i : i \in \mathbb{N}\}$ is a countable basis of the topology of $\mathcal{M}^k$.

2. We prove now that for each $x_o \in \mathcal{M}^k$ there exists $j_o \in \mathbb{N}$ such that $x_o \in V_{j_o}$, $\bar{V}_{j_o}$ is compact and $\bar{V}_{j_o} \subset \mathcal{M}^k$. Let $x_o \in \mathcal{M}^k$. Then there is an open $\mathcal{O} \subset \mathbb{R}^k$, an open $\mathcal{U} \subset \mathbb{R}^N$, $x_o \in \mathcal{U}$, and a bi-Lipschitz mapping $\psi : \mathcal{O} \to \psi(0)$ such that $\psi(0) = \mathcal{M} \cap \mathcal{U}$. There exists a unique $y_o \in \mathcal{O}$ such that $x_o = \psi(y_o)$. Since $\mathcal{O}$ is open there exists $\varepsilon > 0$

such that $B_\varepsilon(y_o) \subset \overline{B_\varepsilon(y_o)} \subset \mathcal{O}$. Since $\overline{B_\varepsilon(y_o)}$ is compact, $\psi(\overline{B_\varepsilon(y_o)}) \subset \mathcal{M}^k \cap U$ is compact too. Because of the continuity of ψ^{-1} and $\psi(B_\varepsilon(y_o)) = (\psi^{-1})^{-1}(B_\varepsilon(y_o)) \subset \mathcal{M}^k \cap \mathcal{U}$ the set $\psi(B_\varepsilon(y_o))$ is open in $M^k \cap U$. Since $(V_i)_{i\in\mathbb{N}}$ is a basis of the topology of $\mathcal{M}^k \cap \mathcal{U}$ there exists $j_o \in \mathbb{N}$, $V_{j_o} \subset \mathcal{B}^k$ such that $x_o \in V_{j_o} \subset \psi(B_\varepsilon(y_o))$. Then $\overline{V_{j_o}} \subset \psi(\overline{B_\varepsilon(y_o)}) \subset \mathcal{M}^k \cap \mathcal{U}$ and $\overline{V_{j_o}}$ is compact.

3. The set $\mathcal{W} := \{V_j \in \mathcal{B}^k : \bar{V}_j \text{ compact}, \bar{V}_j \subset \mathcal{M}^k\}$ is either finite or at most countable infinite. Let

$$\mathcal{K}_n := \bigcup_{\substack{j=1 \\ V_j \in \mathcal{W}}}^{n} \bar{V}_j.$$

Then $\mathcal{K}_n$ is compact, $\mathcal{K}_n \subset \mathcal{M}^k$ and $\mathcal{K}_n \subset \mathcal{K}_{n+1}$. Therefore $\bigcup\limits_{n=1}^{\infty} \mathcal{K}_n \subset \mathcal{M}^k$. If conversely $x_o \in \mathcal{M}^k$, then by part 2 of proof there exists $V_{j_o} \in \mathcal{W}$ such that $x_o \in V_{j_o} \subset \bar{V}_{j_o} \subset \bigcup\limits_{n=1}^{\infty} \mathcal{K}_n$, whence $\mathcal{M}^k \subset \bigcup\limits_{n=1}^{\infty} \mathcal{K}_n$. ■

Theorem 2.15 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold. Then there is an at most countable set $\Lambda \subset \mathbb{N}$ such that*

1. *for each $i \in \Lambda$ there exists an open set $\mathcal{O}_i \subset \mathbb{R}^k$, an open set $\mathcal{U}_i \subset \mathbb{R}^N$ and a bi-Lipschitz mapping $\psi_i : \mathcal{O}_i \to \psi_i(\mathcal{O}_i)$ such that $\psi_i(\mathcal{O}_i) = \mathcal{M}^k \cap \mathcal{U}_i$*
2. $\mathcal{M}^k = \bigcup\limits_{i\in\Lambda} \mathcal{M}^k \cap \mathcal{U}_i = \bigcup\limits_{i\in\Lambda} \psi_i(\mathcal{O}_i)$

Proof Let the sequence $(\mathcal{K}_n)_{n\in\mathbb{N}} \subset \mathcal{M}^k$ be according Lemma 2.14. For each $x_o \in \mathcal{K}_n$ there exists an open $\mathcal{O}_{x_o} \subset \mathbb{R}^k$, an open $\mathcal{U}_{x_o} \subset \mathbb{R}^N$ and a bi-Lipschitz $\psi_{x_o} : \mathcal{O}_{x_o} \to \psi(\mathcal{O}_{x_o})$ such that $\psi_{x_o}(\mathcal{O}_{x_o}) = \mathcal{M}^k \cap \mathcal{U}_{x_o}$. Then for each $n \in \mathbb{N}$

$$\{\mathcal{M}^k \cap \mathcal{U}_{x_o} : x_o \in \mathcal{K}_n\}$$

is an open covering of the compact set $\mathcal{K}_n$. Therefore there exists $p_n \in \mathbb{N}$ and $x_j^{(n)} \in \mathcal{K}_n$, $j = 1, \dots, p_n$, such that

$$\mathcal{K}_n \subset \bigcup_{j=1}^{p_n} \mathcal{M}^k \cap \mathcal{U}_{x_j^{(n)}}.$$

Then

$$\mathcal{M}^k = \bigcup_{n=1}^{\infty} \mathcal{K}_n = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{p_n} \mathcal{M}^k \cap \mathcal{U}_{x_j^{(n)}}.$$

The set $\left\{x_j^{(n)} : n \in \mathbb{N}, j = 1, \dots, p_n\right\}$ is at most countable infinite. Let the set of corresponding pairs of indizes be numbered consecutively which gives Λ. If $i \in \Lambda$, $x_i = x_j^{(n)}$, $j \in \{1, \dots, p_n\}$, then let $\mathcal{U}_i := \mathcal{U}_{x_i^{(n)}}$, $\psi_i := \psi_{x_j^{(n)}}$, $\mathcal{O}_i := \mathcal{O}_{x_j^{(n)}}$. ■

Theorem 2.15 justifies the following definition.

Definition 2.16 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz manifold.*

1. *A pair $(\mathcal{O}, \psi)$ with an open set $\mathcal{O} \subset \mathbb{R}^k$ and a bi-Lipschitz mapping $\psi : \mathcal{O} \to \psi(\mathcal{O})$ such that there exists an open set $\mathcal{U} \subset \mathbb{R}^N$ with the property $\psi(\mathcal{O}) = \mathcal{M}^k \cap \mathcal{U}$ is called a (local)* **parametric representation** *of $\mathcal{M}^k$ or* **chart** *of $\mathcal{M}^k \cap \mathcal{U}$.*

2. *Let either $\Lambda = \{1, \dots, s\}$ $(s \in \mathbb{N})$ or $\Lambda = \mathbb{N}$. A system $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$, each $(\mathcal{O}_i, \psi_i)$ being a chart of $\mathcal{M}^k$, is called a* **parametric representation** *or* **atlas** *of $\mathcal{M}^k$.*

Remark 2.17 *Let $\mathcal{M}^k$ be a k-dimensional Lipschitz manifold. For $i = 1, 2$ let $\mathcal{O}_i \subset \mathbb{R}^k$ and $\mathcal{U}_i \subset \mathbb{R}^N$ be open, let $\psi_i : \mathcal{O}_i \to \psi(\mathcal{O}_i)$ be bi-Lipschitz such that*

$$\psi_i(\mathcal{O}_i) = \mathcal{M}^k \cap \mathcal{U}_i \qquad (i = 1, 2).$$

Then $\mathcal{U} := \psi_1(\mathcal{O}_1) \cap \psi_2(\mathcal{O}_2)$ is a relatively open subset of $\mathcal{M}^k$. Then the sets $\psi_i^{-1}(\mathcal{U}) \subset \mathbb{R}^k$ are open $(i = 1, 2)$. The mapping $\psi_2^{-1} \circ \psi_1$ is a bi-Lipschitz mapping from $\psi_1^{-1}(\mathcal{U})$ onto $\psi_2^{-1}(\mathcal{U})$ (as a mapping from a subset of $\mathbb{R}^k$ into $\mathbb{R}^k$).

2.2 Examples

Example 1

For $k \in \mathbb{Z}$ let

$$\begin{aligned} I_1^{(k)} &:=]2^{-2k-2}, 2^{-2k-1}], \\ I_2^{(k)} &:=]2^{-2k-1}, 2^{-2k}], \\ I^{(k)} &:= I_1^{(k)} \cup I_2^{(k)} \end{aligned}$$

Let $g_i : \mathbb{R}_+ \to \mathbb{R}$ $(\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}, i = 1, 2)$ be defined by

(E.1)
$$g_1(t) := \begin{cases} 0 & \text{if } t = 0 \\ 0 & \text{if } t \in I_1^{(k)} \\ 1 & \text{if } t \in I_2^{(k)} \end{cases}$$

(E.2)
$$g_2(t) := \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \in I_1^{(k)} \\ 0 & \text{if } t \in I_2^{(k)} \end{cases}$$

Then g_i are measurable and bounded. Let for $t \in \mathbb{R}_+$

(E.3)
$$f_i(t) := \int_0^t g_i(s)ds, \quad i = 1, 2$$

Then f_i is differentiable in $I_1^{(k)}$ and $I_2^{(k)}$ (at the right endpoint of $I_j^{(k)}$ from the left side, $j = 1, 2$)

Denote by $g_{i\varepsilon}$ the mollification of $g_i(\varepsilon > 0)$ and define

$$f_i^{(\varepsilon)}(t) := \int_0^t g_{i\varepsilon}(s)ds$$

Then $f_i^{(\varepsilon)} \in C^\infty(\mathbb{R}_+)$, $f_i^{(\varepsilon)'}(t) = g_{i\varepsilon}(t)$

$$|f_i(t) - f_i^{(\varepsilon)}(t)| \le \int_0^t |g_i(s) - g_{i\varepsilon}(s)|ds \to 0 \quad (\varepsilon \to 0)$$

and for all $t \in \mathbb{R}$. Furthermore, for $R > 0$

$$\int_0^R |g_i(t) - g_{i\varepsilon}(t)|dt \to 0 \quad (\varepsilon \to 0)$$

Let $\varphi \in C_0^\infty(R_+)$ and choose $R > 0$ such that $\operatorname{supp} \varphi \subset [0, R]$. Then

$$\int_{\mathbb{R}_+} f_i(t)\varphi'(t)dt = \int_0^R f_i(t)\varphi'(t)dt = \lim_{\varepsilon \to 0} \int_0^R f_i^{(\varepsilon)}(t)\varphi'(t)dt = -\lim_{\varepsilon \to 0} \int_0^R g_{i\varepsilon}(t)\varphi(t)dt =$$

$$= -\int_0^R g_i(t)\varphi(t)dt$$

whence g_i is the weak derivative of f_i, f_i, $f_i' = g_i \in L^1([0, R])$ for all $R > 0$, $i = 1, 2$. Let $t, t' \in \mathbb{R}_+$. Then

$$|f_i(t) - f_i(t')| \le \left| \int_{t'}^t |g_i(s)|ds \right| \le |t - t'| \tag{E.4}$$

We write $f(t) := (f_1(t), f_2(t))$. Then $f : \mathbb{R}_+ \to \mathbb{R}^2$. For $x = (x_1, x_2) \in \mathbb{R}^2$ we denote by

$$\|x\| := \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$$

the Euclidean norm of x. By (E.4) we see

$$\|f(t) - f(t')\| \le \sqrt{2}|t - t'| \qquad \forall t, t' \in \mathbb{R}_+. \tag{E.5}$$

If $t \in \mathbb{R}_+$, $t > 0$ then there is a unique $k_o \in \mathbb{Z}$ such that $t \in I^{(k_o)}$. From the definition (E.3) we calculate easily f_i. Let $i = 1$ and $t \in I_1^{(k_o)}$. Observing (E.1) we see

$$f_1(t) = \int_0^{2^{-2k_o-2}} g_1(s)ds = \sum_{k=k_o+1}^{\infty} \int_{2^{-2k-1}}^{2^{-2k}} ds = \frac{1}{3} 2^{-2k_o-1}$$

Let $t \in I_2^{(k_o)}$. Then

$$f_1(t) = \frac{1}{3}2^{-2k_o-1} + \int\limits_{2^{-2k_o-1}}^{t} ds = t - \frac{1}{3}2^{-2k_o}$$

Similarly we calculate f_2. The result is

(E.6)
$$f(t) = (f_1(t), f_2(t)) = \begin{cases} 0 & \text{if } t = 0 \\ \left(\frac{1}{3}2^{-2k-1}, t - \frac{1}{3}2^{-2k-1}\right) & \text{if } t \in I_1^{(k)} \\ \left(t - \frac{1}{3}2^{-2k}, \frac{1}{3}2^{-2k}\right) & \text{if } t \in I_2^{(k)} \end{cases}$$

Let for $t \in \mathbb{R}_+$ $y = (y_1, y_2) := (f_1(t), f_2(t))$. Then we see from (E.6)

(E.7)
$$t = y_1 + y_2 = f_1(t) + f_2(t) \qquad \forall t \in \mathbb{R}_+,$$

whence

$$t - t' = f_1(t) + f_2(t) - f_1(t') - f_2(t')$$

and

$$\begin{aligned} |t - t'| &\leq |f_1(t) - f_1(t')| + |f_2(t) - f_2(t')| \\ &\leq \sqrt{2}\left[(f_1(t) - f_1(t'))^2 + (f_2(t) - f_2(t'))^2\right]^{\frac{1}{2}} = \sqrt{2}\|f(t) - f(t')\|. \end{aligned}$$

Because of (E.5) we finally see

(E.8)
$$\frac{1}{\sqrt{2}}|t - t'| \leq \|f(t) - f(t')\| \leq \sqrt{2}|t - t'| \qquad \forall t, t' \in \mathbb{R}_+$$

We extend now f to $\mathbb{R}$. Let

(E.9)
$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \geq 0 \\ -f(-t) & \text{if } t < 0 \end{cases}$$

Because of (E.6), (E.7) we immediately see that $\tilde{f}\,|_{\mathbb{R}_+}$ and $\tilde{f}\,|_{\{x \in \mathbb{R}, x < 0\}}$ are bi-Lipschitz.

By (E.9) we see that (E.7) continues to hold for $t < 0$ and f_i replaced by $\tilde{f}_i$, whence the first inequality in (E.8) holds true for all $t, t' \in \mathbb{R}$. Let $t > 0 > t'$. Then by (E.5)

$$\|\tilde{f}(t) - \tilde{f}(t')\| = \|f(t) + f(-t')\| \leq \|f(t)\| + \|f(-t')\| \leq \sqrt{2}(t + (-t')) = \\ = \sqrt{2}|t - t'|$$

Therefore (E.8) is satisfied with $\tilde{f}$ in place of f and for all $t, t' \in \mathbb{R}$.

It follows immediately from (E.6$_2$) that f is not a graph of a map $h : \mathbb{R} \to \mathbb{R}^2$. This can also be seen from the continuous line in figure 1. Further for every $\varepsilon > 0$ the interval $]-\varepsilon, \varepsilon[$ contains infinitely many intervals $I_i^{(k)}$ of constancy of f_i ($k \geq k_o(\varepsilon) \in \mathbb{N}$) whence f is not in regular parametrization in a neighborhood of zero. Then, because of Theorem 2.10 it can't be in implicit representation too.

But let now $a := \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and we set

$$S := \begin{pmatrix} a & a \\ -a & a \end{pmatrix}$$

Then S is an orthogonal matrix. Let

$$h(t) := \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} := S \begin{pmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \end{pmatrix} = \begin{pmatrix} a\left(\tilde{f}_1(t) + \tilde{f}_2(t)\right) \\ a\left(\tilde{f}_2(t) - \tilde{f}_1(t)\right) \end{pmatrix}$$

Because of (E.7) we see $h_1(t) = at$ for $t \in \mathbb{R}$. Further by (E.6) for $t > 0$.

(E.10)
$$h_2(t) = \begin{cases} a\left(t - \frac{1}{3}2^{-2k}\right) & \text{if } t \in I_1^{(k)} \\ a\left(\frac{1}{3}2^{-2k+1} - t\right) & \text{if } t \in I_2^{(k)} \end{cases}, t > 0$$

If $t < 0$ because of $\tilde{f}_i(t) = -\tilde{f}_i(-t)$

(E.11)
$$h_2(t) = \begin{cases} a\left(t + \frac{1}{3}2^{-2k}\right) & \text{if } -t \in I_1^{(k)} \\ a\left(-t - \frac{1}{3}2^{-2k+1}\right) & \text{if } -t \in I_2^{(k)} \end{cases} = -h_2(-t)$$

Let $s := at$. Then

$$|t| \in I_1^{(k)} \Leftrightarrow |s| \in J_1^{(k)} := \left]a2^{-2k-2}, a2^{-2k-1}\right]$$
$$|t| \in I_2^{(k)} \Leftrightarrow |s| \in J_2^{(k)} := \left]a2^{-2k-1}, a2^{-2k}\right]$$

Let $\tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}$,

$$\tilde{\varphi}(s) = h_2\left(\frac{s}{a}\right) = \begin{cases} 0 & \text{if } s = 0 \\ s - \frac{a}{3}2^{-2k} & \text{if } s \in J_1^{(k)} \\ \frac{a}{3}2^{-2k+1} - s & \text{if } s \in J_2^{(k)} \end{cases}$$

We set

$$\varphi(s) := \begin{cases} \tilde{\varphi}(s) & \text{for } s \geq 0 \\ -\tilde{\varphi}(-s) & \text{for } s < 0 \end{cases}$$

For $t, t' \in \mathbb{R}$ by (E.8) we see

$$|h_2(t) - h_2(t')| \leq \|h(t) - h(t')\| = \|S\,(f(t) - f(t'))\,\| \leq \sqrt{2}|t - t'|$$

Therefore ($s = at, s' = at'$)

$$|\varphi(s) - \varphi(s')| \leq \frac{\sqrt{2}}{a}|s - s'| \tag{E.12}$$

and $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Further

$$H := \{h(t) : t \in \mathbb{R}\} = \{(s, \varphi(s)) : s \in \mathbb{R}\}$$

and H is the graph of the function φ. See in addition the interrupted line in figure 1. Clearly, H is in regular parametric representation too. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x_1, x_2) := \varphi(x_1) - x_2.$$

Then $(x_1, x_2) \in H$ if and only if $F(x_1, x_2) = 0$. Let $V' = V'' = \mathbb{R}$. Then $(0, 0) \in V$, $F(0, 0) = 0$. Further

$$\begin{aligned}|F(x_1, x_2) - F(y_1, y_2)| &\leq |\varphi(x_1) - \varphi(y_1)| + (x_2 - y_2) \leq \\ &\leq \max\left(\frac{\sqrt{2}}{a}, 1\right)(|x_1 - y_1| + |x_2 - y_2|)\end{aligned}$$

and

$$|F(x_1, y_2) - F(x_1, z_2)| = |y_2 - z_2|$$

that is, (2.1) and (2.2) of Definition 2.6 are satisfied too, and H is given in implicit representation.

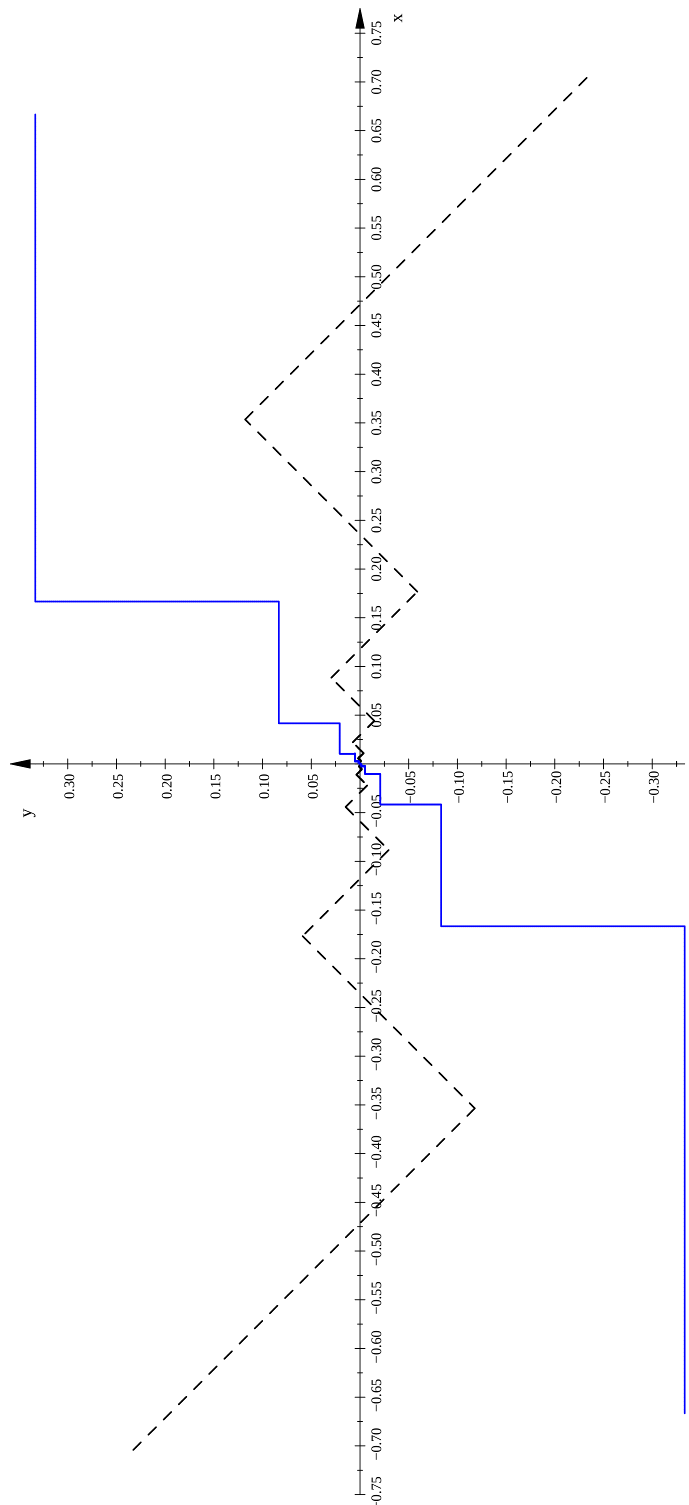
x
y
0.75 0.70 0.65 0.60 0.55 0.50 0.45 0.40 0.35 0.30 0.25 0.20 0.15 0.10 0.05 −0.05 −0.10 −0.15 −0.20 −0.25 −0.30 −0.35 −0.40 −0.45 −0.50 −0.55 −0.60 −0.65 −0.70 −0.75
0.30 0.25 0.20 0.15 0.10 0.05 −0.05 −0.10 −0.15 −0.20 −0.25 −0.30

Figure 1

Example 2

We want now to construct a bi-Lipschitz curve $\tilde{f} : \mathbb{R} \to \mathbb{R}^2$ passing through $x_o = 0 \in \mathbb{R}^2$ such that there doesn't exist a local Euclidean coordinate system with origin at $x_o = 0$ such that $\tilde{f}$ could be represented as a graph.

Let $a := 2^{-4}$. Then

$$-\frac{\pi}{2\ln 2} \ln a^k = 2k\pi \text{ for } k \in \mathbb{Z} \tag{E.13}$$

For $t > 0$ let

$$f(t) = (f_1(t), f_2(t)) = t\left(\cos\left(\frac{-\pi}{2\ln 2}\ln t\right), \sin\left(-\frac{\pi}{2\ln 2}\ln t\right)\right) \tag{E.14}$$

Because of (E.13), (E.14) we see immediately the following scaling property

$$f(a^k f) = a^k f(t) \qquad \forall t > 0, \quad \forall k \in \mathbb{Z} \tag{E.15}$$

We prove now that $f : \mathbb{R}_+ \to f(\mathbb{R}_+)$ (where $\mathbb{R}_\pm := \{x \in \mathbb{R} : x \underset{(<)}{>} 0\}$) is bi-Lipschitz. From the definition (E.14) of f it follows

$$\|f(t)\| = t \text{ for } 0 < t \in \mathbb{R} \tag{E.16}$$

Let now $t, t' > 0$. Then

$$|t - t'| = |\|f(t)\| - \|f(t')\|| \leq \|f(t) - f(t')\| \tag{E.17}$$

Further

$$f_1'(t) = \cos\left(-\frac{\pi}{2\ln 2}\ln t\right) + t\sin\left(-\frac{\pi}{2\ln 2}\ln t\right) \cdot \frac{\pi}{2\ln 2} \cdot \frac{1}{t}$$

whence

$$|f_1'(t)| \leq 1 + \frac{\pi}{2\ln 2} =: C$$

The same estimate holds true for $f_2'(t)$. If $t > t' > 0$ then for $i = 1, 2$

$$|f_i(t) - f_i(t')| = \left|\int_t^{t'} f_i'(s)ds\right| \leq C|t' - t|$$

and therefore

$$\|f(t) - f(t')\| = \left(|f_1(t) - f_1(t')|^2 + |f_2(t) - f_2(t')|^2\right)^{\frac{1}{2}} \leq \sqrt{2}C|t - t'|$$

Because of (E.17) we see

$$|t-t'| \le \|f(t)-f(t')\| \le \sqrt{2}C|t-t'| \quad \forall t,t'>0 \tag{E.18}$$

Let now $\tilde f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\tilde f(t) := \begin{cases} f(t) & \text{if } t>0 \\ 0 & \text{if } t=0 \\ \frac{1}{2}f(-t) & \text{if } t<0 \end{cases} \tag{E.19}$$

Trivially $\tilde f\,|_{\mathbb{R}_+}$ and $\tilde f\,|_{\mathbb{R}_-}$ are both bi-Lipschitz and

$$\frac{1}{2}|t-t'| \le \|\tilde f(t)-\tilde f(t')\| \le \frac{\sqrt{2}C}{2}|t-t'| \text{ for } t,t' \in \mathbb{R}_-. \tag{E.20}$$

It remains to consider the case $s>0>t$. Let $\alpha := -\frac{\pi}{2\ln 2}$. Then

$$\begin{aligned} \|\tilde f(s)-\tilde f(t)\|^2 &= \left[s\cos(\alpha\ln s)+\frac{1}{2}t\cos(\alpha\ln(-t))\right]^2 + \\ &\quad + \left[s\sin(\alpha\ln s)+\frac{1}{2}t\sin(\alpha\ln(-t))\right]^2 = \\ &= s^2+\frac{t^2}{4}+st\left[\cos(\alpha\ln s)\cos(\alpha\ln(-t))+\sin(\alpha\ln s)\sin(\alpha\ln(-t))\right] = \\ &= s^2+\frac{t^2}{4}+st\cos\left(\alpha\ln\left(-\frac{t}{s}\right)\right) = \\ &= s^2-2st+t^2+2st-\frac{3}{4}t^2+st\cos\left(\alpha\ln\left(-\frac{t}{s}\right)\right) = \\ &= (s-t)^2+st\left[2+\cos\left(\alpha\ln\left(-\frac{t}{s}\right)\right)\right]-\frac{3}{4}t^2 \end{aligned}$$

Since $2+\cos\left(\alpha\ln\left(-\frac{t}{s}\right)\right)\ge 1$, $s>0$ and $t<0$ we see

$$\|\tilde f(s)-\tilde f(t)\|^2 \le (s-t)^2$$

whence

$$\|\tilde f(s)-\tilde f(t)\| \le |s-t| \tag{E.21}$$

For the estimate from below we observe

$$\begin{aligned} \frac{\|\tilde f(s)-\tilde f(t)\|^2}{|s-t|^2} &= 1+\frac{st\left[2+\cos\left(\alpha\ln\left(-\frac{t}{s}\right)\right)\right]-\frac{3}{4}t^2}{s^2-2st+t^2} = \\ &= 1+\frac{\frac{t}{s}\left[2+\cos\left(\alpha\ln\left(-\frac{t}{s}\right)\right)\right]-\frac{3}{4}\left(\frac{t}{s}\right)^2}{1-2\frac{t}{s}+\left(\frac{t}{s}\right)^2} \end{aligned} \tag{E.22}$$

Let $z := -\frac{t}{s} > 0$ and let

$$h(z) := \begin{cases} \frac{-z[2+\cos(\alpha \ln z)] - \frac{3}{4}z^2}{(1+z)^2} & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases}$$

We prove now that there exists a constant $C_o > -1$ such that

$$\text{(E.23)} \qquad h(z) \geq C_o > -1 \qquad \forall z \geq 0$$

For $z > 0$ we see

$$h(z) \geq \frac{-3z - \frac{3}{4}z^2}{(1+z)^2} = \frac{-3(z + \frac{1}{4}z^2)}{(1+z)^2} := w(z)$$

and

$$w'(z) = \frac{3\left(\frac{1}{2}z - 1\right)}{(1+z)^3} \text{ for } z \geq 0$$

$$w'(z) \begin{cases} < 0 & \text{for } 0 \leq z < 2 \\ = 0 & \text{for } z = 2 \\ > 0 & \text{for } z > 2. \end{cases}$$

Therefore w has at $z = 2$ an isolated minimum, $w(2) = -1$. On the other hand

$$h(2) = \frac{-2\left[2 + \cos\frac{\pi}{2}\right] - 3}{3^2} = -\frac{7}{9} > -1$$

Since h is continuous and $h(z) \geq w(z) > -1$ for $z \neq 2$, h attains its minimum at a point $z_o \in [0, 3]$, $h(z_o) > -1$. By strong monotonicity of w in $[3, \infty]$,

$$h(z) \geq w(z) \geq w(3) = -\frac{63}{64} > -1.$$

With $C_o := \min\left(h(z_o), -\frac{63}{64}\right) > -1$ we see $h(z) \geq C_o$ for $0 \leq z < \infty$ and by (E.22)

$$\text{(E.24)} \qquad \frac{\|\tilde{f}(s) - \tilde{f}(t)\|^2}{|s-t|^2} \geq 1 + C_o > 0$$

With $L_1 := \min\left(\frac{1}{2}, \sqrt{1 + C_o}\right) > 0$ and $L_2 := \max\left(1, \sqrt{2}C\right) > 0$ we get from (E.18), (E.20) and (E.23)

$$\text{(E.25)} \qquad L_1|s-t| \leq \|\tilde{f}(s) - \tilde{f}(t)\| \leq L_2|s-t| \text{ for all } s, t \in \mathbb{R}.$$

Clearly every straight line starting from $0 \in \mathbb{R}^2$ cuts the curve $\tilde{f}$ at infinitely many points, whence $\tilde{f}$ is not a graph of a function (see figure 2).

Because of the equivalences proved in Theorem 2.10, this two dimensional manifold can't be in regular parametric representation too. But this can be seen directly. Any orthogonal matrix is either of type ($\beta \in [0, 2\pi[$)

$$A(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \quad , \det A(\beta) = 1$$

or of type

$$B(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} \quad , \det B(\beta) = -1.$$

We consider e.g.

$$A(\beta)\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t)\cos\beta - f_2(t)\sin\beta \\ f_1(t)\sin\beta + f_2(t)\cos\beta \end{pmatrix} =: \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

If we assume that g_2 is bi-Lipschitz in a neighborhood of zero, then there exists $\varepsilon > 0$ and $C > 0$ such that

$$C|t - t'| \leq |g_2(t) - g_2(t')| \tag{E.26}$$

for all t, t' with $|t|, |t'| \leq \varepsilon$. Since $\left(\alpha := -\frac{\pi}{2\ln 2}\right)$

$$\begin{aligned} g_2(t) - g_2(t') = t\sin\beta\cos\alpha t + t\cos\beta\sin\alpha t - t'\sin\beta\cos\alpha t' - t'\cos\beta\sin\alpha t' = \\ = t\sin(\beta + \alpha\ln t) - t'\sin(\beta + \alpha\ln t'). \end{aligned}$$

We choose $k_o \in \mathbb{N}$ such that $2^{-4k_o} \leq \varepsilon$ and $j > k_o + 1$, $r = j + k$ with $k \in \mathbb{N}$. Let

$$t := 2^{-4j + \frac{2\beta}{\pi}}, \qquad t' = 2^{-4r-1+\frac{2\beta}{\pi}}.$$

Then $t, t' \leq \varepsilon$,

$$\begin{aligned} \sin(\beta + \alpha\ln t) &= \sin\left(\beta + \frac{\pi}{2\ln 2}\left(4j - \frac{2\beta}{\pi}\right)\ln 2\right) = \sin 2\pi j = 0 \\ \sin(\beta + \alpha\ln t') &= \sin\left(\beta + \frac{\pi}{2\ln 2}\left(4r + 1 - \frac{2\beta}{\pi}\right)\ln 2\right) = \sin\frac{\pi}{2} = 1 \end{aligned}$$

and by (E.25)

$$2^{2\frac{\beta}{\pi}}\left|2^{-4j} - 2^{-4r-1}\right| \leq C^{-1}|g_2(t')| = C^{-1}t' = 2^{2\frac{\beta}{\pi}}2^{-4r-1}$$

whence

$$\left|2^{-4j+4r+1} - 1\right| \leq 1.$$

Since $-4j + 4r + 1 = 4k + 1$ ($k \in \mathbb{N}$) and $2^{4k+1} \to \infty$ ($k \to \infty$) we get a contradiction. The other cases can be handled similarly.

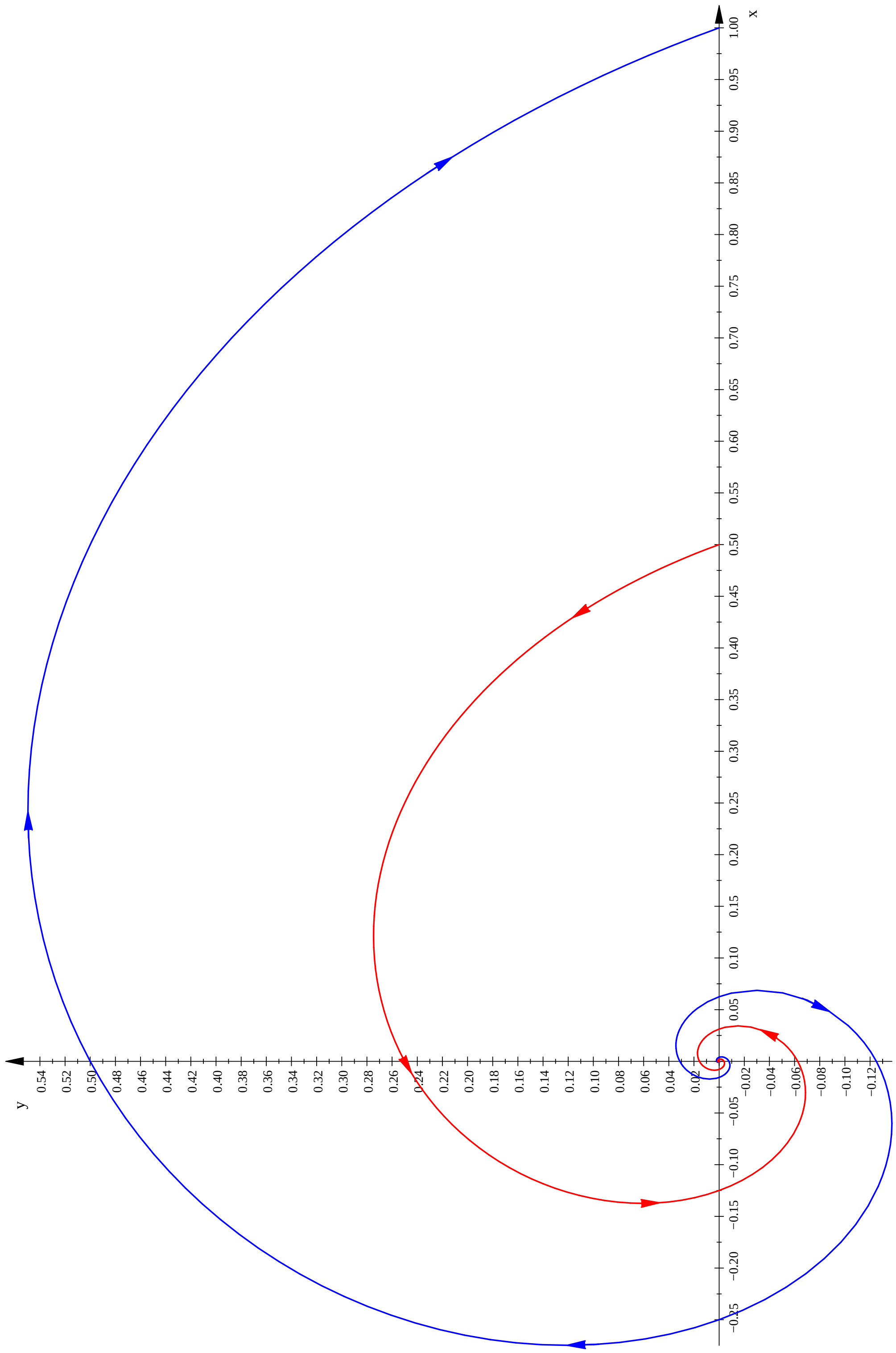

Figure 2

3 The measure space $\left(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k), \mu_k\right)$

Throughout this section, we use the following notations:

$$\begin{aligned} \mathbb{L}(\mathbb{R}^k) &= \sigma\text{-algebra of Lebesgue-measurable subsets of } \mathbb{R}^k, \\ \lambda_k &= \text{Lebesgue-measure on } \mathbb{L}(\mathbb{R}^k). \end{aligned}$$

3.1 The σ-algebra $\mathcal{L}(\mathcal{M}^k)$

We begin by proving

Proposition 3.1 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz manifold. Let*

$$\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}, \qquad \left\{(\hat{\mathcal{O}}_j, \hat{\psi}_j) : j \in \hat{\Lambda}\right\}$$

be two parametric representations of $\mathcal{M}^k$. For $\mathcal{E} \subseteq \mathcal{M}^k$, the following statements 1. and 2. are equivalent:

1. $\psi_i^{-1}\left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right) \in \mathbb{L}(\mathbb{R}^k)\ \forall i \in \Lambda$;
2. $\hat{\psi}_j^{-1}\left(\mathcal{E} \cap \hat{\psi}_j(\hat{\mathcal{O}}_j)\right) \in \mathbb{L}(\mathbb{R}^k)\ \forall j \in \hat{\Lambda}$.

Proof 1. $\Rightarrow$ 2. Observing that $\mathcal{M}^k = \bigcup\limits_{i\in\Lambda} \psi_i(\mathcal{O}_i)$ we obtain for any $j \in \hat{\Lambda}$

$$\mathcal{E} \cap \hat{\psi}_j(\hat{\mathcal{O}}_j) = \left(\mathcal{E} \cap \mathcal{M}^k\right) \cap \hat{\psi}_j(\hat{\mathcal{O}}_j) = \bigcup_{i\in\Lambda} \left(\mathcal{E} \cap \psi_i(\mathcal{O}_i) \cap \hat{\psi}_j(\hat{\mathcal{O}}_j)\right) = \bigcup_{i\in\Lambda} \left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right) \cap \mathcal{U}_{ij},$$

where

$$\mathcal{U}_{ij} := \psi_i(\mathcal{O}_i) \cap \hat{\psi}_j(\hat{\mathcal{O}}_j).$$

It follows

$$\psi_i^{-1}\left(\mathcal{E} \cap \hat{\psi}_j(\mathcal{O}_j)\right) = \bigcup_{i\in\Lambda} \psi_i^{-1}\left[\left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right) \cap \mathcal{U}_{ij}\right] = \bigcup_{i\in\Lambda} \psi_i^{-1}\left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right) \cap \psi_i^{-1}(\mathcal{U}_{ij})$$
$$\left[\text{for } \psi_i^{-1} \text{ is injective}\right],$$

and therefore

$$\begin{aligned} (3.1)\quad \hat{\psi}_j^{-1}\left(\mathcal{E} \cap \hat{\psi}_j(\mathcal{O}_j)\right) &= \left(\hat{\psi}_j^{-1} \circ \psi_i\right)\left[\psi_i^{-1}\left(\mathcal{E} \cap \hat{\psi}_j(\mathcal{O}_j)\right)\right] \\ &= \bigcup_{i\in\Lambda} \left\{\left(\hat{\psi}_j^{-1} \circ \psi_i\right)\left[\psi_i^{-1}\left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right)\right]\right\} \cap \left(\hat{\psi}_j^{-1} \circ \psi_i\right)\left[\psi_i^{-1}(\mathcal{U}_i j)\right] \\ &\qquad \left[\text{for } \hat{\psi}_j^{-1} \circ \psi_i \text{ is injective}\right]. \end{aligned}$$

By 1., $\psi_i^{-1}\left(\mathcal{E}\cap\psi_i(\mathcal{O}_i)\right)$ is a measurable subset of $\mathbb{R}^k$ which is contained in $\mathcal{O}_i$. The mapping $\hat{\psi}_j^{-1}\circ\psi_i$ being bi-Lipschitz from $\mathcal{O}_i(\subset\mathbb{R}^k)$ into $\mathbb{R}^k$, it follows that

$$\left(\hat{\psi}_j^{-1}\circ\psi_i\right)\left[\psi_i^{-1}\left(\mathcal{E}\cap\psi_i(\mathcal{O}_i)\right)\right]\in\mathbb{L}(\mathbb{R}^k)\quad\forall i\in\Lambda.$$

Finally, by construction, for every $i\in\Lambda$, the set

$$\left(\hat{\psi}_j^{-1}\circ\psi_i\right)\left[\psi_i^{-1}(\mathcal{U}_{ij})\right]=\hat{\psi}_j^{-1}(\mathcal{U}_{ij})$$

is open in $\mathbb{R}^k$. Now (3.1) implies

$$\hat{\psi}_j^{-1}\left(\mathcal{E}\cap\hat{\psi}_j(\mathcal{O}_j)\right)\in\mathbb{L}(\mathbb{R}^k).$$

Whence the claim.

The implication 2. $\Longrightarrow$ 1. is established by changing the roles of $\{(\mathcal{O}_i\psi_i):i\in\Lambda\}$ and $\left\{(\hat{\mathcal{O}}_j,\hat{\psi}_j):j\in\hat{\Lambda}\right\}$ in the proof above. ■

Definition 3.2 *Let $\mathcal{M}^k\subset\mathbb{R}^n$ be a k-dimensional Lipschitz-manifold. Let $\{(\mathcal{O}_i,\psi_i):i\in\Lambda\}$ be any parametric representation of $\mathcal{M}^k$.*

Define

$$\mathcal{L}(\mathcal{M}^k):=\left\{\mathcal{E}\subseteq\mathcal{M}^k:\psi_i^{-1}(\mathcal{E}\cap\psi_i(\mathcal{O}_i))\in\mathbb{L}(\mathbb{R}^k)\quad\forall i\in\Lambda\right\}.$$

By Proposition 3.1, the system $\mathcal{L}(\mathcal{M}^k)$ of subsets of $\mathcal{M}^k$ is intrinsically defined, i.e. $\mathcal{L}(\mathcal{M}^k)$ *is independent of the parametric representation of $\mathcal{M}^k$ under consideration. Thus, $(\mathcal{M}^k,\mathcal{L}(\mathcal{M}^k))$ is a measurable space.*

Remark 3.3 An analogous Definition is given in [1].

Theorem 3.4 *$\mathcal{L}(\mathcal{M}^k)$ is a σ-algebra of subsets of $\mathcal{M}^k$.*

Proof Clearly, the empty set is in $\mathcal{L}(\mathcal{M}^k)$. Next, given $l\in\Lambda$, for every $i\in\Lambda$ the set $\psi_i^{-1}\left(\psi_i(\mathcal{O}_i)\cap\psi_i(\mathcal{O}_i)\right)$ is open in $\mathbb{R}^k$. Thus

$$\psi_l(\mathcal{O}_l)\in\mathcal{L}(\mathcal{M}^k)\qquad\forall l\in\Lambda. \tag{3.2}$$

Let $\mathcal{E}\in\mathcal{L}(\mathcal{M}^k)$. Define $\mathcal{E}^C:=\mathcal{M}^k\setminus\mathcal{E}$: We prove $\mathcal{E}^C\in\mathcal{L}(\mathcal{M}^k)$. Indeed, for any $i\in\Lambda$,

$$\psi_i(\mathcal{O}_i)=(\mathcal{E}\cap\psi_i(\mathcal{O}_i))\cup\left(\mathcal{E}^C\cap\psi_i(\mathcal{O}_i)\right),$$

and therefore

$$\mathcal{O}_i=\left[\psi_i^{-1}\left(\mathcal{E}\cap\psi_i(\mathcal{O}_i)\right)\right]\cup\left[\psi_i^{-1}\left(\mathcal{E}^C\cap\psi_i(\mathcal{O}_i)\right)\right].$$

Here the two sets in brackets on the right hand side are disjoint (for ψ_i^{-1} is injective). Hence

$$\psi_i^{-1}\left(\mathcal{E}^C \cap \psi_i(\mathcal{O}_i)\right) = \mathcal{O}_i \setminus \left[\psi_i^{-1}\left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right)\right] \in \mathbb{L}(\mathbb{R}^k),$$

i.e. $\mathcal{E}^C \in \mathcal{L}(\mathcal{M}^k)$.

Let $\mathcal{E}_l \in \mathcal{L}(\mathcal{M}^k)$ $(l = 1, 2, \ldots)$. Define $\mathcal{E} := \bigcup_{l=1}^{\infty} \mathcal{E}_l$. Then, for any $i \in \Lambda$,

$$\mathcal{E} \cap \psi_i(\mathcal{O}_i) = \bigcup_{l=1}^{\infty} \left(\mathcal{E}_l \cap \psi_i(\mathcal{O}_i)\right).$$

It follows

$$\psi_i^{-1}\left(\mathcal{E} \cap \psi_i(\mathcal{O}_i)\right) = \bigcup_{l=1}^{\infty} \psi_i^{-1}\left(\mathcal{E}_l \cap \psi_i(\mathcal{O}_i)\right) \in \mathbb{L}(\mathbb{R}^k),$$

i.e. $\mathcal{E} \in \mathcal{L}(\mathcal{M}^k)$. ■

Representation of $\mathcal{M}^k$ by a disjoint union of sets of $\mathcal{L}(\mathcal{M}^k)$

Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold, and let $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$ be a parametric representation of $\mathcal{M}^k$. We pass from the sets $\psi_i(\mathcal{O}_i) \in \mathcal{L}(\mathcal{M}^k)$ to a disjoint system of sets in $\mathcal{L}(\mathcal{M}^k)$ with union $\mathcal{M}^k$. Define

$$\begin{cases} \mathcal{U}_1 & := \psi_1(\mathcal{O}_1), \\ \mathcal{U}_i & := \psi_i(\mathcal{O}_i) \setminus \bigcup_{l=1}^{i-1} \psi_l(\mathcal{O}_l) \quad (i = 2, 3, \ldots). \end{cases} \tag{3.3}$$

By (3.2), $\mathcal{U}_i \in \mathcal{L}(\mathcal{M}^k)$ for all $i \in \Lambda$. On the other hand, the following properties of the system $\{\mathcal{U}_i : i \in \Lambda\}$ are readily seen:

1. $\mathcal{U}_i \cap \mathcal{U}_{i'} = \phi$ for $i, i' \in \Lambda$, $i \neq i'$;
2. $\bigcup_{l=1}^{i} \mathcal{U}_l = \bigcup_{l=1}^{i} \psi_l(\mathcal{O}_l)$ $\forall i \in \Lambda$;
3. $\mathcal{M}^k = \bigcup_{i \in \Lambda} \mathcal{U}_i$.

Thus, for every $\mathcal{E} \in \mathcal{L}(\mathcal{M}^k)$ we have the disjoint union

$$\mathcal{E} = \bigcup_{i \in \Lambda} (\mathcal{E} \cap \mathcal{U}_i), \quad (\mathcal{E} \cap \mathcal{U}_i) \in \mathcal{L}(\mathcal{M}^k). \tag{3.4}$$

■

Measurable functions

Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz manifold. A function $f : \mathcal{M}^k \to \bar{\mathbb{R}}$ is called *measurable* (with respect to the measure space $(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k))$) if

$$\forall a \in \mathbb{R}: \ \ \left\{\xi \in \mathcal{M}^k : f(\xi) \geq a\right\} \in \mathcal{L}(\mathcal{M}^k).$$

■

Let $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$ be a parametric representation of $\mathcal{M}^k$. Observing that, for any $i \in \Lambda$ and any $a \in \mathbb{R}$,

$$\psi_i^{-1}\left(\left\{\xi \in \mathcal{M}^k : f(\xi) \geq a\right\} \cap \psi_i(\mathcal{O}_i)\right) = \left\{x \in \mathcal{O}_i : f\left(\psi_i(x)\right) \geq a\right\},$$

we obtain:

$$\begin{aligned} & f : \mathcal{M}^k \to \bar{\mathbb{R}} \textit{ is measurable } \Leftrightarrow \\ & \forall i \in \Lambda, \quad \forall a \in \mathbb{R}: \ \left\{x \in \mathcal{O}_i : f\left(\psi_i(x)\right) \geq a\right\} \in \mathbb{L}(\mathbb{R}^k) \end{aligned}$$

■

3.2 The measure μ_k

Preliminaries (I)

Let $1 \leq k < N$. We consider the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \cdots & \cdots & \cdots \\ a_{N1} & \cdots & a_{Nk} \end{pmatrix}.$$

Define

$$a_r := \begin{pmatrix} a_{1r} \\ \vdots \\ a_{Nr} \end{pmatrix} \qquad (r = 1, \ldots, k),$$

and

$$\langle a_r, a_s \rangle_N := \sum_{l=1}^{N} a_{lr} a_{ls} \qquad (r, s = 1, \ldots, k).$$

Then

$$G(a_1, \ldots, a_k) := \det(A^\top A) = \det \begin{pmatrix} \langle a_1, a_1 \rangle_N & \cdots & \langle a_1, a_k \rangle_N \\ \cdots & \cdots & \cdots \\ \langle a_k, a_1 \rangle_N & \cdots & \langle a_k, a_k \rangle_N \end{pmatrix}$$

is called Gram's determinant of $\{a_1, \ldots, a_k\}$. The following properties of $G(a_1, \ldots, a_k)$ are well-known.

1. Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ be related by the $(k \times k)$-matrix $M = (m_{rs})_{r,s=1,\ldots,k}$, i.e.

$$a_r = \sum_{l=1}^{k} m_{rl} b_l \qquad (r = 1, \ldots, k).$$

Then

$$G(a_1, \ldots, a_k) = (\det M)^2 G(b_1, \ldots, b_k).$$

2. Let A be an $(N \times k)$-matrix as above. For any k-tuple $\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$ with $1 \le i_1 < i_2 < \ldots < i_k \le N$, define

$$A_{i_1,\ldots,i_k} := \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \cdots & \cdots & \cdots \\ a_{i_k,1} & \cdots & a_{i_k,i_k} \end{pmatrix}.$$

Then

$$G(a_1, \ldots, a_k) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le N} \left(\det A_{i_1,\ldots,i_k}\right)^2.$$

■

Let $\mathcal{O} \subset \mathbb{R}^k$ be open. Let

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} : \mathcal{O} \longrightarrow \mathbb{R}^N$$

be Lipschitzian. This is equivalent to the Lipschitz-continuity of each component $\psi_l : \mathcal{O} \to \mathbb{R}$ $(l = 1, \ldots, N)$. By a theorem of Rademacher, ψ_l is differentiable a.e. in $\mathcal{O}$. The partial derivatives $\dfrac{\partial \psi_l}{\partial x_r}$ $(l = 1, \ldots, N; r = 1, \ldots, k)$ are bounded measurable functions in $\mathcal{O}$; $\left(\dfrac{\partial \psi_l}{\partial x_1}(x), \ldots, \dfrac{\partial \psi_l}{\partial x_k}(x)\right)$ represent the tangential vectors to $\mathcal{M}^k$ at $x \in \mathcal{O}$.

Next, define

$$\psi'(x) := \begin{pmatrix} \dfrac{\partial \psi_1}{\partial x_1}(x) & \cdots & \dfrac{\partial \psi_1}{\partial x_k}(x) \\ \cdots & \cdots & \cdots \\ \dfrac{\partial \psi_N}{\partial x_1}(x) & \cdots & \dfrac{\partial \psi_N}{\partial x_k}(x) \end{pmatrix}$$

for a.e. $x \in \mathcal{O}$, and

$$G_\psi := G_\psi(x) = \det\left(\left(\psi'(x)\right)^\top \psi'(x)\right)$$

for a.e. $x \in \mathcal{O}$. The function G_ψ is bounded and measurable in $\mathcal{O}$. ■

Preliminaries (II)

Let $\mathcal{O} \subset \mathbb{R}^k$ be open. Let $\psi : \mathcal{O} \to \mathbb{R}^N$ be bi-Lipschitz, i.e. there exists $L_i = \text{const} > 0$ $(i = 1, 2)$ such that

$$L_1\|x - y\|_k \le \|\psi(x) - \psi(y)\|_N \le L_2\|x - y\|_k \qquad \forall x, y \in \mathcal{O}.$$

As above, by a theorem of Rademacher, there exists $\mathcal{N} \subset \mathcal{O}$ with $\lambda_k(\mathcal{N}) = 0$ such that ψ is differentiable at every $x \in \mathcal{O} \setminus \mathcal{N}$. The matrix $\psi'(x)$ satisfies

$$L_1\|\xi\|_k \le \|\psi'(x)\xi\|_N \le L_2\|\xi\|_k \qquad \forall \xi \in \mathbb{R}^k,\ \forall x \in \mathcal{O} \setminus \mathcal{N}$$

(see [7]).

Next, fix any $x \in \mathcal{O} \setminus \mathcal{N}$. There exist

$$D = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_k \end{pmatrix}$$

and $S \in O(\mathbb{R}^k)$ (= set of orthogonal k-matrices) such that

$$\psi'(x)^\top \psi(x) = S^\top DS.$$

Thus, given $\eta \in \mathbb{R}^k$ there exists $\xi \in \mathbb{R}^k$ with $S\xi = \eta$, and therefore

$$\langle \psi'(x)^\top \psi(x)\xi, \xi\rangle_k = \langle DS\xi, S\xi\rangle_k = \sum_{l=1}^k \sigma_l \eta_l^2.$$

It follows that

$$L_1^2\|\eta\|_k^2 \le \sum_{l=1}^k \sigma_l \eta_l^2 \le L_2^2\|\eta\|_k^2.$$

Hence

$$L_1^2 \le \sigma_l \le L_2^2, \qquad l = 1, \dots, k.$$

Observing that

$$G_\psi(x) = \det\left(\psi'(x)^\top \psi'(x)\right) = (\det S)^2 \det D = \prod_{l=1}^k \sigma_l,$$

we obtain

$$L_1^{2k} \le G_\psi(x) \le L_2^{2k}, \qquad x \in \mathcal{O} \setminus \mathcal{N}. \tag{3.5}$$

■

The following result forms the basis for the definition of the measure on $\mathcal{L}(\mathcal{M}^k)$.

Theorem 3.5 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold. Let $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$ and $\left\{\left(\hat{\mathcal{O}}_j, \hat{\psi}_j\right) : j \in \hat{\Lambda}\right\}$ be two parametric representations of $\mathcal{M}^k$, and let $\{\mathcal{U}_i : i \in \Lambda\}$ resp. $\left\{\hat{\mathcal{U}}_j : j \in \hat{\Lambda}\right\}$ denote the system of disjoint sets associated with the parametric representation according to (3.3).*

Then, for every $\mathcal{E} \in \mathcal{L}(\mathcal{M}^k)$,

$$\sum_{i\in\Lambda} \int_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}} d\lambda_k = \sum_{j\in\hat{\Lambda}} \int_{\hat{\psi}_j^{-1}(\mathcal{E}\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\hat{\psi}_j}} d\lambda_k, \tag{3.6}$$

i.e. if the left (resp. right) hand side of (3.6) is finite then the other side does and there holds equality, or if the left (resp. right) hand side of (3.6) is equal to $+\infty$ then the other does.

Proof We divide the proof into two parts.

[1] For any $i \in \Lambda$ and any $j \in \hat{\Lambda}$, we have

$$\int_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\psi_i}} d\lambda_k = \int_{\hat{\psi}_i^{-1}(\mathcal{E}\cap\mathcal{U}_i\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\hat{\psi}_j}} d\lambda_k \tag{3.7}$$

Indeed, define

$$T_{ij} := \psi_i^{-1} \circ \hat{\psi}_j.$$

Then $T_{ij} : \hat{\psi}_j^{-1}\left(\psi_i(\mathcal{O}_i) \cap \hat{\psi}_j(\hat{\mathcal{O}}_j)\right) \to \psi_i^{-1}\left(\psi_i(\mathcal{O}_i) \cap \hat{\psi}_j(\hat{\mathcal{O}}_j)\right)$ is bi Lipschitz. Observing that

$$\psi_i^{-1}\left(\mathcal{E} \cap \mathcal{U}_i \cap \hat{\mathcal{U}}_j\right) = T_{ij}\left(\hat{\psi}_j^{-1}(\mathcal{E} \cap \mathcal{U}_i \cap \hat{\mathcal{U}}_j)\right),$$

the change of variables formula reads

$$\int_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\psi_i}} d\lambda_k = \int_{\hat{\psi}_i^{-1}(\mathcal{E}\cap\mathcal{U}_i\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\psi_i} \circ T_{ij}} |\det T'_{ij}| d\lambda_k \tag{3.8}$$

(see [6], [7] for a detailed discussion of the transformation of Lebensgue measure and integral under bi-Lipschitz mappings; these works contain also many references to this topic).

On the other hand, the definition of T_{ij} is equivalent to $\hat{\psi}_j = \psi_i \circ T_{ij}$. Hence, by the chain rule,

$$\hat{\psi}'_j(x) = \psi'_i(T_{ij}(x))T'_{ij}(x) \quad \text{for a.e. } x \in V_{ij}$$

[or, in coordinate form,

$$\frac{\partial\hat{\psi}_{jm}}{\partial x_r}(x) = \sum_{l=1}^{k} \frac{\partial\psi_{im}}{\partial\xi_l}(T_{ij}(x))\frac{\partial T_{ij,l}}{\partial x_r}(x)$$

$(m = 1, \ldots, N; r = 1, \ldots, k)$]. Now from preliminaries (I)/1., it follows that

$$G_{\hat{\psi}_j} = (\det T'_{ij})^2 G_{\psi_i}(T_{ij}(.)). \tag{3.9}$$

Taking the square root on both sides of this equality and inserting this into (3.8) implies (3.7).

$\boxed{2}$ Let $i \in \Lambda$ and $j \in \hat{\Lambda}$ be arbitrary. We have

$$\mathcal{E} \cap \mathcal{U}_i = \bigcup_{j\in\hat{\Lambda}} (\mathcal{E} \cap \mathcal{U}_i \cap \hat{\mathcal{U}}_j), \quad \mathcal{E} \cap \hat{\mathcal{U}}_j = \bigcup_{i\in\Lambda} (\mathcal{E} \cap \mathcal{U}_i \cap \hat{\mathcal{U}}_j).$$

Therefore

$$\psi_i^{-1}(\mathcal{E} \cap \mathcal{U}_i) = \bigcup_{j\in\hat{\Lambda}} \psi_i^{-1}(\mathcal{E} \cap \mathcal{U}_i \cap \hat{\mathcal{U}}_j),$$
$$\hat{\psi}_i^{-1}(\mathcal{E} \cap \mathcal{U}_j) = \bigcup_{j\in\Lambda} \hat{\psi}_i^{-1}(\mathcal{E} \cap \mathcal{U}_i \cap \hat{\mathcal{U}}_j).$$

Here both unions on the right hand side are disjoint (for ψ_i^{-1} and $\hat{\psi}_j^{-1}$ are injective). Observing the countable additivity of the integral, we obtain

$$\int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}}\,d\lambda_k = \sum_{j\in\hat{\Lambda}} \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\psi_i}}\,d\lambda_k =$$
$$= \sum_{j\in\hat{\Lambda}} \int\limits_{\hat{\psi}_j^{-1}(\mathcal{E}\cap\mathcal{U}_i\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\hat{\psi}_j}}\,d\lambda_k \quad \text{[by (3.7)]}$$
$$\leq \sum_{l\in\Lambda} \sum_{j\in\hat{\Lambda}} \int\limits_{\hat{\psi}_j^{-1}(\mathcal{E}\cap\mathcal{U}_l\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\hat{\psi}_j}}\,d\lambda_k = \sum_{j\in\hat{\Lambda}} \sum_{l\in\Lambda} \int\limits_{\hat{\psi}_j^{-1}(\mathcal{E}\cap\mathcal{U}_l\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\hat{\psi}_j}}\,d\lambda_k =$$
$$= \sum_{j\in\hat{\Lambda}} \int\limits_{\hat{\psi}_j^{-1}(\mathcal{E}\cap\hat{\mathcal{U}}_j)} \sqrt{G_{\hat{\psi}_j}}\,d\lambda_k.$$

An analogous reverse inequality is readily obtained by the same reasoning.

The assertion of the theorem is now easily seen by a standard argument. ■

Definition of the measure μ_k

We now introduce

Definition 3.6 *Let $\mathcal{M}^k \subset \mathbb{R}^N$ be an k-dimensional Lipschitz-manifold. Let $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$ be a parametric representation of $\mathcal{M}^k$, and let $\{\mathcal{U}_i : i \in \Lambda\}$ be the associated system of disjoint subsets of $\mathcal{L}(\mathcal{M}^k)$ according to (3.3) Define*

$$\mu_k(\emptyset) := 0$$
$$\mu_k(\mathcal{E}) := \sum_{i\in\Lambda} \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}} d\lambda_k, \quad \mathcal{E} \in \mathcal{L}(\mathcal{M}^k).$$

By Theorem 3.5, for $\mathcal{E} \in \mathcal{L}(\mathcal{M}^k)$ the number $\mu_k(\mathcal{E})$ is intrinsically defined for the measurable space $(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k))$, i.e. *$\mu_k(\mathcal{E})$ does not depend on the parametric representation of $\mathcal{M}^k$ under consideration.*

Theorem 3.7 *μ_k is a measure on the σ-algebra $\mathcal{L}(\mathcal{M}^k)$.*

Proof By definition, $\mu_k(\mathcal{E}) \geq 0$ for all $\mathcal{E} \in \mathcal{L}(\mathcal{M}^k)$. Now, fix any parametric representation $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$. Let $\{\mathcal{U}_i : i \in \Lambda\}$ denote the associated system of disjoint subsets of $\mathcal{L}(\mathcal{M}^k)$ according to (3.3).

Let $\mathcal{E}_l \in \mathcal{L}(\mathcal{M}^k)$ $(l \in \mathbb{N})$ be a family of disjoint sets. Define $\mathcal{E} := \bigcup\limits_{l=1}^{\infty} \mathcal{E}_l$. Then, for every $i \in \Lambda$

$$\mathcal{E} \cap \mathcal{U}_i = \bigcup_{l=1}^{\infty} (\mathcal{E}_l \cap \mathcal{U}_i) \text{ disjoint.}$$

Hence

$$\psi_i^{-1}(\mathcal{E} \cap \mathcal{U}_i) = \bigcup_{l=1}^{\infty} \psi_i^{-1}(\mathcal{E}_l \cap \mathcal{U}_i) \text{ disjoint,}$$

and therefore

$$\mu_k(\mathcal{E}) = \sum_{i\in\Lambda} \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}} d\lambda_{N-1} = \sum_{i\in\Lambda} \sum_{l=1}^{\infty} \int\limits_{\psi_i^{-1}(\mathcal{E}_l\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}} d\lambda_{N-1} =$$
$$= \sum_{l=1}^{\infty} \sum_{i\in\Lambda} \int\limits_{\psi_i^{-1}(\mathcal{E}_l\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}} d\lambda_{N-1} = \sum_{l=1}^{\infty} \mu_k(\mathcal{E}_l)$$

■

Combining Theorems 3.5 and 3.7 we obtain:
$(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k), \mu_k)$ *is a measure space* (for general notation see e.g. [2], [3]). This measure space is a well-defined intrinsic object associated with the manifold $\mathcal{M}^k$. It follows that for any μ_k-integrable function f: $\mathcal{M}^k \to \bar{\mathbb{R}}$ the real number

$$\int\limits_{\mathcal{M}^k} f d\mu_k$$

is well defined in the sense of the theory of the integral.

We have:

- $\int\limits_{\mathcal{E}} f d\mu_k := \int\limits_{\mathcal{M}^k} f \chi_{\mathcal{E}} d\mu_k, \quad \mathcal{E} \in \mathcal{L}(\mathcal{M}^k);$
- *if* $\mathcal{M}^k = \left(\bigcup\limits_{l=1}^{m} \mathcal{E}_l \right) \cup \mathcal{N}$, *with* $\mathcal{E}_l$ *disjoint and* $\mu_k(\mathcal{N}) = 0$, *then*

$$\int\limits_{\mathcal{M}^k} f d\mu_k = \sum_{l=1}^{m} \int\limits_{\mathcal{E}_l} f d\mu_k.$$

■

Sets of measure zero

Theorem 3.8 *Let* $\mathcal{M}^k \subset \mathbb{R}^N$ *be an k-dimensional Lipschitz-manifold. Let* $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$ *be a parametric representation of* $\mathcal{M}^k$, *and let* $\{\mathcal{U}_i : i \in \Lambda\}$ *be the associated system of disjoint subsets of* $\mathcal{L}(\mathcal{M}^k)$ *according to (3.3).*

Then, for a set $\mathcal{E} \subset \mathcal{L}(\mathcal{M}^k)$ *the following statements 1., 2., 3. are equivalent:*

1. $\lambda_k \left(\psi_i^{-1} \left(\mathcal{E} \cap \psi_i(\mathcal{O}_i) \right) \right) = 0 \ \forall i \in \Lambda$;
2. $\lambda_k \left(\psi_i^{-1} (\mathcal{E} \cap \mathcal{U}_i) \right) = 0 \ \forall i \in \Lambda$;
3. $\mu_k(\mathcal{E}) = 0$.

Proof 1. $\iff$ 2. The implication 1. $\Longrightarrow$ 2. is obvious, since $\mathcal{U}_i \subseteq \mathcal{O}_i$, $\mathcal{U}_i \in \mathcal{L}(\mathcal{M}^k)$ for all $i \in \Lambda$.

To prove 2. $\Longrightarrow$ 1., note that for every $i \in \Lambda$

$$\mathcal{E} \cap \psi_i(\mathcal{O}_i) = \bigcup_{l \in \Lambda} (\mathcal{E} \cap \mathcal{U}_l \cap \psi_i(\mathcal{O}_i)) \quad \text{[see (3.4)]}.$$

Hence

$$\psi_i^{-1} \left(\mathcal{E} \cap \psi_i(\mathcal{O}_i) \right) = \bigcup_{l \in \Lambda} \left(\psi_i^{-1} \circ \psi_l \right) \psi_l^{-1} \left(\mathcal{E} \cap \mathcal{U}_l \cap \psi_i(\mathcal{O}_i) \right).$$

Now $\psi_l^{-1} \left(\mathcal{E} \cap \mathcal{U}_l \cap \psi_i(\mathcal{O}_i) \right) \leq \psi_l^{-1} \left(\mathcal{E} \cap \mathcal{U}_l \right)$, and 2. implies:

$$\psi_l^{-1} \left(\mathcal{E} \cap \mathcal{U}_l \cap \psi_i(\mathcal{O}_i) \right) \text{ is Lebesgue-measurable,}$$
$$\lambda_k \left(\psi_l^{-1} \left(\mathcal{E} \cap \mathcal{U}_l \cap \psi_i(\mathcal{O}_i) \right) \right) = 0$$

Therefore

$$\lambda_k\left[\left(\psi_i^{-1}\circ\psi_l\right)\psi_l^{-1}\left(\mathcal{E}\cap\mathcal{U}_l\cap\psi_i(\mathcal{O}_i)\right)\right]=0,\quad l\in\Lambda.$$

Whence the implication 2. $\Longrightarrow$ 1.

2. $\Longleftrightarrow$ 3. The implication 2. $\Longrightarrow$ 3. is an immediate consequence of the definition of $\mu_k(\mathcal{E})$.

We prove 3. $\Longrightarrow$ 2. From

$$L_{i1}\|\xi\|_k\le\|\psi'(x)\xi\|_N\le L_{i2}\|\xi\|_k\quad\forall\xi\in\mathbb{R}^k,\ \text{for a.e. } x\in\mathcal{O}_i$$

($L_{i1}, L_{i2}=\text{const}>0$; $i\in\Lambda$; see Preliminaries (II)) it follows

$$L_{i1}^{2k}\le G_{\psi_i}(x)\le L_{i2}^{2k}\quad\text{for a.e. } x\in\mathcal{O}_i$$

(see (3.6)). Thus

$$\mu_k(\mathcal{E})\ge L_{i1}^k\lambda_k\left(\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)\right),\quad i\in\Lambda.$$

Now 2. follows. ■

With the above notations (see Preliminaries (II)), define

$$\alpha_1:=\inf_{i\in\Lambda}L_{i1}^k,\quad\alpha_2:=\sup_{i\in\Lambda}L_{i2}^k.$$

Assume $\alpha_1>0$, $\alpha_2<+\infty$. Then, for every $\mathcal{E}\in\mathcal{L}(\mathcal{M}^k)$,

$$\alpha_1\sum_{i\in\Lambda}\lambda_k\left(\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)\right)\le\mu_k(\mathcal{E})\le\alpha_2\sum_{i\in\Lambda}\lambda_k\left(\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)\right).$$

■

We note that the measure μ_k is *complete*, i.e. for $\mathcal{E}\in\mathcal{L}(\mathcal{M}^k)$, $\mu_k(\mathcal{E})=0$ and $\mathcal{F}\subset\mathcal{E}$ it follows $\mathcal{F}\in\mathcal{L}(\mathcal{M}^k)$. Indeed, we have

$$\psi_i^{-1}\left(\mathcal{F}\cap\psi_i(\mathcal{O}_i)\right)\subseteq\psi_i^{-1}\left(\mathcal{E}\cap\psi_i(\mathcal{O}_i)\right)\quad\forall i\in\Lambda.$$

The Lebesgue measure λ_k on $\mathbb{L}(\mathbb{R}^k)$ being complete, we obtain $\psi_i^{-1}\left(\mathcal{F}\cap\psi_i(\mathcal{O}_i)\right)\in\mathbb{L}(\mathbb{R}^k)$ for all $i\in\Lambda$. ■

3.3 Integration

Theorem 3.9 *Let $\mathcal{M}^k\subset\mathbb{R}^N$ be a k-dimensional Lipschitzmanifold. Let $\{(\mathcal{O}_i,\psi_i): i\in\Lambda\}$ be a parametric representation of $\mathcal{M}^k$, and let $\{\mathcal{U}_i : i\in\Lambda\}$ be the associated system of disjoint subsets of $\mathcal{L}(\mathcal{M}^k)$ according to (3.3).*
Then, for any $\mathcal{L}(\mathcal{M}^k)$-measurable function $f:\mathcal{M}^k\to[0,+\infty]$ and any $\mathcal{E}\in\mathcal{L}(\mathcal{M}^k)$ the following statements 1. and 2. are equivalent:

1. $\int\limits_{\mathcal{E}} f d\mu_k < +\infty$;

2. $\exists\ C_0 = \text{const}$:

$$\sum_{i=1}^{m} \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k \ \le\ C_0 \qquad \forall m\in\mathbb{N}.$$

In either case,

$$\int\limits_{\mathcal{E}} f d\mu_k = \sum_{i\in\Lambda} \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k. \tag{3.10}$$

Proof Let $\mathcal{E}\in\mathcal{L}(\mathcal{M}^k)$. We divide the proof into two parts.

1 Assume there exists $i\in\Lambda$ such that $\mathcal{E}\subseteq\mathcal{U}_i$.

1.1 Assume $f:\mathcal{M}^k\to[0,+\infty]$ a step function, i.e. $f=\sum\limits_{l=1}^{m} a_l\chi_{\mathcal{F}_l}$, where $a_l\in\mathbb{R}$, $\mathcal{F}_l\in\mathcal{L}(\mathcal{M}^k)$. We obtain

$$\int\limits_{\mathcal{E}} f d\mu_k = \int\limits_{\mathcal{M}^k} f\chi_{\mathcal{E}} d\mu_k = \sum_{l=1}^{m} a_l\mu_k(\mathcal{E}\cap\mathcal{F}_l).$$

By the definition of μ_k,

$$\begin{aligned}\mu_k(\mathcal{E}\cap\mathcal{F}_l) &= \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\ \mathcal{F}_l\cap\mathcal{U}_i)} \sqrt{G_{\psi_i}}d\lambda_k \quad [\text{ for } \mathcal{E}\cap\mathcal{U}_j=\phi\ \forall j\neq i]\\ &= \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{F}_l)} \sqrt{G_{\psi_i}}d\lambda_k = \int\limits_{\psi_i^{-1}(\mathcal{E})} \chi_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{F}_l)}(\psi_i(x))\sqrt{G_{\psi_i}(x)}d\lambda_k =\\ &= \int\limits_{\psi_i^{-1}(\mathcal{E})} \chi_{\mathcal{F}_l}(\psi_i(x))\sqrt{G_{\psi_i}(x)}d\lambda_k,\end{aligned}$$

for

$$\chi_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{F}_l)}(\psi_i(x)) = \chi_{\mathcal{F}_l}(\psi_i(x)) \quad \forall x\in\psi_i^{-1}(\mathcal{E}).$$

It follows that

$$\sum_{l=1}^{m} a_l\mu_k(\mathcal{E}\cap\mathcal{F}_l) = \int\limits_{\psi_i^{-1}(\mathcal{E})} \sum_{l=1}^{m} a_l\chi_{\mathcal{F}_l}(\psi_i(x))\sqrt{G_{\psi_i}(x)}d\lambda_k = \int\limits_{\psi_i^{-1}(\mathcal{E})} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k.$$

Thus,

$$\int_{\mathcal{E}} f d\mu_k < +\infty \iff \int_{\psi_i^{-1}(\mathcal{E})} f \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k < \infty, \tag{3.11}$$

$$\int_{\mathcal{E}} f d\mu_k = +\infty \iff \exists\, l \in \{1, \dots, m\} : \int_{\psi_i^{-1}(\mathcal{E})} \chi_{\mathcal{F}_l} \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k = +\infty. \tag{3.12}$$

$\boxed{1.2}$ Assume $f : \mathcal{M}^k \to [0, +\infty]$ is $\mathcal{L}(\mathcal{M}^k)$-measurable. Then there exist step functions $f_S : \mathcal{M}^k \to [0, +\infty[$ $(s = 1, 2, \dots)$ such that

$$f_s(\xi) \le f_{s+1}(\xi), \quad \lim_{s\to\infty} f_s(\xi) = f(\xi) \quad \forall \xi \in \mathcal{M}^k.$$

We obtain

$$f_s(\psi_i(x))\sqrt{G_{\psi_i}(x)} \le f_{s+1}(\psi_i(x))\sqrt{G_{\psi_i}(x)} \quad \text{for a.e. } x \in \mathcal{O}_i,$$
$$\lim_{s\to\infty} f_s(\psi_i(x))\sqrt{G_{\psi_i}(x)} = f(\psi_i(x))\sqrt{G_{\psi_i}(x)} \quad \text{for a.e. } x \in \mathcal{O}_i.$$

By part $\boxed{1.1}$, (3.11) and (3.12) hold with f_s in place of f, and

$$\int_{\psi_i^{-1}(\mathcal{E})} f_s(\psi_i(x))\sqrt{G_{\psi_i}(x)} d\lambda_k = \int_{\mathcal{E}} f_s d\mu_k \quad (s = 1, 2 \dots).$$

The claim now follows from the monotone convergence theorem.

$\boxed{2}$ For any $\mathcal{E} \in \mathcal{L}(\mathcal{M}^k)$,

$$\mathcal{E} = \bigcup_{i\in\Lambda} (\mathcal{E} \cap \mathcal{U}_i) \text{ disjoint}, \quad (\mathcal{E} \cap \mathcal{U}_i) \in \mathcal{L}(\mathcal{M}^k)$$

(see (3.4)). Let $f : \mathcal{M}^k \to [0, +\infty]$ be any $\mathcal{L}(\mathcal{M}^k)$-measurable function. By part $\boxed{1}$, for every $m \in \mathbb{N}$,

$$\int_{\bigcup_{i=1}^m (\mathcal{E}\cap\mathcal{U}_i)} f d\mu_k = \sum_{i=1}^m \int_{\mathcal{E}\cap\mathcal{U}_i} f d\mu_k = \sum_{i=1}^m \int_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} f \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k.$$

Thus

$$\int_{\mathcal{E}} f d\mu_k \ge \sum_{i=1}^m \int_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} f \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k,$$

resp.

$$\int\limits_{\bigcup_{i=1}^{m}(\mathcal{E}\cap\mathcal{U}_i)} f d\mu_k \leq \sum_{l\in\Lambda} \int\limits_{\psi_l^{-1}(\mathcal{E}\cap\mathcal{U}_l)} f\circ\psi_l\sqrt{G_{\psi_l}}d\lambda_k.$$

Whence the claim. ■

Corollary 3.10 *Notations as in Theorem 3.9.*

Let $\mathcal{E}\in\mathcal{L}(\mathcal{M}^k)$, and let $f:\mathcal{E}\to\bar{\mathbb{R}}$ be μ_k-integrable. Then

$$\int\limits_{\mathcal{E}} f d\mu_k = \sum_{i\in\Lambda} \int\limits_{\psi_i^{-1}(\mathcal{E}\cap\mathcal{U}_i)} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k.$$

Proof By a standard argument, we write $f=f^+-f^-$ and apply Theorem 3.9 to both f^+ and f^- to obtain the claim. ■

Let $M^k\subset\mathbb{R}^N$ be a k-dimensional Lipschitz-manifold. Let $\{(\mathcal{O}_i,\psi_i): i\in\Lambda\}$ be a parametric representation of $\mathcal{M}^k$, an let $\{\mathcal{U}_i : i\in\Lambda\}$ be the associated system of disjoint subsets of $\mathcal{L}(\mathcal{M}^k)$ according to (3.3). Let $f:\mathcal{M}^k\to\mathbb{R}$ be μ_k-integrable. Then, by Theorem 3.9,

$$\int\limits_{\mathcal{M}^k} f\mu_k = \sum_{i\in\Lambda} \int\limits_{\psi_i^{-1}(\mathcal{U}_i)} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k.$$

■

For what follows, assume

$$\boxed{\Lambda=\{1,\ldots,s\}}\ .$$

Estimates of $\int\limits_{\mathcal{M}^k} f d\mu_k$ from below and above for nonnegative f.

Theorem 3.11 *Let $\mathcal{M}^k\subset\mathbb{R}^N$ be a k-dimensional Lipschitz-manifold with parametric representation $\{(\mathcal{O}_i,\psi_i): i=1,\ldots,s\}$. Let $\{\mathcal{U}_i : i=1,\ldots,s\}$ denote the associated system of disjoint subsets of $\mathcal{L}(\mathcal{M}^k)$ according to (3.3).*
Then, there exists $c_o=\mathrm{const}>0$ (depending on $\{(\mathcal{O}_i,\psi_i): i=1,\ldots,s\}$) such that, for every μ_k-integrable $f:\mathcal{M}^k\to[0,+\infty]$,

$$c_o\sum_{i=1}^{s}\int\limits_{\mathcal{O}_i} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k \leq \sum_{i=1}^{s}\int\limits_{\psi_i^{-1}(\mathcal{U}_i)} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k \leq \sum_{i=1}^{s}\int\limits_{\mathcal{O}_i} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k. \tag{3.13}$$

Proof To begin with, we note

$$\mathcal{O}_i=\bigcup_{l=1}^{s}\psi_i^{-1}\left(\mathcal{U}_l\cap\psi_i(\mathcal{O}_i)\right),\quad i=1,\ldots,s.$$

It follows that

$$\int_{\mathcal{O}_i} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k = \sum_{l=1}^{s} \int_{\psi_i^{-1}(\mathcal{U}_l\cap\psi_i(\mathcal{O}_i))} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k.$$

Next, as above define

$$T_{il} := \psi_i^{-1}\circ\psi_l, \qquad l=1,\dots,s.$$

Then T_{il} is a bi-Lipschitz-mapping of $\psi_l^{-1}(\psi_l(\mathcal{O}_l)\cap\psi_i(\mathcal{O}_i))$ onto $\psi_i^{-1}(\psi_l(\mathcal{O}_l)\cap\psi_i(\mathcal{O}_i))$. Observing that $\psi_i\circ T_{il}=\psi_l$, we obtain

$$\begin{aligned}
\int_{\psi_i^{-1}(\mathcal{U}_l\cap\psi_i(\mathcal{O}_i))} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k &= \int_{\psi_l^{-1}(\mathcal{U}_l\cap\psi_i(\mathcal{O}_i))} f\circ\psi_l\sqrt{G_{\psi_i}\circ T_{il}}|\det T'_{il}|d\lambda_k \\
&\qquad\qquad\qquad\qquad [\text{by change of variables}] \\
&\le L_{i2}^k \operatorname*{ess\,sup}_{\psi_l^{-1}(\mathcal{U}_l\cap\psi_i(\mathcal{O}_i))}|\det T'_{il}|\int_{\psi_l^{-1}(\mathcal{U}_l)} f\circ\psi_l d\lambda_k \qquad [L_{i2}^k \text{ from (3.6)}] \\
&\le \frac{L_{i2}^k}{L_{l1}^k}\operatorname*{ess\,sup}_{\psi_l^{-1}(\mathcal{U}_l\cap\psi_i(\mathcal{O}_i))}|\det T'_{il}|\int_{\psi_l^{-1}(\mathcal{U}_l)} f\circ\psi_l\sqrt{G_{\psi_l}}d\lambda_k
\end{aligned}$$

(see Preliminaries (II)).

Thus,

$$\int_{\mathcal{O}_i} f\circ\psi_i\sqrt{G_{\psi_i}}d\lambda_k \le L_{i2}^k M_i\sum_{l=1}^{s}\int_{\psi_l^{-1}(\mathcal{U}_l)} f\circ\psi_l\sqrt{G_{\psi_l}}d\lambda_k,$$

where

$$M_i := \max_{j=1,\dots,s}\frac{1}{L_{j1}^k}\operatorname*{ess\,sup}_{\psi_j^{-1}(\mathcal{U}_j\cap\psi_i(\mathcal{O}_i))}|\det T'_{ij}|.$$

Then the first inequality (3.11) follows with

$$\frac{1}{c_o} := \sum_{i=1}^{s} L_{i2}^k M_i.$$

The second inequality is obvious. ■

From Theorem 3.11 we obtain

Corollary 3.12 *Notations as in Theorem 3.11. Then*

$$c_o \sum_{i=1}^{s} \int\limits_{\mathcal{O}_i} f \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k \le \int\limits_{\mathcal{M}^k} f d\mu_k \le \sum_{i=1}^{s} \int\limits_{\mathcal{O}_i} f \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k \tag{3.14}$$

($c_o = \text{const}$ as in (3.11)), and

$$\begin{cases} \exists \;\; c_1, c_2 = \text{const} > 0 \text{ such that} \\ c_1 \sum_{i=1}^{s} \int\limits_{\mathcal{O}_i} f \circ \psi_i d\lambda_k \le \int\limits_{\mathcal{M}^k} f d\mu_k \le c_2 \sum_{i=1}^{s} \int\limits_{\mathcal{O}_i} f \circ \psi_i d\lambda_k. \end{cases} \tag{3.15}$$

Proof Inequality (3.13) is identical to (3.12). To prove the first inequality in (3.14) we note that, for $i = 1, \ldots, s$,

$$\begin{aligned} \int\limits_{\mathcal{O}_i} f \circ \psi_i d\lambda_k &\le \frac{1}{L^k_{i1}} \int\limits_{\mathcal{O}_i} f \circ \psi_i \sqrt{G_{\psi_i}} d\lambda_k \quad \text{[by (3.6)]} \\ &\le \frac{L^k_{i2} M_i}{L^k_{i1}} \sum_{l=1}^{s} \int\limits_{\psi_l^{-1}(\mathcal{U}_l)} f \circ \psi_l \sqrt{G_{\psi_l}} d\lambda_k = \frac{L^k_{i2} M_i}{L^k_{i1}} \int\limits_{\mathcal{M}^k} f d\mu_k \end{aligned}$$

The first inequality in (3.14) follows with

$$\frac{1}{c_1} := \sum_{i=1}^{s} \frac{L^k_{i2} M_i}{L^k_{i2}}.$$

The second inequality in (3.14) is readily seen with

$$c_2 := \max_{j=1,\ldots,s} L^k_{j2}$$

■

Calculation of $\int\limits_{\mathcal{M}^k} f d\mu_k$ by a partition of unity

Let $\{(\mathcal{O}_i, \psi_i) : i = 1, \ldots, s\}$ be a parametric representation of $\mathcal{M}^k$. Then

$$\psi_i(\mathcal{O}_i) = \mathcal{M}^k \cap U_i, \quad U_i \subset \mathbb{R}^N \text{ open } (i = 1, \ldots, s).$$

Let $\mathcal{M}^k$ be compact. Then there exists a partition of unity subordinated to $\{U_1, \ldots, U_s\}$, i.e. there exist $\zeta_i \in C^\infty_c(U_i)$ $(i = 1, \ldots, s)$, such that

$$\zeta_i(\xi) \ge 0 \quad \forall \xi \in U_i \quad (i = 1, \ldots, s), \qquad \sum_{i=1}^{s} \zeta_i(\xi) = 1 \quad \forall \xi \in \mathcal{M}^k.$$

With these notations we have

Theorem 3.13 *Let $f : \mathcal{M}^k \to \bar{\mathbb{R}}$ be μ_k-integrable. Then*

$$\int\limits_{\mathcal{M}^k} f d\mu_k = \sum_{i=1}^{s} \int\limits_{\mathcal{O}_i} (f \circ \psi_i)(\zeta_i \circ \psi_i)\sqrt{G_{\psi_i}} d\lambda_k.$$

Proof First, we have

$$\int\limits_{\mathcal{M}^k} f d\mu_k = \sum_{i=1}^{s} \int\limits_{\psi_i(\mathcal{O}_i)} f\zeta_i d\mu_k.$$

By Corollary 3.10,

$$\int\limits_{\psi_i(\mathcal{O}_i)} f\zeta_i d\mu_k = \sum_{l=1}^{s} \int\limits_{\psi_l^{-1}(\psi_i(\mathcal{O}_i)\cap\mathcal{U}_l)} (f \circ \psi_l)(\zeta_i \circ \psi_l)\sqrt{G_{\psi_l}} d\lambda_k.$$

The mapping $T_{li} := \psi_l^{-1} \circ \psi_i$ is bi-Lipschitz continuous of $\psi_i^{-1}(\psi_i(\mathcal{O}_i) \cap \psi_l(\mathcal{O}_l))$ onto $\psi_l^{-1}(\psi_i(\mathcal{O}_i) \cap \psi_l(\mathcal{O}_l))$. We obtain

$$\begin{aligned}
&\int\limits_{\psi_l^{-1}(\psi_i(\mathcal{O}_i)\cap\mathcal{U}_l)} (f \circ \psi_l)(\zeta_i \circ \psi_l)\sqrt{G_{\psi_l}} d\lambda_k \\
&\quad = \int\limits_{\psi_i^{-1}(\psi_i(\mathcal{O}_i)\cap\mathcal{U}_l)} (f \circ \psi_i)(\zeta_i \circ \psi_i)\sqrt{G_{\psi_l}(T_{li})}|\det T'_{li}| d\lambda_k \quad \text{[by change of variables]} \\
&\quad = \int\limits_{\psi_i^{-1}(\psi_i(\mathcal{O}_i)\cap\mathcal{U}_l)} (f \circ \psi_i)(\zeta_i \circ \psi_i)\sqrt{G_{\psi_i}} d\lambda_k \quad \text{[by the chain rule, and (3.9)].}
\end{aligned}$$

Observing that

$$\mathcal{O}_i = \bigcup_{l=1}^{s} \psi_i^{-1}(\psi_i(\mathcal{O}_i) \cap \mathcal{U}_l) \quad \text{disjoint,}$$

we obtain

$$\begin{aligned}
\int\limits_{\psi_i(\mathcal{O}_i)} f\zeta_i d\mu_k &= \sum_{l=1}^{s} \int\limits_{\psi_i^{-1}(\psi_i(\mathcal{O}_i)\cap\mathcal{U}_l)} (f \circ \psi_i)(\zeta_i \circ \psi_i)\sqrt{G_{\psi_i}} d\lambda_k = \\
&= \int\limits_{\mathcal{O}_i} (f \circ \psi_i)(\zeta_i \circ \psi_i)\sqrt{G_{\psi_i}} d\lambda_k.
\end{aligned}$$

The claim follows. ■

3.4 The space $L^p(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k), \mu_k)$

Let $\mathcal{M}^k \subset \mathbb{R}^N$ be a k-dimensional Lipschitz-manifold, and let $\{(\mathcal{O}_i, \psi_i) : i \in \Lambda\}$ be a parametric representation of $\mathcal{M}^k$. Let $1 \leq p < +\infty$. As usual, define

$$L^p(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k), \mu_k) := \text{vector space of all equivalence classes of } \mathcal{L}(\mathcal{M}^k)\text{-measurable functions } f : \mathcal{M}^k \to \bar{\mathbb{R}} \text{ such that } \int_{\mathcal{M}^k} |f|^p d\mu_k < +\infty$$

[recall that a function $f : \mathcal{M}^k \to \bar{\mathbb{R}}$ is $\mathcal{L}(\mathcal{M}^k)$-measurable if

$$\forall i \in \Lambda,\ \forall a \in \mathbb{R} :\ \psi_i^{-1}(\{\xi : f(\xi) \geq a\} \cap \psi_i(\mathcal{O}_i)) \in \mathbb{L}(\mathbb{R}^k)].$$

Further, two measurable functions $f, g : \mathcal{M}^k \to \bar{\mathbb{R}}$ are called equivalent if there is $N \subset \mathcal{M}^k$, $\mu_k(N) = 0$ and $f(x) = g(x)\ \forall x \in \mathcal{M}^k \setminus N$]. $L^p(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k), \mu_k)$ is a normed vector space with respect to

$$\|f\|_{L^p} := \left(\int_{\mathcal{M}^k} |f|^p d\mu_k \right)^{\frac{1}{p}}.$$

By (3.9),

$$\int_{\mathcal{M}^k} |f|^p d\mu_k = \sum_{i \in \Lambda} \int_{\psi_i^{-1}(\mathcal{U}_i)} |f \circ \psi_i|^p \sqrt{G_{\psi_i}} d\lambda_k.$$

We have: $L^p(\mathcal{M}^k, \mathcal{L}(\mathcal{M}^k), \mu_k)$ is *complete* (see [2], [3]). ■

Let $\Lambda = \{1, \ldots, s\}$. Then from Corollary 3.12 it follows that

$$c_o \sum_{i=1}^{s} \int_{\mathcal{O}_i} |f \circ \psi_i|^p \sqrt{G_{\psi_i}} d\lambda_k \leq \int_{\mathcal{M}^k} |f|^p d\mu_k \leq \sum_{i=1}^{s} \int_{\mathcal{O}_i} |f \circ \psi_i|^p \sqrt{G_{\psi_i}} d\lambda_k, \tag{3.16}$$

and

$$c_1 \sum_{i=1}^{s} \int_{\mathcal{O}_i} |f \circ \psi_i|^p d\lambda_k \leq \int_{\mathcal{M}^k} |f|^p d\mu_k \leq c_2 \sum_{i=1}^{s} \int_{\mathcal{O}_i} |f \circ \psi_i|^p d\lambda_k \tag{3.17}$$

(with c_o, c_1, c_2 as in Corollary 3.12). ■

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