SURFACES ISOGENOUS TO A PRODUCT: THEIR AUTOMORPHISMS AND DEGENERATIONS

DISSERTATION
zur Erlangung
des DOKTORGRADES (DR. RER. NAT.)
der MATHEMATISCH-NATURWISSENSCHAFTLICHEN FAKULTÄT
der UNIVERSITÄT BAYREUTH

vorgelegt von
WENFEI LIU
aus Zhejiang

BAYREUTH
JANUAR 2010

UNIVERSITÄT BAYREUTH
FAKULTÄT VON MATHEMATIK

Angefertigt mit der Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Bayreuth.

Datum: Januar 2010

1. Referent: Fabrizio Catanese

2. Referent: Jin-Xing Cai

3. Referent: Nikolaos Tziolas

to my parents.

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Abstract

In this thesis, we consider the automorphisms and stable degenerations of surfaces isogenous to a product.

First we consider the action of the automorphisms on cohomology in the case where the group G is abelian. It is shown that, if the irregularity of the surface is ≥ 2, the action of $(G \times G)/G$ on the second cohomology is mostly faithful (Theorems 2.3 and 2.4). For surfaces with irregularity 0 or 1, examples are given (Examples 2.7 and 2.8).

Then we consider the stable degenerations of surfaces isogenous to a product and classify the possible singularities on them (Corollaries 3.12 and 3.20). As a result, we show that the $\mathbb{Q}$-Gorenstein deformations of the degenerations with certain singuarities are unobstructed and get some connected components of the moduli space of stable surfaces (Corollary 4.6).

Zusammenfassung

Komplexe algebraische Flächen isogen zu einem Produkt von Kurven wurden von Catanese in [Cat00] einführt. Diese Flächen sind von der Form $(C \times D)/G$, wobei C und D zwei glatte Kurven von Geschlecht ≥ 2 sind und G eine endliche Gruppe ist, die auf $C \times D$ frei wirkt. Eine besondere Eigenschaft von einer Fläche isogen zu einem Produkt von Kurven ist folgende: die Fläche kann durch die topologischen Invarianten charakterisiert werden. Gegeben sei eine Fläche S isogen zu einem Produkt, dann ist der Modulraum $M_S^{top} = M_S^{diff}$ von Flächen homömorph zu S entweder irreduzibel und zusammenhängend oder er enthält zwei zusammenhängende Komponenten, die durch komplexer Konjugation ineinander übergeführt werden (Theorem 1.7). Diese Flächen geben viele ziemlich einfache Beispiele von Flächen, die diffeomorph aber nicht deformationsäquivalent sind. Es gibt auch andere Anwendungen, zum Beispiel sind sie wichtig, um Flächen mit kleinen Invarianten zu studieren. Viele Autoren haben Flächen isogen zu einem Produkt studiert.

In dieser Dissertation betrachten wir Wirkung der Automorphismengruppen von Flächen isogen zu einem Produkt auf der Kohomologie, und stabile Degeneration von solchen Flächen. Wir bemerken, dass es eine starke Beziehung zwischen Automorphismen und der Existenz von feinen Modulräumen gibt, und zwischen Degenerationen und Kompaktifizierungen von Modulräumen von Flächen mit kanonischen Singularitäten auch. Im Fall, dass die Irregularität $q(S) \geq 2$ ist und G ist abelsch, zeigen wir, dass die Wirkung von $(G \times G)/G$ auf der zweiten Kohomologiegruppe

meistens effektiv ist. Wir klassifizieren alle möglichen Singularitäten auf diesen stabilen Degenerationen. Als weitere Ergebnisse können wir zeigen, dass die Deformationen von den Degenerationen mit besonderen Singularitäten ohne Obstruktionen sind, und wir bekommen einige Zusammenhängskomponenten des Modulraums von stabilen Flächen.

Der Inhalt dieser Dissertation ist in 4 Kapitel gegliedert. In Kapitel 1 geben wir eine Einleitung über Flächen isogen zu einem Produkt, und über stabile Flächen. Wir erinnern an Cartans Lemma, das unerlässlich zum Studium von Glättung von Varietäten mit Gruppenwirkung ist. Wir erklären die notwendige Theorie über $\mathbb{Q}$-Gorenstein Deformationen, die später für die Kompaktifizierung von Modulraum benutzt wird. In Kapitel 2 betrachten wir Untergruppen der Automorphismengruppe einer Fläche isogen zu einem Produkt, die auf die Kohomologiegruppen wirken. Falls die Gruppe abelsch ist und falls die Irregularität $q(S)$ groß ist, dann zeigen wir, dass der Kern der Wirkung trivial ist. In Kapitel 3 geben wir eine vollständige Klassifikation von Singularitäten auf Degenerationen von Flächen isogen zu einem Produkt. Wir studieren in Sektion 3.1 die Glättung von stabilen Kurven mit einer Gruppenwirkung, in den Sektionen 3.2 und 3.3 die Glättung von einem Produkt von zwei stabilen Kurven mit einer Gruppenwirkung. Die Glättbarkeit ist charakterisiert durch möglichen Stabilisatoren der Wirkung, und die Singularitäten der Degeneration sind nur Quotienten von gewissen vollständigen Durchschnittssingularitäten, modulo die Stabilisatoren. In Kapitel 4 betrachten wir $\mathbb{Q}$-Gorenstein Deformationen der Degenerationen, die wir im Kapitel 3 bekommen haben. Wir sehen, dass die $\mathbb{Q}$-Gorenstein Deformationen ohne Obstruktion sind, falls die stabile Fläche, die wir betrachten, keine Singularitäten von Typ (U_{2c}) oder (M) enthält (Korollar 3.12 und 3.20). Damit können wir zeigen, dass die stabile Kompaktifizierung von manchen Modulräumen von Flächen isogen zu einem Produkt schon eine Zusammenhangskomponenten des Modulraums von stabilen Flächen ist.

Acknowledgments

I would like to thank my advisor Professor Fabrizio Catanese for suggesting this research and for his patience and encouragement during many discussions. The idea of classifying the singularities of degenerations of surfaces isogenous to a product originates from him. I learned a lot from him about the way to do mathematics.

The staff in Lehrstuhl VIII Mathematik of Bayreuth University made an excellent atmosphere to do research and discuss there. I am very grateful for their hospitality and support. I want to thank Fabio Perroni for several discussions during the preparation of this thesis. I also benefited from talks with Mario Chan, Matteo Penegini, Michael Lönne, Professor Ingrid Bauer and Stephen Coughlan on many occasions. It is a great pleasure to express my sincere gratitude for Frau Rostock, the secretary of Lehrstuhl VIII. She can always help me with urgent practical issues. Also during the process of my application for the doctor degree in Bayreuth University, Matteo, Professor Fabrizio Catanese and Professor Ingrid Bauer helped me a lot to communicate with the committee. Matteo spent much time helping me write a German translation of the introduction of the thesis. Stephen saved me from many errors in English. I am very grateful for all these.

I want to thank my domestic advisor Professor Jinxing Cai for constant encouragement and for teaching me how to do mathematics.

Thanks also go to Sönke Rollenske for several conversations and for a careful reading of my note.

This work was completed at Bayreuth University under the financial support of China Scholarship Council "High-level university graduate program"and DFG

Forschergruppe 790 “Classification of algebraic surfaces and compact complex manifolds ”. Thanks to their support, I had the opportunities to attend several conferences and summer schools both in Germany and Italy, which greatly expanded my view of the international mathematics community.

Erklärung

Ich bestätige, dass ich diese Arbeit selbständig verfasst habe, und ich keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt habe.

Introduction

Surfaces isogenous to a (higher) product were introduced by Catanese [Cat00]. These are surfaces of the form $(C \times D)/G$, where C, D are two smooth curves of genus ≥ 2 and G is a finite group acting freely on $C \times D$. A remarkable property of surfaces isogenous to a product is that they can be characterized by topological invariants. Given a surface isogenous to a product S, the moduli space $M_S^{top} = M_S^{diff}$ of surfaces with the same topological type as S is either irreducible and connected or contains two connected components which are interchanged by complex conjugation (Theorem 1.9). These surfaces give a rather simple series of examples of surfaces which are diffeomorphic but not deformation equivalent. There are also other applications, for example, in the study of surfaces with small invariants. Intensive efforts are being made in the topic of surfaces isogenous to a product.

In this thesis, we are interested in the automorphisms of a surface isogenous to a product, their action on cohomology, and also the stable degenerations of such surfaces. Note that automorphisms are related to the existence of certain fine moduli spaces (cf. [Po77, Lecture 10]), while stable degenerations concern the compactification of the moduli space of surfaces with only canonical singularities and ample dualizing sheaf. It is shown in this work that, if the group G is abelian and the irregularity of the surface is ≥ 2, the action of $(G \times G)/G$ on the second cohomology is mostly faithful. For the degenerations of surfaces isogenous to a product, we classify the possible singularities on them. As a result, we can show that the $\mathbb{Q}$-Gorenstein deformations of the degenerations with certain singularities are unobstructed and hence get some connected components of the moduli space of stable surfaces.

The content of the thesis is as follows. Chapter 1 gives first some preliminaries on surfaces isogenous to a product and stable surfaces. We also recall Cartan's lemma which is indispensable in the smoothings of varieties with group actions. For the later compactification of moduli space, we include the necessary $\mathbb{Q}$-Gorenstein deformation theory which has already been used by Hacking to compactify the moduli of plane curves. Then we define a moduli stack of stable surfaces for which we can use the $\mathbb{Q}$-Gorenstein deformation theory.

In Chapter 2 we consider the automorphism group of a surface isogenous to a product and its action on cohomology. We restrict our attention to the case when the group G is abelian. In this case $G \cong (G \times G)/G$ is a subgroup of automorphisms and we can consider the action of G on cohomology. We show among other things that the kernel of this action is trivial if the irregularity of the surface is large (Theorems 2.3 and 2.4). We also construct surfaces with irregularity 1 or 0 such that the kernel of the action is $\cong \mathbb{Z}_2^{\oplus 2}$, giving examples of one extremal case in [Cai04, Theorem A] (Examples 2.7 and 2.8).

Chapter 3 gives a complete classification of singularities on the stable degenerations of surfaces isogenous to a product. In Section 3.1, we study the smoothings of stable curves with a group action and, in Sections 3.2, 3.3, the smoothing of a product of two stable curves with a group action. The smoothability is characterized in term of the possible stabilizers of the action (Propositions 3.10 and 3.18) and the singularities on the degenerations are just quotients of certain complete intersection singularities by the stabilizers (Corollaries 3.12 and 3.20). Then we give examples for each type of singularity (Examples 3.13–3.16 and 3.21).

Chapter 4 considers the $\mathbb{Q}$-Gorenstein deformations of the degenerations obtained in Chapter 3. We see that the $\mathbb{Q}$-Gorenstein deformations are unobstructed if the stable surface under consideration does not contain singularities of type (U_{2c}) or (M) (Theorem 4.5). Therefore we can show that the stable compactifications of some moduli spaces of surfaces isogenous to a product already yield connected components of the moduli space of stable surfaces.

Chapter 1

Preliminaries

1.1 Notation

The following are some notations and conventions that we will use in the text.

Let G be a finite group acting on a set A.

$|G|$ is the order of G. For $\sigma \in G$, $|\sigma|$ is the order of σ.

For a subset $A' \subset A$ and a subset $G' \subset G$, $G'A' := \{ga|g \in G', a \in A'\}$. If $GA' \subset A'$, we say that A' is G-invariant.

For $a \in A$, $G_a := \{g \in G|g \cdot a = a\}$ is the stabilizer of a.

If $G_a \neq \{1\}$, we say that a is a fixed point of the action, or that the action of G is not free in a.

If $G_a = \{1\}$ for every $a \in A$, we say that G acts freely on A.

$\mathbb{C}^*$ is the group of nonzero complex numbers.

For a finite abelian group G, G^* denotes the character group $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$.

$\mathbb{Z}_n$ denotes the cyclic group of order n.

A variety is a projective, reduced, connected scheme of finite type over $\mathbb{C}$. A surface (resp. curve) is a variety of pure dimension two (resp. one).

S usually denotes a surface isogenous to a product while X usually denotes a singular surface.

A one-parameter family of varieties means a flat family over the unit disk.

Finally, we work over the field $\mathbb{C}$ of complex numbers.

1.2 Surfaces isogenous to a product

Definition 1.1 ([Cat00], Definition 3.1)**.** *A smooth projective surface S is isogenous to a (higher) product if it is a quotient $S = (C \times D)/G$, where C, D are smooth curves of genus at least two, and G is a finite group acting freely on $C \times D$.*

The following rigidity lemma is important in studying the action of G on $C \times D$ and the automorphism group $\mathrm{Aut}(S)$.

Lemma 1.2 ([Cat00], Lemma 3.8)**.** *Let $f\colon C_1 \times C_2 \to B_1 \times B_2$ be a surjective holomorphic map between products of curves. Assume that both B_1, B_2 have genus ≥ 2. Then, after possibly interchanging B_1 and B_2, there are holomorphic maps $f_i\colon C_i \to B_i$ such that $f(x, y) = (f_1(x), f_2(y))$.*

Remark 1.3. With the same proof as in [Cat00], this lemma is still valid if the curves C_i, B_i are stable curves (see [vO05, Lemma 4.1]).

Corollary 1.4 ([Cat00], Corollary 3.9)**.** *Assume that C, D are curves of genus ≥ 2. Then the inclusion $Aut(C) \times Aut(D) \subset Aut(C \times D)$ is an equality if $C \ncong D$, whereas $Aut(C \times C)$ is a semidirect product of $Aut(C)^2$ with the $\mathbb{Z}_2$ given by the involution interchanging the two factors.*

Let $S = (C \times D)/G$ be a surface isogenous to a product. Let $G^\circ := G \cap (\mathrm{Aut}(C) \times \mathrm{Aut}(D))$; then G° acts on the two factors C, D and acts on $C \times D$ via the diagonal action. If G° acts faithfully on both C and D, we say that $(C \times D)/G$ is a minimal realization of S. By [Cat00, Propostion 3.13], a minimal realization exists and is unique, i.e., if $(E \times F)/\Gamma \cong S$ is another minimal realization, then $\Gamma \cong G, E \cong C, F \cong D$ (up to relabelling), and the actions of Γ and G are the same under the above identifications. In the following we always assume $S = (C \times D)/G$ is *the* minimal realization.

Definition 1.5. *Let $S = (C \times D)/G$ be a surface isogenous to a product. Let G° be the subgroup of G defined above. We say that S is of nonmixed type if $G = G^\circ$. Otherwise S is said to be of mixed type.*

Remark 1.6. Let K_S be a canonical divisor of S and $\chi(\mathcal{O}_S)$ the holomorphic Euler characteristic of S. Then we can see that

$$K_S^2 = \frac{8(g(C)-1)(g(D)-1)}{|G|} \text{ and } \chi(\mathcal{O}_S) = \frac{(g(C)-1)(g(D)-1)}{|G|},$$

where $g(C), g(D)$ are the genus of C, D respectively.

To construct surfaces isogenous to a product of unmixed type with group G, we start with two smooth curves C' and D' and then try to find G-coverings $C \to C'$ and $D \to D'$ such that G acts freely on $C \times D$ via the diagonal action. For this, we recall how to construct a covering of a curve following [BCGP09].

Definition 1.7 ([BCGP09], Definition 0.11)**.** *An orbifold surface group of genus g' and multiplicities $m_1, \ldots, m_r$ is the group presented as follows:*

$$\mathbb{T}(g'; m_1, \ldots, m_r) := \langle a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_r | \\ c_1^{m_1}, \ldots, c_r^{m_r}, \prod_{i=1}^{g'} [a_i, b_i] \cdot c_1 \cdot \ldots \cdot c_r \rangle.$$

If $m_1 \geq \cdots \geq m_r$, the sequence $(g'; m_1, \ldots, m_r)$ is called the signature of the orbifold surface group.

By Riemann's existence theorem, to give a smooth curve C with a G-action and with quotient C' of genus g' is equivalent to giving:

(1) the branch point set $\{P_1, \ldots P_r\} \subset C'$,

(2) a surjection of the fundamental group $\pi_1(C' \backslash \{P_1, \ldots P_r\})$ onto $\mathbb{T}(g'; m_1, \ldots, m_r)$ such that $a_1, b_1, \ldots, a_{g'}, b_{g'} \in \mathbb{T}(g'; m_1, \ldots, m_r)$ are image elements of a symplectic basis of the fundamental group of C', while each c_i is the image of a simple geometric loop around the point p_i.

(3) a surjective homomorphism $\varphi\colon \mathbb{T}(g'; m_1, \ldots, m_r) \to G$ such that

(4) $\varphi(c_i)$ is an element of order exactly m_i and

(5) *Hurwitz's formula* holds: $2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right)$.

If the above data exist, we say that $(\varphi(a_1), \ldots, \varphi(b_{g'}); \varphi(c_1), \ldots, \varphi(c_r))$ is a generating vector for G of type $(g'|m_1, \ldots, m_r)$ ([Pe09R, Definition 2.1]). Let $\sigma_i := \varphi(c_i)$. Then the sets $\{\sigma \langle \sigma_i \rangle \sigma^{-1}\}_{\sigma \in G}$ are just the stabilizers of the points lying over $P_i \in C'$ and $\Sigma := \bigcup_{\sigma \in G} \bigcup_{k=0}^{\infty} \{\sigma \sigma_1^k \sigma^{-1}, \ldots, \sigma \sigma_r^k \sigma^{-1}\}$ is the set of all the elements of G fixing some point of C.

If G is abelian, then the situation is much simpler: $\sigma \sigma_i \sigma^{-1} = \sigma_i$ and the set of stabilizers is just $\Sigma = \bigcup_{k=0}^{\infty} \{\sigma_1^k, \ldots, \sigma_r^k\}$; moreover, since $\prod_{i=1}^{g'} [a_i, b_i] \cdot c_1 \cdots c_r = 1$, we have $\prod_{i=1}^{g'}[\varphi(a_i),\varphi(b_i)]\cdot\varphi(c_1)\cdots\varphi(c_r)=1$ and hence

$$\sigma_1\cdots\sigma_r=1. \tag{1.1}$$

Now, suppose $C\to C'$ and $D\to D'$ are two G-coverings of smooth curves. By the above construction, we have two generating vectors for G, say $(\lambda_1,\dots,\lambda_{2g_1'};\sigma_1,\dots,\sigma_r)$ and $(\psi_1,\dots,\psi_{2g_2'};\tau_1,\dots,\tau_s)$. And we can also consider the two subsets of G

$$\Sigma_1=\bigcup_{\sigma\in G}\bigcup_{k=0}^{\infty}\{\sigma\sigma_1^k\sigma^{-1},\dots,\sigma\sigma_r^k\sigma^{-1}\},$$
$$\Sigma_2=\bigcup_{\sigma\in G}\bigcup_{k=0}^{\infty}\{\sigma\tau_1^k\sigma^{-1},\dots,\sigma\tau_s^k\sigma^{-1}\}.$$

Note that Σ_1 (resp. Σ_2) consists of the elements of G fixing some point on C (resp. D). Let G act on $C\times D$ via the diagonal action. Then the fact that G acts freely on $C\times D$ amounts to saying that

$$\Sigma_1\cap\Sigma_2=\{1\} \tag{1.2}$$

and in this case $S:=(C\times D)/G$ is a surface isogenous to a product. On the other hand, every surface isogenous to a product of unmixed type can be obtained in this way from two sets of data as above satisfying (1.2).

In the mixed type case, we have $C\cong D$ and there is an exact sequence of groups

$$1\to G^\circ\to G\to\mathbb{Z}_2\to 1.$$

We have the following description of surfaces of mixed type:

Proposition 1.8 ([Cat00]). *Surfaces S isogenous to a product and of mixed type are obtained as follows. There is a (faithful) action of a finite group G° on a curve C of genus at least 2 and a nonsplit extension*

$$1\to G^\circ\to G\to\mathbb{Z}_2\to 1,$$

yielding a class $[\varphi]$ in $Out(G^\circ)=Aut(G^\circ)/Int(G^\circ)$, which is of order ≤ 2. Once we fix a representative φ of the above class, there exists an element τ' in $G\setminus G^\circ$ such that, setting $\tau=\tau'^2$, we have:

(I) $\varphi(\gamma) = \tau'\gamma\tau'^{-1}$,

(II) *G acts, under a suitable isomorphism of* C *and* D*, by the formulae:* $\gamma(P,Q) = (\gamma P, (\varphi\gamma)Q)$ *for* γ *in* G°*; whereas the lateral class of* G° *consists of the transformations*

$$\tau'\gamma(P,Q) = ((\varphi\gamma)Q, \tau\gamma P).$$

Let Γ *be the subset of* G° *consisting of the transformations having some fixed point. Then the condition that* G *acts freely amounts to:*

(A) $\Gamma \cap \varphi(\Gamma) = \{1\}$.

(B) *there is no* γ *in* G° *such that* $\varphi(\gamma)\tau\gamma$ *is in* Γ.

The structure of the moduli space of surfaces isogenous to a product is illustrated in the following theorem:

Theorem 1.9 ([Cat03])**.** *Let* S *be a surface isogenous to a product. Then any surface* S' *with the same fundamental group and Euler number as* S *is diffeomorphic to* S*. The corresponding moduli space* $M_S^{top} = M_S^{diff}$ *is either irreducible and connected or it contains two connected components which are interchanged by complex conjugation. There are infinitely many examples of the latter case, and moreover these moduli spaces are almost all of general type.*

1.3 Cartan's lemma

The following lemma is used throughout Chapters 2 and 3 for the (analytically) local analysis of the group actions.

Lemma 1.10 (Cartan's lemma)**.** *Let (X,x) be an analytic singularity with Zariski tangent space* T *and let* G *be a finite group of automorphisms of* (X,x)*. Then there exists a* G*-equivariant embedding* $(X,x) \to (T,0)$*.*

Proof. See [Cat87] or [M08, Lemma 2.5]. □

1.4 $\mathbb{Q}$-Gorenstein deformation theory of semi log canonical surfaces

We recall the $\mathbb{Q}$-Gorenstein deformation theory of semi log canonical surfaces set out by Hacking ([Hac01], [Hac04]). This section is mostly taken from Hacking's two aforementioned articles.

Let $\mathcal{F}$ be a coherent sheaf on a variety X satisfying Serre's condition S_2. If n is a positive integer, we define the n-th reflexive power of $\mathcal{F}$ by

$$\mathcal{F}^{[n]} := (\mathcal{F}^{\otimes n})^{\vee\vee},$$

the double dual of the n-th tensor product. If n is negative, we define the n-th reflexive power of $\mathcal{F}$ by

$$\mathcal{F}^{[n]} := (\mathcal{F}^{\otimes(-n)})^{\vee}.$$

Definition 1.11. *A surface X is said to have semi log canonical (slc) singularities if*

(i) X is Cohen–Macaulay.

(ii) X has at most normal crossing singularities in codimension 1.

(iii) The dualizing sheaf ω_X is a $\mathbb{Q}$-line bundle, i.e., there is some $n \in \mathbb{N}$ such that $\omega_X^{[n]}$ is an invertible sheaf.

(iv) if $\tilde{X} \to X$ is the normalization and $\tilde{D} \subset \tilde{X}$ is the preimage of the part of X_{sing}, then the pair $(\tilde{X}, \tilde{D})$ is log canonical, i.e., for any resolution $\mu\colon \hat{X} \to \tilde{X}$, we have

$$K_{\hat{X}} + \mu_*^{-1}\tilde{D} \equiv \mu^*(K_{\tilde{X}} + \tilde{D}) + \sum a_i E_i$$

with all $a_i \geq -1$.

Remark 1.12. Since a slc surface X has at most normal crossing crossing singularities in codimesion 1, X is Gorenstein in codimension 1. So we can associate a Weil divisor K_X to the dualizing sheaf ω_X, which does not contain any double curve of X as an irreducible component. In general, for $n \in \mathbb{Z}$, $\omega_X^{[n]}$ is a divisorial sheaf and nK_X is its associated Weil divisor (see [Har77, Chapter II.6] and [R80, Pages 281–285]). We usually write $\mathcal{O}_X(nK_X)$ for $\omega_X^{[n]}$.

Let $P \in X$ be a slc surface germ. Let n be the index of P, i.e., the smallest positive integer such that $\omega_X^{[n]}$ is invertible. We define the canonical covering $\pi\colon Z \to X$ by

$$Z = \underline{\mathrm{Spec}}_X(\mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \cdots \oplus \mathcal{O}_X((n-1)K_X)),$$

where the multiplication is given by fixing an isomorphism $\mathcal{O}_X(nK_X) \tilde{\to} \mathcal{O}_X$ ([R87, Proposition 3.6]). It is characterized by the following properties:

(1) The morphism π is cyclic quotient of degree n which is étale in codimension 1.

(2) The surface Z is Gorenstein; that is, it is Cohen–Macaulay, and the Weil divisor K_Z is Cartier.

For X a slc surface, the canonical covering at a point $P \in X$ is uniquely determined in the étale topology. Hence the data of canonical coverings everywhere locally on X defines a Deligne–Mumford stack $\mathfrak{X}$ with coarse moduli space X, the canonical covering stack of X.

Definition 1.13 (The usual deformations)**.** *Let X be a scheme and $(R, \mathbb{C})$ a noetherian local $\mathbb{C}$-algebra. A deformation $\mathcal{X}/R$ of X over R is a flat morphism $f\colon \mathcal{X} \to \mathrm{Spec} R$ with an isomorphism $\mathcal{X} \otimes_R \mathbb{C} \cong X$, where $\mathbb{C}$ is the residue field of R. Similarly we can define the deformations of a family of schemes $\mathcal{X}/A$, where A is a noetherian $\mathbb{C}$-algebra.*

Let $B = \mathrm{Spec} R$ and $0 \in B$ the closed point. Then we also denote a deformation $\mathcal{X}/R$ by $\mathcal{X}/(0 \in B)$.

Let $\underline{C}$ be the category of noetherian $\mathbb{C}$-algebras. Given an infinitesimal extension (i.e., a surjection with a nilpotent kernel) $A' \to A$ in $\underline{C}$, write $\mathcal{D}ef_{\mathcal{X}/A}(A')$ for the set of deformations of $\mathcal{X}/A$ over A'. Given a family $\mathcal{X}'/A'$ extending $\mathcal{X}/A$, write $\mathcal{A}ut_{\mathcal{X}/A}(\mathcal{X}'/A')$ for the group of automorphisms of $\mathcal{X}'/A'$ over A' which restrict to the identity on $\mathcal{X}/A$.

Definition 1.14. *Let $A \in \underline{C}$ and let $\mathcal{X}/A$ be a family of schemes over A. Let $\mathcal{L}_{\mathcal{X}/A}$ be a cotangent complex for $\mathcal{X}/A$. For a finite A-module M, we define*

$$T^i(\mathcal{X}/A, M) = Ext^i_{\mathcal{O}_\mathcal{X}}(\mathcal{L}_{\mathcal{X}/A}, \mathcal{O}_\mathcal{X} \otimes_A M),$$
$$\mathcal{T}^i(\mathcal{X}/A, M) = \mathcal{E}xt^i_{\mathcal{O}_\mathcal{X}}(\mathcal{L}_{\mathcal{X}/A}, \mathcal{O}_\mathcal{X} \otimes_A M).$$

Remark 1.15. These vector spaces and $\mathcal{O}_\mathcal{X}$-modules govern the usual deformations of $\mathcal{X}/A$. There is a local-to-global spectral sequence relating the $\mathcal{T}^i$ and the T^i:

$$E_2^{pq} = H^p(\mathcal{X}, \mathcal{T}^q(\mathcal{X}/A, M)) \Rightarrow T^{p+q}(\mathcal{X}/A, M).$$

Theorem 1.16. *Let $\mathcal{X}/A$ be a family of schemes and let M be a finite A-module. Then*

(1) $\mathcal{T}^0(\mathcal{X}/A, M) = \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\Omega_{\mathcal{X}/A}, \mathcal{O}_\mathcal{X} \otimes_A M)$.

(2) $\mathcal{T}^1(\mathcal{X}/A, M)$ is supported on the locus where $\mathcal{X}/A$ is not a smooth morphism.

(3) $\mathcal{T}^2(\mathcal{X}/A, M)$ *is supported on the locus where* $\mathcal{X}/A$ *is not a local complete intersection morphism.*

Theorem 1.17. *Let* $A_0 \in \underline{C}$*, let* $A \to A_0$ *be an extension, and* $A' \to A$ *an infinitesimal extension with kernel* M*, where* M *is a finite* A_0*-module (i.e., writing* N *for the kernel of* $A' \to A_0$*, we have* $MN = 0$ *in* A'*). Let* $\mathcal{X}_0/A_0$ *be a family of schemes and* $\mathcal{X}/A$ *a family extending* $\mathcal{X}_0/A_0$*.*

(1) *There exists a canonical element* $o_{\mathcal{X}/A}(A') \in T^2(\mathcal{X}_0/A_0, M)$ *such that* $\mathcal{D}ef_{\mathcal{X}/A}(A') \neq \emptyset$ *if and only if* $o_{\mathcal{X}/A}(A') = 0$*.*

(2) *If* $o_{\mathcal{X}/A}(A') = 0$*,* $\mathcal{D}ef_{\mathcal{X}/A}(A')$ *is a principal homogeneous space under* $T^1(\mathcal{X}_0/A_0, M)$*.*

(3) *Given* $\mathcal{X}'/A'$ *extending* $\mathcal{X}/A$*,* $\mathcal{A}ut_{\mathcal{X}/A}(\mathcal{X}'/A')$ *is naturally isomorphic to* $T^0(\mathcal{X}_0/A_0, M)$*.*

Hacking exploited the so-called $\mathbb{Q}$-Gorenstein deformation theory of slc surfaces (Definitions 1.18 and 1.19). It turns out that this has good properties similar to those of the usual deformation theory above (Theorems 1.25 and 1.26). This new deformation theory enables the construction of Deligne–Mumford stacks for a moduli problem of stable surfaces.

Definition 1.18. *Let* $\mathcal{X}/A$ *be a family of slc surfaces. We say that* $\mathcal{X}/A$ *is weakly* $\mathbb{Q}$*-Gorenstein if the relative dualizing sheaf* $\omega_{\mathcal{X}/A}$ *is* $\mathbb{Q}$*-Cartier.*

Definition 1.19. *Let* $\mathcal{X}/A$ *be a family of slc surfaces. We say that* $\mathcal{X}/A$ *is* $\mathbb{Q}$*-Gorenstein if* $\omega^{[i]}_{\mathcal{X}/A}$ *commutes with base change for all* $i \in \mathbb{Z}$*.*

Given a $\mathbb{Q}$*-Gorenstein family of slc surfaces* $\mathcal{X}/A$ *and an infinitesimal extension* $A' \to A$ *in* $\underline{C}$*, write* $\mathcal{D}ef^{QG}_{\mathcal{X}/A}(A')$ *for the set of* $\mathbb{Q}$*-Gorenstein deformations of* $\mathcal{X}/A$ *over* A'*.*

Let $\mathcal{X}/R$ be a $\mathbb{Q}$-Gorenstein deformation over a local noetherian $\mathbb{C}$-algebra with residue field $\mathbb{C}$. We say that $\mathcal{X}/R$ is *versal* if the natural map $\lambda_A\colon \mathrm{Hom}(R, A) \to \mathcal{D}ef^{QG}_{X/\mathbb{C}}(A)$ is surjective for any local Artin $\mathbb{C}$-algebra A. We will see that every stable surface X admits a *veral* $\mathbb{Q}$-Gorenstein deformation (cf. Page 13). If, in addition, the map λ_A is an isomorphism for $A = \mathbb{C}[\epsilon]/(\epsilon^2)$, then we say that $\mathcal{X}/R$ is *semiuniversal.* A semiuniversal $\mathbb{Q}$-Gorenstein deformation is unique up to isomorphism, provided it exists. If it does exist, we denote its base by Def^{QG}_X. When we refer to the *usual* deformation of X, we denote the base of *the* semiuniversal deformation by Def_X.

Remark 1.20. If a stable surface X is Gorenstein, then the $\mathbb{Q}$-Gorenstein deformation theory of X coincides with the usual one. This is because in this case every deformation $\mathcal{X}/(0 \in B)$ of X has invertible relative dualizing sheaf $\omega_{\mathcal{X}/B}$. So we have $\mathrm{Def}_X^{QG} = \mathrm{Def}_X$. Note also that Def_X always exists for a stable surface X.

Remark 1.21. The $\mathbb{Q}$-Gorenstein deformations of a slc surface germ X are precisely those deformations which lift to deformations of the canonical covering $Z \to X$. So we can transform the $\mathbb{Q}$-Gorenstein deformations of a slc surface X into the deformations of its canonical covering stack.

Lemma 1.22 ([Hac04], Lemma 3.3). *Let $P \in X$ be a slc surface germ of index n. Let $\mathcal{X}/(0 \in B)$ be a $\mathbb{Q}$-Gorenstein deformation of X. Then $\mathcal{X}/B$ is weakly $\mathbb{Q}$-Gorenstein of index n.*

Lemma 1.23 ([Hac04], Lemma 3.5). *Let $P \in X$ be a slc surface germ of index n, and let $Z \to X$ be the canonical covering with group $G \cong \mathbb{Z}_n$. Let $\mathcal{Z}/(0 \in B)$ be a G-equivariant deformation of Z inducing a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/(0 \in B)$ of X. Then there is an isomorphism*

$$\mathcal{Z} \cong \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}} \oplus \omega_{\mathcal{X}/B} \oplus \cdots \oplus \omega_{\mathcal{X}/B}^{[n-1]}),$$

where the multiplication is given by fixing a trivialization of $\omega_{\mathcal{X}/B}^{[n]}$. In particular, $\mathcal{Z}/B$ is determined by $\mathcal{X}/B$.

Let $\mathcal{X}/B$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces. For $P \in \mathcal{X}/B$ a point of index n, we define the canonical covering $\pi\colon \mathcal{Z} \to \mathcal{X}$ of $P \in \mathcal{X}/B$ by

$$\mathcal{Z} \cong \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}} \oplus \omega_{\mathcal{X}/B} \oplus \cdots \oplus \omega_{\mathcal{X}/B}^{[n-1]}),$$

where the multiplication is given by fixing a trivialization of $\omega_{\mathcal{X}/B}^{[n]}$ at P. The canonical covering of $P \in \mathcal{X}/B$ is uniquely determined in the étale topology. Hence the data of canonical coverings everywhere locally on $\mathcal{X}/B$ defines a Deligne–Mumford stack $\mathfrak{X}/B$ with coarse moduli space $\mathcal{X}/B$, the canonical covering stack of $\mathcal{X}/B$.

The stack $\mathfrak{X}/B$ is flat over B by the above lemma. Moreover, for any base change $B' \to B$, let $\mathfrak{X}_{B'}$ denote the canonical covering stack of $\mathcal{X} \times_B B'$; then there is a canonical isomorphism $\mathfrak{X}_{B'} \tilde{\to} \mathfrak{X} \times_B B'$.

We consider sheaves on $\mathfrak{X}$. Let $\pi\colon \mathcal{Z} \to \mathcal{X}$ be a local canonical covering at $P \in \mathcal{X}/B$, with group $G \cong \mathbb{Z}_n$. Then $\mathfrak{X}$ has local patch $[\mathcal{Z}/G]$ over $P \in \mathcal{X}$. Sheaves on $[\mathcal{Z}/G]$ correspond to G-equivariant sheaves on $\mathcal{Z}$. Let $p\colon \mathfrak{X} \to \mathcal{X}$ be the induced map to the coarse moduli space. Thus, locally, p is the map $[\mathcal{Z}/G] \to \mathcal{Z}/G$. If $\mathcal{F}$ is a sheaf on $[\mathcal{Z}/G]$ and $\mathcal{F}_{\mathcal{Z}}$ is the corresponding G-equivariant sheaf on $\mathcal{Z}$, then $p_*\mathcal{F} = (\pi_*\mathcal{F}_{\mathcal{Z}})^G$. In particular, the functor p_* is exact because the map π is finite and $(\pi_*\mathcal{F}_{\mathcal{Z}})^G$ is a direct summand of $\pi_*\mathcal{F}_{\mathcal{Z}}$.

Let A be a noetherian $\mathbb{C}$-algebra, and let $A' \to A$ be an infinitesimal extension. Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces, and let $\mathfrak{X}/A$ be the canonical covering stack of $\mathcal{X}/A$. A *deformation* of $\mathfrak{X}/A$ over A' is a Deligne–Mumford stack $\mathfrak{X}'/A'$, flat over A', together with an isomorphism $\mathfrak{X}' \times_{\mathrm{Spec} A'} \mathrm{Spec} A \cong \mathfrak{X}$. Observe that, since the extension $A' \to A$ is infinitesimal, we may identify the étale sites of $\mathfrak{X}'$ and $\mathfrak{X}$. Thus, equivalently, a deformation $\mathfrak{X}'/A'$ of $\mathfrak{X}/A$ is a sheaf $\mathcal{O}_{\mathfrak{X}'}$ of flat A'-algebras on the étale site of $\mathfrak{X}$, together with an isomorphism $\mathcal{O}_{\mathfrak{X}'} \otimes_{A'} A \cong \mathcal{O}_{\mathfrak{X}}$.

In our calculations, we use the local-to-global spectral sequence for *Ext* and the Leray spectral sequence for stacks. In particular, if $\mathfrak{X}/A$ is the canonical covering stack of a $\mathbb{Q}$-Gorenstein family $\mathcal{X}/A$ and $p\colon \mathfrak{X} \to \mathcal{X}$ is the induced map, then $H^i(\mathfrak{X}, \mathcal{F}) = H^i(\mathcal{X}, p_*\mathcal{F})$ for $\mathcal{F}$ a sheaf on $\mathfrak{X}$ since p_* is exact.

Let A be a noetherian $\mathbb{C}$-algebra, and let M be a finite A-module. For $\mathcal{X}/A$ a flat family of schemes over A, let $\mathfrak{X}/A$ be the canonical covering stack and $p\colon \mathfrak{X} \to \mathcal{X}$ the induced map. Then a cotangent complex $\mathcal{L}_{\mathfrak{X}/A}$ is defined. Set

$$T^i_{QG}(\mathcal{X}/A, M) = \mathrm{Ext}^i_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{L}_{\mathfrak{X}/A}, \mathcal{O}_{\mathfrak{X}} \otimes_A M),$$
$$\mathcal{T}^i_{QG}(\mathcal{X}/A, M) = p_*\mathcal{E}xt^i_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{L}_{\mathfrak{X}/A}, \mathcal{O}_{\mathfrak{X}} \otimes_A M).$$

Proposition 1.24 ([Hac04], Proposition 3.7)**.** *Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces, and let $\mathfrak{X}/A$ be the canonical covering stack. Let $A' \to A$ be an infinitesimal extension of A. For a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}'/A'$ of $\mathcal{X}/A$, let $\mathfrak{X}'/A'$ denote the canonical covering stack of $\mathcal{X}'/A'$. Then the map $\mathcal{X}'/A' \mapsto \mathfrak{X}'/A'$ gives a bijection between the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}/A$ over A' and the set of isomorphism classes of deformations of $\mathfrak{X}/A$ over A'.*

Theorem 1.25 ([Hac04], Lemma 3.8)**.** *Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces, and let M be a finite A-module. Then the natural map $\mathcal{T}^0_{QG}(\mathcal{X}/A, M) \to \mathcal{T}^0(\mathcal{X}/A, M)$ is an isomorphism.*

Theorem 1.26 ([Hac04], Theorem 3.9)**.** *Let $\mathcal{X}_0/A_0$ be a $\mathbb{Q}$-Gorenstein family of slc surfaces. Let M be a finite A_0-module.*

(1) The set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of $\mathcal{X}_0/A_0$ over A_0+M is naturally an A_0-module and is canonically isomorphic to $T^1_{QG}(\mathcal{X}_0/A_0, M)$. Here $A_0 + M$ denotes the ring $A_0[M]$ with $M^2 = 0$.

(2) Let $A \to A_0$ be an infinitesimal extension, and let $A' \to A$ be a further extension with kernel the A_0-module M. Let $\mathcal{X}/A$ be a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}_0/A_0$.

(a) There is a canonical element $o(\mathcal{X}/A, A') \in T^2_{QG}(\mathcal{X}_0/A_0, M)$ which vanishes if and only if there exists a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}'/A'$ of $\mathcal{X}/A$ over A'.

(b) If $o(\mathcal{X}/A, A') = 0$, the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations $\mathcal{X}'/A'$ is an affine space under $T^1_{QG}(\mathcal{X}_0/A_0, M)$.

The above theorems for $\mathbb{Q}$-Gorenstein deformations over general noetherian algebras guarantee that there are versal $\mathbb{Q}$-Gorenstein deformations of our surfaces ([Hac01, Sections 9-11]) and we can construct an algebraic stack out of the $\mathbb{Q}$-Gorenstein deformation theory (cf. [Ar74]). In practice, we need only calculate first-order deformations of a slc surface $X/\mathbb{C}$. So we may assume that $A_0 = \mathbb{C}$ and $M \cong \mathbb{C}$ in Theorems 1.25, 1.26 and define $T^i_X, \mathcal{T}^i_X, T^i_{QG,X}, \mathcal{T}^i_{QG,X}$ by $T^i_X = T^i_X(X/\mathbb{C}, \mathbb{C})$, etc. By the above theorems, first-order $\mathbb{Q}$-Gorenstein deformations of $X/\mathbb{C}$ are identified with $T^1_{QG,X}$, and the obstructions to extending $\mathbb{Q}$-Gorenstein deformations lie in $T^2_{QG,X}$. By Theorem 1.25, we also have $\mathcal{T}^0_{QG,X} = \mathcal{T}^0_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, the tangent sheaf of X. Working locally at $P \in X$, let $\pi\colon Z \to X$ be the canonical covering, with group G; then $\mathcal{T}^i_{QG,X} = (\pi_*\mathcal{T}^i_Z)^G$. Finally, very important is a local-to-global spectral sequence

$$E_2^{pq} = H^p(\mathcal{T}^q_{QG,X}) \Rightarrow T^{p+q}_{QG,X},$$

given by the local-to-global spectral sequence for *Ext* on the canonical covering stack of X. In particular, we have an exact sequence ([Hac04, Page 227])

$$0 \to H^1(\mathcal{T}_X) \to T^1_{QG,X} \to H^0(\mathcal{T}^1_{QG,X}) \to H^2(\mathcal{T}_X).$$

We refer to [T09, Definition 2.3] and the discussion thereafter for a similar sequence of *weakly* $\mathbb{Q}$-Gorenstein deformations (in our sense).

1.5 Stable surfaces and their moduli

Definition 1.27. *A stable surface is a slc surface with an ample dualizing sheaf.*

We are mostly interested in stable surfaces that are quotients of a product of two stable curves.

We reproduce a proof of the following proposition to get a feeling what a product of two stable curves is.

Proposition 1.28 ([vO05], Proposition 3.1)**.** *Let C, D be stable curves. Then $Z := C \times D$ is a stable surface.*

Proof. To start with, we note that C, D are local complete intersections, which implies that Z is a local complete intersection and hence Cohen–Macaulay.

Now pick a point $(P, Q) \in Z$. There are three cases:

(1) P, Q are both smooth points of the respective curves. Then (P, Q) is a smooth point of Z.

(2) One of P, Q is a node. Then the local equation of Z around (P, Q) can be taken as $xy = 0$ in $\mathbb{C}^3$ with $(P, Q) = (0, 0, 0)$ and Z has normal crossing singularities around (P, Q).

(3) Both of P, Q are nodes. Then the set $\{(P, Q)|P, Q \text{ are both nodes}\}$ is finite. The local equation of Z around (P, Q) are $xy = 0, zw = 0$ in $\mathbb{C}^4$ with $(P, Q) = (0, 0, 0, 0)$. So locally
$$Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4$$
where $Z_1 : y = w = 0, Z_2 : y = z = 0, Z_3 : x = w = 0, Z_4 : x = z = 0$. Take the normalization $\tilde{Z} \to Z$. Then
$$\tilde{Z} = \bigsqcup_{1 \le i \le 4} \tilde{Z}_i \text{ and } \tilde{Z}_i \tilde{\to} Z_i \cong \mathbb{C}^2, 1 \le i \le 4.$$
The inverse image of the 1-dimensional part of Z_{sing} is $\tilde{D} = \bigsqcup_{1 \le i \le 4} \tilde{D}_i \subset \tilde{Z}$ where $\tilde{D}_i \subset \tilde{Z}_i$, e.g., $\tilde{D}_1 : xz = y = w = 0$ under the identification of $\tilde{Z}_1$ with Z_1. Now look at the connected components $(\tilde{Z}_i, \tilde{D}_i), (1 \le i \le 4)$. Let $\sigma_i \colon \hat{Z}_i \to \tilde{Z}_i$ be the blow-up at $Q_i := (0, 0) \in \tilde{Z}_i$. Let $\hat{D}_i$ be the strict transform

of $\tilde{D}_i$ and E_i the exceptional divisor. Then $\hat{D}_i \cup E_i$ is simple normal crossing and

$$K_{\hat{Z}_i} + \hat{D}_i = \sigma_i^*(K_{\tilde{Z}_i} + \tilde{D}_i) - E_i,$$

so the pair $(\tilde{Z}_i, \tilde{D}_i)$ is log canonical.

For the ampleness of K_Z, we note that $K_Z = \pi_1^* K_C + \pi_2^* K_D$, where $\pi_i \colon Z \to C_i, i = 1, 2$ are the projections. Since C, D are stable curves, K_C and K_D are ample. So K_Z is ample by Segre embedding. This concludes the proof of Propostition 1.26. □

For lack of appropriate reference, we also prove the following more or less known statement.

Proposition 1.29. *Let Z be a stable surface and G a finite group acting on Z with finitely many fixed points. Then Z/G is also a stable surface.*

Proof. Let $X = Z/G$. We proceed in several steps according to the definition of a stable surface.

Step 1: X is Cohen–Macaulay.

For any affine open $U = \mathrm{Spec} A \subset X$, we show that A is Cohen–Macaulay. If $\pi \colon Z \to X$ denotes the quotient map, then $\pi^{-1}(U)$ is a G-invariant open affine subset of Z, say $\pi^{-1}(U) = \mathrm{Spec} B$. Note that G acts on B and $A = B^G$. Now the assertion follows from a theorem of Eagon–Hochster ([BH93, Theorem 6.4.5]).

Step 2: X has normal crossing singularities in codimension 1.

This is because Z has normal crossing singularities in codimension 1 and $Z \to X$ is étale in codimension 1.

Step 3: ω_X is $\mathbb{Q}$-Cartier.

By the GAGA principle, we can prove this assertion analytically. Since Z is a stable surface, K_Z is $\mathbb{Q}$-Cartier, i.e., there is $m \in \mathbb{N}$ such that $\omega_Z^{[m]}$ is Cartier. Pick a point $x \in X$. Suppose $\pi^{-1}(x) = \{z_1, \cdots, z_k\}$. Taking an open neighborhood V of x, which is small enough, we can assume that $U := \pi^{-1}(V) = \bigcup_{1 \le i \le k} U_i$ satisfies the following three conditions:

(1) $z_i \in U_i$ and U_i is G_{z_i}-invariant;

(2) the U_i's are pairwisely disjoint;

(3) $\omega_Z^{[m]}|U = \mathcal{O}_U \cdot s \cong \mathcal{O}_U$, for some $s \in \Gamma(U, \omega_Z^{[m]})$.

Now let $n := |G|$ and consider the invertible sheaf $(\omega_Z^{[m]})^{\otimes n}$. We have

$$(\omega_Z^{[m]})^{\otimes n}|_U = \mathcal{O}_U \cdot s'$$

where $s' := \otimes_{\tau \in G} \tau(s) \in \Gamma(U, (\omega_Z^{[m]}))^{\otimes n}$ is G-invariant. So the isomorphism

$$\begin{array}{ccc} (\omega_Z^{[m]})^{\otimes n}|_U & \tilde{\rightarrow} & \mathcal{O}_U \\ s' & \mapsto & 1 \end{array}$$

is G-equivariant and

$$(\pi_*((\omega_Z^{[m]})^{\otimes n})|_U)^G \cong (\pi_* \mathcal{O}_U)^G = \mathcal{O}_V.$$

Therefore $(\pi_*((\omega_Z^{[m]})^{\otimes n}))^G$ is invertible.

Since $\pi \colon Z \to X$ is étale in codimension 1, $(\pi_*((\omega_Z^{[m]})^{\otimes n}))^G$ and $\omega_X^{[mn]}$ coincide in codimension 1. On the other hand, $\omega_X^{[mn]}$ and $(\pi_*((\omega_Z^{[m]})^{\otimes n}))^G$ are both S_2 $\mathcal{O}_X$-modules, so we have $(\pi_*((\omega_Z^{[m]})^{\otimes n}))^G \cong \omega_X^{[mn]}$. Therefore $\omega_X^{[mn]}$ is invertible and X is $\mathbb{Q}$-Gorenstein.

Step 4: X has semi log canonical singularities.

Since we have already seen that X has normal crossings in codimension 1, by [KoSB88, Propostion 4.30] X is semi log canonical if and only if $(\tilde{X}, \tilde{D})$ is log canonical, where $\tilde{X} \to X$ is the normalization, and $\tilde{D} \subset \tilde{X}$ is the inverse image of the 1-dimensional part of X_{sing}. Similarly the fact that Z is stable implies that $(\tilde{Z}, \tilde{E})$ is log canonical, where $\tilde{Z} \to Z$ is the normalization, and $\tilde{E} \subset \tilde{Z}$ is the inverse image of the 1-dimensional part of Z_{sing}. Note that the group action of G on Z lifts to $\tilde{Z}$ and $\tilde{Z}/G = \tilde{X}$. Let $\tilde{\pi} \colon \tilde{Z} \to \tilde{X}$ be the quotient map. We have $K_{\tilde{Z}} + \tilde{E} = \tilde{\pi}^*(K_{\tilde{X}} + \tilde{D})$. Now we can apply [KoM98, Proposition 5.20]: the pair $(\tilde{Z}, \tilde{E})$ is log canonical if and only if $(\tilde{X}, \tilde{D})$ is log canonical. So X has semi log canonical singularities.

Step 5: ω_X is ample.

We know from the proof of Step 3 that $\omega_X^{[mn]}$ is Cartier, where m is the index of Z and $n = |G|$. Since $\pi \colon Z \to X$ is étale in codimension 1, we have

$$\pi^*(\omega_X^{[mn]}) = \omega_Z^{\otimes mn}.$$

Since π is a finite morphism, the fact that $\omega_Z^{\otimes mn}$ is ample implies that $\omega_X^{[mn]}$ is ample.

In conclusion, $X = Z/G$ is a stable surface. $\square$

Corollary 1.30. *Let C, D be stable curves. Let $Z := C \times D$ and G a group acting on Z with finitely many fixed points. Then Z/G is a stable surface.*

Definition 1.31. *Let C, D be two stable curves and G a finite group acting on $C \times D$ with finitely many fixed points. We shall say that $X := (C \times D)/G$ is a surface stably isogenous to a product. As in the case of surfaces isogenous to a product, X is said to be of unmixed type if $G < Aut(C) \times Aut(D)$, and of mixed type otherwise.*

We also say that the pair $(C \times D, G)$ is of unmixed type if $G < Aut(C) \times Aut(D)$, and of mixed type otherwise.

The stable degenerations of surfaces isogenous to a product are surfaces stably isogenous to a product, as the following result shows:

Theorem 1.32 (van Opstall). *Suppose $\mathcal{X} \to \Delta^*$ is a family of surfaces isogenous to a product over a punctured disk. Then, possibly after a finite change of base, totally ramified over the origin in the disk, $\mathcal{X}$ can be completed to a family of stable surfaces over the disk whose central fibre is a quotient of a product of stable curves (under a possibly nonfree group action.)*

According to the proof of the above theorem in [vO06b, Theorem 3.1], we give an explicit description of the stable degenerations of surfaces isogenous to a product here. There are two cases:

(i) (unmixed case) In this case, the general fibre $\mathcal{X}_t$ of $\mathcal{X} \to \Delta^*$ in the above theorem is a surface isogenous to a product of unmixed type. We have, up to finite base change, G-equivariant smoothings of stable curves (cf. Section 3.1) $\mathcal{C} \to \Delta$ and $\mathcal{D} \to \Delta$ such that the completion $\tilde{\mathcal{X}} \to \Delta$ of $\mathcal{X} \to \Delta^*$ is of the form $(\mathcal{C} \times_\Delta \mathcal{D})/G \to \Delta$. In particular, setting $C := \mathcal{C}_0, D := \mathcal{D}_0$, the central fibre of the completion is of the form $(C \times D)/G$ where G acts faithfully on C, D and acts diagonally on $C \times D$.

(ii) (mixed case) In this case, there exists a finite group G°, a G°-equivariant smoothing $\mathcal{C} \to \Delta$ of stable curves and a nonsplit extension

$$1 \to G^\circ \to G \to \mathbb{Z}_2 \to 1$$

yielding an automorphism φ of G°, such that the pairs $(\mathcal{C}_t, G)$ with $t \neq 0$ satisfy all the properties, namely $(I), (II), (A), (B)$ in Proposition 1.7. On the central fibre $\mathcal{C}_0$ of $\mathcal{C} \to \Delta$, we still have a G-action on $\mathcal{C}_0 \times \mathcal{C}_0$ that enjoy properties $(I), (II)$ in propostion 1.7, but not necessarily $(A), (B)$, i.e., the action of G on $\mathcal{C}_0 \times \mathcal{C}_0$ is not necessarily free. Now the completion $\tilde{\mathcal{X}} \to \Delta$ of $\mathcal{X} \to \Delta^*$ is of the form $(\mathcal{C} \times_\Delta \mathcal{C})/G \to \Delta$.

In both cases, the degeneration $\mathcal{X}_0$ is of the form $(C \times D)/G$, where C, D are stable curves and G acts in the way described above. Tautologically the pair $(C \times D, G)$

admits a free smoothing, i.e., a one-parameter family $\mathcal{C} \times_\Delta \mathcal{D} \to \Delta$ such that the following hold:

(i) $\mathcal{C}_0 \times \mathcal{D}_0 \cong C \times D$;

(ii) The fibre $\mathcal{C}_t \times \mathcal{D}_t$ over $t \neq 0$ is smooth;

(iii) G acts on $\mathcal{C} \times_\Delta \mathcal{D}$ preserving the fibres and the action of G on the central fibre coincides with the given action of G on $C \times D$;

(iv) G acts freely on the general fibres $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$.

Now we consider the following moduli stack of stable surfaces over the category $(Sch)/\mathbb{C}$ of noetherian $\mathbb{C}$-schemes: for any $B \in (Sch)/\mathbb{C}$,

$$\mathcal{M}^{st}_{a,b}(B) = \{\mathcal{X}/B \mid \mathcal{X}/B \text{ is a } \mathbb{Q}\text{-Gorenstein family of stable surfaces over } B$$
$$\text{and for any closed point } t \in B, K^2_{\mathcal{X}_t} = a, \chi(\mathcal{O}_{\mathcal{X}_t}) = b\}.$$

Theorem 1.33. *$\mathcal{M}^{st}_{a,b}$ is a separated and proper Deligne–Mumford stack of finite type. The underlying coarse moduli space $M^{st}_{a,b}$ is compact and it contains the moduli space of stable surfaces X with at most canonical singularities and $K^2_X = a, \chi(\mathcal{O}_X) = b$.*

Proof. The proof for the assertion that $\mathcal{M}^{st}_{a,b}$ is a Deligne–Mumford stack of finite type is the same as [Hac04, Theorem 4.4]. For the separatedness and properness of $\mathcal{M}^{st}_{a,b}$, we refer to the arguments of [HK04, Remark 2.13]. □

Remark 1.34. By [KeM97, Corollary 1.3], a separated Deligne-Mumford stack has a separated algebraic space as coarse moduli space. In particular, our $M^{st}_{a,b}$ is a separated algebraic space of finite type. Moreover it is complete. [Ko90, Theorem 4.12] says that $M^{st}_{a,b}$ is in fact a projective scheme. We shall get some connected components of this moduli scheme by studying the $\mathbb{Q}$-Gorenstein deformations of degenerations of surfaces isogenous to a product in Chapter 4.

On the other hand, there is Viehweg's moduli functor of stable surfaces

$$\mathcal{M}^V_{a,b} \colon (Sch)/\mathbb{C} \to (Sets)$$

such that, for any $B \in (Sch)/\mathbb{C}$,

$$\begin{aligned}\mathcal{M}^V_{a,b}(B) = \{ &\text{ isomorphism classes of families of stable surfaces } \mathcal{X}/B \\ &\text{such that the relative dualizing sheaf } \omega_{\mathcal{X}/B} \text{ is } \mathbb{Q}\text{-Cartier} \\ &\text{and for any closed point } t \in B, K^2_{\mathcal{X}_t} = a, \chi(\mathcal{O}_{\mathcal{X}_t}) = b\}.\end{aligned}$$

Viehweg's moduli functor also turns out to have a projective scheme $M^V_{a,b}$ as coarse moduli space ([HK04]).

Since the condition of $\mathbb{Q}$-Gorenstein family is a priori stronger than Viehweg's, there is an inclusion of moduli spaces $M^{st}_{a,b} \subset M^V_{a,b}$ which induces a bijection between their closed points. This already implies that $(M^{st}_{a,b})_{red} = (M^V_{a,b})_{red}$, i.e., they have the same reduced scheme structure. So the topological structures of $M^{st}_{a,b}$ and $M^V_{a,b}$ are the same. In particular, the connected components are the same for both moduli spaces. We refer to [Kov09, Section 7] for a nice discussion of the two moduli spaces.

Chapter 2

Automorphisms and their action on cohomology

Let V be a variety. We can consider the induced action of $\mathrm{Aut}(V)$ on $H^*(V, R)$ and get a homomorphism $\varphi\colon \mathrm{Aut}(V) \to \mathrm{Aut}(H^*(V, R))$. Here $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}$. We have the following questions: Is φ injective? If not, what is the kernel? We can also consider the actions $\varphi_i\colon \mathrm{Aut}(V) \to \mathrm{Aut}(H^i(V, R))$ on the i-th cohomology. The problem is closely related to the existence of fine moduli space of varieties with level-n structures ([Po77, Lecture 10]).

It is well known that, if $V = C$ is a smooth curve of genus ≥ 2, then $\mathrm{Aut}(C)$ acts faithfully on $H^1(C, \mathbb{C})$, i.e., φ_1 is injective for $R = \mathbb{C}$.

For smooth surfaces $V = S$, the above problem has been studied by many authors (cf. [Cai04]). In the case of $K3$ surfaces and Enriques surfaces, φ_2 is injective if the ring of coefficients is $R = \mathbb{Z}$ ([BR75, U76]). However if $R = \mathbb{C}$, then there is an example of Enriques surface S such that φ_2 is not injective ([P79]). In the following, we will assume the ring of coefficients $R = \mathbb{C}$. For surfaces of general type with base point free $|K_S|$, Peters [P79] proved among other things that, if φ_2 is not injective, then either $K_S^2 = 8\chi(\mathcal{O}_S)$ and $\mathrm{Ker}(\varphi_2)$ is a 2-group, or $K_S^2 = 9\chi(\mathcal{O}_S)$ and $\mathrm{Ker}(\varphi_2)$ is a 3-group. Then in a series of papers ([Cai04, Cai06a, Cai06b, Cai07]), Cai studied the case of surfaces of general type more systematically. He showed that if $\chi(\mathcal{O}_S) > 188$, then $|\mathrm{Ker}(\varphi_2)| \leq 4$ ([Cai04, Theorem A]). If the surfaces have a fibration of curves of genus 2 and $\chi(\mathcal{O}_S)$ is ≥ 5, then $|\mathrm{Ker}(\varphi_2)| \leq 2$ and he can effectively classify the surfaces with $|\mathrm{Ker}(\varphi_2)| = 2$ ([Cai07, Theorems 1.1 and 1.2]).

In this chapter, I study the action of automorphisms on the second cohomology (with $\mathbb{C}$-coefficient) in the case of surfaces isogenous to a product. Moreover we only consider a subgroup of automorphisms of the surface which is easily derived from the construction of the surface $S = (C \times D)/G$ with G abelian.

Back to the general situation, if V is a variety and $G < \mathrm{Aut}(V)$ is a subgroup of the automorphism group, then we have the action of G on $H^*(V, \mathbb{C})$, i.e., a homomorphism $\phi\colon G \to \mathrm{Aut}(H^*(V, \mathbb{C}))$. We can also consider the action of G on the i-th cohomology $\phi_i\colon G \to \mathrm{Aut}(H^i(V, \mathbb{C}))$.

Assuming G is abelian, then $H^i(V, \mathbb{C})$ decomposes into $\bigoplus_\chi H^i(V, \mathbb{C})^\chi$, where $\chi \in G^*$ runs through the characters of G and $H^i(V, \mathbb{C})^\chi$ is the eigenspace of χ. Denote $h^i(V, \mathbb{C})^\chi := dim_{\mathbb{C}} H^i(V, \mathbb{C})^\chi$.

Theorem 2.1 ([B87], Proposition 2, or [B91], p.244)**.** *Let C be a smooth curve of genus $g(C) \geq 2$ and $G < Aut(C)$ a finite abelian group of automorphisms. Let $\pi\colon C \to C/G$ be the quotient map and r the number of branch points on C/G. Then, for $1 \neq \chi \in G^*$,*

$$h^1(C, \mathbb{C})^\chi = (2g(C/G) - 2 + r) - \sum_{j=1}^{r} l_{\sigma_j}(\chi) \tag{2.1}$$

where $\langle\sigma_1\rangle, \cdots, \langle\sigma_r\rangle$ are the stabilizers of the points lying over the r branch points of $C \to C/G$ and for any $\sigma \in G$,

$$l_\sigma(\chi) = \begin{cases} 1 & \text{if } \chi(\sigma) = 1; \\ 0 & \text{if } \chi(\sigma) \neq 1. \end{cases}$$

For the automorphism group of a surface isogenous to a product, we have the following result.

Proposition 2.2. *Let $S = (C \times D)/G$ be (the minimal realization of) a surface isogenous to a product. Then*

$$Aut(S) = N_G/G,$$

where N_G is the normalizer of G in $Aut(C \times D)$.

Proof. Let $\pi\colon C \times D \to S$ be the quotient map. Given a $\sigma \in G$, we have a lift $\tilde{\sigma}$ of σ with $\tilde{\sigma} \in \mathrm{Aut}(C \times D)$ such that

$$\begin{array}{ccc} C \times D & \xrightarrow{\tilde{\sigma}} & C \times D \\ \pi \big\downarrow & & \big\downarrow \pi \\ S & \xrightarrow{\sigma} & S \end{array}$$

is commutative. This is simply because of the uniqueness of minimal realization of S. On the other hand, given $\tilde{\sigma} \in \mathrm{Aut}(C \times D)$, $\tilde{\sigma}$ descends to an automorphism $\sigma \in \mathrm{Aut}(S)$ if and only if it is in the normalizer N_G of G in $\mathrm{Aut}(C \times D)$. Hence we have a surjective homomorphism of groups $N_G \twoheadrightarrow \mathrm{Aut}(S)$ and the kernel is easily seen to be G. So $\mathrm{Aut}(S) = N_G/G$. $\square$

Now we assume $S = (C \times D)/G$ is a surface isogenous to a product of unmixed type and G is abelian. Let N_G be the normalizer of G in $\mathrm{Aut}(C \times D)$. By Proposition 2.2, $\mathrm{Aut}(S) = N_G/G$. Since G is abelian, it is easy to see that $G \times G < N_G$. So we have $G \cong (G \times G)/G < N_G/G = \mathrm{Aut}(S)$.

Consider the induced action of G on $H^2(S, \mathbb{C})$, i.e., the homomorphism $\phi_2\colon G \to \mathrm{Aut}(H^2(S, \mathbb{C}))$. Note that

$$\begin{aligned} H^2(S, \mathbb{C}) &= H^2(C \times D, \mathbb{C})^G \\ &= \left(H^2(C, \mathbb{C}) \otimes_{\mathbb{C}} H^0(D, \mathbb{C})\right) \bigoplus \left(H^0(C, \mathbb{C}) \otimes_{\mathbb{C}} H^2(D, \mathbb{C})\right) \\ &\qquad \bigoplus_{\chi \in G^*} \left(H^1(C, \mathbb{C})^{\chi} \otimes_{\mathbb{C}} H^1(D, \mathbb{C})^{\chi^{-1}}\right). \end{aligned}$$

For any $\sigma \in (G \times G)/G$, σ acts trivially on the summands

$$\left(H^2(C, \mathbb{C}) \otimes_{\mathbb{C}} H^0(D, \mathbb{C})\right) \bigoplus \left(H^0(C, \mathbb{C}) \otimes_{\mathbb{C}} H^2(D, \mathbb{C})\right).$$

So σ acts nontrivially on $H^2(S, \mathbb{C})$ if and only if σ acts nontrivially on $H^1(C, \mathbb{C})^{\chi} \otimes_{\mathbb{C}} H^1(D, \mathbb{C})^{\chi^{-1}}$ for some $\chi \in G^*$. The above condition further translates to the existence of $\chi \in G^*$ such that

$$\chi(\sigma) \neq 1, H^1(C, \mathbb{C})^{\chi} \neq 0 \text{ and } H^1(D, \mathbb{C})^{\chi^{-1}} \neq 0.$$

Theorem 2.3. *Assume the genus $g(C/G)$ of C/G is ≥ 2. Then $\phi_2\colon G \to Aut(H^2(S, \mathbb{C}))$ is injective.*

Proof. We have to show that for any $\sigma \in G$, $\phi_2(\sigma) \neq 1$, i.e., σ acts nontrivially on $H^2(S, \mathbb{C})$. By the above discussion, it suffices to find a $\chi_0 \in G^*$, such that

$$\chi_0(\sigma) \neq 1, H^1(C, \mathbb{C})^{\chi_0} \neq 0 \text{ and } H^1(D, \mathbb{C})^{\chi_0^{-1}} \neq 0.$$

For any $\chi \in G^*$, we have

$$h^1(C, \mathbb{C})^{\chi} = (2g(C/G) - 2 + r) - \sum_{j=1}^{r} l_{\sigma_j}(\chi) \geq 2g(C/G) - 2 \geq 2$$

by Theorem 2.1. On the other hand, since $\sum_{\chi} h^1(D, \mathbb{C})^{\chi} = 2g(D) > 2g(D/\sigma) = \sum_{\chi(\sigma)=1} h^1(D, \mathbb{C})^{\chi}$, there is at least one χ_0 such that $h^1(D, \mathbb{C})^{\chi_0^{-1}} > 0$ and $\chi_0(\sigma) \neq 1$. This χ_0 is what we want and the proof is complete. □

Theorem 2.4. *Assume $g(C/G) = g(D/G) = 1$. For the two quotient maps $C \to C/G$ and $D \to D/G$, we have two respective generating vectors, say $(a, b; \sigma_1, \ldots, \sigma_r)$ and $(c, d; \tau_1, \ldots, \tau_s)$ (cf. Page 5). Then either*

(i) $\phi_2 \colon G \to Aut(H^2(S, \mathbb{C}))$ is injective, or

(ii) $\sigma_1 = \cdots = \sigma_r, \tau_1 = \cdots = \tau_s$ and $|\sigma_1| = |\tau_1| = 2$. In this case, $Ker(\phi_2) = \langle \sigma_1 \tau_1 \rangle$.

Proof. By the previous discussion, given $1 \neq \sigma \in G$, σ acts nontrivially on $H^2(S, \mathbb{C})$ if and only there is a $\chi \in G^*$ such that

$$\chi(\sigma) \neq 1, h^1(C, \mathbb{C})^{\chi} > 0 \text{ and } h^1(D, \mathbb{C})^{\chi^{-1}} > 0. \tag{2.2}$$

Now Broughton's formula (2.1) gives

$$\begin{aligned} h^1(C, \mathbb{C})^{\chi} &= (2g(C/G) - 2 + r) - \sum_{i=1}^{r} l_{\sigma_i}(\chi) \\ &= r - \sum_{i=1}^{r} l_{\sigma_i}(\chi). \end{aligned}$$

Similarly

$$h^1(D, \mathbb{C})^{\chi} = s - \sum_{j=1}^{s} l_{\tau_j}(\chi).$$

Taking the definition of $l_{\sigma_i}(\chi)$ (cf. Theorem 2.1) into consideration, we see that $h^1(C, \mathbb{C})^{\chi} > 0$ if and only if $\chi(\sigma_i) \neq 1$ for at least one i. A similar argument using the definition of $l_{\tau_j}(\chi)$ proves that $h^1(D, \mathbb{C})^{\chi^{-1}} > 0$ if and only if $\chi(\tau_j) \neq 1$ for at least one j. So (2.2) is equivalent to the following conditions:

$$\chi(\sigma) \neq 1, \chi(\sigma_i) \neq 1 \text{ and } \chi(\tau_j) \neq 1, \text{ for some } i \text{ and } j. \tag{2.3}$$

Set

$$G_{ij} := \langle \sigma_i, \tau_j \rangle, 1 \le i \le r, 1 \le j \le s.$$

The fact that G acts freely on $C \times D$ means $\langle \sigma_i \rangle \cap \langle \tau_j \rangle = \{1\}$, for any $1 \le i \le r, 1 \le j \le s$ (cf. (1.2) on Page 6). So G_{ij} is in fact isomorphic to the direct sum $\langle \sigma_i \rangle \oplus \langle \tau_j \rangle$. As a result, we can always find a $\chi'_{ij} \in G^*_{ij}$ such that

$$\chi'_{ij}(\sigma_i) = \xi_i \text{ and } \chi'_{ij}(\tau_j) = \eta_j$$

where ξ_i and η_j are roots of unity of order $|\sigma_i|$ and $|\tau_j|$ respectively.

Consider the exact sequence

$$1 \to G_{ij} \to G \to G/G_{ij} \to 1.$$

Since $\mathbb{C}^*$ is an injective $\mathbb{Z}$-module, we get the dual exact sequence

$$1 \to (G/G_{ij})^* \to G^* \to G^*_{ij} \to 1,$$

by applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)$. In particular, we can find a $\chi_{ij} \in G^*$ restricting to χ'_{ij} by the surjection $G^* \to G^*_{ij}$. So we have $\chi_{ij} \in G^*$ such that

$$\chi_{ij}(\sigma_i) = \xi_i \text{ and } \chi_{ij}(\tau_j) = \eta_j.$$

If σ is in some $\langle \sigma_i \rangle$ or $\langle \tau_j \rangle$, then at least one of the χ_{ij}'s satisfies (2.3) and hence σ acts nontrivially on $H^2(S, \mathbb{C})$.

In the following, we assume $\sigma \notin \langle \sigma_i \rangle$ or $\langle \tau_j \rangle$, for any $1 \le i \le r$ and $1 \le j \le s$.

Suppose $\sigma \notin G_{i,j}$ for some i, j. Then the image $\overline{\sigma}$ of σ in G/G_{ij} is nontrivial, so we have a character $\overline{\chi_0} \in (G/G_{ij})^*$ such that $\overline{\chi_0}(\overline{\sigma}) \neq \chi_{ij}(\sigma)^{-1}$. Let $\chi_0 \in G^*$ be the image of $\overline{\chi_0}$ under the natural lifting map $(G/G_{ij})^* \to G^*$. Setting $\chi = \chi_{ij} \cdot \chi_0$, we have

$$\chi(\sigma) = \overline{\chi_0}(\overline{\sigma}) \cdot \chi_{ij}(\sigma) \neq 1, \chi(\sigma_i) = \xi_i \neq 1, \chi(\tau_j) = \eta_j \neq 1.$$

So this χ satisfies (2.3) and hence σ acts nontrivially on $H^2(S, \mathbb{C})$..

In the following, we assume further that $\sigma \in G_{i,j}$ for every i, j.

Assume $|\sigma_i| = |\tau_j| = 2$ for any $1 \le i \le r, 1 \le j \le s$. Due to the assumption that $\sigma \notin \langle \sigma_i \rangle$ or $\langle \tau_j \rangle$, for any $1 \le i \le r$ and $1 \le j \le s$, this is only possible when $\sigma_1 = \cdots = \sigma_r$, and $\tau_1 = \cdots = \tau_s$, i.e., we are in case (ii) of the theorem. In this case, $\sigma = \sigma_1 \tau_1$. For any $\chi \in G^*$ such that $\chi(\sigma_1) \neq 1$ and $\chi(\tau_1) \neq 1$, we have

$$\chi(\sigma_1) = -1, \chi(\tau_1) = -1.$$

Hence $\chi(\sigma) = \chi(\sigma_1 \tau_1) = 1$ and σ is in $Ker(\phi_2)$. Moreover there is no other nontrivial element in $Ker(\phi_2)$ as we have seen above. So $Ker(\phi_2) = \langle \sigma_1 \tau_1 \rangle$.

If there is some σ_i or τ_j, say σ_i whose order is ≥ 3, then we can adjust χ'_{ij} of the value at σ_i to find a $\chi' \in (G_{ij})^*$ such that $\chi'(\sigma) \neq 1, \chi'(\sigma_i) = \xi'_i$ and $\chi'(\tau_j) = \eta_j$ where ξ'_i is a $|\sigma_i|$-th root of unity. More precisely, if $\chi'_{ij}(\sigma) \neq 1$, we simply set $\chi' = \chi_{ij}$; otherwise $\chi'_{ij}(\sigma) = 1$. In the later case, we write $\sigma = \sigma_i^a \tau_j^b$. Since $\sigma \notin \langle \sigma_i \rangle$ by assumption, we have $a \neq 0$. Let $\chi' \in (G_{ij})^*$ be such that $\chi'(\sigma_i) = \xi'_i := \xi_i^{-1}$ and $\chi'(\tau_j) = \eta_j$. Then it is easily seen that $\chi'(\sigma) \neq 1$. By the surjection $G^* \to (G_{ij})^*$, we can get a χ such that $\chi(\sigma) \neq 1, \chi(\sigma_i) = \xi'_i$ and $\chi(\tau_j) = \eta_j$. In particular, χ satisfies (2.3) above and the proof is complete. $\square$

Remark 2.5. When case (ii) of the above theorem occurs, we have $\sigma_1 \cdots \sigma_r = \tau_1 \cdots \tau_s = 1$ by (1.1) on Page 5. So both r and s are even.

We will give examples of case (ii) in Theorem 2.4.

Example 2.6. Let $G = \langle \sigma, \tau \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$: so σ and τ are elements of order 2. Let C', D' be two elliptic curves and $\{P_1, \cdots, P_{2r'}\} \subset C'$ (resp. $\{Q_1, \cdots, Q_{2s'}\} \subset D'$) be a set of distinct $2r'$ (resp. $2s'$) points. Then the fundamental group of the punctured curve $C' \setminus \{P_1, \cdots, P_{2r'}\}$ is

$$\pi_1(C' \setminus \{P_1, \cdots, P_{2r'}\}) = \langle \alpha, \beta, \gamma_1, \cdots, \gamma_{2r'} |\ [\alpha, \beta]\gamma_1 \cdots \gamma_{2r'} = 1 \rangle.$$

where α, β is the standard basis of $\pi_1(C)$. There is a surjective homomorphism

$$\begin{aligned} \pi_1(C' \setminus \{P_1, \cdots, P_{2r'}\}) &\to G \\ \alpha, \beta &\mapsto \tau \\ \gamma_i &\mapsto \sigma, \qquad \text{for all } 1 \leq i \leq 2r'. \end{aligned}$$

By Riemann's existence theorem (cf. Page 5), we have a G-covering of smooth curves $C \to C'$ such that $P_1, \cdots, P_{2r'}$ are the branch points and the stabilizer of the points lying over P_i is $\langle \sigma \rangle$ for any $1 \leq i \leq 2r'$.

Similarly we can construct a G-covering $D \to D'$ such that $Q_1, \cdots, Q_{2s'}$ are the branch points and the stabilizer of the points lying over Q_j is $\langle \tau \rangle$ for any $1 \leq j \leq 2s'$.

Let G act diagonally on $C \times D$. Since $\langle \sigma \rangle \cap \langle \tau \rangle = \{1\}$, G acts freely on $C \times D$ and hence $S := (C \times D)/G$ is a surface isogenous to a product in case (ii) of Theorem 2.4 (cf. Pages 5 and 6). By Hurwitz's formula, we have

$$2g(C) - 2 = |G|(2g(C') - 2) + |G| \cdot r',$$

hence

$$g(C) = \frac{|G| \cdot r'}{2} + 1 = 2r' + 1.$$

Similarly $g(D) = 2s' + 1$. Then it is easy to see that (cf. Remark 1.6)

$$K_S^2 = \frac{8(g(C) - 1)(g(D) - 1)}{|G|} = 8r's',$$

which can be arbitrarily large as r', s' grow. So such surfaces S form an infinite series of surfaces of general type.

Now we consider examples with $g(C/G) = 1$ and $g(D/G) = 0$.

Example 2.7. Let $G = \langle e_1, e_2, e_3 \rangle \cong \mathbb{Z}_2^{\oplus 3}$. We have

$$|e_1| = |e_2| = |e_3| = 2 \text{ and } \langle e_i \rangle \cap \langle e_j \rangle = 1, \text{ for all } 1 \le i < j \le 3.$$

Let C', D' be two smooth curves of genera $g(C') = 1, g(D') = 0$. Let $\{P_1, \cdots, P_{2r'}\} \subset C'$ be a set of distinct $2r'$ points. Then

$$\pi_1(C' \setminus \{P_1, \cdots, P_{2r'}\}) = \langle \alpha, \beta, \gamma_1, \cdots, \gamma_{2r'} |\ [\alpha, \beta]\gamma_1 \cdots \gamma_{2r'} = 1.\rangle$$

and we have a surjective homomorphism

$$f \colon \pi_1(C' \setminus \{P_1, \cdots, P_{2r'}\}) \to G$$

such that $f(\alpha) = e_2, f(\beta) = e_3$ and $f(\gamma_i) = e_1$ for any $1 \le i \le 2r'$. By Riemann's existence theorem, we get a G-covering $C \to C'$ such that $P_1, \cdots, P_{2r'}$ are the branch points and $\langle e_1 \rangle$ is the stabilizer of the points lying over P_i for any $1 \le i \le 2r'$.

Let $Q_1, \cdots, Q_6$ be six distinct points on D'. Then

$$\pi_1(D' \setminus \{Q_1, \cdots, Q_6\}) = \langle \delta_1, \cdots, \delta_6 |\ \delta_1 \cdots \delta_6 = 1 \rangle.$$

So we have a surjective homomorphism

$$h : \pi_1(D' \setminus \{Q_1, \cdots, Q_6\}) \to G$$

such that

$$h(\delta_1) = h(\delta_2) = e_2, h(\delta_3) = h(\delta_4) = e_3, h(\delta_5) = h(\delta_6) = e_1 + e_2 + e_3.$$

Again by Riemann's existence theorem, we get a G-covering $D \to D'$ such that $Q_1, \cdots, Q_6$ are the branch points and the stabilizers of the points lying over Q_1, Q_2 (resp. Q_3, Q_4, resp. Q_5, Q_6) are both $\langle e_2 \rangle$ (resp. $\langle e_3 \rangle$, resp. $\langle e_1 + e_2 + e_3 \rangle$).

Let G act on $C \times D$ diagonally. Since

$$\langle e_1 \rangle \cap \langle e_2 \rangle = \langle e_1 \rangle \cap \langle e_3 \rangle = \langle e_1 \rangle \cap \langle e_1 + e_2 + e_3 \rangle = \{1\},$$

we see that G acts freely on $C \times D$ (cf. Pages 5 and 6), and hence $S := (C \times D)/G$ is a surface isogenous to a product.

Set

$$\tau_1 = \tau_2 = e_2, \tau_3 = \tau_4 = e_3, \tau_5 = \tau_6 = e_1 + e_2 + e_3.$$

By Theorem 2.1, given a $\chi \in G^*$, $H^1(C,\mathbb{C})^\chi \neq 0$ (resp. $H^1(D,\mathbb{C})^{\chi^{-1}} \neq 0$) if and only if $\chi(e_1) \neq 1$ (resp. $\chi(\tau_i) \neq 1$ for at least three $i, 1 \leq i \leq 6$) (see the proof of Theorem 2.4). Consider the character χ such that $\chi(e_1) = \chi(e_2) = \chi(e_3) = -1$. This is the only character χ satisfying the following conditions:

$$H^1(C,\mathbb{C})^\chi \neq 0 \text{ and } H^1(D,\mathbb{C})^{\chi^{-1}} \neq 0.$$

Let $G_\chi = \mathrm{Ker}(\chi \colon G \to \mathbb{C}^*)$. Then

$$\mathrm{Ker}(\phi_2 \colon G \to H^2(S,\mathbb{C})) = G_\chi = \langle e_1+e_2, e_1+e_3\rangle \cong \mathbb{Z}_2^{\oplus 2}.$$

By Hurwitz's formula,

$$2g(C) - 2 = |G|(2g(C') - 2) + |G| \cdot r',$$

hence $g(C) = 4r' + 1$. Similarly $g(D) = 5$. So

$$K_S^2 = \frac{8(g(C)-1)(g(D)-1)}{|G|} = 16r',$$

and our surfaces S form an infinite series as r' varies, giving the existence of one case in [Cai04, Theorem A]. Note that the irregularity of S is $q(S) = g(C') + g(D') = 1$.

Following the line of Example 2.7, we give examples with $g(C/G) = g(D/G) = 0$.

Example 2.8. Let $G = \langle e_1, e_2, e_3\rangle \cong \mathbb{Z}_2^{\oplus 3}$. Let C', D' be two smooth curves of genus 0. Let $\{P_1, \cdots, P_{2r'+6}\}, r' \geq 1$ be a set of distinct $2r'+6$ points on C'. Then

$$\pi_1(C' \setminus \{P_1, \cdots, P_{2r'+6}\}) = \langle \gamma_1, \cdots, \gamma_{2r'+6} |\ \gamma_1 \cdots \gamma_{2r'+6} = 1.\rangle$$

and we have a surjective homomorphism

$$f \colon \pi_1(C' \setminus \{P_1, \cdots, P_{2r'+6}\}) \to G$$

such that

$$\begin{aligned} f(\gamma_1) = \cdots = f(\gamma_{2r'+2}) &= e_1 + e_3, \\ f(\gamma_{2r'+3}) = f(\gamma_{2r'+4}) &= e_3, \\ f(\gamma_{2r'+5}) = f(\gamma_{2r'+6}) &= e_1 + e_2 + e_3. \end{aligned}$$

By Riemann's existence theorem, we get a G-covering $C \to C'$ such that $P_1, \cdots, P_{2r'+6}$ are the branch points and $\langle e_1 + e_3\rangle$ (resp. $\langle e_3\rangle$, $\langle e_1+e_2+e_3\rangle$) is the stabilizer of the points lying over P_i for $1 \leq i \leq 2r'+2$ (resp. $P_{2r'+3}$ and $P_{2r'+4}$, $P_{2r'+5}$ and $P_{2r'+6}$). We have $g(C) = 4r' + 5$.

Let $Q_1, \cdots, Q_5$ be five distinct points on D'. Then

$$\pi_1(D' \setminus \{Q_1, \cdots, Q_5\}) = \langle \delta_1, \cdots, \delta_5 | \ \delta_1 \cdots \delta_5 = 1 \rangle.$$

So we have a surjective homomorphism

$$h : \pi_1(D' \setminus \{Q_1, \cdots, Q_5\}) \to G$$

such that

$$h(\delta_1) = h(\delta_2) = e_2 + e_3, h(\delta_3) = e_1, h(\delta_4) = e_2, h(\delta_5) = e_1 + e_2.$$

Again by Riemann's existence theorem, we get a G-covering $D \to D'$ such that $Q_1, \cdots, Q_5$ are the branch points and the stabilizers of the points lying over Q_1 and Q_2 (resp. Q_3, Q_4, Q_5) are $\langle e_2 + e_3 \rangle$ (resp. $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_1 + e_2 \rangle$). We have $g(D) = 3$.

Let G act on $C \times D$ diagonally. We see that G acts freely on $C \times D$ (cf. Pages 5 and 6), and hence $S := (C \times D)/G$ is a surface isogenous to a product.

Set

$$\begin{aligned} \sigma_1 = \cdots = \sigma_{2r'+2} &= e_1 + e_3, \\ \sigma_{2r'+3} = \sigma_{2r'+4} = e_3, \sigma_{2r'+5} = \sigma_{2r'+6} &= e_1 + e_2 + e_3; \\ \tau_1 = \tau_2 = e_2 + e_3, \tau_3 = e_1, \tau_4 = e_2, \tau_5 &= e_1 + e_2. \end{aligned}$$

By Theorem 2.1, given a $\chi \in G^*$, $H^1(C, \mathbb{C})^\chi \neq 0$ (resp. $H^1(D, \mathbb{C})^{\chi^{-1}} \neq 0$) if and only if $\chi(\sigma_i) \neq 1$ for at least three $i, 1 \leq i \leq 2r' + 6$ (resp. $\chi(\tau_j) \neq 1$ for at least three $j, 1 \leq j \leq 5$). Consider the character χ such that $\chi(e_1) = \chi(e_2) = \chi(e_2 + e_3) = -1$. This is the only character χ satisfying the following conditions:

$$H^1(C, \mathbb{C})^\chi \neq 0 \text{ and } H^1(D, \mathbb{C})^{\chi^{-1}} \neq 0.$$

Let $G_\chi = \mathrm{Ker}(\chi \colon G \to \mathbb{C}^*)$. Then

$$\mathrm{Ker}(\phi_2 \colon G \to H^2(S, \mathbb{C})) = G_\chi = \langle e_1 + e_2, e_3 \rangle \cong \mathbb{Z}_2^{\oplus 2}.$$

We have

$$K_S^2 = \frac{8(g(C) - 1)(g(D) - 1)}{|G|} = r' + 1,$$

and the surfaces S form an infinite series as r' varies, giving the existence of one case in [Cai04, Theorem A]. Since $q(S) = g(C') + g(D') = 0$, the surfaces are regular.

Remark 2.9. It seems that the picture is far from being complete. It is very interesting to know when $G \times G = N_G$ in $\mathrm{Aut}(C \times D)$. In that case, the group G considered in Theorems 2.3 and 2.4 will be the whole automorphism group of S (Theorem 2.2) and our results will contribute more to the problem posed in the beginning of this chapter.

It is also interesting to consider the case where G is not abelian, but the situation there is much more complicated. For example, $G \times G$ is not contained in N_G any more and we cannot get an automorphism group of S as in the abelian case.

Chapter 3

Stable degenerations of surfaces isogenous to a product

In this chapter I will classify the possible singularities on a stable degeneration of surfaces isogenous to a product.

3.1 Smoothings of stable curves with group actions

Our surfaces can be constructed by taking finite quotients of products of two stable curves, so their geometry is closely related to that of stable curves. In this section, we will establish some facts about smoothings of stable curves with group actions. More precisely, in the case when the group action admits a smoothing, we will show what the stabilizers on the central fibre can be and how they act locally analytically. These facts are used in Sections 3.2 and 3.3 for the smoothing of a product of stable curves.

We will recall the definition of stable curves first ([DM69]):

Definition 3.1. *Let $g \geq 2$ be an integer. A stable curve of genus g is a reduced, connected, 1-dimensional scheme C over $\mathbb{C}$ such that:*

(i) C has only ordinary double points as singularities;

(ii) if E is a non-singular rational component of C, then E meets the other components of C in more than 2 points;

(iii) dim $H^1(\mathcal{O}_C) = g$.

Then we give the definition of smoothing of a stable curve with a group action.

Definition 3.2. *Let G be a finite group acting faithfully on a stable curve C. A smoothing of the pair (C, G) is a (flat) family of stable curves $\mathcal{C} \to \Delta$ over the unit disk such that*

(i) the central fibre $\mathcal{C}_0$ is isomorphic to C;

(ii) The fibre $\mathcal{C}_t$ of the family over $t \neq 0$ is smooth;

(iii) G acts on $\mathcal{C}$ preserving the fibres, and the action on $\mathcal{C}_0$ coincides with the given one on C under the isomorphism of (i).

Remark 3.3. We also call $\mathcal{C} \to \Delta$ a G-equivariant smoothing of C.

Now let (C, G) be as in the definition and assume $\mathcal{C} \to \Delta$ is a smoothing of (C, G).

Lemma 3.4. *There are only finitely many points on C having non-trivial stabilizers, or, equivalently, there are only finitely many fixed points for the G-action.*

Proof. Otherwise there is a $\tau \neq 1 \in G$ acting as identity on some irreducible component D of C. Pick a smooth point P of D, the germ of $\mathcal{C}$ around P can be viewed as a deformation of the germ of C around P. Since this deformation must be trivial, $\mathcal{C}$ is smooth around P and we can take local coordinate z of D and local coordinate t of Δ such that (z, t) form local coordinates of $\mathcal{C}$ around P and $\mathcal{C} \to \Delta$ is given by $(z, t) \mapsto t$. Therefore τ acts as $(z, t) \mapsto (z, t)$, i.e., τ acts as identity on $\mathcal{C}$ around P. So $\tau = 1 \in G$, a contradiction. □

Lemma 3.5. *If $P \in C$ is a smooth point, then G_P is cyclic.*

Proof. There is an embedding $G_P \hookrightarrow GL(T_P C)$, where $T_P C \cong \mathbb{C}$ is the tangent space of C at P. Since $T_P C$ is a 1-dimensional vector space, $GL(T_P C) \cong \mathbb{C}^*$. So G_P, being a finite subgroup of $\mathbb{C}^*$, is cyclic. □

Lemma 3.6. *If $P \in C$ is a smooth point, and $1 \neq \tau \in G_P$, then τ also fixes points on $\mathcal{C}_t$, for $t \neq 0$.*

Proof. As in the proof of Lemma 3.4, we can find local coordinates (z, t) for $\mathcal{C}$ around P such that τ acts as $(z, t) \mapsto (\xi(\tau) z, t)$, where $\xi(\tau) \in \mathbb{C}^*$ is a primitive $|\tau|$-th root of unity. So τ fixes $z = 0 \subset \mathcal{C}$, which maps onto Δ. Now it is evident that $\mathcal{C}_t \cap \{z = 0\} \neq \varnothing$, i.e., τ fixes points on $\mathcal{C}_t$, for $t \neq 0$. □

Lemma 3.7. *If $P \in C$ is a node, then G_P is either cyclic or dihedral.*

Proof. The germ of $\mathcal{C}$ around P can be seen as a deformation of a node. We can find an embedding of the germ into $(\mathbb{C}^3, 0)$ such that the equation of the germ is $xy - t^k = 0, k \geq 1$. In fact, let

$$\begin{array}{ccc} \{xy - s = 0\} & \to & \Delta \\ (x, y, s) & \mapsto & s \end{array}$$

be the semiuniversal family of a node. Then locally around P, $\mathcal{C} \to \Delta$ is just the pull-back by

$$\begin{array}{ccc} \Delta & \to & \Delta \\ t & \mapsto & s = t^k \end{array}$$

where $k \geq 1$.

By Cartan's lemma, we can assume the action of G_P is given by

$$\tau\colon (x, y, t) \mapsto (a_1 x + a_2 y, b_1 x + b_2 y, t).$$

for any $\tau \in G_P$. Since G_P acts on the central fibre $\mathcal{C}_0 : xy = 0$, it is easy to see that

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} \xi(\tau) & 0 \\ 0 & \xi(\tau)^{-1} \end{pmatrix} \text{or} \begin{pmatrix} 0 & \eta(\tau) \\ \eta(\tau)^{-1} & 0 \end{pmatrix}$$

where $\xi(\tau)$ is a primitive $|\tau|$-th root of unity and $\eta(\tau)$ is some non-zero number. So we have an exact sequence of groups

$$1 \to H \to G_P \xrightarrow{\pi} \mathbb{Z}_2$$

where for any $\tau \in G_P$,

$$\pi(\tau) := det(\tau) = \begin{cases} 1, & \text{if } \tau \text{ does not interchange the branches at } P, \\ -1, & \text{if } \tau \text{ interchanges the branches at } P. \end{cases}$$

and $H := \mathrm{Ker}(\pi)$. Note that H consists of $\tau \in G_P$ whose action is given by

$$\begin{pmatrix} \xi(\tau) & 0 \\ 0 & \xi(\tau)^{-1} \end{pmatrix}.$$

So H embeds into $\mathbb{C}^*$:

$$\begin{array}{ccc} H & \to & \mathbb{C}^* \\ \tau & \mapsto & \xi(\tau), \end{array}$$

which implies that H is cyclic. Let τ_0 be a generator of H.

If $\mathrm{im}(\pi) = \{1\}$, then $G_P = H$ is cyclic.

If $\mathrm{im}\pi = \mathbb{Z}_2$, then there exists $\tau_1 \in G_P$ such that

$$\tau_1(x, y, t) = (\eta(\tau_1) y, \eta(\tau_1)^{-1} x, t).$$

Obviously $\tau_1^2 = 1$ and $\tau_1 \cdot \tau_0 \cdot \tau_1^{-1}(x, y, t) = (\xi(\tau_0)^{-1}x, \xi(\tau_0)y) = \tau_0^{-1}(x, y, t)$. So if $\mathrm{im}\pi = \mathbb{Z}_2$,

$$G_P = \begin{cases} \text{a dihedral group}, & \text{if } |H| \geq 2; \\ \mathbb{Z}_2, & \text{if } |H| = 1. \end{cases}$$

□

Lemma 3.8. *Let $P \in C$ be a node. Suppose $1 \neq \tau \in G_P$: then τ fixes points on $\mathcal{C}_t$ near P, for $t \neq 0$ if and only if $\pi(\tau) = -1$, where $\pi\colon G_P \to \mathbb{Z}_2$ is as in the proof of the previous lemma.*

Proof. We adopt the notation in Lemma 3.7, so the germ of $\mathcal{C}$ around P is defined by $xy - t^k = 0, k \geq 1$ and the action of G_P is a linear one.

Let $\tau \in G_P$. If $\pi(\tau) = 1$, then

$$\tau(x, y, t) = (\xi(\tau)x, \xi(\tau)^{-1}y, t)$$

where $\xi(\tau) \in \mathbb{C}^*$ is a $|\tau|$-th primitive root of unity. Suppose $(x, y, t) \in \mathcal{C}$ is a fixed point of τ. Then

$$\tau(x, y, t) = (\xi(\tau)x, \xi(\tau)^{-1}y, t) = (x, y, t) \Rightarrow x = y = 0.$$

and $xy = t^k$ implies $t = 0$. So τ fixes only $(0, 0, 0)$ on $\mathcal{C}$ and does not fix any point on $\mathcal{C}_t$ for $t \neq 0$.

On the other hand, if $\pi(\tau) = -1$, then $\tau(x, y, t) = (\eta(\tau)y, \eta(\tau)^{-1}x, t)$ with $\eta(\tau) \in \mathbb{C}^*$. So $\tau(x, y, t) = (x, y, t)$ if and only if $\eta(\tau)y = x$. Taking the equation $xy = t^k$ into consideration, τ fixes 2 points: $(\eta(\tau)\sqrt{\frac{t^k}{\eta(\tau)}}, t)$ and $(-\eta(\tau)\sqrt{\frac{t^k}{\eta(\tau)}}, t)$ on $\mathcal{C}_t$, for $t \neq 0$. □

Now we can state our main theorem in this section:

Theorem 3.9. *A pair (C,G) admits a smoothing if and only if for any node $P \in C$, we can find local (analytic) embedding of $C\colon (xy = 0) \subset \mathbb{C}^2$ such that, for any $\tau \in G_P$, the action of τ is given by either*

(i) $(x, y) \mapsto (\xi(\tau)x, \xi(\tau)^{-1}y)$ *where $\xi(\tau)$ is a $|\tau|$-th root of unity; or*

(ii) $(x, y) \mapsto (\eta(\tau)y, \eta(\tau)^{-1}x)$ *where $\eta(\tau) \in \mathbb{C}^*$ is a nonzero number.*

Proof. The "only if"part is shown in the proof of Lemma 3.7.

For the "if "part, we divide the proof into two steps.

Step 1: The germ $P \in C$ has a local G-equivariant smoothing. More precisely, let $U \subset C$ be a neighborhood around P defined by $xy = 0 \subset \mathbb{C}^2$ as in the hypothesis, we will show that the pair (U, G_P) is G_P-smoothable. In fact we can consider the family $\mathcal{U} : (xy - s = 0 \subset \mathbb{C}^2 \times \Delta) \to \Delta$ with $s \in \Delta$ as the parameter. For any $\tau \in G_P$,

$$\tau(x, y) = (\xi(\tau)x, \xi(\tau)^{-1}y) \text{ or } (\eta(\tau)y, \eta(\tau)^{-1}x),$$

and it is easily seen that the action of G_P on U extends to the family $\mathcal{U} \to \Delta$.

Note that $\mathcal{U} \to \Delta$ is the semiuniversal deformation of the node $P \in U$ and the tangent space of the base space at 0 is

$$\mathcal{E}xt^1_{\mathcal{O}_U}(\Omega_U, \mathcal{O}_U) \cong T_0\Delta \cong \mathbb{C}.$$

The fact that (U, G_P) is smoothable means exactly that $\mathcal{E}xt^1_{\mathcal{O}_U}(\Omega_U, \mathcal{O}_U)$ is G_P-invariant.

Step 2: We will use the local-to-global exact sequence

$$0 \to H^1(C, \mathcal{T}_C) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \xrightarrow{\pi} H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) \to 0$$

to prove that local smoothings of nodes with stabilizers lift to smoothings of (C, G). To do this, first note that

$$H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_{P \text{ node}} \mathrm{Ext}^1_{\mathcal{O}_{C,P}}(\Omega_{C,P}, \mathcal{O}_{C,P}), \tag{3.1}$$

where, for any coherent sheaf $\mathcal{F}$ on C, $\mathcal{F}_P$ denotes the stalk of $\mathcal{F}$ at P. For any $\tau \in G$, τ acts on $H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$ and maps the $\mathrm{Ext}^1_{\mathcal{O}_{C,P}}(\Omega_{C,P}, \mathcal{O}_{C,P})$ summand isomorphically to the $\mathrm{Ext}^1_{\mathcal{O}_{C,\tau(P)}}(\Omega_{C,\tau(P)}, \mathcal{O}_{C,\tau(P)})$ summand.

Let $n(P) := |G/G_P|$ and $\tau_1, \cdots, \tau_{n(P)} \in G$ representatives of elements of G/G_P. Then $\tau_1(P), \cdots, \tau_{n(P)}(P)$ is the orbit of P under the action of G. And G acts on the vector space

$$V_P := \bigoplus_{j=1}^{n(P)} \mathrm{Ext}^1_{\mathcal{O}_{C,\tau_j(P)}}(\Omega_{C,\tau_j(P)}, \mathcal{O}_{C,\tau_j(P)}).$$

The invariant subspace V_P^G is 1-dimensional, spanned by

$$(\tau_1(\sigma), \cdots, \tau_{n(P)}(\sigma))$$

where σ is an element spanning $\mathrm{Ext}^1_{\mathcal{O}_{C,P}}(\Omega_{C,P}, \mathcal{O}_{C,P}) \cong \mathbb{C}$. In view of (3.1), the dimension of $H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C))$ is exactly the number of node orbits under the action of G. Taking the G-invariants of the local-to-global sequence, we get

$$0 \to H^1(C, \mathcal{T}_C)^G \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)^G \xrightarrow{\pi} H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))^G \to 0.$$

In particular, there exists $\lambda \in \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)^G$ such that the $\pi(\lambda)$'s P-summand is nonzero for any node $P \in C$. Then λ gives a smoothing of (C, G). $\square$

3.2 Singularities of degenerations of surfaces isogenous to a product of unmixed type

We can study smoothings of a product of two stable curves with a group action in a similar way as the G-equivariant smoothings of curves in Section 3.1.

We treat the unmixed case first. Let C, D be two stable curves. Assume a finite G acts (faithfully) on C, D and acts diagonally on $C \times D$. Suppose $(C \times D, G)$ admits a free smoothing $(\mathcal{C} \times_\Delta \mathcal{D}, G) \to \Delta$, where $\mathcal{C} \to \Delta$ and $\mathcal{D} \to \Delta$ are smoothings of C, D respectively (cf. the discussion after Theorem 1.32). Let $(P, Q) \in C \times D$, and $G_{(P,Q)}$ the stabilizer as usual. First observe that

$$\begin{aligned} \tau \in G_{(P,Q)} &\Leftrightarrow \tau(P,Q) = (P,Q) \\ &\Leftrightarrow (\tau P, \tau Q) = (P,Q) \\ &\Leftrightarrow \tau P = P, \tau Q = Q \\ &\Leftrightarrow \tau \in G_P \cap G_Q, \end{aligned}$$

hence $G_{(P,Q)} = G_P \cap G_Q$.

Proposition 3.10 (Criterion for free smoothings in the unmixed case). *Let C, D be two stable curves and let G be a finite group acting on C and D. Let G act on $Z := C \times D$ diagonally. Then the pair (Z, G) admits a free smoothing if and only if for any $(P, Q) \in C \times D$, we have one of the following:*

(U_0) *if both P, Q are smooth points on C, D respectively, then $G_{(P,Q)} = \{1\}$.*

(U_1) *if one of P, Q, say P, is a node and the other is a smooth point, then $G_{(P,Q)} = \langle\tau\rangle$ is cyclic, and we can find a local embedding of $C : (xy = 0) \subset \mathbb{C}^2$ as well as a local coordinate z of D such that $\tau(x, y, z) = (\xi x, \xi^{-1} y, \xi^q z)$, where ξ is a primitive root of unity of order $|\tau|$ and $(q, |\tau|) = 1$.*

(U_2) *if both P, Q are nodes of respective curves, then $G_{(P,Q)} = \langle\tau\rangle$ is cyclic and τ interchanges the branches of at most one of C and D. In this case, we have one of the following*

- (U_{2a}) $G_{(P,Q)} = \{1\}$.
- (U_{2b}) *if τ does interchange the branches of C or D, say C, then the order of τ is 2 and we can choose local embeddings $C : (xy = 0) \subset \mathbb{C}^2$ and $D : (zw = 0) \subset \mathbb{C}^2$ such that $\tau(x, y, z, w) = (y, x, -z, -w)$.*

(U_{2c}) *if τ does not interchange any branches of C, D, then we can choose local embeddings $C : (xy = 0) \subset \mathbb{C}^2$ and $D : (zw = 0) \subset \mathbb{C}^2$ such that $\tau(x, y, z, w) = (\xi x, \xi^{-1} y, \xi^q z, \xi^{-q} w)$, where ξ is a primitive root of unity of order $|\tau|$ and $(q, |\tau|) = 1$.*

Proof. For the " $\Rightarrow$ " direction, suppose $\mathcal{Z} = \mathcal{C} \times_\Delta \mathcal{D} \to \Delta$ is a free smoothing of $(C \times D, G)$. Let (P, Q) be any point on $C \times D$. We divide our further discussion into 3 cases:

(U_0) P, Q are both smooth points on C, D respectively.

We will show in this case that $G_{(P,Q)} = \{1\}$. Suppose $1 \neq \tau \in G_{(P,Q)}$: then P, Q are both fixed points of τ. Since P, Q are both smooth points on C, D respectively, τ fixes points on $\mathcal{C}_t, \mathcal{D}_t$ for $t \neq 0$ by Lemma 3.6. So τ fixes points on $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$, which contradicts the assumption that G acts freely on $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$.

(U_1) One of P, Q, say P, is a node and the other is a smooth point.

Suppose $G_{(P,Q)} \neq \{1\}$. By Lemma 3.5, G_Q is cyclic and hence its subgroup $G_{(P,Q)}$ is also cyclic. Let $G_{(P,Q)} = \langle \tau \rangle, \tau \neq 1$. By Lemma 3.6, τ fixes points of $\mathcal{D}_t$ for $t \neq 0$. Since G acts freely on $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$, τ does not fix any point of $\mathcal{C}_t, t \neq 0$. By Lemma 3.8, τ does not interchange the (analytic) branches of C around P. Hence there are local embeddings $C : (xy = 0) \subset \mathbb{C}^2$ with $P = (0, 0)$ and local coordinate z of D around Q such that the action of τ on $C \times D$ is

$$(x, y, z) \mapsto (\xi(\tau) x, \xi(\tau)^{-1} y, \xi(\tau)^q z)$$

where $\xi(\tau)$ is a primitive root of unity of order $|\tau|$ and $(q, |\tau|) = 1$.

(U_2) P, Q are both nodes on C, D.

Suppose $G_{(P,Q)} \neq \{1\}$. By Lemma 3.7, we have that G_P and G_Q are either cyclic or dihedral. This implies that $G_{(P,Q)} = G_P \cap G_Q$ is either cyclic or dihedral. Suppose $G_{(P,Q)}$ is dihedral. Then G_P and G_Q are both dihedral and, by the proof of Lemma 3.7, there is $\tau_1 \in G_{(P,Q)}$ (resp. $\tau_2 \in G_{(P,Q)}$) such that τ_1 (resp. τ_2) interchanges the branches of C at P (resp. the branches of D at Q). By Lemma 3.8, τ_1 (resp. τ_2) fixes points of $\mathcal{C}_t$ (resp. $\mathcal{D}_t$) for $t \neq 0$. Since neither τ_1 nor τ_2 fixes points on $\mathcal{C}_t \times \mathcal{D}_t$, τ_1 (resp. τ_2) does not fix points on $\mathcal{D}_t$ (resp. $\mathcal{C}_t$). Again by Lemma 3.8, τ_1 (resp. τ_2) does not interchange the branches of D at Q (resp. the branches of C at P). Now set $\tau := \tau_1 \tau_2$, then τ interchanges the branches of C as well as those of D. This implies that τ fixes points on $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$, a contradiction.

So $G_{(P,Q)}$ is cyclic and we can assume that $G_{(P,Q)} = \langle \tau \rangle$. Since τ does not fix any point on $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$, τ interchanges the branches of at most one of C and D. If τ does interchange the branches of one of C and D, say C, then the order of τ is 2 (Lemma 3.7) and we can choose local embeddings $C\colon (xy = 0) \subset \mathbb{C}^2$ and $D\colon (zw = 0) \subset \mathbb{C}^2$ such that τ acts as

$$(x, y, z, w) \to (y, x, -z, -w).$$

If τ does not interchange any branches of C, D, then we can choose local embeddings $C : (xy = 0) \subset \mathbb{C}^2, D : (zw = 0) \subset \mathbb{C}^2$ such that

$$\tau(x, y, z, w) = (\xi(\tau)x, \xi(\tau)^{-1}y, \xi(\tau)^q z, \xi(\tau)^{-q} w)$$

where $\xi(\tau)$ is a primitive root of unity of order $|\tau|$ and $(q, |\tau|) = 1$.

For the other direction, note that C and D admit G-equivariant smoothings $\mathcal{C} \to \Delta, \mathcal{D} \to \Delta$ by Theorem 3.9. In each of the cases (U_0), (U_1), (U_2), any non-trivial element $\tau^k \in G_{(P,Q)} = \langle \tau \rangle$ interchanges at most the local branches of one of the factors. This guarantees that τ^k acts locally freely on at least one of the factors of $\mathcal{C}_t \times \mathcal{D}_t$ for $t \neq 0$ (Lemma 3.8). So $\mathcal{Z} := \mathcal{C} \times_\Delta \mathcal{D} \to \Delta$ is a required free smoothing. $\square$

Remark 3.11. In the unmixed case, $G_{(P,Q)}$ is always cyclic.

According to Theorem 1.32 and the discussion thereafter, a surface X is a stable degeneration of surfaces isogenous to a product of unmixed type if and only if $X = Z/G$ where $Z := C \times D$ is a product of 2 stable curves and G is a finite group acting diagonally on Z such that (Z, G) admits a free smoothing. So we have

Corollary 3.12. *The possible singularities of a surface X which is a stable degeneration of surfaces isogenous to a product of unmixed type are as follows:*

(U_{1a}) *Normal crossing singularities: $(xy = 0) \subset \mathbb{C}^3$. These are the general singularities of X.*

(U_{1b}) *Quotients of the above singularities under the group action:*

$$(x, y, z) \mapsto (\xi x, \xi^{-1} y, \xi^q z)$$

where ξ is a primitive n-th root of unity, $(q, n) = 1$. In this case, the index of the singularity is n and the canonical covering is a singularity of type (U_{1a}).

(U_{2a}) *The degenerate cusp: $(xy = 0, zw = 0) \subset \mathbb{C}^4$.*

(U_{2b}) *A $\mathbb{Z}_2$-quotient of the degenerate cusp in (U_{2a}) under the group action:*

$$(x, y, z, w) \mapsto (y, x, -z, -w)$$

In this case, the index of the singularity is 2 and the canonical covering is the degenerate cusp in (U_{2a}).

(U_{2c}) *Other quotients of the degenerate cusp in U2a) under the group action:*

$$(x, y, z, w) \mapsto (\xi x, \xi^{-1} y, \xi^q z, \xi^{-q} w),$$

where ξ is a primitive n-th root of unity, $(q, n) = 1$. In this case, the singularity is still a (Gorenstein) degenerate cusp.

We give some examples of singularities in Corollary 3.12.

Example 3.13. Let $G = \langle \sigma \rangle \cong \mathbb{Z}_2$. Let C, D' be two hyperelliptic curves. Suppose σ acts on C and D' as the respective hyperelliptic involutions. Let $\{Q'_1, \ldots, Q'_{2k}\}$ be the fixed points of σ on D'. We obtain a stable curve D from D' by identifying Q'_{2i-1} and Q'_{2i} for any $1 \le i \le k$. Note that σ also acts on D. Let G act on $C \times D$ diagonally. Then the quotient $(C \times D)/G$ has singularities of type (U_{1a}) or (U_{1b}).

Example 3.14. Let C, D be two stable curves. Let G be a finite group acting freely on $C \times D$. Then $(C \times D)/G$ has singularities of type (U_{1a}) or (U_{2a}) (Proposition 1.28).

Example 3.15. Let $G = \langle \sigma \rangle \cong \mathbb{Z}_2$. Let C' and D' be two smooth curves of genera ≥ 1 such that G acts (faithfully) on both. Assume σ fixes $2k$ points $P'_1, P'_2, \cdots, P'_{2k}$ on C'. Let C be the stable curve obtained by identifying P'_{2i-1} and P'_{2i} for $1 \le i \le k$. Denote by P_i the image on C of P'_{2i-1} and P'_{2i} for $1 \le i \le k$. Then σ acts on C and $P_1, \cdots, P_k$ are the fixed points. Note that σ does not interchange the local branches of C around P_i for any $1 \le i \le k$.

Assume σ acts freely on D'. Pick a point $Q' \in D'$. Let D be the stable curve obtained by identifying Q' and $\sigma(Q')$. Denote by Q the image on D of Q' and $\sigma(Q')$. Then σ acts on D and Q is the only fixed point. Moreover σ interchanges the local branches of D around Q.

Now let G acts on $C \times D$ diagonally. Then $(P_1, Q), \cdots, (P_k, Q)$ are the fixed points and the quotient $(C \times D)/G$ only has singularities of type (U_{1a}) or (U_{2b}).

Example 3.16. Let $G = \langle \sigma \rangle \cong \mathbb{Z}_2$. Let C' and D' be two smooth curves of genera ≥ 1 such that G acts (faithfully) on both. Assume σ fixes $2k$ points $P'_1, P'_2, \cdots, P'_{2k}$ on C'. We obtain a stable curve C from C' by identifying P'_{2i-1} and P'_{2i} for $1 \le i \le k$. Similarly, we can obtain a stable curve D from D'. Note that σ also acts on C and D. Let G acts on $C \times D$ diagonally. Then the quotient $(C \times D)/G$ has singularities of type (U_{1a}) or (U_{2c}).

Remark 3.17. In general, we can consider the singular locus X_{sing} of a surface stably isogenous to a product $X = (C \times D)/G$ as in Definition 1.31. Suppose $\{P_1, \cdots, P_{m_1}\}$ (resp. $\{Q_1, \cdots, Q_{m_2}\}$) is the set of nodes of C (resp. D). Then the singular locus of Z is

$$Z_{sing} = \Delta_1 \cup \Delta_2$$

where $\Delta_1 = C \times \{Q_1, \cdots, Q_{m_2}\}$ and $\Delta_2 = \{P_1, \cdots, P_{m_1}\} \times D$. And the singular locus of X is

$$X_{sing} = \pi(\Delta_1) \cup \pi(\Delta_2) = \pi(Z_{sing})$$

where $\pi : Z \to X$ is the quotient map. Note that Z_{sing} is a one-dimensional variety and G has an action on Z_{sing}. We have $X_{sing} = Z_{sing}/G$. Since X is of unmixed type, $\pi(\Delta_1)$ and $\pi(\Delta_2)$ do not have common components. Note that X always has singularities of type (U_{1a}), i.e., normal crossing singularities.

3.3 Singularities of degenerations of surfaces isogenous to a product of mixed type

Now we consider the mixed case.

Proposition 3.18 (Criterion for free smoothings in the mixed case)**.** *Let C be a stable curve and $G^\circ < Aut(C)$ a finite group. Let*

$$1 \to G^\circ \to G \to \mathbb{Z}_2 \to 1,$$

be a non-split extension, yielding a class $[\varphi]$ in $Out(G^\circ) = Aut(G^\circ)/Int(G^\circ)$, which is of order ≤ 2. Fix a representative φ of the above class. Suppose there exists an element $\tau' \in G \setminus G^\circ$ such that

(I) $\varphi(\gamma) = \tau'\gamma\tau'^{-1}$,

(II) *G acts on $Z := C \times C$ by the formulae: $\gamma(P, Q) = (\gamma P, (\varphi\gamma)Q)$ for γ in G°; whereas the lateral class of G° consists of the transformations*

$$\tau'\gamma(P, Q) = ((\varphi\gamma)Q, \tau\gamma P),$$

where $\tau := \tau'^2 \in G^\circ$.

Then (Z, G) admits a free smoothing $\mathcal{Z} = \mathcal{C} \times_\Delta \mathcal{C} \to \Delta$ if and only if the following hold:

(i) *The pair (Z, G°) satisfies one of the properties (U_0), (U_1), (U_2) for any point on Z, as described in Proposition 3.10.*

(ii) *There are only finitely many points with nontrivial stabilizers on $C \times C$.*

(iii) *If $(P, Q) \in C \times C$ is such that $G_{(P,Q)} \nsubseteq G^\circ$, then P, Q are both nodes on C.*

Proof. "$\Rightarrow$ "If (Z, G) admits a free smoothing $\mathcal{Z} \to \Delta$, then $\mathcal{Z} \to \Delta$ is also a free smoothing of (Z, G°). Hence (Z, G°) satisfies (U_0), (U_1), (U_2) in Proposition 3.10.

Let Γ be the subset of G° consisting of the transformations having fixed points on $\mathcal{C}_t$ for $t \neq 0$. Since $\mathcal{C} \times_\Delta \mathcal{C} \to \Delta$ is a free smoothing of $C \times C$, we have

(A) $\Gamma \cap \varphi(\Gamma) = \{1\}$.

(B) there is no γ in G° such that $\varphi(\gamma)\tau\gamma$ is in Γ. In particular, $\varphi(\gamma)\tau\gamma \neq 1$.

The above two conditions simply say that G acts freely on $\mathcal{C}_t \times \mathcal{C}_t$ for $t \neq 0$ (cf. Proposition 1.8).

If there were infinitely many points with non-trivial stabilizers on $C \times C$, then some $1 \neq \sigma \in G$ fixes infinitely many points. If $\sigma \in G^\circ$, then σ acts on C and σ must fix a component of C. Now Lemma 3.4 implies that $\sigma = 1$, which is a contradiction. Hence $\sigma = \tau'\gamma \in G \setminus G^\circ$ for some $\gamma \in G^\circ$. Since $\sigma^2 \in G^\circ$ also fixes infinitely many points, we have $\varphi(\gamma)\tau\gamma = \sigma^2 = 1$ by Lemma 3.4 again, which contradicts (B) above. This proves (ii).

For (iii), we discuss the possible stabilizer of a point $(P, Q) \in C \times C$ in the following 2 cases.

(M0) P, Q are both smooth points of C.

We will show $G_{(P,Q)} = \{1\}$ in this case. Since G° acts freely on $\mathcal{C}_t \times \mathcal{C}_t$ for $t \neq 0$. We see that $G^\circ \cap G_{(P,Q)} = \{1\}$ by the claim for the unmixed (U_0) case. Suppose on the contrary that there is a $1 \neq \tau_1 \in G_{(P,Q)}$, then $\tau_1 = \tau'\gamma \in G \setminus G^\circ$ for some $\gamma \in G^\circ$. Now $\tau_1^2 \in G^\circ \cap G_{(P,Q)}$ implies that $\tau_1^2 = 1$ and hence

$$\tau'\gamma\tau'\gamma = 1 \Rightarrow (\tau'\gamma\tau'^{-1})\tau'^2\gamma = 1 \Rightarrow \varphi(\gamma)\tau\gamma = 1.$$

This contradicts (B) above.

(M1) One of P, Q is a node, while the other is a smooth point.

We will show that $G_{(P,Q)} \subset G^\circ$ in this case. Otherwise (P, Q) is fixed by $\tau'\gamma \in G \setminus G^\circ$ for some $\gamma \in G^\circ$, i.e.,

$$(P, Q) = \tau'\gamma(P, Q) = ((\varphi\gamma)Q, \tau\gamma P),$$

so $P = (\varphi\gamma)Q, Q = \tau\gamma P$. In particular, either P, Q are both nodes or they are both smooth points of C, a contradiction.

Hence (iii) follows.

"$\Leftarrow$ "By Theorem 3.8, condition (i) implies that (C, G°) has a free smoothing $\mathcal{C} \to \Delta$. Set $\mathcal{Z} := \mathcal{C} \times_\Delta \mathcal{C} \to \Delta$. We can introduce an action of G on $\mathcal{Z} \to \Delta$ by the formulae in (II): for any $(P, Q) \in \mathcal{C}_t \times \mathcal{C}_t, \gamma(P, Q) = (\gamma P, (\varphi\gamma)Q)$ if $\gamma \in G^\circ$; whereas for $\tau'\gamma \in G \setminus G^\circ$,

$$\tau'\gamma(P, Q) = ((\varphi\gamma)Q, \tau\gamma P).$$

It remains to check that G acts freely on $\mathcal{Z}_t$ for $t \neq 0$. Note that G° acts freely by hypothesis (i) and Proposition 3.10. Now let $\tau'\gamma \in G \setminus G^\circ$ for some $\gamma \in G^\circ$. If $\tau'\gamma$ does not fix points on $C \times C$, then obviously $\tau'\gamma$ does not fix points on $\mathcal{C}_t \times \mathcal{C}_t$ for $t \neq 0$. If $\tau'\gamma \in G_{(P,Q)}$ for some $(P, Q) \in C \times C$, then both P, Q must be nodes by (iii) and we can find local embeddings of the first factor: $xy = t^n$ (resp. of the second factor: $zw = t^m$) such that the action of $\tau'\gamma$ around (P, Q) is given by:

$$(x, y, z, w, t) \mapsto (az, bw, x, y, t)$$

where $a, b \in \mathbb{C}^*$ are nonzero numbers.

Hence

$$(\tau'\gamma)^2(x, y, z, w, t) = (ax, by, az, bw, t).$$

If $t \neq 0$, then $xy = t^n, zw = t^m$ implies that $xyzw \neq 0$. Suppose $\tau'\gamma$ fixes some $(x, y, z, w, t) \in \mathcal{C}_t \times \mathcal{C}_t$ for $t \neq 0$, then

$$az = x, bw = y, x = z, y = w$$

and this implies that

$$a = 1, b = 1.$$

So $(\tau'\gamma)^2 = 1$. Now, for any $(P', Q') \in \mathcal{Z}_t$, $(\tau'\gamma)^2(P', Q') = ((\varphi\gamma)\tau\gamma P', \tau\gamma(\varphi\gamma)Q')$, so we have $\tau\gamma(\varphi\gamma) = (\varphi\gamma)\tau\gamma = 1$. This implies that

$$\{((\varphi\gamma)Q', Q') | Q' \in \mathcal{C}_0\} \subset C \times C$$

is fixed by $\tau'\gamma$, which contradicts hypothesis (ii) that there are only finitely many points with nontrivial stabilizers.

This contradiction shows that $\tau'\gamma$ does not fix any points on $\mathcal{C}_t \times \mathcal{C}_t, t \neq 0$ and hence G acts freely on $\mathcal{C}_t \times \mathcal{C}_t$ for $t \neq 0$. $\square$

Remark 3.19. Let the notation be as in Proposition 3.18. Then the statement (ii) in the proposition is equivalent to the assertion that any $\tau'\gamma \in G \setminus G^\circ$ has order > 2. This equivalence can be proved as follows: if there are infinitely many points with nontrivial stabilizers, then some $1 \neq \sigma \in G$ fixes infinitely many points. Note that $\sigma \in G \setminus G^\circ$ by Lemma 3.4. Since $\sigma^2 \in G^\circ$ also fixes infinitely many points, we have $\sigma^2 = 1$ by Lemma 3.4 again. On the other hand, suppose $\sigma = \tau'\gamma \in G \setminus G^\circ$ is of order 2. Then $(\varphi\gamma)\tau\gamma = \tau\gamma(\varphi\gamma) = 1$ and σ fixes every point on the curve $\{((\varphi\gamma)Q, Q)|\ Q \in C\}$.

According to Theorem 1.32, a surface X is a stable degeneration of surfaces isogenous to a product of mixed type if and only if $X = Z/G$ where $Z = C\times C$ is a product of two identical stable curves and G is a finite group acting in the way described in Proposition 3.18.

Corollary 3.20. *The possible singularities of a surface X which is a stable degeneration of surfaces isogenous to a product of mixed type are as follows:*

- (U) *The singularities of type (U_{1a}), (U_{1b}), (U_{2a}), (U_{2b}), (U_{2c}) occurring on a stable degeneration of surfaces isogenous to a product of unmixed type (see Corollary 3.12).*
- (M) *A quotient of the degenerate cusp of type (U_{2a}) under an action of automorphisms τ_1 and τ_2:*

$$\begin{aligned}\tau_1\colon (x,y,z,w) &\mapsto (\xi x, \xi^{-1}y, \xi^{q}z, \xi^{-q}w),\\ \tau_2\colon (x,y,z,w) &\mapsto (az, a^{-1}w, bx, b^{-1}y),\end{aligned}$$

where ξ is a primitive n-th root of unity, $(q,n)=1$ and $ab \in \langle\xi\rangle \setminus \langle \xi^{q+1}\rangle$. In this case, the index of the singularity is 2 and the canonical covering is the singularity of type (U_{2c}).

Proof. Let (Z,G) be as in Proposition 3.18 such that $X = Z/G$. If $(P,Q)\in Z = C\times C$ is such that $G_{(P,Q)} \subset G^\circ$, then the singularity $\pi(P,Q)\in X$ is of type (U_{1a}), (U_{1b}), (U_{2a}), (U_{2b}) or (U_{2c}), where $\pi\colon Z\to X$ is the quotient map.

If $(P,Q)\in Z = C\times C$ is such that $G_{(P,Q)} \not\subseteq G^\circ$, then P,Q are both nodes of C. We want to know the action of $G_{(P,Q)}$ around (P,Q). Note that $(C\times C, G^\circ)$ is of unmixed type and $G^\circ\cap G_{(P,Q)}$ is just the stabilizer of $(P,Q)\in C\times C$ under the action of G°. By the analysis done for the unmixed type, $G^\circ\cap G_{(P,Q)} = \langle\tau_1\rangle$ for some $\tau_1\in G^\circ$ and τ_1 interchanges at most the branches of one factors of $C\times C$. We will show that τ_1 does not interchange any branches at P or Q.

By assumption there is a $\gamma\in G^\circ$ such that $\sigma := \tau'\gamma \in G_{(P,Q)}$. Suppose τ_1 interchanges the branches at one of P,Q, say P, then $|\tau_1| = 2$ (Propostion 3.10, Case (U_{2b}). Note that $(\tau'\gamma)^2 \in G_{(P,Q)}\cap G^\circ = \langle\tau_1\rangle$ and for any $(P',Q')\in C\times C$,

$$(\tau'\gamma)^2(P',Q') = (\tau'\gamma)((\varphi\gamma)Q', \tau\gamma P') = ((\varphi\gamma)\tau\gamma P', \tau\gamma(\varphi\gamma)Q'),$$

so $(\varphi\gamma)\tau\gamma\in\langle\tau_1\rangle$. By condition (B) in the proof of Proposition 3.18, $(\varphi\gamma)\tau\gamma\neq 1$. On the other hand $|\tau_1| = 2$, so $(\varphi\gamma)\tau\gamma = \tau_1$. Since $\tau'\gamma\in G_{(P,Q)}$, we have $\tau'\gamma(P,Q) = (P,Q)$, i.e., $(\varphi\gamma)Q = P$ and $(\tau\gamma)P = Q$. Now the fact that τ_1 interchanges the branches of C at P implies that $\tau\gamma(\varphi\gamma) = \tau\gamma\tau_1(\tau\gamma)^{-1}$ interchanges the branches of the first factor C at Q. Since τ_1 acts on the second factor of $C \times C$ via $\tau\gamma(\varphi\gamma)$, τ_1 also interchanges the branches of the second factor C at Q, a contradiction.

So the actions of τ_1, σ are of the form

$$\begin{aligned} \tau_1 &\colon (x,y,z,w) \mapsto (\xi x, \xi^{-1} y, \xi^q z, \xi^{-q} w), \\ \sigma &\colon (x,y,z,w) \mapsto (az, a^{-1} w, bx, b^{-1} y), \end{aligned}$$

where $C : (xy = 0) \subset \mathbb{C}^2$ and $C : (zw = 0) \subset \mathbb{C}^2$ are suitable local embeddings of C around P, Q and ξ is a primitive n-th root of unity with $n = |\tau_1|$, $(q,n) = 1$. Since $\tau_2 \in \langle \tau_1 \rangle$ and $(\tau_1^k \tau_2)^2 \neq 1$ for any k (cf. Remark 3.19), we can easily see that $ab \in \langle \xi \rangle \setminus \langle \xi^{q+1} \rangle$. Let $\pi\colon Z \to X$ be the quotient map. Then the singularity $\pi(P,Q) \in X$ is of type (M). $\square$

We give an example of singularity of type (M).

Example 3.21. Let $G = \langle \sigma \rangle \cong \mathbb{Z}_4$. Then $\tau_1 := \sigma^2$ has order 2. Let C' be a smooth curve of genus ≥ 2. Suppose τ_1 acts on C' so that there are exactly two fixed points P_1' and P_2'. We obtain a stable curve C from C' by identifying P_1' and P_2'. Let $P \in C$ denote the image of P_1' and P_2'. Then τ_1 also acts on C and has exactly one fixed point P.

We can give an action of G on $C \times C$ as follows:

$$\sigma(P_1, P_2) := (P_2, \tau_1 P_1), \text{ for any point } (P_1, P_2) \in C \times C.$$

Then $\tau_1(P_1, P_2) = (\tau_1 P_1, \tau_1 P_2)$ and $\sigma\tau_1(P_1, P_2) = (\tau_1 P_2, P_1)$. It is easy to see that $(P,P) \in C \times C$ is the only point with a nontrivial stabilizer which is G. The quotient $(C \times C)/G$ has singularities of type (U_{1a}) or (M).

Chapter 4

Connected components of the moduli space

In this chapter we will study the $\mathbb{Q}$-Gorenstein deformations of surfaces stably isogenous to a product. As a result, we get some connected components of the moduli space of stable surfaces $M^{st}_{a,b}$ defined in Section 1.5. We use the $\mathbb{Q}$-Gorenstein deformation theory which is carefully recalled in Section 1.4.

Let $Z := C \times D$ be a product of stable curves and $X = Z/G$ a surface stably isogenous to a product. Let $\pi\colon Z \to X$ be the quotient map. For any G-equivariant $\mathcal{O}_Z$-module $\mathcal{F}$, we define an $\mathcal{O}_X$-module $\pi^G_*\mathcal{F} := (\pi_*\mathcal{F})^G$. Note that both π_* and π^G_* are exact functors from the category of G-equivariant $\mathcal{O}_Z$-modules to the category of $\mathcal{O}_X$-modules.

Lemma 4.1. *Let $\mathcal{F}$ be a G-equivariant $\mathcal{O}_Z$-module. Then for any $p \geq 0$, we have $H^p(Z, \mathcal{F})^G = H^p(X, \pi^G_*\mathcal{F})$.*

Proof. Let

$$\begin{array}{rcl} (\cdot)^G\colon & \{G\text{-vector spaces}\} & \to \{\text{vector spaces}\} \\ & V & \to V^G \end{array}$$

be the (exact) functor taking G-invariants. Let Γ be the global section functor from the category of $\mathcal{O}_X$-modules to the category of vector spaces. We can define a composite functor

$$F := (\cdot)^G \circ (\Gamma \circ \pi_*)\colon \{G\text{-equivariant } \mathcal{O}_Z\text{-modules}\} \to \{\text{vector spaces}\}$$

Note that there is another decomposition of F into two functors: $F = \Gamma \circ \pi_*^G$. So there are two spectral sequences corresponding to the two decompositions:

$$H^p(G, H^q(X, \pi_* \mathcal{F})) \Rightarrow R^{p+q}F(\mathcal{F})$$
$$H^p(X, R^q\pi_*^G \mathcal{F}) \Rightarrow R^{p+q}F(\mathcal{F}), \text{ for all } p, q \geq 0.$$

Since $(\cdot)^G$ is an exact functor, $H^p(G, H^q(X, \pi_*\mathcal{F})) = 0,$ for all $p > 0$. Hence

$$R^pF(\mathcal{F}) = H^0(G, H^p(X, \pi_*\mathcal{F})) = H^p(X, \pi_*\mathcal{F})^G. \tag{4.1}$$

Similarly, since π_*^G is an exact functor, $R^p\pi_*^G\mathcal{F} = 0,$ for all $p > 0$. Hence

$$R^pF(\mathcal{F}) = H^p(X, \pi_*^G\mathcal{F}). \tag{4.2}$$

We also have

$$H^p(X, \pi_*\mathcal{F}) = H^p(Z, \mathcal{F}). \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we get, for any $p \geq 0$,

$$H^p(Z, \mathcal{F})^G = H^p(X, \pi_*^G\mathcal{F}), \text{ for any } G\text{-equivariant } \mathcal{O}_Z\text{-module } \mathcal{F}.$$

□

Lemma 4.2. *If all the (possible) singularities on X are of type (U_{1a}), (U_{1b}), (U_{2a}) or (U_{2b}), then*

$$\pi_*^G\mathcal{T}_Z^i = \mathcal{T}_{QG,X}^i, i = 0, 1, 2.$$

Proof. Let P be a point on X and let $\{Q_j\}_j = \pi^{-1}(P)$ be the inverse image of P. Let $Q \in \{Q_j\}_j$ be a point over P. By the description of singularity types given in Corollary 3.12 or 3.20, the germ $Q \in Z$ is the canonical covering of $P \in X$ with group G_Q, given that the singularity $P \in X$ is of type (U_{1a}), (U_{1b}), (U_{2a}) or (U_{2b}). Moreover G/G_Q acts transitively on the set of germs $\{Q_j \in Z\}_j$.

Let $\mathfrak{X}$ be the canonical covering stack of X. Then we have a morphism $\tilde{\pi}\colon Z \to \mathfrak{X}$ by the disccussion in the previous paragraph. Since $X = Z/G$, it is easy to see that $\mathfrak{X} = [Z/G]$ is the quotient stack and $\tilde{\pi}$ is an étale morphism. Now, as in the situation of a germ $P \in X$ (cf. Page 13), we have $\pi_*^G\mathcal{T}_Z^i = \mathcal{T}_{QG,X}^i$. □

Corollary 4.3. *Suppose all the (possible) singularities on X are of type (U_{1a}), (U_{1b}), (U_{2a}) or (U_{2b}). Then $\pi_*^G\mathcal{T}_Z = \mathcal{T}_X$ and $\pi_*^G\mathcal{T}_Z^1 = \mathcal{T}_{QG,X}^1$.*

Proof. This is part of the statement in Lemma 4.2, noting that $\mathcal{T}_Z = \mathcal{T}_Z^0$ and $\mathcal{T}_X = \mathcal{T}_X^0$.

We give an alternate proof here.

First observe that both $\pi_*^G\mathcal{T}_Z$ and $\mathcal{T}_X$ are S_2-sheaves of $\mathcal{O}_X$-modules ([AbH09, Lemma 5.1.1]). Since $\pi\colon Z \to X$ is étale off a finite subset, $\pi_*^G\mathcal{T}_Z$ and $\mathcal{T}_X$ coincide off the finite subset. Then the S_2-property guarantees that $\pi_*^G\mathcal{T}_Z$ and $\mathcal{T}_X$ are isomorphic on the whole of X.

For $\pi_*^G\mathcal{T}_Z^1 = \mathcal{T}_{QG,X}^1$, we view $\pi_*^G\mathcal{T}_Z^1$ (resp. $\mathcal{T}_{QG,X}^1$) as the sheaf of first-order G-equivariant local deformations of Z (resp. first-order $\mathbb{Q}$-Gorenstein local deformations of X). Let P be any point on X and let $\pi^{-1}(P) = \{Q_j\}_j$ be inverse image of P. Every germ $Q_j \in Z$ is a canonical covering of $P \in X$ and they are permuted under the action of G, because the possible singularities on X are of type (U_{1a}), (U_{1b}), (U_{2a}) or (U_{2b}). Since $\mathbb{Q}$-Gorenstein deformations of the germ $P \in X$ are precisely those deformations which lift to deformations of the canonical covering (cf. Remark 1.21), we have a natural identification $\pi_*^G\mathcal{T}_Z^1 = \mathcal{T}_{QG,X}^1$ sending a first-order G-equivariant local deformations of Z to its quotient under G. $\square$

We give a more down-to-earth proof of the following proposition.

Proposition 4.4 ([vO05], Corollary 2.3)**.** $\mathrm{Def}_Z = \mathrm{Def}_C \times \mathrm{Def}_D$.

Proof. First we will show that

$$\mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \oplus \mathrm{Ext}^1_{\mathcal{O}_D}(\Omega_D, \mathcal{O}_D). \tag{4.4}$$

Let $\pi_1 \colon Z \to C$ and $\pi_2 \colon Z \to D$ be the projections. We have $\Omega_Z = \pi_1^*\Omega_C \oplus \pi_2^*\Omega_D$, so

$$\mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z) = \mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_C, \mathcal{O}_Z) \oplus \mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_2^*\Omega_D, \mathcal{O}_Z).$$

We want to identify $\mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_C, \mathcal{O}_Z)$ (resp. $\mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_D, \mathcal{O}_Z)$) with $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (resp. $\mathrm{Ext}^1_{\mathcal{O}_D}(\Omega_D, \mathcal{O}_D)$) in a natural way.

An element of $\mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_C, \mathcal{O}_Z)$ is given by an extension of $\pi_1^*\Omega_C$ by $\mathcal{O}_Z$

$$0 \to \mathcal{O}_Z \to \mathcal{F} \to \pi_1^*\Omega_C \to 0. \tag{4.5}$$

Applying π_{1*}, we get the derived long exact sequence of $\mathcal{O}_C$-modules:

$$0 \to \pi_{1*}\mathcal{O}_Z \to \pi_{1*}\mathcal{F} \to \pi_{1*}\pi_1^*\Omega_C \to R^1\pi_{1*}\mathcal{O}_Z. \tag{4.6}$$

Note that, for any coherent sheaf $\mathcal{G}$ on C, we can use Čech cohomology to show that

$$\begin{aligned} R^i\pi_{1*}\pi_1^*\mathcal{G} &= \mathcal{G} \otimes_{\mathcal{O}_C} R^i\pi_{1*}\mathcal{O}_Z \\ &= \mathcal{G} \otimes_{\mathcal{O}_C} (\mathcal{O}_C \otimes_{\mathbb{C}} H^i(D, \mathcal{O}_D)) \\ &= \mathcal{G} \otimes_{\mathbb{C}} H^i(D, \mathcal{O}_D). \end{aligned}$$

Therefore

$$\pi_{1*}\mathcal{O}_Z = \mathcal{O}_C, \pi_{1*}\pi_1^*\Omega_C = \Omega_C, R^1\pi_{1*}\mathcal{O}_Z = \mathcal{O}_C \otimes_{\mathbb{C}} H^1(D, \mathcal{O}_D),$$

and we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_C}(\pi_{1*}\pi_1^*\Omega_C, R^1\pi_{1*}\mathcal{O}_Z) &= \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C \otimes_{\mathbb{C}} H^1(D, \mathcal{O}_D)) \\ &= H^0(C, \mathcal{T}_C) \otimes_{\mathbb{C}} H^1(D, \mathcal{O}_D) = 0.\end{aligned}$$

The last equation is because of the fact that C is a stable curve. So any morphism $\pi_{1*}\pi_1^*\Omega_C \to R^1\pi_{1*}\mathcal{O}_Z$ is a zero morphism and (4.6) gives an exact sequence of $\mathcal{O}_C$-modules

$$0 \to \mathcal{O}_C \to \pi_{1*}\mathcal{F} \to \Omega_C \to 0, \tag{4.7}$$

namely an element of $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$. Now we have a map

$$f\colon \mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_C, \mathcal{O}_Z) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$$

sending an extension of $\mathcal{O}_Z$-modules (4.5) to an extension of $\mathcal{O}_C$-modules (4.7).

Conversely, an element of $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$ is given by an exact sequence of $\mathcal{O}_C$-modules

$$0 \to \mathcal{O}_C \to \mathcal{G} \to \Omega_C \to 0. \tag{4.8}$$

Applying π_1^*, we get an exact sequence of $\mathcal{O}_Z$-modules

$$0 \to \mathcal{O}_Z \to \pi_1^*\mathcal{G} \to \pi_1^*\Omega_C \to 0, \tag{4.9}$$

since $\pi_1\colon Z \to C$ is a flat morphism. So we also establish a map

$$h\colon \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_C, \mathcal{O}_Z)$$

sending an extension of $\mathcal{O}_C$-modules (4.8) to an extension of $\mathcal{O}_Z$-modules (4.9).

It is readily seen that f and h are inverse to each other and hence

$$\mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_C, \mathcal{O}_Z) \cong \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C).$$

Similarly, we can prove that $\mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_1^*\Omega_D, \mathcal{O}_Z) \cong \mathrm{Ext}^1_{\mathcal{O}_D}(\Omega_D, \mathcal{O}_D)$ and the isomorphism (4.4) is established.

Now let $\lambda_1\colon \mathcal{C} \to \mathrm{Def}_C$ be a semiuniversal deformation of C and $\lambda_2\colon \mathcal{D} \to \mathrm{Def}_D$ a semiuniversal deformation D. Then $\lambda_1 \times \lambda_2\colon \mathcal{C} \times \mathcal{D} \to \mathrm{Def}_C \times \mathrm{Def}_D$ is a deformation of $Z = C \times D$, inducing a morphism

$$g\colon \mathrm{Def}_C \times \mathrm{Def}_D \to \mathrm{Def}_Z.$$

The corresponding tangential map dg is just the natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \oplus \mathrm{Ext}^1_{\mathcal{O}_D}(\Omega_D, \mathcal{O}_D) \cong \mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z).$$

established above. Since $\mathrm{Def}_C \times \mathrm{Def}_D$ is smooth (cf. [DM69, Page 79]) and the tangential map dg is an isomorphism, we have $\mathrm{Def}_Z = \mathrm{Def}_C \times \mathrm{Def}_D$.

$\square$

Theorem 4.5. *If all the (possible) singularities of X are of type (U_{1a}), (U_{1b}), (U_{2a}) or (U_{2b}), then*

(i) a semiuniversal $\mathbb{Q}$-Gorenstein deformation of X exists, hence the base Def_X^{QG} is defined;

(ii) G acts on Def_Z and there is an isomorphism $(\mathrm{Def}_Z)^G \cong \mathrm{Def}_X^{QG}$;

(iii) Def_X^{QG} is smooth.

Proof. Note that (iii) is a consequence of (ii) and Proposition 4.4.

Since $Z = C \times D$ is Gorenstein, Def_Z^{QG} exists and is just Def_Z (cf. Remark 1.20). Let $f\colon \mathcal{Z} \to \mathrm{Def}_Z$ be a semiuniversal deformation of Z. Then the action of G on Z induces actions of G on $\mathcal{Z}$ and Def_Z such that f becomes a G-equivariant morphism. Taking the G-invariant part Def_Z^G of Def_Z and the G-quotient of $f^{-1}(\mathrm{Def}_Z^G)$, we get a deformation

$$f^G\colon (f^{-1}(\mathrm{Def}_Z)^G)/G \to (\mathrm{Def}_Z)^G \tag{4.10}$$

of $Z/G = X$.

Since all the possible singularities of X are of type (U_{1a}), (U_{1b}), (U_{2a}) or (U_{2b}), for any $P \in X$ and $Q \in \pi^{-1}(P)$, the germ $Q \in Z$ is the canonical covering of $P \in X$ (cf. Corollary 3.12). So (4.10) is in fact a $\mathbb{Q}$-Gorenstein deformation of X (cf. Remark 1.21). To prove the theorem, it suffices to show that (4.10) is a semiuniversal $\mathbb{Q}$-Gorenstein deformation of X.

Note that $\mathrm{Def}_Z = \mathrm{Def}_C \times \mathrm{Def}_D$ is smooth (cf. the proof of Proposition 4.4), so $(\mathrm{Def}_Z)^G$ is also smooth by Cartan's lemma. By the infinitesimal lifting property of a smooth variety ([Har77, Chap. II, Exercise 8.6]), if we can show that the natural map

$$d\lambda\colon (T_Z^1)^G \to T^1_{QG,X}$$

is an isomorphism, then $f^G\colon (f^{-1}(\mathrm{Def}_Z)^G)/G \to (\mathrm{Def}_Z)^G$ is an unobstructed semiuniversal $\mathbb{Q}$-Gorenstein deformation of X and hence $\mathrm{Def}_X^{QG} \cong (\mathrm{Def}_Z)^G$.

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(Z,\mathcal{T}_Z)^G \arrow[r] \arrow[d, "\alpha"] & (T_Z^1)^G \arrow[r] \arrow[d, "d\lambda"] & H^0(Z,\mathcal{T}_Z^1)^G \arrow[r] \arrow[d, "\beta"] & H^2(Z,\mathcal{T}_Z)^G \arrow[d, "\gamma"] \\
0 \arrow[r] & H^1(X,\mathcal{T}_X) \arrow[r] & T^1_{QG,X} \arrow[r] & H^0(X,\mathcal{T}^1_{QG,X}) \arrow[r] & H^2(X,\mathcal{T}_X)
\end{tikzcd}$$

in which the rows are exact. We will prove that α, β, γ are isomorphisms, then $d\lambda\colon (T_Z^1)^G \to T^1_{QG,X}$ is also an isomorphism by the Five Lemma.

Let $\mathcal{F} = \mathcal{T}_Z$ or $\mathcal{T}_Z^1$ in Lemma 4.1, we get the following three equations

$$\begin{aligned} H^1(Z, \mathcal{T}_Z)^G &= H^1(X, \pi_*^G \mathcal{T}_Z), \\ H^0(Z, \mathcal{T}_Z^1)^G &= H^0(X, \pi_*^G \mathcal{T}_Z^1), \\ H^2(Z, \mathcal{T}_Z)^G &= H^2(X, \pi_*^G \mathcal{T}_Z). \end{aligned}$$

By Corollary 4.3, we have

$$\pi_*^G \mathcal{T}_Z = \mathcal{T}_X, \pi_*^G \mathcal{T}_Z^1 = \mathcal{T}_{QG,X}^1.$$

So

$$\begin{aligned} H^1(Z, \mathcal{T}_Z)^G &= H^1(X, \mathcal{T}_X), \\ H^0(Z, \mathcal{T}_Z^1)^G &= H^0(X, \mathcal{T}_{QG,X}^1), \\ H^2(Z, \mathcal{T}_Z)^G &= H^2(X, \mathcal{T}_X), \end{aligned}$$

and α, β, γ are isomorphisms. □

Corollary 4.6. *Let S be a surface isogenous to a product of unmixed type with minimal representation $(C \times D)/G$. Assume the pair (C, G) is a triangle curve (i.e., $C/G \cong \mathbb{P}^1$, and $C \to C/G$ is branched over 3 points). Let M_S^{top} be the moduli space of smooth surfaces with the same topological type of S and $\overline{M_S^{top}}$ the stable compactification in $M_{a,b}^{st}$ with $a = K_S^2, b = \chi(\mathcal{O}_S)$. Then for any surface $[X]$ in $\overline{M_S^{top}}$, Def_X^{QG} is defined and is smooth, hence $\overline{M_X^{top}}$ is already a connected component of the moduli space $M_{a,b}^{st}$.*

Proof. By [Cat03], every point in M_S^{top} corresponds to a surface S' isogenous to a product with minimal representation $(C' \times D')/G$. Moreover, (C', G) is a triangle curve. In fact, $(C', G) = (C, G)$ or $(\overline{C}, G)$. Since a triangle curve is rigid, in the process of degeneration, it remains the same. If $[X]$ is in $\overline{M_S^{\mathrm{top}}}$, then $X = (C' \times D)/G$, where D is a stable curve. By Proposition 3.10 and Corollary 3.12, the possible singularities of X are of type (U_{1a}) or (U_{1b}). Now Theorem 4.5 applies. □

Remark 4.7. It remains to address the case where X has singularities of type (U_{2c}) or (M). The canonical coverings of these two types of singularities are not complete intersections, which results in a more difficult $\mathbb{Q}$-Gorenstein deformation theory. In contrast to the infinitesimal consideration, there might be some hope for good properties of a one-parameter family of such singularities.

Bibliography

[A94] Alexeev, V. Boundedness and K^2 for log surfaces. Internat. J. Math. 5 (1994), no. 6, 779–810.

[AbH09] Abramovich, D; Hassett, B. Stable varieties with a twist, preprint 2009, arXiv:0904.2797v1.

[Ar74] Artin, M. Versal deformations and algebraic stacks. Invent. Math. 27 (1974), 165–189.

[B87] Broughton, S. The homology and higher representations of the automorphism group of a Riemann surface. Trans. Amer. Math. Soc. 300 (1987), no. 1, 153–158.

[B91] Broughton, S. Classifying finite group actions on surfaces of low genus. J. Pure Appl. Algebra 69 (1991), no. 3, 233–270.

[BCGP09] Bauer, I.; Catanese, F.; Grunewald, F; Pignatelli, R. Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups, preprint 2009, arXiv:0809.3420v3.

[BH93] Bruns, W.; Herzog, J.; Cohen-Macaulay rings. Camb. Stud. in Adv. Math., 39. Cambridge University Press, Cambridge, 1993.

[BHPV04] Barth, W.; Hulek, K.; Peters, C.; Van de Ven, A. Compact complex surfaces. Second edition. Ergeb. Math. Grenzgeb. 3. vol 4. Springer-Verlag, Berlin, 2004.

[BR75] Burns, D.; Rapoport, M. On the Torelli problem for kählerian $K3$ surfaces. Ann. Sci. École Norm. Sup. (4) 8 (1975), no. 2, 235–273.

[Cai04] Cai, J. Automorphisms of a surface of general type acting trivially in cohomology. Tohoku Math. J. (2) 56 (2004), no. 3, 341–355.

[Cai06a] Cai, J. Automorphisms of fiber surfaces of genus 2, inducing the identity in cohomology. Trans. Amer. Math. Soc. 358 (2006), no. 3, 1187–1201.

[Cai06b] Cai, J. Automorphisms of fiber surfaces of genus 2, inducing the identity in cohomology. II. Internat. J. Math. 17 (2006), no. 2, 183–193.

[Cai07] Cai, J. Classification of fiber surfaces of genus 2 with automorphisms acting trivially in cohomology. Pacific J. Math. 232 (2007), no. 1, 43–59.

[Cat84] Catanese, F. On the moduli spaces of surfaces of general type. J. Differential Geom. 19 (1984), no. 2, 483–515.

[Cat87] Catanese, F. Automorphisms of rational double points and moduli spaces of surfaces of general type. Compositio Math. 61 (1987), no. 1, 81–102.

[Cat89] Catanese, F. Everywhere nonreduced moduli spaces. Invent. Math. 98 (1989), no. 2, 293–310.

[Cat00] Catanese, F. Fibred surfaces, varieties isogenous to a product and related moduli spaces. Amer. J. Math. 122 (2000), no. 1, 1–44.

[Cat03] Catanese, F. Moduli spaces of surfaces and real structures. Ann. of Math. (2) 158 (2003), no. 2, 577–592.

[Co00] Conrad, B. Grothendieck duality and base change. Lecture Notes in Mathematics, 1750. Springer-Verlag, Berlin, 2000.

[DM69] Deligne, P.; Mumford, D. The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. No. 36 (1969), 75–109.

[Hac01] Hacking, P. A Compactification of the spaces of plane curves. Thesis, 2001, arXiv:math/0104193v1.

[Hac04] Hacking, P. Compact moduli of plane curves. Duke Math. J. 124 (2004), no. 2, 213–257.

[Har77] Hartshorne, R. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[HK04] Hassett, B.; Kovács, S. J. Reflexive pull-backs and base extension. J. Algebraic Geom. 13 (2004), no. 2, 233–247.

[Ka00] Karu, K. Minimal models and boundedness of stable varieties. J. Algebraic Geom. 9 (2000), no. 1, 93–109.

[KeM97] Keel, S.; Mori, S. Quotients by groupoids. Ann. of Math. (2) 145 (1997), no. 1, 193–213.

[Ko90] Kollár, J. Projectivity of complete moduli. J. Differential Geom. 32 (1990), no. 1, 235–268.

[Ko08] Kollár, J. Hulls and husks. preprint arXiv:0805.0576v3, 2008.

[KoM98] Kollár, J.; Mori, S. Birational geometry of algebraic varieties. Cambridge Tracts in Mathematics, 134. Cambridge University Press, 1998.

[KoSB88] Kollár, J.; Shepherd-Barron, N. I. Threefolds and deformations of surface singularities. Invent. Math. 91 (1988), no. 2, 299–338.

[Kov09] Kovács, S. J. Young person's guide to moduli of higher dimensional varieties. Algebraic geometry–Seattle 2005. Part 2, 711–743, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., Providence, RI, 2009.

[M97] Manetti, M. Iterated double covers and connected components of moduli spaces. Topology 36 (1997), no. 3, 745–764.

[M08] Manetti, M. Smoothings of singularities and deformation types of surfaces. Symplectic 4-manifolds and algebraic surfaces, 169–230, Lecture Notes in Math., 1938, Springer, Berlin, 2008.

[P79] Peters, C. A. M. Holomorphic automorphisms of compact Kähler surfaces and their induced actions in cohomology. Invent. Math. 52 (1979), no. 2, 143–148.

[Pe09R] Penegini, M.; Rollenske, S. The classification of isotrivial fibred surfaces with $p_g = q = 2$. preprint 2009, arXiv:0904.1352v1.

[Po77] Popp, H. Moduli theory and classification theory of algebraic varieties. Lecture Notes in Mathematics, Vol. 620. Springer-Verlag, Berlin-New York, 1977.

[R80] Reid, Miles Canonical 3-folds. Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 273–310, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[R87] Reid, M. Young person's guide to canonical singularities. Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[T09] Tziolas, N. $\mathbb{Q}$-Gorenstein deformations of nonnormal surfaces. Amer. J. Math. 131 (2009), no. 1, 171–193.

[U76] Ueno, K. A remark on automorphisms of Enriques surfaces. J. Fac. Sci. Univ. Tokyo Sect. I A Math. 23 (1976), no. 1, 149–165.

[vO05] van Opstall, M. Moduli of products of curves. Arch. Math. (Basel) 84 (2005), no. 2, 148–154.

[vO06a] van Opstall, M. On the properness of the moduli space of stable surfaces, preprint 2006, arXiv:math/0601345v2.

[vO06b] van Opstall, M. Stable degenerations of surfaces isogenous to a product of curves. Proc. Amer. Math. Soc. 134 (2006), no. 10, 2801–2806.