THE INDEX THEOREM FOR QUASI-TORI

DISSERTATION
zur Erlangung
des DOKTORGRADES (DR. RER. NAT.)
der FAKULTÄT FÜR MATHEMATIK, PHYSIK UND INFORMATIK
der UNIVERSITÄT BAYREUTH

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BAYREUTH
Tag der Einreichung: 27. November, 2012
Tag der Kolloquiums: 15. Februar, 2013

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Unterschrift des Autors

Acknowledgements

It is my pleasure to express here my gratitude to my supervisor Prof. Fabrizio Catanese for suggesting me this research problem and for his continual guidance, as well as sharing his point of view about Mathematics and a lot of his personal experience in life.

My gratitude also goes to Prof. Ingrid Bauer for encouraging me to explore different fields of Mathematics. Moreover, her care to me during my sickness made me feel like home while I was staying in a country distant from mine.

Many thanks to all current and former colleagues in the Lehrstuhl Mathematik VIII of Universität Bayreuth, in particular to Michael Lönne, Fabio Perroni, Masaaki Murakami, Stephen Coughlan, Matteo Penegini, Wenfei Liu and Yifan Chen, for their help on my thesis, inspiring discussions on Mathematical ideas, sharing about the cultures and lifestyles of their own countries, and, most importantly, their encouragements which helped me to get through the most depressing period of my Ph.D. study. Thanks also to our secretary Leni Rostock who helped to sort out all the troubles during my stay in Bayreuth, from getting the residence permit to finding a medical doctor. Thanks to her, we have never missed the birthday of anybody in Lehrstuhl VIII. Wish that she would enjoy her life after retirement.

Special thanks to my M.Phil. supervisor Prof. Ngaiming Mok, who taught me the basics about the Bochner–Kodaira formulas; and to Michael Lönne, Florian Schrack, Sascha Weigl and Christian Gleißner who helped me to translate the abstract and summary into German.

I would also like to thank DAAD for their support under the Forschungsstipendien für Doktoranden.

Lastly, I would like to declare that I owe my friends outside the Mathematics community in both Hong Kong and Germany a lot. Without their comforts and encouragements, this thesis could never be finished. My debts to them can never be fully redeemed. I am also badly indebted to my parents, who have given me freedom to do whatever I wish.

Abstract

The Index theorem for holomorphic line bundles on complex tori asserts that some cohomology groups of a line bundle vanish according to the signature of the associated hermitian form. In this article, this theorem is generalized to quasi-tori, i.e. connected complex abelian Lie groups which are not necessarily compact. In view of the Remmert–Morimoto decomposition of quasi-tori as well as the Künneth formula, it suffices to consider only Cousin-quasi-tori, i.e. quasi-tori which have no non-constant holomorphic functions. The Index theorem is generalized to holomorphic line bundles, both linearizable and non-linearizable, on Cousin-quasi-tori using L^2-methods coupled with the Kazama–Dolbeault isomorphism and Bochner–Kodaira formulas.

Zusammenfassung

Ein *Quasi-Torus* ist eine zusammenhängende komplexe abelsche Lie-Gruppe $X = \mathbb{C}^n/\Gamma$, wobei Γ eine diskrete Untergruppe von $\mathbb{C}^n$ ist. X heißt *Cousin-Quasi-Torus*, wenn alle holomorphen Funktionen auf X konstant sind. Ist X kompakt, so ist X ein *komplexer Torus*.

Nach einem Satz von Remmert und Morimoto (vgl. [**Mo2**] oder [**CC1**, Prop. 1.1]) gibt es für jeden Quasi-Torus X eine Zerlegung $X \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X'$, wobei X' ein Cousin-Quasi-Torus ist. Das Ziel des vorliegenden Artikels ist, das Verschwinden von Kohomologiegruppen von Geradenbündeln auf X zu untersuchen. Die Künnethformel (vgl. [**Kau**]) besagt, dass sich die Kohomologiegruppen von X in direkte Summen von topologischen Tensorprodukten von Kohomologiegruppen von $\mathbb{C}^a \times (\mathbb{C}^*)^b$ und des Cousin-Quasi-Torus X' zerlegen lassen. Man wird dadurch auf den Fall geführt, dass X ein Cousin-Quasi-Torus ist, da $\mathbb{C}^a \times (\mathbb{C}^*)^b$ Steinsch ist und somit alle höheren Kohomologiegruppen (mit Grad ≥ 1) von kohärenten Garben verschwinden. Es wird also im vorliegenden Artikel angenommen, dass X ein Cousin-Quasi-Torus ist.

Sei F der maximale komplexe Unterraum von $\mathbb{R}\Gamma$ und $m := \dim_{\mathbb{C}} F$. Wie im kompakten Fall kann jedem holomorphen Geradenbündel L eine hermitesche Form $\mathcal{H}$ auf $\mathbb{C}^n$ zugeordnet werden, deren Imaginärteil $\operatorname{Im}\mathcal{H}$ mit der ersten Chernklasse $c_1(L)$ von L assoziiert ist und ganzzahlige Werte in $\Gamma \times \Gamma$ annimmt. Im Unterschied zum kompakten Fall ist $\mathcal{H}$ nicht eindeutig. Lediglich die Einschränkung von $\operatorname{Im}\mathcal{H}$ auf $\mathbb{R}\Gamma \times \mathbb{R}\Gamma$, und somit $\mathcal{H}|_{F\times F}$, ist eindeutig bestimmt. Dies macht zumindest plausibel, dass nur $\mathcal{H}|_{F\times F}$ anstelle von $\mathcal{H}$ für die Eigenschaften von L verantwortlich ist. Die vorliegende Dissertation widmet sich dem Beweis des folgenden Satzes:

Index-Satz für Cousin-Quasi-Tori. *Sei $X = \mathbb{C}^n/\Gamma$ ein Cousin-Quasi-Torus, F der maximale komplexe Unterraum von $\mathbb{R}\Gamma$, L ein holomorphes Geradenbündel auf X und $\mathcal{H}$ eine mit L assoziierte hermitesche Form auf $\mathbb{C}^n \times \mathbb{C}^n$. Sei $m := \dim_{\mathbb{C}} F$. Die Einschränkung $\mathcal{H}|_{F\times F}$ habe s_F^- negative und s_F^+ positive Eigenwerte. Dann gilt*

$$H^q(X, L) = 0 \quad \textit{für } q < s_F^- \textit{ oder } q > m - s_F^+ .$$

Dieser Satz wird zurückgeführt auf den Index-Satz für komplexe Tori, wie er von Mumford [**Mum**], Kempf [**Kem**], Umemura [**U**], Matsushima [**Ma**] und Murakami [**Mur**] für kompakte X bewiesen wurde. Da X stark $(m+1)$-vollständig ist (vgl. [**Kaz1**]; siehe auch §2.2), enthält der Satz auch einen Spezialfall des Resultats von Andreotti und Grauert, das besagt, dass $H^q(X, \mathscr{F}) = 0$ ist für alle $q \geq m+1$ und für jede kohärente analytische Garbe $\mathscr{F}$ auf X (vgl. [**AGr**]).

Das Verschwinden von $H^q(X, L)$ kann unter Verwendung der Dolbeault-Isomorphismen auf gewisse $\overline{\partial}$-Gleichungen für L-wertige $(0, q)$-Formen zurckgeführt werden. Diese können mit L^2-Methoden gelöst werden. Man zeigt zunächst die Existenz einer formalen Lösung einer $\overline{\partial}$-Gleichung in einem Hilbertraum, indem man die benötigte L^2-Abschätzung nachweist, und beweist dann die Glattheit der Lösung. Letzteres kann mit Hilfe der Regularitätstheorie von $\overline{\partial}$-Operatoren erledigt werden, also ist der entscheidende Schritt der Nachweis der benötigten L^2-Abschätzungen. Diese kann man durch Anwendung der Bochner–Kodaira-Ungleichungen bekommen.

Jeder Cousin Quasi-Torus X hat eine Faserbündelstruktur über einem komplexen Torus T mit steinschen Fasern (siehe §2.1 und (eq 2.3)). Mit Hilfe der Lerayschen Spektralsequenz folgt

$$H^q(X, L) \cong H^q(T, p_*\mathscr{O}_X(L)) \quad \text{für alle } q \geq 0 \,,$$

wobei $p\colon X \to T$ die Projektion aus (eq 2.3) ist. Die Idee ist jetzt zu zeigen, dass der Dolbeault Komplex der Garben $\big(\mathscr{A}_T^{0,\bullet} \otimes_{\mathscr{O}_T} p_*\mathscr{O}_X(L), \overline{\partial}\big)$, eine azyklische Auflösung von $p_*\mathscr{O}_X(L)$ auf T ist und das Verschwinden der Kohomologie durch Lösen der $\overline{\partial}$-Gleichungen zu zeigen. Kazama [**Kaz2**] und Kazama–Umeno [**KU2**] geben eine leicht veränderte Formulierung, sie betrachten die Auflösung von $\mathscr{O}_X(L)$ durch einen Unterkomplex $\big(\mathscr{H}^{0,\bullet}(L), \overline{\partial}\big)$ von $\big(\mathscr{A}_X^{0,\bullet}(L), \overline{\partial}\big)$ (siehe §2.3 für die Definition von $\mathscr{H}^{0,q}(L)$). Der Teilkomplex ist ebenfalls eine azyklische Auflösung von $\mathscr{O}_X(L)$ auf X und liefert damit den Kazama–Dolbeault Isomorphismus (vgl. [**KU2**], siehe auch Theorem 2.3.1). Letzterer Ansatz wird hier aufgegriffen. Das Ziel der Darstellung ist dann die Lösung der $\overline{\partial}$-Gleichung $\overline{\partial}\xi = \psi$ für ein gegebenes $\psi \in \Gamma(X, \mathscr{H}^{0,q}(L))$ mit $\overline{\partial}\psi = 0$.

Jedes Geradenbündel L auf X kann durch ein System von Automorphiefaktoren definiert werden, die in eine zur Appell–Humbert-Normalform analoge Normalform übergeführt werden können, die gegeben ist durch (vgl. [**CC1**, §2.2] und [**V**, §2])

$$\varrho(\gamma) e^{\pi\mathcal{H}(z,\gamma) + \frac{\pi}{2}\mathcal{H}(\gamma,\gamma) + f_\gamma(z)} \qquad \forall\, \gamma \in \Gamma \,,$$

wobei ϱ ein Halbcharakter auf Γ und $\{f_\gamma(z)\}_{\gamma\in\Gamma}$ ein additiver Kozykel ist (vgl. [**CC1**, §2.2] und [**V**, §2], siehe auch (eq 2.8)). Wenn $\{f_\gamma(z)\}_{\gamma\in\Gamma}$ ein Korand ist, so wird L als *linearisierbar* bezeichnet; andernfalls als *nicht linearisierbar*. Indem man den Trick verwendet, den Murakami in [**Mur**] für den kompakten Fall benutzt hat (siehe §3.3), nämlich die Metrik g so abzuändern, dass der vom linearen Teil (dem zahmen Teil) von L in den Basisrichtungen kommende Krümmungsterm von unten beschränkt ist, wenn q im gegebenen Bereich liegt, kann man die benötigten L^2-Abschätzungen erhalten, wenn L linearisierbar ist (siehe §4). Dies beweist den Index-Satz für linearisierbare L (siehe Theorem 4.1.1).

Beim Nachweis der benötigten L^2-Abschätzungen für nicht linearisierbare L auf X gibt eine zusätzliche technische Schwierigkeit, die von dem vom nichtlinearen Teil (dem wilden Teil) von L kommenden Krümmungsterm herrührt. Für diesen wird Takayama's schwaches $\partial\overline{\partial}$-Lemma ([**Taka2**, Lemma 3.14]; siehe auch §5.1) angewandt, um den Term auf relativ kompakten Teilmengen von X zu beschränken. Dadurch erhält man die benötigten L^2-Abschätzungen nicht auf X, sondern lediglich auf der ausschöpfenden Familie $\{K_c\}_{c\in\mathbb{R}_{>0}}$ von pseudokonvexen relativ kompakten Teilmengen. Man erhält dann eine Folge $\{\xi_\nu\}_{\nu\geq 1}$ von lokalen Lösungen, so dass $\overline{\partial}\xi_\nu = \psi|_{\overline{K_\nu}}$ ist für ein gegebenes $\psi \in \Gamma(X, \mathscr{H}^{0,q}(L)) \cap \ker\overline{\partial}$ und für alle ganzen Zahlen $\nu \geq 1$. Indem man ein Argument im Beweis von Theorem B für Steinsche Räume in [**GR**, Ch. IV, §5] nachvollzieht, speziell indem man eine Approximation vom Runge-Typ verwendet, kann man die lokalen Lösungen ξ_ν so korrigieren, dass sie auf jedem K_c konvergieren, was dann eine globale Lösung für alle q im gegebenen Bereich liefert (siehe §5.4). Der Beweis des Index-Satzes ist damit vollständig.

Contents

Erklärung i

Acknowledgements ii

Abstract iii

Zusammenfassung iv

Chapter 1. Introduction and the main theorem 1
1.1. The main theorem 2
1.2. Methodology 3

Chapter 2. Preliminaries 4
2.1. A $(\mathbb{C}^*)^{n-m}$-principal bundle structure on X 4
2.2. An exhaustive family of pseudoconvex subsets 5
2.3. Kazama sheaves and Kazama–Dolbeault isomorphism 5
2.4. Holomorphic line bundles on X 6
2.5. A hermitian metric on L 7
2.6. An L^2-norm, the L^2-spaces $L_2{}^{0,(q',q'')}_{c,X}$ and differential operators 9

Chapter 3. L^2 estimates 11
3.1. Existence of a solution of $\overline{\partial}\xi = \psi$ 11
3.2. Bochner–Kodaira formulas 16
3.3. Murakami's trick 20

Chapter 4. The linearizable case 27
4.1. Proof of Theorem 1.1.1 for linearizable L 27

Chapter 5. The non-linearizable case 28
5.1. Bounds on the wild curvature terms 28
5.2. Existence of weak solutions on K_c 29
5.3. A Runge-type approximation 30
5.4. Proof of Theorem 1.1.1 for general L 32

List of Symbols 35

Bibliography 36

CHAPTER 1

Introduction and the main theorem

A *quasi-torus* is a complex abelian Lie group $X = \mathbb{C}^n/\Gamma$, where Γ is a discrete subgroup of $\mathbb{C}^n$. X is said to be a *Cousin-quasi-torus* if all holomorphic functions on X are constant functions.[1] X is the familiar *complex torus* when it is compact, i.e. when $\operatorname{rk}\Gamma = 2n$.

The study of quasi-tori dates back to the early 20th century when Cousin studied the triply periodic functions of two complex variables ([**Cou**]). There he showed the existence of 2-dimensional quasi-tori without non-constant holomorphic functions. He also gave, among other things, a complete description of holomorphic line bundles on quasi-tori of dimension 2 and their sections using a method of asymptotic counting of zeros of the sections. In the 60's, Kopfermann ([**Kop**]) studied systematically toroidal groups of arbitrary dimensions with a view to generalize the theory of abelian functions on complex tori. He also gave an example of a non-compact toroidal group with no non-constant meromorphic functions. Morimoto ([**Mo1**] and [**Mo2**]) studied Cousin-quasi-torus as the maximal toroidal subgroup of a complex (not necessarily abelian) Lie group, aiming to classify non-compact complex Lie groups. He classified all 3-dimensional abelian complex Lie groups. In the early 70's, Andreotti and Gherardelli gave seminars on quasi-abelian varieties, i.e. Cousin-quasi-tori which possess structures of quasi-projective algebraic varieties ([**AGh**]). They showed that, among other things, a Cousin-quasi-torus is a quasi-abelian variety if and only if the Generalized Riemann Relations are satisfied on it. Later on, among other contributors, Kazama ([**Kaz1**] and [**Kaz2**]), Pothering ([**P**]), Hefez ([**Hef**]), Vogt ([**V**]), Huckleberry and Margulis ([**HM**]), Abe ([**Ab1**] and [**Ab2**]), Capocasa and Catanese ([**CC1**] and [**CC2**]), and Takayama ([**Taka2**]) made some direct contributions to the theory of quasi-tori and Cousin-quasi-tori. A brief exposition of the historical development of the Generalized Riemann Relations can be found in [**CC1**, p. 29], and the Introduction of [**AK**] describes a brief chronology of the study of toroidal groups in general.

The current research stems from the study of Capocasa and Catanese (ref. [**CC1**] and [**CC2**]). In [**CC1**], they gave an affirmative answer to a long standing problem of whether the existence of a non-degenerate meromorphic function on a quasi-torus is equivalent to the Generalized Riemann Relations. In [**CC2**], they moved on to prove the Lefschetz type theorems on quasi-tori in the best form, based on a statement of Abe with an erroneous proof in [**Ab3**, Thm. 6.4] (see [**CC2**, Corollary 1.2]).[2] Abe's statement is then substituted by a result proven by Takayama ([**Taka1**, Thm. 1.3 and

[1]A Cousin-quasi-torus is also called a *toroidal group* or (H,C)-*group* in literature, where the latter means that all holomorphic functions are constant (ref. [**AK**, Def. 1.1.1]).

[2]Théorème 6.4 in [**Ab3**] asserts that, on a non-compact toroidal group X, there exists a constant $c > 0$ such that, for any holomorphic line bundle L with an associated hermitian form $\mathcal{H}$ on $\mathbb{C}^n$ such that $\mathcal{H}|_{F\times F} > cI_m$ (where I_m is the $m \times m$-identity matrix and F is the maximal complex subspace of $\mathbb{R}\Gamma$; see §2), $H^0(X, L)$ is non-trivial, and in fact infinite-dimensional.

Thm. 6.1]).[3] These results clarify some basic properties of meromorphic functions and global sections of holomorphic line bundles on quasi-tori. This article goes a step further into the investigation of the higher cohomology groups of holomorphic line bundles on quasi-tori. The aim is to generalize the Index theorem on tori to quasi-tori.

1.1. The main theorem

Denote the $\mathbb{C}$-span and $\mathbb{R}$-span of Γ by $\mathbb{C}\Gamma$ and $\mathbb{R}\Gamma$ respectively. Let $\pi\colon \mathbb{C}^n \to X$ be the natural projection. Then $K := \pi(\mathbb{R}\Gamma) = \mathbb{R}\Gamma/\Gamma$ is the maximal compact subgroup of X, and $F := \mathbb{R}\Gamma \cap \sqrt{-1}\mathbb{R}\Gamma$ is the maximal complex subspace in $\mathbb{R}\Gamma$.

By a theorem of Remmert and Morimoto (ref. [**Mo2**], see also [**CC1**, Prop. 1.1]), if X is a quasi-torus, there is a decomposition $X \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X'$, where X' is Cousin. The aim of this article is to investigate the vanishing of cohomology groups of holomorphic line bundles on X. The Künneth formula (ref. [**Kau**]) asserts that the cohomology groups on X decompose into direct sum of topological tensor products of cohomology groups on $\mathbb{C}^a \times (\mathbb{C}^*)^b$ and the Cousin-quasi-torus X'. In view of this, since $\mathbb{C}^a \times (\mathbb{C}^*)^b$ is Stein and thus all higher cohomology groups (with degree ≥ 1) of coherent sheaves vanish, one is reduced to the case where X is Cousin. In what follows, X *is assumed to be a non-compact Cousin-quasi-torus* unless otherwise stated. In this case, $\mathbb{C}\Gamma = \mathbb{C}^n$, and $\operatorname{rk}\Gamma = \dim_{\mathbb{R}} \mathbb{R}\Gamma = n + m$ for some integer m such that $0 < m < n$. Note that m is the complex dimension of F.

Given a holomorphic line bundle L on X, it is analogous to the compact case that there is a hermitian form $\mathcal{H}$ on $\mathbb{C}^n \times \mathbb{C}^n$ associated to L, whose imaginary part $\operatorname{Im}\mathcal{H}$ takes integral values on $\Gamma \times \Gamma$ and corresponds to the first Chern class $c_1(L)$ of L (ref. [**CC1**]). $\operatorname{Im}\mathcal{H}$ is uniquely determined only on $\mathbb{R}\Gamma \times \mathbb{R}\Gamma$, so $\mathcal{H}$ is uniquely determined only on $F \times F$.

The following theorem is a generalization of the Index theorem on complex tori (ref. [**Mum**, p. 150], [**Mur**] or [**BL**, §3.4])[4] to Cousin-quasi-tori, which is the main result of this article.

THEOREM 1.1.1. *Let $X = \mathbb{C}^n/\Gamma$ be a Cousin-quasi-torus, F the maximal complex subspace of $\mathbb{R}\Gamma$, L a holomorphic line bundle on X, and $\mathcal{H}$ a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ associated to L. Let $m := \dim_{\mathbb{C}} F$. Suppose $\mathcal{H}|_{F\times F}$ has respectively s_F^- negative and s_F^+ positive eigenvalues. Then one has*

$$H^q(X, L) = 0 \quad \text{for } q < s_F^- \text{ or } q > m - s_F^+ .$$

Let $\mathbf{\Omega}_X^p$ be the sheaf of germs of holomorphic p-forms on X, and set $\mathbf{\Omega}_X^p(L) := \mathbf{\Omega}_X^p \otimes_{\mathscr{O}_X} \mathscr{O}_X(L)$. Since the cotangent bundle of X is trivial, one has $\mathbf{\Omega}_X^p(L) \cong \bigoplus^{\binom{n}{p}} \mathscr{O}_X(L)$, and thus $H^q(X, \mathbf{\Omega}_X^p(L)) \cong \bigoplus^{\binom{n}{p}} H^q(X, L)$. Therefore, one has the following

[3] Theorem 1.3 and 6.1 in [**Taka1**] together asserts that, for any positive line bundle L on a non-compact toroidal group X, there exists an explicitly given integer $\mu_0 > 0$ such that $H^0(X, L^{\otimes\mu})$ is non-trivial for all $\mu \geq \mu_0$. Corollary 1.2 in [**CC2**] holds true by applying Takayama's result and Proposition 1.1 in [**CC2**]. Takayama also gives a different proof of a weaker form of Lefschetz type theorems in [**Taka2**].

[4] The Index theorem on complex tori was first proven by Mumford [**Mum**] and Kempf [**Kem**] in the algebraic case, and later by Umemura [**U**], Matsushima [**Ma**] and Murakami [**Mur**] in the analytic case.

Corollary 1.1.2. *With the same assumptions as in Theorem 1.1.1, one has, for any* $p \geq 0$,

$$H^q(X, \mathbf{\Omega}_X^p(L)) = 0 \quad \text{for } q < s_F^- \text{ or } q > m - s_F^+ .$$

Note that the statement is reduced to the original Index theorem when X is a compact complex torus, in which case $m = n$. Moreover, it can be shown that X is strongly $(m+1)$-complete (ref. [**Kaz1**] and [**Take**]; convention of the numbering here following [**D1**, pp. 512]; see also §2.2), so Theorem 1.1.1 includes a special case of the result of Andreotti and Grauert, which asserts that $H^q(X, \mathscr{F}) = 0$ for all $q \geq m+1$ and for any coherent analytic sheaf $\mathscr{F}$ on X (ref. [**AGr**]). The remaining part of this article is devoted to proving Theorem 1.1.1.

1.2. Methodology

Let L be a holomorphic line bundle on X. Since every Cousin-quasi-torus X has a fibre bundle structure over a complex torus T with Stein fibres (see §2.1 and (eq 2.3)), it follows from a Leray spectral sequence argument that

$$H^q(X, L) \cong H^q(T, p_* \mathscr{O}_X(L)) \quad \text{for all } q \geq 0 ,$$

where $p \colon X \to T$ is the projection in (eq 2.3). Let $\mathscr{A}_T^{0,q}$ (resp. $\mathscr{A}_X^{0,q}$) be the sheaf of germs of smooth differential $(0,q)$-forms on T (resp. on X). The idea is then to show that the Dolbeault complex of sheaves $\left(\mathscr{A}_T^{0,\bullet} \otimes_{\mathscr{O}_T} p_* \mathscr{O}_X(L), \overline{\partial}\right)$, is an acyclic resolution of $p_* \mathscr{O}_X(L)$ on T, and to prove vanishing by solving $\overline{\partial}$-equations. A slightly different formulation is given by Kazama [**Kaz2**] and Kazama–Umeno [**KU2**], who consider the resolution of $\mathscr{O}_X(L)$ by a subcomplex $\left(\mathscr{H}^{0,\bullet}(L), \overline{\partial}\right)$ of $\left(\mathscr{A}_X^{0,\bullet}(L), \overline{\partial}\right)$ (see §2.3 for the definition of $\mathscr{H}^{0,q}(L)$). The subcomplex is also an acyclic resolution of $\mathscr{O}_X(L)$ on X, thus yielding the Kazama–Dolbeault isomorphism (ref. [**KU2**], see also Theorem 2.3.1). This latter formulation is adopted in this article, so, to prove Theorem 1.1.1 is to solve the $\overline{\partial}$-equations $\overline{\partial}\xi = \psi$ for any $\psi \in \Gamma(X, \mathscr{H}^{0,q}(L))$ such that $\overline{\partial}\psi = 0$ and for all q's in the range given in the Theorem.

The required $\overline{\partial}$-equations are solved by exhibiting L^2 estimates (eq 3.4) for certain L-valued forms on X. When L is linearizable (see Definition 2.4.1), these estimates can be obtained from Bochner–Kodaira formulas together with a trick employed by Murakami for the case of tori (ref. [**Mur**]) (see §3.3 and §4).

For non-linearizable L, the required L^2 estimates can only be obtained on compact subsets of X via Takayama's Weak $\partial\overline{\partial}$-Lemma (ref. [**Taka2**], see also §5.1). Then, given $\psi \in \Gamma(X, \mathscr{H}^{0,q}(L))$ such that $\overline{\partial}\psi = 0$ and an exhaustive sequence $\{K_\nu\}_{\nu \in \mathbb{N}_{>0}}$ of pseudoconvex relatively compact open subsets of X, a sequence $\{\xi_\nu\}_{\nu \in \mathbb{N}_{>0}}$ of weak solutions of $\overline{\partial}\xi_\nu = \psi|_{K_\nu}$ is obtained. Using a Runge-type approximation (see §5.3) and following an argument in [**GR**, Ch. IV, §1, Thm. 7], the solutions ξ_ν's can be adjusted so that they converge to a weak global solution of $\overline{\partial}\xi = \psi$. A strong solution in $\Gamma(X, \mathscr{H}^{0,q-1}(L))$ then exists by the regularity theory for $\overline{\partial}$ or elliptic operators (ref. [**Hör3**, Thm. 4.2.5 and Cor. 4.2.6] or [**Hör2**, Thm. 4.1.5 and Cor. 4.1.2]) and the Kazama–Dolbeault isomorphism (ref. [**KU2**], see also Theorem 2.3.1).

CHAPTER 2

Preliminaries

2.1. A $(\mathbb{C}^*)^{n-m}$-principal bundle structure on X

Let $X = \mathbb{C}^n/\Gamma$ be a Cousin-quasi-torus. Then one has $\mathbb{C}\Gamma = \mathbb{C}^n$ and $\operatorname{rk}\Gamma = n+m$ with $m > 0$. Define $K := \pi(\mathbb{R}\Gamma) = \mathbb{R}\Gamma/\Gamma$ and $F := \mathbb{R}\Gamma \cap \sqrt{-1}\mathbb{R}\Gamma$ as before. Fix a basis of $\mathbb{C}^n$ such that the period matrix of X is given by

$$\begin{bmatrix} I_{n-m} & & A_1 + \sqrt{-1}B_1 \\ & I_m & A_2 + \sqrt{-1}B_2 \end{bmatrix} , \tag{eq 2.1}$$

where an empty entry means a zero entry, I_r denotes the identity matrix of rank r, A_i and B_i denotes real matrices such that A_1 and B_1 are of size $(n-m)\times m$, and A_2 and B_2 are square matrices of size $m \times m$. By re-ordering the basis of $\mathbb{C}^n$ and respectively the basis of Γ, B_2 can be assumed to be invertible (since $\operatorname{rk}\Gamma = n+m$). Take a change of coordinates given by the matrix

$$\begin{bmatrix} I_{n-m} & -B_1B_2^{-1} \\ & B_2^{-1} \end{bmatrix} ,$$

the period matrix under the new coordinates is then given by

$$\begin{bmatrix} I_{n-m} & \beta_1 & \alpha_1 \\ & \beta_2 & \alpha_2 \end{bmatrix} , \tag{eq 2.2}$$

where

$$\begin{aligned} \beta_1 &= -B_1B_2^{-1} , & \alpha_1 &= A_1 - B_1B_2^{-1}A_2 , \\ \beta_2 &= B_2^{-1} , & \alpha_2 &= B_2^{-1}A_2 + \sqrt{-1}I_m , \end{aligned}$$

which are all real matrices except for α_2. Let the new coordinates of $\mathbb{C}^n$ be denoted by $(u, v) := (u^1, \ldots, u^{n-m}, v^1, \ldots, v^m)$, or simply by $z := (z^1, \ldots, z^n)$. This new coordinate system is called an *apt coordinate system (with respect to* Γ*)* (see [**CC1**, Def. 2.3]; also called an *toroidal coordinate system*, see [**AK**, §1.1.12]), which is characterized by the properties

(1) $F = \{(u, v) \in \mathbb{C}^n : u = 0\}$;
(2) each coordinate of the imaginary part $\operatorname{Im} u$ of u is a global function on X and $K = \{(u, v) \bmod \Gamma \in X : \operatorname{Im} u = 0\}$;
(3) the standard basic vectors $e_1, \ldots, e_{n-m}$ in $\mathbb{C}^n$ can be completed to a basis of Γ.

The choice of an apt coordinate system fixes a decomposition $\mathbb{C}^n = E \oplus F$, where E is the complex vector subspace of $\mathbb{C}^n$ spanned by $e_1, \ldots, e_{n-m}$ with u as the coordinate vector. Set $\Gamma' := \Gamma \cap E = \mathbb{Z}\langle e_1, \ldots, e_{n-m}\rangle = \mathbb{Z}^{n-m}$. Let $\tilde{p}\colon \mathbb{C}^n \to F$ be the projection $(u, v) \mapsto v$. It can be seen from (eq 2.2) that $\tilde{p}(\Gamma)$ is a lattice in F, i.e. a discrete subgroup of F of rank $2m$. Let $T^m := F/\tilde{p}(\Gamma)$, which is a complex torus of dimension m. Then $\tilde{p}$ induces a holomorphic epimorphism $p\colon X \to T^m$ with kernel $E/\Gamma' \cong (\mathbb{C}^*)^{n-m}$. Therefore, X has a $(\mathbb{C}^*)^{n-m}$-principal bundle structure given by the exact sequence of groups

(eq 2.3) $$0 \longrightarrow (\mathbb{C}^*)^{n-m} \xrightarrow{\iota} X \xrightarrow{p} T^m \longrightarrow 0$$

(ref. [**St**, §7.4] and [**Hir**, Thm. 3.4.3]). In local coordinates, ι is given by $u \bmod \Gamma' \mapsto (u, 0) \bmod \Gamma$ and p by $(u, v) \bmod \Gamma \mapsto v \bmod \tilde{p}(\Gamma)$. In view of the fibre bundle structure, the tangential directions with respect to the u-coordinates are called the *fibre directions*, while those of the v-coordinates are called the *base directions*. These terminologies are used throughout this article to simplify description.

Since the cotangent bundle of X is trivial, the decomposition $\mathbb{C}^n = E \oplus F$ induces a decomposition of the holomorphic cotangent bundle $\mathbf{T}^{*\,1,0} := \mathbf{T}_X^{*\,1,0}$ of X with respect to the fibre and base directions, i.e.

(eq 2.4) $$\mathbf{T}^{*\,1,0} = \mathbf{T}_u^{*\,1,0} \oplus \mathbf{T}_v^{*\,1,0} ,$$

where $\mathbf{T}_u^{*\,1,0}$ and $\mathbf{T}_v^{*\,1,0}$ are holomorphic subbundles generated at every point of X respectively by du^i for $i = 1, \ldots n-m$ and dv^j for $j = 1, \ldots, m$. For later use, define as usual $\mathbf{T}_v^{*\,p,q} := \bigwedge^p \mathbf{T}_v^{*\,1,0} \wedge \bigwedge^q \overline{\mathbf{T}_v^{*\,1,0}}$ for any integers $p, q \geq 0$, where $\mathbf{T}_v^{*\,0,0} = \bigwedge^0 \mathbf{T}_v^{*\,1,0} = \bigwedge^0 \overline{\mathbf{T}_v^{*\,1,0}}$ denotes the trivial line bundle on X. Define $\mathbf{T}_u^{*\,p,q}$ similarly with $\mathbf{T}_u^*$ in place of $\mathbf{T}_v^*$.

2.2. An exhaustive family of pseudoconvex subsets

Every Cousin-quasi-torus is pseudoconvex and strongly $(m+1)$-complete (cf. [**Kaz1**] and [**Take**]; convention of the numbering here following [**D1**, pp. 512]). Indeed, define $\varphi(z) := \varphi(\operatorname{Im} u) := \|\operatorname{Im} u\|^2$ ($\|\cdot\|$ is the Euclidean 2-norm here). Then φ is an exhaustion function on X whose Levi form is given by

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-m} du^i \wedge \overline{du^i} ,$$

which is semi-positive definite with exactly $n-m$ positive eigenvalues everywhere on X. Therefore, X is pseudoconvex and strongly $(m+1)$-complete.

For any $c > 0$, set $K_c := \{z \in X \ : \ \varphi(z) < c\}$. Then $\{K_c\}_{c>0}$ forms an exhaustive family of open relatively compact subsets of X. Set also $K_\infty := X$, and $K_0 := K$, the maximal compact subgroup of X. For every $c > 0$, K_c is of course itself pseudoconvex.

2.3. Kazama sheaves and Kazama–Dolbeault isomorphism

Let $\mathscr{A} := \mathscr{A}_X$ be the sheaf of germs of smooth functions on X. Fix a choice of an apt coordinate system. Let V be any holomorphic vector bundle on X. Define on X the *Kazama sheaves* as in [**KU2**] to be

$$\mathscr{H} := \left\{ f \in \mathscr{A} \ : \ \frac{\partial f}{\partial \overline{u^i}} \equiv 0 \ \text{ for } 1 \leq i \leq n-m \right\} \quad \text{and}$$

$$\mathscr{H}^{0,q} := \mathscr{H} \otimes_{p^{-1}\mathscr{A}_{T^m}} p^{-1}\mathscr{A}_{T^m}^{0,q} , \quad \mathscr{H}^{0,q}(V) := \mathscr{H}^{0,q} \otimes_{\mathscr{O}_X} \mathscr{O}_X(V) \quad \text{for } 1 \leq q \leq m,$$

where p is the projection given in (eq 2.3) and $\mathscr{A}_{T^m}^{0,q}$ is the sheaf of germs of $(0,q)$-forms on the base torus T^m. In words, Kazama sheaf $\mathscr{H}$ consists of germs of sections of $\mathscr{A}$ which are *holomorphic in the fibre directions*, and $\mathscr{H}^{0,q}$ consists of $\mathscr{H}$-valued *$(0,q)$-forms in the base directions*. Note that the definitions of the sheaves depend on the choice of the decomposition (eq 2.4). Set also $\mathscr{H}^{0,0}(V) := \mathscr{H}(V)$. For notational convenience, the space of sections $\Gamma(U, \mathscr{H}^{0,q}(V))$ over any subset U of X

is also denoted by $\mathscr{H}^{0,q}(U;V)$, and similarly for spaces of sections of other sheaves. The following *Kazama–Dolbeault isomorphism* is proven in [**KU1**] and [**KU2**] (see also [**Kaz2**]).

THEOREM 2.3.1. *The complex*

$$0 \longrightarrow \mathscr{O}_X(V) \longrightarrow \mathscr{H}^{0,0}(V) \xrightarrow{\overline{\partial}} \mathscr{H}^{0,1}(V) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathscr{H}^{0,m}(V) \longrightarrow 0 \tag{eq 2.5}$$

is an acyclic resolution of $\mathscr{O}_X(V)$ over X, i.e. $H^p(X, \mathscr{H}^{0,q}(V)) = 0$ for any $p \geq 1$ and $0 \leq q \leq m$. Consequently, the natural injection of complexes

$$\left(\mathscr{H}^{0,\bullet}(X;V), \overline{\partial}\right) \hookrightarrow \left(\mathscr{A}^{0,\bullet}(X;V), \overline{\partial}\right)$$

induces the isomorphisms

$$H^q_{\overline{\partial}}(\mathscr{H}^{0,\bullet}(X;V)) \cong H^q_{\overline{\partial}}(\mathscr{A}^{0,\bullet}(X;V)) \cong H^q(X,V)$$

for all $q \geq 0$.

In view of the Kazama–Dolbeault isomorphism, to show the vanishing of $H^q(X,V)$ it suffices to show that for any $\overline{\partial}$-closed $\psi \in \mathscr{H}^{0,q}(X;V)$ there exists $\xi \in \mathscr{A}^{0,q-1}(X;V)$ such that

$$\overline{\partial}\xi = \psi \ . \tag{eq 2.6}$$

In fact, (eq 2.6) means that the class ψ mod $\overline{\partial}\mathscr{A}^{0,q-1}(X;V)$ is the zero class in $H^q_{\overline{\partial}}(\mathscr{A}^{0,\bullet}(X;V))$, so, by the isomorphism, the class ψ mod $\overline{\partial}\mathscr{H}^{0,q-1}(X;V)$ is also the zero class in $H^q_{\overline{\partial}}(\mathscr{H}^{0,\bullet}(X;V))$. Therefore, ξ in (eq 2.6) can be chosen in $\mathscr{H}^{0,q-1}(X;V)$.

2.4. Holomorphic line bundles on X

Every holomorphic line bundle L on X can be defined by a system of factors of automorphy, which can be taken into a normal form analogous to the Appell–Humbert normal form, given by (ref. [**CC1**, Remark 1.11 and §2.2] and [**V**, §2])

$$\varrho(\gamma)e^{\pi\mathcal{H}(z,\gamma)+\frac{\pi}{2}\mathcal{H}(\gamma,\gamma)+f_\gamma(z)} \qquad \forall\, \gamma \in \Gamma \ , \tag{eq 2.7}$$

where

- $\mathcal{H}$ is a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$, whose imaginary part $\operatorname{Im}\mathcal{H}$ takes integral values on $\Gamma \times \Gamma$ and corresponds to the first Chern class $c_1(L)$ of L;
- ϱ is a semi-character for $\operatorname{Im}\mathcal{H}$ on Γ, i.e.
$$\varrho(\gamma+\gamma') = \varrho(\gamma)\varrho(\gamma')e^{\pi\sqrt{-1}\operatorname{Im}\mathcal{H}(\gamma,\gamma')}$$
for all $\gamma, \gamma' \in \Gamma$, and $|\varrho(\gamma)| = 1$ for all $\gamma \in \Gamma$; and
- $\{f_\gamma\}_{\gamma\in\Gamma}$ is an additive 1-cocycle with values in $\mathscr{O}_{\mathbb{C}^n}(\mathbb{C}^n)$, i.e. $f_\gamma \in \mathscr{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ for all $\gamma \in \Gamma$ and
$$f_{\gamma+\gamma'}(z) = f_{\gamma'}(z+\gamma) + f_\gamma(z)$$
for all $\gamma, \gamma' \in \Gamma$.

According to [**V**, Prop. 8], under a fixed apt coordinate system, $f_\gamma(z)$ can be taken to be independent of the variable v for every $\gamma \in \Gamma$. Denote by γ_u the image of $\gamma \in \Gamma$ under the projection $\mathbb{C}^n \ni (u,v) \mapsto u \in E$ (see page 4 for the definition of E). Also according to [**V**, Prop. 8] (cf. also [**CC1**, §1.2]), for any $u \in E$, one has

$$\begin{cases} f_{\gamma'}(u) = 0 \\ f_\gamma(u+\gamma'_u) = f_\gamma(u) \end{cases} \qquad \text{for all } \gamma' \in \Gamma' \text{ and } \gamma \in \Gamma \ , \tag{eq 2.8}$$

where $\Gamma' := \Gamma \cap E = \mathbb{Z}\langle e_1, \ldots, e_{n-m}\rangle$ as in §2.1.

It is apparent that L can be decomposed into $L_{\mathsf{t}} \otimes L_{\mathsf{w}}$, where L_{t} is defined by the linear part

$$\varrho(\gamma)e^{\pi\mathcal{H}(z,\gamma)+\frac{\pi}{2}\mathcal{H}(\gamma,\gamma)}$$

of the factor of automorphy in (eq 2.7), while L_{w} is defined by the non-linear part

$$e^{f_\gamma(z)} .$$

Call L_{t} and L_{w} the *tame part* and *wild part* of L respectively.

Definition 2.4.1. L is said to be *linearizable* if L_{w} is trivial, i.e. there exists a holomorphic function g on $\mathbb{C}^n$ such that $g(z+\gamma) - g(z) = f_\gamma(z)$ for all $\gamma \in \Gamma$ and $z \in \mathbb{C}^n$. L is said to be *non-linearizable* otherwise.

$\operatorname{Im}\mathcal{H}$ is uniquely determined only on $\mathbb{R}\Gamma \times \mathbb{R}\Gamma$ (see [**CC1**, Remark 1.11] and also [**AGh**]). Then one has the following proposition.

Proposition 2.4.2. *Let $\mathcal{H}$ be a hermitian form associated to L. Suppose in a chosen apt coordinate system the matrix associated to $\mathcal{H}|_{E\times E}$ is given by H_E. Then, $\operatorname{Re} H_E$ can be chosen arbitrarily by multiplying the cocycle defining L by a suitable coboundary.*

Proof. Fix $\mathcal{H}$ and an apt coordinate system. Let $\mathcal{B}(u,u)$ be any symmetric $\mathbb{C}$-bilinear form with *real* coefficients on $E \times E$ and denote the corresponding $(n-m)\times(n-m)$-matrix under the chosen apt coordinates by B. Note that γ_u is a real vector by the choice of coordinates (see (eq 2.2)). Then multiplying $e^{\frac{\pi}{2}\mathcal{B}(u+\gamma_u,u+\gamma_u)-\frac{\pi}{2}\mathcal{B}(u,u)}$ (which is a component of a 1-coboundary) to (eq 2.7) gives rise to a system of factors of automorphy defining a line bundle isomorphic to L. The new system of factors of automorphy is of the same form as in (eq 2.7) with $\mathcal{H}$ replaced by $\mathcal{H}'$, where $\mathcal{H}'$ is a hermitian form such that $\mathcal{H}'(z,\gamma) = \mathcal{H}(z,\gamma) + \mathcal{B}(u,\gamma_u)$ (note that such hermitian $\mathcal{H}'$ exists since all γ_u's as well as B are real). Then $\operatorname{Re} H'_E = \operatorname{Re} H_E + B$, while the other entries of the matrix of $\operatorname{Im}\mathcal{H}'$ are the same as the respective entries of $\operatorname{Im}\mathcal{H}$. Therefore, since B is arbitrary, $\operatorname{Re} H_E$ can be chosen arbitrarily. ☐

This shows that one cannot, in general, replace s_F^+ and s_F^- in Theorem 1.1.1 by s^+ and s^-, the numbers of positive and negative eigenvalues of $\mathcal{H}$ (instead of $\mathcal{H}|_{F\times F}$) respectively. In fact, if L is the trivial line bundle, $\mathcal{H}$ can be chosen such that $\operatorname{Re} H_E$ is negative definite and the other entries of the matrix associated to $\operatorname{Im}\mathcal{H}$ are zero. Such $\mathcal{H}$ has at least 1 negative eigenvalue. However, $\dim H^0(X,\mathscr{O}_X)$ cannot be 0 since there exist constant functions on X (which is true even for any complex manifold). In fact, Kazama has shown in [**Kaz2**, Thm. 4.3] that $H^q(X,\mathscr{O}_X)$ are non-trivial for all $1 \le q \le m$ for any Cousin-quasi-torus X.

2.5. A hermitian metric on L

Given a holomorphic line bundle L, hermitian metrics η_{t} on the tame part L_{t} and η_{w} on the wild part L_{w} of L are defined below. The product $\eta := \eta_{\mathsf{t}}\eta_{\mathsf{w}}$ then defines a hermitian metric on L.

Define a hermitian metric on L_{t} by $\eta_{\mathsf{t}}(z) := e^{-\pi\mathcal{H}(z,z)}$ as in the compact case. Then the corresponding curvature form on X, called the *tame part of the curvature form of L*, is given by

$$\Theta_{\mathfrak{T}} := -\sqrt{-1}\partial\overline{\partial}\log\eta_{\mathsf{t}} = \pi\sqrt{-1}\partial\overline{\partial}\mathcal{H}(z,z) .$$

Next is to define a hermitian metric η_{w} on L_{w}. An apt coordinate system is fixed in what follows. First notice the following

PROPOSITION 2.5.1. *There exists a smooth function $\hbar$ on $\mathbb{C}^n$ which is holomorphic along the fibre directions under the chosen apt coordinate system and satisfies*

$$\hbar(z+\gamma)-\hbar(z)=f_\gamma(u) \quad \text{for all } \gamma\in\Gamma\ . \tag{eq 2.9}$$

PROOF. This follows from the fact that $H^1(X,\mathscr{H})=0$ (ref. [**KU2**]). A direct proof is given as follows.

Let Γ'' be the subgroup of Γ generated by the last $2m$ column vectors of the period matrix (eq 2.2) of Γ. Then $\Gamma=\Gamma'\oplus\Gamma''$ (Γ' defined as in §2.1). Write $\gamma_v:=\tilde p(\gamma)$ for all $\gamma\in\Gamma$ ($\tilde p$ defined as in §2.1). Note that $\gamma'_v=0$ for all $\gamma'\in\Gamma'$. Recall that $\tilde p(\Gamma)=\tilde p(\Gamma'')$ is the lattice defining T^m in (eq 2.3), therefore discrete in F. Take a suitable smooth function ρ with compact support on F with variable v such that $\sum_{\gamma''\in\Gamma''}\rho(v+\gamma''_v)\equiv 1$. Note that the sum is a sum of finitely many non-zero terms at each $v\in F$ due to the discreteness of Γ''. Define

$$\hbar(z):=-\sum_{\gamma''\in\Gamma''}\rho(v+\gamma''_v)f_{\gamma''}(u)\ .$$

Then $\hbar$ is holomorphic along the fibre directions. To see that it satisfies (eq 2.9), note that, for any $\gamma_0=\gamma'_0+\gamma''_0\in\Gamma$ where $\gamma'_0\in\Gamma'$ and $\gamma''_0\in\Gamma''$,

$$\begin{aligned}\hbar(z+\gamma_0)&=-\sum_{\gamma''\in\Gamma''}\rho(v+\gamma''_v+(\gamma_0)_v)\ f_{\gamma''}(u+(\gamma_0)_u)\\&=-\sum_{\gamma''\in\Gamma''}\rho(v+\gamma''_v+(\gamma''_0)_v)\ f_{\gamma''}(u+(\gamma''_0)_u)\\&=-\sum_{\gamma''\in\Gamma''}\rho(v+\gamma''_v+(\gamma''_0)_v)\left(f_{\gamma''+\gamma''_0}(u)-f_{\gamma''_0}(u)\right)\\&=\hbar(z)+f_{\gamma''_0}(u)\ ,\end{aligned}$$

using the fact that $f_{\gamma''}(u+\gamma'_u)=f_{\gamma''}(u)$ for all $\gamma'\in\Gamma'$ (see (eq 2.8)) and $\Gamma''=\Gamma''+\gamma''_0$. Applying (eq 2.8) again, one obtains

$$f_{\gamma_0}(u)=f_{\gamma''_0}(u+(\gamma'_0)_u)+f_{\gamma'_0}(u)=f_{\gamma''_0}(u)\ .$$

This $\hbar$ therefore satisfies (eq 2.9). □

It follows from (eq 2.9) that $\frac{\partial}{\partial\overline{v^j}}\hbar$ and $\frac{\partial}{\partial\overline{u^i}}\hbar$ define smooth functions on X (note that $f_\gamma(u)$ are holomorphic). Therefore, $\overline\partial\hbar$ is a (smooth) 1-form on X, so is $\partial\overline{\hbar}$.

Take any $\delta\in\mathscr{H}(X)$, and let $\hbar_\delta:=\hbar-\delta$ for notational convenience. Define a hermitian metric on the wild part L_{w} of L by $\eta_{\mathsf{w}}(z):=e^{-2\operatorname{Re}\hbar_\delta(z)}$. The corresponding curvature form, called the *wild part of the curvature form of L*, is given by

$$\begin{aligned}\Theta_{\mathfrak{W}}&:=-\sqrt{-1}\partial\overline\partial\log\eta_{\mathsf{w}}=2\sqrt{-1}\partial\overline\partial\operatorname{Re}\hbar_\delta\\&=\sqrt{-1}d\left(\overline\partial\hbar_\delta-\partial\overline{\hbar}_\delta\right)\ .\end{aligned}$$

Note that, since $\overline\partial\hbar$ is a smooth $(0,1)$-form on X, $\sqrt{-1}d\left(\overline\partial\hbar_\delta-\partial\overline{\hbar}_\delta\right)$ is a d-exact smooth real $(1,1)$-form on X.

The function δ is an auxiliary function which will be chosen suitably according to the Weak $\partial\overline\partial$-Lemma of Takayama [**Taka2**, Lemma 3.14] (see also Lemma 5.1.1) in order to obtain the required L^2 estimates. Details are given in §5.1.

With the chosen η_{t} and η_{w}, a hermitian metric on L is defined by

$$\eta(z) := \eta_{\mathsf{t}}(z)\eta_{\mathsf{w}}(z) = e^{-\pi\mathcal{H}(z,z)-2\operatorname{Re}\hbar_\delta(z)} \ . \tag{eq 2.10}$$

The *curvature form of L with respect to η* is then given by

$$\Theta_{\mathfrak{T}} + \Theta_{\mathfrak{W}} \ ,$$

which represents the class $2\pi c_1(L)$ in $H^2(X,\mathbb{R})$ (while $\Theta_{\mathfrak{T}}$ represents $2\pi c_1(L)$ in $2\pi H^2(X,\mathbb{Z})$).

2.6. An L^2-norm, the L^2-spaces $L^{0,(q',q'')}_{2\,c,\chi}$ and differential operators

Let g be a hermitian metric on X. Fix an apt coordinate system. For the purpose of this article, *g is chosen to be a translational invariant metric such that the decomposition $\mathbf{T}^{1,0} = \mathbf{T}^{1,0}_u \oplus \mathbf{T}^{1,0}_v$ is orthogonal.* Denote by $\omega := -\operatorname{Im} g$ the associated $(1,1)$-form as usual.

Fix any holomorphic line bundle L. Consider any $0 < c \leq \infty$ and $0 \leq q \leq n$. Denote the pointwise 2-norm on $\mathscr{A}^{0,q}(K_c;L)$ induced from the hermitian metrics g and η by $|\cdot|_{g,\eta}$. Let also $\widetilde{\chi}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth function and set $\chi := \widetilde{\chi}\circ\varphi$. For the purpose of this article, *$\widetilde{\chi}$ is always assumed to be a non-negative convex increasing function.* In this case, χ is plurisubharmonic. Set $|\zeta|^2_{g,\eta,\chi} := |\zeta|^2_{g,\eta} e^{-\chi}$. Let μ be the measure induced from the volume form $\frac{\omega^{\wedge n}}{n!}$. Define

$$\|\zeta\|_{K_c,\chi} := \sqrt{\int_{K_c} |\zeta|^2_{g,\eta,\chi}\, d\mu} \qquad \text{for any } \zeta \in \mathscr{A}^{0,q}(K_c;L) \ .$$

Then $\|\cdot\|_{K_c,\chi}$ defines an L^2-norm with weight $e^{-\chi}$ (or simply χ) on $\mathscr{A}^{0,q}_0(K_c;L)$, the space of sections in $\mathscr{A}^{0,q}(K_c;L)$ with compact support. To simplify notation, $d\mu$ in the integral is made implicit in what follows. The inner product corresponding to $\|\cdot\|_{K_c,\chi}$ is denoted by $\langle\cdot,\cdot\rangle_{K_c,\chi}$. The norm is written as $\|\cdot\|_{K_c,g,\eta,\chi}$ to emphasize its dependence on g and η when necessary.

Denote by $L^{0,q}_{2\,c,\chi} := L^{0,q}_{2\,\chi}(K_c;L)$ the Hilbert space of (μ-)measurable L-valued $(0,q)$-forms ζ on K_c such that $\|\zeta\|_{K_c,\chi} < \infty$. It is well known that $\mathscr{A}^{0,q}_0(K_c;L) \subset L^{0,q}_{2\,c,\chi}$ is a dense subspace under the norm $\|\cdot\|_{K_c,\chi}$.

For any $0 \leq p',q' \leq n-m$ and $0 \leq p'',q'' \leq m$, define

$$\mathscr{A}^{(p',p''),(q',q'')} := \mathscr{A}\left(\mathbf{T}^{*\,p',q'}_u \wedge \mathbf{T}^{*\,p'',q''}_v\right) \ ,$$

i.e. a sheaf of germs of smooth sections of $\mathbf{T}^{*\,p',q'}_u \wedge \mathbf{T}^{*\,p'',q''}_v$ (defined in §2.1). For other values of p',p'',q' and q'', set $\mathscr{A}^{(p',p''),(q',q'')} := 0$. Note that, for $0 \leq p,q \leq n$, there is a decomposition

$$\mathscr{A}^{p,q} = \bigoplus_{\substack{p'+p''=p\\ q'+q''=q}} \mathscr{A}^{(p',p''),(q',q'')} \ . \tag{eq 2.11}$$

This decomposition depends on the choice of the decomposition (eq 2.4). Since the fibre and base directions are orthogonal to each other with respect to g, the decomposition is also orthogonal with respect to g. As only those sheaves with $p'+p''=0$ are considered in what follows, set

$$\mathscr{A}^{0,(q',q'')} := \mathscr{A}^{(0,0),(q',q'')}$$

for notational convenience. Notice that $\mathscr{H}^{0,q''}(L)$ is a subsheaf of $\mathscr{A}^{0,(0,q'')}(L)$ for $0 \leq q'' \leq m$. For any $c > 0$, denote also the space of sections in $\mathscr{A}^{0,(q',q'')}(K_c; L)$ with compact support by $\mathscr{A}_0^{0,(q',q'')}(K_c; L)$. Define

$$L^{0,(q',q'')}_{2\,c,\chi} := L^{0,(q',q'')}_{2\,\chi}(K_c; L) := \overline{\mathscr{A}_0^{0,(q',q'')}(K_c; L)} \ ,$$

i.e. the closure of $\mathscr{A}_0^{0,(q',q'')}(K_c; L)$ in $\left(L^{0,q'+q''}_{2\,c,\chi}, \|\cdot\|_{K_c,\chi}\right)$. Note that the decomposition

$$L^{0,q}_{2\,c,\chi} = \bigoplus_{q'+q''=q} L^{0,(q',q'')}_{2\,c,\chi} \tag{eq 2.12}$$

induced from (eq 2.11) is also an orthogonal decomposition.

The operator $\overline{\partial}$ is decomposed into $\overline{\partial}_{[u]} + \overline{\partial}_{[v]}$ according to the decomposition (eq 2.4), where $\overline{\partial}_{[u]}$ and $\overline{\partial}_{[v]}$ are operators such that

$$\overline{\partial}_{[u]} \colon \mathscr{A}^{0,(q',q'')}(K_c; L) \to \mathscr{A}^{0,(q'+1,q'')}(K_c; L) \quad \text{and}$$
$$\overline{\partial}_{[v]} \colon \mathscr{A}^{0,(q',q'')}(K_c; L) \to \mathscr{A}^{0,(q',q''+1)}(K_c; L) \ .$$

Denote the *formal adjoints* of $\overline{\partial}_{[u]}$ and $\overline{\partial}_{[v]}$ above respectively by

$$\vartheta_{[u]} \colon \mathscr{A}^{0,(q'+1,q'')}(K_c; L) \to \mathscr{A}^{0,(q',q'')}(K_c; L) \quad \text{and}$$
$$\vartheta_{[v]} \colon \mathscr{A}^{0,(q',q''+1)}(K_c; L) \to \mathscr{A}^{0,(q',q'')}(K_c; L)$$

(see, for example, [**D1**, Ch. VI, 1.5] for the definition).

Some basic facts about differential operators on Hilbert spaces are recalled here. Extend the action of these operators to $L^{0,(q',q'')}_{2\,c,\chi}$ in the sense of distributions (or currents). Then, they define *closed* (i.e. having closed graph) and *densely defined* linear operators on $L^{0,(q',q'')}_{2\,c,\chi}$ (see, for example, [**Hör2**, Ch. 1] and [**D2**, Prop. 4.9]) with *domain* given by

$$\mathrm{Dom}^{(q',q'')}_{K_c,\chi} T \ \ (\text{or } \mathrm{Dom}\, T) := \left\{ \zeta \in L^{0,(q',q'')}_{2\,c,\chi} \ : \ \|T\zeta\|_{K_c,\chi} < \infty \right\} \ , \tag{eq 2.13}$$

where T denotes any of the above operators. Note that T is densely defined since $\mathscr{A}_0^{0,(q',q'')}(K_c; L) \subset \mathrm{Dom}^{(q',q'')}_{K_c,\chi} T$. An operator will be written as $(T, \mathrm{Dom}\, T)$ when the domain is emphasized.

Given $\overline{\partial}_{[u]} \colon L^{0,(q',q'')}_{2\,c,\chi} \to L^{0,(q'+1,q'')}_{2\,c,\chi}$ and $\overline{\partial}_{[v]} \colon L^{0,(q',q'')}_{2\,c,\chi} \to L^{0,(q',q''+1)}_{2\,c,\chi}$ with domains given as in (eq 2.13), their Hilbert space adjoints (also called Von Neumann's adjoints, see for example [**D1**, Ch. VIII, §1] for a discussion on them) are denoted respectively by

$$\overline{\partial}^*_{[u]} \colon L^{0,(q'+1,q'')}_{2\,c,\chi} \to L^{0,(q',q'')}_{2\,c,\chi} \quad \text{and} \quad \overline{\partial}^*_{[v]} \colon L^{0,(q',q''+1)}_{2\,c,\chi} \to L^{0,(q',q'')}_{2\,c,\chi} \ ,$$

which are closed and densely defined operators on $L^{0,(q'+1,q'')}_{2\,c,\chi}$ and $L^{0,(q',q''+1)}_{2\,c,\chi}$ respectively. Denote also their domains of definition respectively by $\mathrm{Dom}^{(q'+1,q'')}_{K_c,\chi} \overline{\partial}^*_{[u]}$ and $\mathrm{Dom}^{(q',q''+1)}_{K_c,\chi} \overline{\partial}^*_{[v]}$.

In general, one has $\mathrm{Dom}^{(q'+1,q'')}_{K_c,\chi} \overline{\partial}^*_{[u]} \subset \mathrm{Dom}^{(q'+1,q'')}_{K_c,\chi} \vartheta_{[u]}$ and $\overline{\partial}^*_{[u]}\zeta = \vartheta_{[u]}\zeta$ for all $\zeta \in \mathrm{Dom}^{(q'+1,q'')}_{K_c,\chi} \overline{\partial}^*_{[u]}$ (see, for example, [**D1**, Ch. VIII, §3]). The same holds true for $\overline{\partial}^*_{[v]}$ and $\vartheta_{[v]}$.

CHAPTER 3

L^2 estimates

3.1. Existence of a solution of $\overline{\partial}\xi = \psi$

The aim of this section is to show that, for $0 \le q \le m$, given $\psi \in \mathscr{H}^{0,q}(K_c; L) \cap L_2{}^{0,(0,q)}_{c,\chi}$ such that $\overline{\partial}\psi = 0$ on K_c, there exists a weak solution $\xi \in L_2{}^{0,(0,q-1)}_{c,\chi}$ of the $\overline{\partial}$-equation $\overline{\partial}\xi = \psi$ provided that an L^2 estimate is satisfied. When $c = \infty$, there exists a strong solution which lies in $\mathscr{H}^{0,q-1}(X; L)$.

First recall the following classical theorems for L^2 estimates (see, for example, [**Hör3**, Lemmas 4.1.1 and 4.1.2] or [**D1**, Ch. VIII, Thm. 1.2]). Let $(\mathfrak{H}_1, \langle\cdot,\cdot\rangle_1)$, $(\mathfrak{H}_2, \langle\cdot,\cdot\rangle_2)$ and $(\mathfrak{H}_3, \langle\cdot,\cdot\rangle_3)$ be some Hilbert spaces, and let $(S, \operatorname{Dom} S)$ and $(T, \operatorname{Dom} T)$ be two closed (i.e. closed graph) and densely defined linear operators with domains $\operatorname{Dom} S \subset \mathfrak{H}_2$ and $\operatorname{Dom} T \subset \mathfrak{H}_1$ respectively such that

$$\mathfrak{H}_1 \xrightarrow{T} \mathfrak{H}_2 \xrightarrow{S} \mathfrak{H}_3$$

and $S \circ T = 0$, i.e. $T(\operatorname{Dom} T) \subset \ker S := \{\zeta \in \operatorname{Dom} S : S\zeta = 0\}$. Let S^* and T^* denote the Hilbert space adjoints of S and T respectively, which are also closed, densely defined and satisfies $T^* \circ S^* = 0$ (see, for example, [**D1**, Ch. VIII, Thm. 1.1]).

Theorem 3.1.1 (see [**Hör3**, Lemmas 4.1.1 and 4.1.2]). *If there exists a constant $C > 0$ such that*

$$\|S\zeta\|_3^2 + \|T^*\zeta\|_1^2 \ge C\,\|\zeta\|_2^2 \quad \text{for all } \zeta \in \operatorname{Dom} S \cap \operatorname{Dom} T^* , \tag{eq 3.1}$$

then

(1) *for every $\psi \in \ker S$, there exists $\xi \in \overline{\operatorname{im} T^*} \cap \operatorname{Dom} T$ such that $T\xi = \psi$ and $\|\xi\|_1^2 \le \frac{1}{C}\|\psi\|_2^2$. In other words, $\ker S = \operatorname{im} T$ (and thus $\operatorname{im} T$ is closed as $\ker S$ is so);*

(2) *for every $\Psi \in (\ker T)^\perp = \overline{\operatorname{im} T^*}$, there exists $\Xi \in \overline{\operatorname{im} T} \cap \operatorname{Dom} T^*$ such that $T^*\Xi = \Psi$ and $\|\Xi\|_2^2 \le \frac{1}{C}\|\Psi\|_1^2$. In other words, $\overline{\operatorname{im} T^*} = \operatorname{im} T^*$.*

Remark 3.1.2. By exchanging the roles of S and T^*, one also gets $\ker T^* = \operatorname{im} S^*$ and $\overline{\operatorname{im} S} = \operatorname{im} S$ if the L^2 estimate (eq 3.1) is satisfied.

When X is compact, consider the complex

$$L_2^{0,q-1}(X; L) \xrightarrow{\overline{\partial}} L_2^{0,q}(X; L) \xrightarrow{\overline{\partial}} L_2^{0,q+1}(X; L) .$$

Murakami [**Mur**] shows that the L^2 estimates (eq 3.1) hold for $q < s^-$ or $q > n - s^+$ by choosing the hermitian metric g suitably. The L^2 estimate on $L_2^{0,q}(X; L)$ implies that the harmonic L-valued $(0,q)$-forms must vanish. Elements in $H^q(X, L)$ are represented by harmonic forms when X is compact, so this proves the vanishing of $H^q(X, L)$ in the compact case.

In the current situation, although elements in $H^q(X, L)$ are not represented by harmonic forms in general, the L^2 estimate (eq 3.1) is still useful in solving $\overline{\partial}$-equations which leads to the vanishing of $H^q(X, L)$ for suitable q's according to Theorem 3.1.1 (1).

Due to the existence of non-linearizable line bundles, it turns out it is necessary to solve $\overline{\partial}$-equation on K_c for any $0 < c < \infty$ (see §5.1). Therefore, the aim now is to solve the $\overline{\partial}$-equation $\overline{\partial}\xi = \psi|_{K_c}$ for a given $\psi \in \mathscr{H}^{0,q}(X; L)$ with $\overline{\partial}\psi = 0$. In view of the fibre bundle structure (eq 2.3), instead of considering the complex $L^{0,q-1}_{2\;c,\chi} \xrightarrow{\overline{\partial}} L^{0,q}_{2\;c,\chi} \xrightarrow{\overline{\partial}} L^{0,q+1}_{2\;c,\chi}$, it is natural (see the discussion in §1.2) to consider the subcomplex

$$L^{0,(0,q-1)}_{2\;c,\chi} \underset{T^*_{q-1}}{\overset{T_{q-1}}{\rightleftarrows}} L^{0,q}_{2\;c,\chi\,<2>} \underset{S^*_q}{\overset{S_q}{\rightleftarrows}} L^{0,q+1}_{2\;c,\chi\,<3>} , \tag{eq 3.2}$$

where T_{q-1} and S_q act as $\overline{\partial}$ on $L^{0,(0,q-1)}_{2\;c,\chi}$ and $L^{0,q}_{2\;c,\chi\,<2>}$ respectively, and T^*_{q-1} and S^*_q are their Hilbert space adjoints.[1] The Hilbert spaces in the complex are defined as

$$\begin{aligned}
\mathscr{A}^{0,q}_{<2>}(K_c; L) &:= \mathscr{A}^{0,(1,q-1)} \oplus \mathscr{A}^{0,(0,q)}(K_c; L) , \\
\mathscr{A}^{0,q+1}_{<3>}(K_c; L) &:= \mathscr{A}^{0,(2,q-1)} \oplus \mathscr{A}^{0,(1,q)} \oplus \mathscr{A}^{0,(0,q+1)}(K_c; L) ; \\
L^{0,q}_{2\;c,\chi\,<2>} &:= \overline{\mathscr{A}^{0,q}_{0\,<2>}(K_c; L)} = L^{0,(1,q-1)}_{2\;c,\chi} \oplus L^{0,(0,q)}_{2\;c,\chi} , \\
L^{0,q+1}_{2\;c,\chi\,<3>} &:= \overline{\mathscr{A}^{0,q+1}_{0\,<3>}(K_c; L)} = L^{0,(2,q-1)}_{2\;c,\chi} \oplus L^{0,(1,q)}_{2\;c,\chi} \oplus L^{0,(0,q+1)}_{2\;c,\chi} .
\end{aligned}$$

Recall from (eq 2.11) and (eq 2.12) that all the direct sums on the right hand sides above are orthogonal decompositions. Denote the norms on $L^{0,(0,q-1)}_{2\;c,\chi}$, $L^{0,q}_{2\;c,\chi\,<2>}$ and $L^{0,q+1}_{2\;c,\chi\,<3>}$ respectively by $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$, and their inner products by $\langle\cdot,\cdot\rangle$ with the corresponding subscripts.

Write the Hilbert space adjoint of $\overline{\partial}\colon L^{0,q-1}_{2\;c,\chi} \to L^{0,q}_{2\;c,\chi}$ as $\overline{\partial}^*$. Let $\mathrm{pr}\colon L^{0,q-1}_{2\;c,\chi} \to L^{0,(0,q-1)}_{2\;c,\chi}$ be the orthogonal projection. For later use, $(T^*_{q-1}, \operatorname{Dom} T^*_{q-1})$ is described more explicitly.

PROPOSITION 3.1.3. *With the notation described above, one has*

$$\begin{aligned}
\operatorname{Dom} T^*_{q-1} &= \operatorname{Dom}_{K_c,\chi} \overline{\partial}^* \cap L^{0,q}_{2\;c,\chi\,<2>} \\
&= \operatorname{Dom}^{(1,q-1)}_{K_c,\chi} \overline{\partial}^*_{[u]} \oplus \operatorname{Dom}^{(0,q)}_{K_c,\chi} \overline{\partial}^*_{[v]} .
\end{aligned}$$

Moreover, for any $\zeta = \zeta' + \zeta'' \in \operatorname{Dom} T^*_{q-1}$ *where* $\zeta' \in \operatorname{Dom}^{(1,q-1)}_{K_c,\chi} \overline{\partial}^*_{[u]}$ *and* $\zeta'' \in \operatorname{Dom}^{(0,q)}_{K_c,\chi} \overline{\partial}^*_{[v]}$, *one has* $T^*_{q-1}\zeta = \mathrm{pr}\,\overline{\partial}^*\zeta = \overline{\partial}^*_{[u]}\zeta' + \overline{\partial}^*_{[v]}\zeta''$.

PROOF. Define operators $(W_1, \operatorname{Dom} W_1)$ and $(W_2, \operatorname{Dom} W_2)$ from $L^{0,q}_{2\;c,\chi\,<2>}$ into $L^{0,(0,q-1)}_{2\;c,\chi}$ such that

$$\begin{aligned}
\operatorname{Dom} W_1 &:= \operatorname{Dom}_{K_c,\chi} \overline{\partial}^* \cap L^{0,q}_{2\;c,\chi\,<2>} , \\
\operatorname{Dom} W_2 &:= \operatorname{Dom}^{(1,q-1)}_{K_c,\chi} \overline{\partial}^*_{[u]} \oplus \operatorname{Dom}^{(0,q)}_{K_c,\chi} \overline{\partial}^*_{[v]} ,
\end{aligned}$$

and

$$\begin{aligned}
W_1\zeta &:= \mathrm{pr}\,\overline{\partial}^*\zeta && \text{for } \zeta \in \operatorname{Dom} W_1 , \\
W_2\zeta &:= \overline{\partial}^*_{[u]}\zeta' + \overline{\partial}^*_{[v]}\zeta'' && \text{for } \zeta = \zeta' + \zeta'' \in \operatorname{Dom} W_2 .
\end{aligned}$$

[1] The symbol T_{q-1} (resp. S_q) is used instead of $\overline{\partial}$ so that the domains and codomains of the two operators can be distinguished. More precisely, if $\iota\colon L^{0,(0,q-1)}_{2\;c,\chi} \hookrightarrow L^{0,q-1}_{2\;c,\chi}$ and $\mathrm{pr}\colon L^{0,q}_{2\;c,\chi} \to L^{0,q}_{2\;c,\chi\,<2>}$ are respectively the inclusion and projection, then $T_{q-1} = \mathrm{pr} \circ \overline{\partial} \circ \iota$. Therefore, T^*_{q-1} and $\overline{\partial}^*$ are different operators.

These are closed and densely defined linear operators on $L_2{}^{0,q}_{c,\chi\,<2>}$. Since $\|T_{q-1}\zeta\|_2^2 = \left\|\overline{\partial}\zeta\right\|_2^2 = \left\|\overline{\partial}_{[u]}\zeta\right\|_2^2 + \left\|\overline{\partial}_{[v]}\zeta\right\|_2^2$ for all $\zeta \in L_2{}^{0,(0,q-1)}_{c,\chi}$, it follows that

$$\begin{aligned}\operatorname{Dom} T_{q-1} &= \operatorname{Dom}\overline{\partial} \cap L_2{}^{0,(0,q-1)}_{c,\chi} \\ &= \operatorname{Dom}^{(0,q-1)}_{K_c,\chi}\overline{\partial}_{[u]} \cap \operatorname{Dom}^{(0,q-1)}_{K_c,\chi}\overline{\partial}_{[v]}\ .\end{aligned}$$

First is to show that $(T^*_{q-1}, \operatorname{Dom} T^*_{q-1}) = (W_1, \operatorname{Dom} W_1)$. Note that, for any $f \in L_2{}^{0,(0,q-1)}_{c,\chi}$ and any $\zeta \in \operatorname{Dom} W_1$, one has

$$\langle f, W_1\zeta\rangle_1 = \left\langle f, \operatorname{pr}\overline{\partial}^*\zeta\right\rangle_1 = \left\langle f, \overline{\partial}^*\zeta\right\rangle_{K_c,\chi}\ .$$

For any $\tilde{\zeta} \in L_2{}^{0,q}_{c,\chi} = L_2{}^{0,q}_{c,\chi\,<2>} \oplus \left(L_2{}^{0,q}_{c,\chi\,<2>}\right)^\perp$, write $\tilde{\zeta} = \zeta + \zeta^\perp$ where $\zeta \in L_2{}^{0,q}_{c,\chi\,<2>}$ and $\zeta^\perp \in \left(L_2{}^{0,q}_{c,\chi\,<2>}\right)^\perp = \bigoplus_{q'=2}^{q} L_2{}^{0,(q',q-q')}_{c,\chi}$. Note that $\overline{\partial}^*\zeta^\perp \in \bigoplus_{q'=1}^{q-1} L_2{}^{0,(q',q-1-q')}_{c,\chi} = \left(L_2{}^{0,(0,q-1)}_{c,\chi}\right)^\perp$, thus $\left\langle f, \overline{\partial}^*\zeta^\perp\right\rangle_{K_c,\chi} = 0$ for any $f \in L_2{}^{0,(0,q-1)}_{c,\chi}$. Therefore, for any $f \in L_2{}^{0,(0,q-1)}_{c,\chi}$, one has

$$\begin{aligned}& f \in \operatorname{Dom} W_1^* \\ :\iff& \exists\, C > 0\colon\ \forall\, \zeta \in \operatorname{Dom} W_1\ ,\ |\langle f, W_1\zeta\rangle_1| = \left|\left\langle f, \overline{\partial}^*\zeta\right\rangle_{K_c,\chi}\right| \le C\,\|\zeta\|_2 \\ \iff& \exists\, C > 0\colon\ \forall\, \tilde{\zeta} \in \operatorname{Dom}_{K_c,\chi}\overline{\partial}^*\ , \\ &\qquad \left|\left\langle f, \overline{\partial}^*\tilde{\zeta}\right\rangle_{K_c,\chi}\right| = \left|\left\langle f, \overline{\partial}^*\zeta\right\rangle_{K_c,\chi}\right| \le C\left\|\tilde{\zeta}\right\|_{K_c,\chi} \\ \iff& f \in \operatorname{Dom}\overline{\partial} \cap L_2{}^{0,(0,q-1)}_{c,\chi} = \operatorname{Dom} T_{q-1} \qquad \text{as } \left(\overline{\partial}^*\right)^* = \overline{\partial}\end{aligned}$$

(ref. [**D1**, Ch. VIII, §1] for the definition of the domain of Hilbert space adjoints), and thus $\operatorname{Dom} W_1^* = \operatorname{Dom} T_{q-1}$. It follows that $\langle f, W_1\zeta\rangle_1 = \left\langle f, \overline{\partial}^*\zeta\right\rangle_{K_c,\chi} = \left\langle \overline{\partial}f, \zeta\right\rangle_2 = \langle T_{q-1}f, \zeta\rangle_2$ for any $f \in \operatorname{Dom} T_{q-1}$ and $\zeta \in \operatorname{Dom} W_1$. As a result, $(T_{q-1}, \operatorname{Dom} T_{q-1}) = (W_1^*, \operatorname{Dom} W_1^*)$, and hence $(T^*_{q-1}, \operatorname{Dom} T^*_{q-1}) = (W_1, \operatorname{Dom} W_1)$ (ref. [**D1**, Ch. VIII, Thm. 1.1]).

The proof of $(T^*_{q-1}, \operatorname{Dom} T^*_{q-1}) = (W_2, \operatorname{Dom} W_2)$ is similar. Notice that $\|\zeta\|_2^2 = \|\zeta'\|_2^2 + \|\zeta''\|_2^2$ and thus $\|\zeta'\|_2 + \|\zeta''\|_2 \le \sqrt{2}\,\|\zeta\|_2$ for all $\zeta = \zeta' + \zeta'' \in L_2{}^{0,q}_{c,\chi\,<2>}$. Then, for any $f \in L_2{}^{0,(0,q-1)}_{c,\chi}$, one has

$$\begin{aligned}& f \in \operatorname{Dom} W_2^* \\ :\iff& \exists\, C > 0\colon\ \forall\, \zeta = \zeta' + \zeta'' \in \operatorname{Dom} W_2\ , \\ &\qquad |\langle f, W_2\zeta\rangle_1| = \left|\left\langle f, \overline{\partial}^*_{[u]}\zeta' + \overline{\partial}^*_{[v]}\zeta''\right\rangle_1\right| \le C\,\|\zeta\|_2 \\ \iff& \exists\, C > 0\colon\ \forall\, \zeta' \in \operatorname{Dom}^{(1,q-1)}_{K_c,\chi}\overline{\partial}^*_{[u]} \text{ and } \forall\, \zeta'' \in \operatorname{Dom}^{(0,q)}_{K_c,\chi}\overline{\partial}^*_{[v]}\ , \\ &\qquad \left|\left\langle f, \overline{\partial}^*_{[u]}\zeta'\right\rangle_1\right| \le C\,\|\zeta'\|_2 \text{ and } \left|\left\langle f, \overline{\partial}^*_{[v]}\zeta''\right\rangle_1\right| \le C\,\|\zeta''\|_2 \\ \iff& f \in \operatorname{Dom}^{(0,q-1)}_{K_c,\chi}\overline{\partial}_{[u]} \cap \operatorname{Dom}^{(0,q-1)}_{K_c,\chi}\overline{\partial}_{[v]} = \operatorname{Dom} T_{q-1}\ ,\end{aligned}$$

and thus $\operatorname{Dom} W_2^* = \operatorname{Dom} T_{q-1}$. Note that $\langle f, W_2\zeta\rangle_1 = \left\langle \overline{\partial}_{[u]}f, \zeta'\right\rangle_2 + \left\langle \overline{\partial}_{[v]}f, \zeta''\right\rangle_2 = \left\langle \overline{\partial}_{[u]}f + \overline{\partial}_{[v]}f, \zeta' + \zeta''\right\rangle_2 = \langle T_{q-1}f, \zeta\rangle_2$ for $f \in \operatorname{Dom} T_{q-1}$ and $\zeta \in \operatorname{Dom} W_2$, since

$L^{2\,0,(1,q-1)}_{\;c,\chi} \perp L^{2\,0,(0,q)}_{\;c,\chi}$. Therefore, one has $(T_{q-1}, \operatorname{Dom} T_{q-1}) = (W_2^*, \operatorname{Dom} W_2^*)$, and thus $(T_{q-1}^*, \operatorname{Dom} T_{q-1}^*) = (W_2, \operatorname{Dom} W_2)$ (ref. [**D1**, Ch. VIII, Thm. 1.1]). □

Suppose now given $0 < c \leq \infty$ and $\psi \in \mathscr{H}^{0,q}(K_c; L) \cap L^{2\,0,(0,q)}_{\;c,\chi} \subset L^{2\,0,q}_{\;c,\chi<2>}$ such that $S_q\psi = \overline{\partial}\psi = 0$. Theorem 3.1.1 (1) asserts that, if the L^2 estimate (eq 3.1) is satisfied, then there exists $\xi \in \overline{\operatorname{im} T_{q-1}^*} \subset L^{2\,0,(0,q-1)}_{\;c,\chi}$ such that

$$T_{q-1}\xi = \overline{\partial}\xi = \psi \quad \text{in } L^{2\,0,(0,q)}_{\;c,\chi} . \tag{eq 3.3}$$

One can have a further reduction. When $c = \infty$, since (X, g) is complete in the sense of Riemannian geometry, $\mathscr{A}^{0,q}_{0\,<2>}(X; L)$ is dense in $\operatorname{Dom}_X T_{q-1}^* \cap \operatorname{Dom}_X S_q$ under the above graph norm (see, for example, [**D1**, Ch. VIII, Thm. 3.2]). Therefore, it suffices to establish the required L^2 estimates (eq 3.1) for $\zeta \in \mathscr{A}^{0,q}_{0\,<2>}(X; L)$.

Suppose $c < \infty$. Note that $\mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \subset \operatorname{Dom} S_q$. Since ∂K_c is smooth and χ is smooth on a neighborhood of $\overline{K}_c$, using [**Hör1**, Prop. 2.1.1] together with an argument of partition of unity, it yields the following

PROPOSITION 3.1.4. *$\mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom} T_{q-1}^*$ is dense in $\operatorname{Dom} T_{q-1}^* \cap \operatorname{Dom} S_q$ under the graph norm* $\sqrt{\left\|T_{q-1}^*\zeta\right\|_1^2 + \|S_q\zeta\|_3^2 + \|\zeta\|_2^2}$.

PROOF. Note that the statement follows from [**Hör1**, Prop. 2.1.1] when $\mathbf{T}^{*\,0,q}_X$ and L are both trivial by using a partition of unity. The aim now is to handle the case when L is non-trivial.

Take a locally finite open cover $\{U_\alpha\}_{\alpha\in A}$ of X such that every U_α is a coordinate chart of X and L is trivialized on each U_α with transition functions $\sigma_{\alpha\beta} \in \mathscr{O}_X^*(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in A$. Then, for any $\zeta \in L^{2\,0,q}_{\;\chi}(X; L)$ with ζ_α representing ζ over U_α under the trivialization, one has $\zeta_\alpha = \sigma_{\alpha\beta}\zeta_\beta$ on $U_\alpha \cap U_\beta$.

Fix any $\zeta \in \operatorname{Dom} T_{q-1}^* \cap \operatorname{Dom} S_q$. It suffices to show that ζ can be approximated by a sequence $\left\{\zeta^{(\nu)}\right\}_{\nu\in\mathbb{N}} \subset \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom} T_{q-1}^*$ under the given graph norm.

Extend ζ by zero to a section on X. Using a partition of unity which decomposes ζ into a sum of finitely many compactly supported sections, one can assume that ζ is compactly supported in a coordinate chart $U := U_0 \in \{U_\alpha\}_{\alpha\in A}$. Then the hermitian metric η on L can be viewed as a function $\widetilde{\eta} := \eta_0$ on $U = U_0$ (under the given trivialization), and any L-valued form $f \in L^{2\,0,q}_{\;g,\eta,\chi}(U; L)$ can be viewed as a $\mathscr{O}_X$-valued form $\widetilde{f} := f_0 \in L^{2\,0,q}_{\;g,\widetilde{\eta},\chi}(U)$. Let $W := U \cap K_c$. Note that one has $\left\|\widetilde{f}\right\|_{W,g,\widetilde{\eta},\chi} = \|f\|_{W,g,\eta,\chi}$, $\left\|\overline{\partial}\widetilde{f}\right\|_{W,g,\widetilde{\eta},\chi} = \left\|\overline{\partial} f\right\|_{W,g,\eta,\chi}$ and $\left\|\overline{\partial}^*\widetilde{f}\right\|_{W,g,\widetilde{\eta},\chi} = \left\|\overline{\partial}^* f\right\|_{W,g,\eta,\chi}$ for all $f \in L^{2\,0,q}_{\;g,\eta,\chi}(W; L)$. Then $\zeta \in \operatorname{Dom} T_{q-1}^* \cap \operatorname{Dom} S_q$ implies $\widetilde{\zeta} \in \operatorname{Dom}_{W,g,\widetilde{\eta},\chi} \overline{\partial}^* \cap \operatorname{Dom}_{W,g,\widetilde{\eta},\chi} \overline{\partial} \cap L^{2\,0,q}_{\;g,\widetilde{\eta},\chi<2>}(W)$. Since g and χ are fixed in what follows, subscripts of them are omitted from the notations below.

By [**Hör1**, Prop. 2.1.1] (or applying [**Hör1**, Prop. 1.2.4] directly), there exists a sequence $\left\{\widetilde{\zeta}^{(\nu)}\right\}_{\nu\in\mathbb{N}} \subset \mathscr{A}^{0,q}(\overline{W}) \cap \operatorname{Dom}_{W,\widetilde{\eta}} \overline{\partial}^*$ such that

$$\left\|\overline{\partial}^*\left(\widetilde{\zeta}^{(\nu)} - \widetilde{\zeta}\right)\right\|_{W,\widetilde{\eta}}^2 + \left\|\overline{\partial}\left(\widetilde{\zeta}^{(\nu)} - \widetilde{\zeta}\right)\right\|_{W,\widetilde{\eta}}^2 + \left\|\widetilde{\zeta}^{(\nu)} - \widetilde{\zeta}\right\|_{W,\widetilde{\eta}}^2 \to 0$$

as $\nu \to \infty$ and $\operatorname{supp}\widetilde{\zeta}^{(\nu)} \Subset U$ for all $\nu \in \mathbb{N}$. As $\widetilde{\zeta}^{(\nu)}$'s are obtained from convolutions between smoothing kernels and $\widetilde{\zeta}$ which do not change the type of forms, it follows that $\widetilde{\zeta}^{(\nu)} \in \mathscr{A}^{0,q}_{<2>}(\overline{W})$. The sections $\zeta^{(\nu)} \in \mathscr{A}^{0,q}_{<2>}(\overline{W}; L)$ defined by $\zeta_\alpha^{(\nu)} := \frac{1}{\sigma_{0\alpha}}\widetilde{\zeta}^{(\nu)}$ on $U_\alpha \cap U \neq \emptyset$ are compactly supported in U (hence $\zeta^{(\nu)} \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$) and satisfy $\widetilde{\zeta^{(\nu)}} = \widetilde{\zeta}^{(\nu)}$. Therefore, one obtains a sequence $\left\{\zeta^{(\nu)}\right\}_{\nu\in\mathbb{N}} \subset \operatorname{Dom}_{K_c,\eta}\overline{\partial}^* \cap \mathscr{A}^{0,q}_{<2>}(\overline{K}_c;L) = \operatorname{Dom} T^*_{q-1} \cap \mathscr{A}^{0,q}_{<2>}(\overline{K}_c;L)$ (see Proposition 3.1.3) such that

$$
\begin{aligned}
&\left\|T^*_{q-1}\left(\zeta^{(\nu)}-\zeta\right)\right\|_1^2 + \left\|S_q\left(\zeta^{(\nu)}-\zeta\right)\right\|_3^2 + \left\|\zeta^{(\nu)}-\zeta\right\|_2^2 \\
\leq\ & \left\|\overline{\partial}^*\left(\zeta^{(\nu)}-\zeta\right)\right\|_{W,\eta}^2 + \left\|\overline{\partial}\left(\zeta^{(\nu)}-\zeta\right)\right\|_{W,\eta}^2 + \left\|\zeta^{(\nu)}-\zeta\right\|_{W,\eta}^2 && \text{as } T^*_{q-1} = \operatorname{pr}\overline{\partial}^* \text{ by Prop. 3.1.3} \\
=\ & \left\|\overline{\partial}^*\left(\widetilde{\zeta}^{(\nu)}-\widetilde{\zeta}\right)\right\|_{W,\widetilde{\eta}}^2 + \left\|\overline{\partial}\left(\widetilde{\zeta}^{(\nu)}-\widetilde{\zeta}\right)\right\|_{W,\widetilde{\eta}}^2 + \left\|\widetilde{\zeta}^{(\nu)}-\widetilde{\zeta}\right\|_{W,\widetilde{\eta}}^2 \\
\to\ & 0 && \text{as } \nu\to\infty
\end{aligned}
$$

as required. □

As a result, it suffices to establish the required L^2 estimates (eq 3.1) for $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c;L) \cap \operatorname{Dom} T^*_{q-1}$.

The above discussion is summarized in the following

PROPOSITION 3.1.5. *Suppose* $0 < c \leq \infty$. *If there exists a constant* $C > 0$ *such that*

(eq 3.4) $$\left\|S_q\zeta\right\|_3^2 + \left\|T^*_{q-1}\zeta\right\|_1^2 \geq C\left\|\zeta\right\|_2^2$$

$$\textit{for all } \zeta \in \begin{cases} \mathscr{A}^{0,q}_{<2>}(\overline{K}_c;L)\cap \operatorname{Dom} T^*_{q-1} & \textit{when } c<\infty\,, \\ \mathscr{A}^{0,q}_{0\,<2>}(X;L) & \textit{when } c=\infty\,, \end{cases}$$

then, for every $\psi \in \mathscr{H}^{0,q}(K_c;L) \cap L^{0,(0,q)}_{2\ \chi}(K_c;L)$ *such that* $\overline{\partial}\psi = 0$, *there exists* $\xi \in L^{0,(0,q-1)}_{2\ \chi}(K_c;L)$ *such that* $\overline{\partial}\xi = \psi$ *in* $L^{0,(0,q)}_{2\ \chi}(K_c;L)$.

REMARK 3.1.6. Let $L^{0,q-1}_2(K_c;L;\mathrm{loc})$ denote the space of locally L^2 L-valued $(0,q-1)$-forms on K_c, which contains $L^{0,(0,q-1)}_{2\ \chi}(K_c;L)$ as a subspace. It follows from the classical regularity theory for $\overline{\partial}$-operator or elliptic operators (ref. [**Hör3**, Thm. 4.2.5 and Cor. 4.2.6] or [**Hör2**, Thm. 4.1.5 and Cor. 4.1.2]) that the existence of $\xi \in L^{0,q-1}_2(K_c;L;\mathrm{loc})$ satisfying the equation (eq 3.3) in $L^{0,q}_2(K_c;L;\mathrm{loc})$ implies that there exists $\xi \in \mathscr{A}^{0,q-1}(K_c;L)$ (but not necessarily in $\mathscr{A}^{0,(0,q-1)}(K_c;L)$) satisfying the same equation in $\mathscr{A}^{0,q}(K_c;L)$. In case $c=\infty$, Theorem 2.3.1 implies that there even exists a solution $\xi \in \mathscr{H}^{0,q-1}(X;L)$ such that $\overline{\partial}\xi = \psi$ on X.

REMARK 3.1.7. Write $\mathscr{H}^{0,q}_{L^2}(K_c;L) := \mathscr{H}^{0,q}(K_c;L) \cap L^{0,(0,q)}_{2\ c,\chi}$. Following the idea discussed in §1.2, it would be more natural to consider the L^2 estimate on $\mathfrak{H}^{0,q}_{c,\chi} := \overline{\mathscr{H}^{0,q}_{L^2}(K_c;L)}$ rather than $L^{0,q}_{2\ c,\chi<2>}$, where the closure is taken in $L^{0,(0,q)}_{2\ c,\chi}$. However, the author faces the difficulty in obtaining the required estimate from the Bochner–Kodaira inequalities when $\mathfrak{H}^{0,q}_{c,\chi}$ instead of $L^{0,q}_{2\ c,\chi<2>}$ is considered. Write $\overline{\partial}^*_{\mathscr{H}_c}$ as the Hilbert space adjoint of $\overline{\partial} = \overline{\partial}_{[v]}\colon \mathfrak{H}^{0,q}_{c,\chi} \to \mathfrak{H}^{0,q+1}_{c,\chi}$. It can be shown that $\overline{\partial}^*_{\mathscr{H}_c} = \operatorname{pr}_c \circ \overline{\partial}^*_{[v]}$ on $\operatorname{Dom}^{(0,q)}_{K_c,\chi}\overline{\partial}^*_{\mathscr{H}_c}$, where $\operatorname{pr}_c\colon L^{0,(0,q)}_{2\ c,\chi} \to \mathfrak{H}^{0,q}_{c,\chi}$ is the orthogonal projection. Set $\mathfrak{d}^*_{\perp c} := \overline{\partial}^*_{[v]} - \overline{\partial}^*_{\mathscr{H}_c}$, then $\overline{\partial}^*_{\mathscr{H}_c}\zeta$ and $\mathfrak{d}^*_{\perp c}\zeta$ are orthogonal to each other for all $\zeta \in \operatorname{Dom}^{(0,q)}_{K_c,\chi}\overline{\partial}^*_{\mathscr{H}_c}$ and

$$\left\|\overline{\partial}^*_{[v]}\zeta\right\|_{K_c,\chi}^2 = \left\|\overline{\partial}^*_{\mathscr{H}_c}\zeta\right\|_{K_c,\chi}^2 + \left\|\mathfrak{d}^*_{\perp c}\zeta\right\|_{K_c,\chi}^2 .$$

From the Bochner–Kodaira inequalities, one obtains

$$\left\|\overline{\partial}\zeta\right\|^2_{K_c,\chi} + \left\|\overline{\partial}^*_{[v]}\zeta\right\|^2_{K_c,\chi} \geq \int_{K_c} \operatorname{Curv}(\zeta,\zeta)$$

for all $\zeta \in \mathscr{H}^{0,q}_{L^2}(K_c;L) \cap \operatorname{Dom}^{(0,q)}_{K_c,\chi}\overline{\partial} \cap \operatorname{Dom}^{(0,q)}_{K_c,\chi}\overline{\partial}^*_{[v]}$, where $\int_{K_c}\operatorname{Curv}(\zeta,\zeta)$ is the curvature term arising from the curvature of L. By choosing suitably the metrics g and η, the curvature term can be bounded below by $C\left\|\zeta\right\|^2_{K_c,\chi}$ for some constant $C>0$. Therefore, in order to obtain the desired estimate $\left\|\overline{\partial}\zeta\right\|^2_{K_c,\chi} + \left\|\overline{\partial}^*_{\mathscr{H}_c}\zeta\right\|^2_{K_c,\chi} \geq C''\left\|\zeta\right\|^2_{K_c,\chi}$ for some constant $C''>0$, one has to show that $\left\|\overline{\mathfrak{d}}^*_{\perp c}\zeta\right\|^2_{K_c,\chi} \leq C'\left\|\zeta\right\|^2_{K_c,\chi}$ for some constant $C'>0$ such that $C>C'$. However, the constant C' depends on g in general and one may not be able to make C' smaller than C by altering g. That's why the L^2 estimate on $L^{0,q}_{2\,c,\chi\,<2>}$ instead of $\mathfrak{H}^{0,q}_{c,\chi}$ is considered in this article.

3.2. Bochner–Kodaira formulas

Let

$$\nabla\colon \mathscr{A}\left(\mathbf{T}^{*\,\bullet,\bullet}\otimes L\right) \to \mathscr{A}\left(\mathbf{T}^{*\,\mathbb{C}}\otimes\mathbf{T}^{*\,\bullet,\bullet}\otimes L\right),$$

where $\mathbf{T}^{*\,\mathbb{C}} := \mathbf{T}^{*\,1,0}\oplus\mathbf{T}^{*\,0,1}$, be the connection on $\mathbf{T}^{*\,\bullet,\bullet}\otimes L$ induced from the Chern connections on the holomorphic hermitian vector bundles $(\mathbf{T}^{1,0},g)$ and $(L,\eta e^{-\chi})$. Therefore, ∇ is compatible with the pointwise norm $\left|\cdot\right|_{g,\eta,\chi}$.

Under a chosen apt coordinate system, set $\partial_k := \frac{\partial}{\partial z^k}$ and $\partial_{\overline{k}} := \frac{\partial}{\partial\overline{z^k}}$ for $1\leq k\leq n$. These define global vector fields on X. Set $\nabla_k := \nabla_{\partial_k}$ and $\nabla_{\overline{k}} := \nabla_{\partial_{\overline{k}}}$ for $1\leq k\leq n$. Set also $\nabla_{v^j} := \nabla_{n-m+j} = \nabla_{\frac{\partial}{\partial v^j}}$ and $\nabla_{\overline{v^j}} := \nabla_{\overline{n-m+j}} = \nabla_{\frac{\partial}{\partial\overline{v^j}}}$ (and define ∂_{v^j} and $\partial_{\overline{v^j}}$ similarly) for $1\leq j\leq m$ for notational convenience. Since the hermitian metric g is translational invariant on X, the Christoffel symbols given from g vanish and thus one has locally

$$\begin{aligned}\nabla_k &= \partial_k + \partial_k\log\left(\eta e^{-\chi}\right),\\ \nabla_{\overline{k}} &= \partial_{\overline{k}}\end{aligned} \tag{eq 3.5}$$

for $1\leq k\leq n$. For later use, note that the commutator of ∇_k and $\nabla_{\overline{\ell}}$ is given by

$$\Theta_{k\overline{\ell}} := [\nabla_k,\nabla_{\overline{\ell}}] = -\partial_k\partial_{\overline{\ell}}\log\left(\eta e^{-\chi}\right),$$

and the curvature form of L endowed with the metric $\eta e^{-\chi}$ is given by

$$\Theta := -\sqrt{-1}\partial\overline{\partial}\log\left(\eta e^{-\chi}\right) = \sqrt{-1}\sum_{k,\ell=1}^{n}\Theta_{k\overline{\ell}}\,dz^k\wedge d\overline{z^\ell}\,. \tag{eq 3.6}$$

Write the curvature tensor associated to Θ as

$$\mathcal{R} := \sum_{k,\ell=1}^{n}\Theta_{k\overline{\ell}}\,dz^k\otimes d\overline{z^\ell}\,.$$

Since the base and fibre directions are orthogonal to each other with respect to g, the identification between $\mathscr{A}^{p,q}$ and $\overline{\mathscr{A}_{p,q}} = \mathscr{A}_{q,p} := \mathscr{A}\left(\mathbf{T}^{q,p}\right)$ induced from g respects the decomposition (eq 2.4) ($\overline{\mathscr{A}_{p,q}}$ here means the complex conjugate of $\mathscr{A}_{p,q}$). For later use, set $\mathscr{A}_{(p',p''),(q',q'')} := \mathscr{A}\left(\mathbf{T}^{p',q'}_u\wedge\mathbf{T}^{p'',q''}_v\right)$ and $\mathscr{A}_{(p',p''),0} := \mathscr{A}_{(p',p''),(0,0)}$ for $0\leq p',q'\leq n-m$ and $0\leq p'',q''\leq m$. For any $\zeta\in\mathscr{A}^{p,0}\otimes\mathscr{A}^{0,q}$, let $\zeta^\vee$ denote the image of ζ in $\mathscr{A}_{0,p}\otimes\mathscr{A}_{q,0}$ via the isomorphism induced from g. Then, for example, if $\zeta\in\mathscr{A}^{0,(q',q'')}$, one has $\zeta^\vee\in\mathscr{A}_{(q',q''),0}$.

As a bilinear form on $\mathscr{A}_{1,0} \otimes \overline{\mathscr{A}_{1,0}}$, $\mathcal{R}$ can be decomposed according to the decomposition (eq 2.4) into the sum of

$$\mathcal{R}_{u\overline{u}} := \mathcal{R}|_{\mathscr{A}_{(1,0),0}\otimes\overline{\mathscr{A}_{(1,0),0}}} \ , \quad \mathcal{R}_{u\overline{v}} := \mathcal{R}|_{\mathscr{A}_{(1,0),0}\otimes\overline{\mathscr{A}_{(0,1),0}}} \ ,$$
$$\mathcal{R}_{v\overline{u}} := \mathcal{R}|_{\mathscr{A}_{(0,1),0}\otimes\overline{\mathscr{A}_{(1,0),0}}} \ , \quad \mathcal{R}_{v\overline{v}} := \mathcal{R}|_{\mathscr{A}_{(0,1),0}\otimes\overline{\mathscr{A}_{(0,1),0}}} \ .$$

Since $\mathcal{R}$ is a hermitian form, it follows that $\mathcal{R}_{u\overline{u}} = \overline{\mathcal{R}_{u\overline{u}}}$, $\mathcal{R}_{v\overline{v}} = \overline{\mathcal{R}_{v\overline{v}}}$ and $\mathcal{R}_{u\overline{v}} = \overline{\mathcal{R}_{v\overline{u}}}$.

Let $\operatorname{Tr}_g \colon \mathscr{A}^{0,q} \otimes \mathscr{A}^{q,0} \to \mathscr{A}^{0,0}$ be the trace operator which is defined in such a way that $\zeta \otimes \xi \mapsto \xi^\vee \lrcorner \zeta$, where $\zeta \in \mathscr{A}^{0,q}$, $\xi \in \mathscr{A}^{q,0}$ and $\xi^\vee \lrcorner \zeta$ denotes the complete contraction between ζ and $\xi^\vee$. Denote by $\operatorname{Tr}_{g,\eta}$ the similar contraction for L-valued forms.

Fix any $0 < c < \infty$. Denote the Hilbert space adjoint of $\overline{\partial} \colon L^{0,q-1}_{2\,c,\chi} \to L^{0,q}_{2\,c,\chi}$ by $\overline{\partial}^* \colon L^{0,q}_{2\,c,\chi} \to L^{0,q-1}_{2\,c,\chi}$. Identify $\mathscr{A}^{1,1}$ and $\mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1}$ via the isomorphism $dz^k \wedge d\overline{z^\ell} \mapsto dz^k \otimes d\overline{z^\ell}$ for any $1 \leq k, \ell \leq n$. Let $\mathcal{R}^\vee(\zeta \otimes \overline{\zeta})$ (resp. $\left(\partial\overline{\partial}\varphi\right)^\vee(\zeta \otimes \overline{\zeta})$) denotes the natural contraction between $\mathcal{R}^\vee$ (resp. $\left(\partial\overline{\partial}\varphi\right)^\vee$) and $\zeta \otimes \overline{\zeta}$. Let $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ be the decomposition of ∇ into $(1,0)$- and $(0,1)$-types. The $\overline{\nabla}$-Bochner–Kodaira formula (cf. [**Siu**, (2.1.4) and (1.3.3)]) is then given by

$$\begin{aligned} \left\|\overline{\partial}\zeta\right\|^2_{K_c,\chi} + \left\|\overline{\partial}^*\zeta\right\|^2_{K_c,\chi} = &\int_{\partial K_c} \frac{e^{-\chi}}{|d\varphi|_g} \operatorname{Tr}_{g,\eta} \left(\partial\overline{\partial}\varphi\right)^\vee(\zeta \otimes \overline{\zeta}) \\ &+ \left\|\nabla^{(0,1)}\zeta\right\|^2_{K_c,\chi} + \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta \otimes \overline{\zeta}) \end{aligned} \tag{eq 3.7}$$

for all $\zeta \in \mathscr{A}^{0,q}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\chi} \overline{\partial}^*$.

Remark 3.2.1. Note that the measure for the boundary integral is induced from $\left.\left(\frac{(d\varphi)^\vee}{|d\varphi|_g} \lrcorner \frac{\omega^{\wedge n}}{n!}\right)\right|_{\partial K_c}$. In order to compare notations in [**Siu**, (2.1.4)] and those in (eq 3.7), write $[x]_{\mathrm{Siu}}$ to mean the symbol x used in [**Siu**]. Then

$$\left[\overline{\nabla}\right]_{\mathrm{Siu}} = \nabla^{(0,1)} \ , \quad [\nabla]_{\mathrm{Siu}} = \nabla^{(1,0)} \ , \quad [\rho]_{\mathrm{Siu}} = \frac{\varphi - c}{|d\varphi|_g} \ , \quad \left[R_{i\overline{j}k\overline{l}}\right]_{\mathrm{Siu}} = 0 \ ,$$
$$\text{and} \qquad \left[-\Omega_{\alpha\overline{\beta}s\overline{t}}\right]_{\mathrm{Siu}} = \text{components of } \mathcal{R} = \Theta_{k\overline{\ell}} \ .$$

Note that $\left[R_{i\overline{j}k\overline{l}}\right]_{\mathrm{Siu}} = 0$ as the Chern connection on $(\mathbf{T}^{1,0}, g)$ is flat. Also be aware of the typos of the signs preceding the curvature integrals involving $\left[\Omega_{\alpha\overline{\beta}}{}^{\overline{s}}{}_{\overline{t}}\right]_{\mathrm{Siu}}$ and $\left[R^{\overline{s}}{}_{\overline{t}}\right]_{\mathrm{Siu}}$ in [**Siu**, (2.1.4)]. The correct signs can be found in [**Siu**, (1.3.3)]. To see that the boundary term in (eq 3.7) coincides with the one in [**Siu**, (2.1.4)], note that at every $z \in \partial K_c$,

$$\partial\overline{\partial}\left(\frac{\varphi - c}{|d\varphi|_g}\right)(z) = \frac{\partial\overline{\partial}\varphi}{|d\varphi|_g}(z) - \frac{\partial\varphi \wedge \overline{\partial}\,|d\varphi|_g}{|d\varphi|_g^2}(z) - \frac{\partial\,|d\varphi|_g \wedge \overline{\partial}\varphi}{|d\varphi|_g^2}(z) \ .$$

After taking $^\vee$ and contracting with $\zeta \otimes \overline{\zeta}$ where $\zeta \in \mathscr{A}^{0,q}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\chi} \overline{\partial}^*$, the last two terms on the right hand side vanish because, for $\zeta \in \mathscr{A}^{0,q}(\overline{K}_c; L)$, $(\partial\varphi)^\vee \lrcorner \zeta = 0$ on ∂K_c if and only if $\zeta \in \operatorname{Dom}_{K_c,\chi} \overline{\partial}^*$ (ref. [**Hör1**, pg. 101] or [**Siu**, (2.1.1)]). The boundary terms therefore coincides.

When the subcomplex (eq 3.2) is considered, the $\overline{\nabla}$-Bochner–Kodaira formula (eq 3.7) is restricted to $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\chi} \overline{\partial}^* = \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\chi} T^*_{q-1}$ (see Proposition 3.1.3). The $(0,1)$-connection splits into $\nabla^{(0,1)} = \nabla^{(0,1)}_u + \nabla^{(0,1)}_v$

according to the decomposition (eq 2.4). Write $\nabla_{\overline{u}} := \nabla^{(0,1)}_u$ and $\nabla_{\overline{v}} := \nabla^{(0,1)}_v$ for notational convenience. Let also $\mathrm{pr}_F \colon \mathscr{A}^{0,q} \otimes \overline{\mathscr{A}^{0,s}} \to \mathscr{A}^{0,(0,q)} \otimes \overline{\mathscr{A}^{0,(0,s)}}$ be the canonical projection (where $\overline{\mathscr{A}^{0,s}}$ (resp. $\overline{\mathscr{A}^{0,(0,s)}}$) is the complex conjugate of $\mathscr{A}^{0,s}$ (resp. $\mathscr{A}^{0,(0,s)}$)). Set

$$\mathrm{Bd}(\zeta,\zeta) := \int_{\partial K_c} \tfrac{e^{-\chi}}{|d\varphi|_g} \operatorname{Tr}_{g,\eta} \left(\partial\overline{\partial}\varphi\right)^{\vee} (\zeta \otimes \overline{\zeta}) \tag{eq 3.8}$$

for notational convenience. Then (eq 3.7) gives the following

LEMMA 3.2.2. *For any* $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom} T^*_{q-1}$, *where* $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L) \cap \operatorname{Dom} \overline{\partial}^*_{[u]}$ *and* $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$, *one has*

$$\begin{aligned} \|S_q\zeta\|_3^2 + \left\|T^*_{q-1}\zeta\right\|_1^2 = {} & \mathrm{Bd}(\zeta,\zeta) + \left\|\overline{\partial}_{[u]}\zeta''\right\|_3^2 + \left\|\overline{\partial}_{[v]}\zeta'\right\|_3^2 \\ & + \|\nabla_{\overline{u}}\zeta'\|^2_{K_c,\chi} + \|\nabla_{\overline{v}}\zeta''\|^2_{K_c,\chi} \\ & + \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathrm{pr}_F \left(\mathcal{R}^{\vee}(\zeta \otimes \overline{\zeta})\right) . \end{aligned} \tag{eq 3.9}$$

PROOF. On $\operatorname{Dom}_{K_c,\chi} \overline{\partial}^*$, one has $\overline{\partial}^* = \vartheta_{[u]} + \vartheta_{[v]}$. Then, for all $\zeta = \zeta' + \zeta'' \in \operatorname{Dom} T^*_{q-1} = \operatorname{Dom}_{K_c,\chi} \overline{\partial}^* \cap L^{0,q}_{2\ c,\chi\,<2>}$ (see Proposition 3.1.3), one has

$$\overline{\partial}^*\zeta = \vartheta_{[u]}\zeta' + \vartheta_{[u]}\zeta'' + \vartheta_{[v]}\zeta' + \vartheta_{[v]}\zeta'' = T^*_{q-1}\zeta + \vartheta_{[v]}\zeta' ,$$

as $T^*_{q-1}\zeta = \overline{\partial}^*_{[u]}\zeta' + \overline{\partial}^*_{[v]}\zeta''$ (see Proposition 3.1.3) and $\vartheta_{[u]}\zeta'' = 0$. Note also that $\nabla^{(0,1)}\zeta = \nabla_{\overline{u}}\zeta' + \nabla_{\overline{u}}\zeta'' + \nabla_{\overline{v}}\zeta' + \nabla_{\overline{v}}\zeta''$, and $\overline{\partial}\zeta = S_q\zeta$. Since the decomposition (eq 2.4) is orthogonal with respect to g, it follows that

$$\begin{aligned} \left\|\overline{\partial}^*\zeta\right\|^2_{K_c,\chi} &= \left\|T^*_{q-1}\zeta\right\|_1^2 + \|\vartheta_{[v]}\zeta'\|^2_{K_c,\chi} \quad \text{and} \\ \left\|\nabla^{(0,1)}\zeta\right\|^2_{K_c,\chi} &= \|\nabla_{\overline{u}}\zeta'\|^2_{K_c,\chi} + \|\nabla_{\overline{u}}\zeta''\|^2_{K_c,\chi} + \|\nabla_{\overline{v}}\zeta'\|^2_{K_c,\chi} + \|\nabla_{\overline{v}}\zeta''\|^2_{K_c,\chi} . \end{aligned}$$

Note that $\|\nabla_{\overline{u}}\zeta''\|^2_{K_c,\chi} = \left\|\overline{\partial}_{[u]}\zeta''\right\|_3^2$.

Following the argument in [**Hör1**, pg. 101] with $\overline{\partial}_{[v]}$ in place of $\overline{\partial}$, it follows that, for any $\zeta \in \mathscr{A}^{0,(q',q'')}(\overline{K}_c; L)$, $\zeta \in \operatorname{Dom}^{(q',q'')}_{K_c,\chi} \overline{\partial}^*_{[v]}$ if and only if $(\partial_{[v]}\varphi)^{\vee} \lrcorner\, \zeta = 0$ on ∂K_c. Since $\partial_{[v]}\varphi = 0$, it follows that $\mathscr{A}^{0,(q',q'')}(\overline{K}_c; L) \subset \operatorname{Dom}^{(q',q'')}_{K_c,\chi} \overline{\partial}^*_{[v]}$. In particular, $\zeta' \in \operatorname{Dom}^{(1,q-1)}_{K_c,\chi} \overline{\partial}^*_{[v]}$ for all $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L)$. Then, since the decomposition (eq 2.4) is orthogonal with respect to g, by taking the analogy between the decompositions $\mathscr{A}^r = \bigoplus_{p+q=r} \mathscr{A}^{p,q}$ and $\mathscr{A}^{p,q} = \bigoplus_{\substack{p=p'+p'' \\ q=q'+q''}} \mathscr{A}^{(p',p''),(q',q'')}$ and putting $\overline{\partial}_{[v]}$ in place of $\overline{\partial}$, one can follow the derivation of (eq 3.7) as in [**Siu**, §1 and §2] to obtain

$$\begin{aligned} \left\|\overline{\partial}_{[v]}\zeta'\right\|^2_{K_c,\chi} + \|\vartheta_{[v]}\zeta'\|^2_{K_c,\chi} = {} & \int_{\partial K_c} \tfrac{e^{-\chi}}{|d\varphi|_g} \operatorname{Tr}_{g,\eta} \left(\partial_{[v]}\overline{\partial}_{[v]}\varphi\right)^{\vee} \left(\zeta' \otimes \overline{\zeta'}\right) \\ & + \|\nabla_{\overline{v}}\zeta'\|^2_{K_c,\chi} + \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{R}^{\vee}_{v\overline{v}}(\zeta' \otimes \overline{\zeta'}) \end{aligned}$$

for any $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L)$. The boundary term vanishes as $\partial_{[v]}\overline{\partial}_{[v]}\varphi = 0$. Therefore, combining the above results with (eq 3.7) yields

$$\begin{aligned} \|S_q\zeta\|_3^2 + \left\|T^*_{q-1}\zeta\right\|_1^2 = {} & \mathrm{Bd}(\zeta,\zeta) + \left\|\overline{\partial}_{[u]}\zeta''\right\|_3^2 + \left\|\overline{\partial}_{[v]}\zeta'\right\|_3^2 + \|\nabla_{\overline{u}}\zeta'\|^2_{K_c,\chi} + \|\nabla_{\overline{v}}\zeta''\|^2_{K_c,\chi} \\ & + \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{R}^{\vee}(\zeta \otimes \overline{\zeta}) - \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{R}^{\vee}_{v\overline{v}}(\zeta' \otimes \overline{\zeta'}) . \end{aligned}$$

For every fixed $z \in K_c$, $\operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta \otimes \overline{\zeta})$ is a hermitian form in ζ. Again, since the decomposition (eq 2.4) is orthogonal with respect to g, it follows that

$$\begin{aligned}
\operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta' \otimes \overline{\zeta'}) &= \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{u\overline{u}}(\zeta' \otimes \overline{\zeta'}) + \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{v\overline{v}}(\zeta' \otimes \overline{\zeta'}) \ , \\
\overline{\operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta' \otimes \overline{\zeta''})} &= \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta'' \otimes \overline{\zeta'}) = \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{v\overline{u}}(\zeta'' \otimes \overline{\zeta'}) \ , \\
\operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta'' \otimes \overline{\zeta''}) &= \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{v\overline{v}}(\zeta'' \otimes \overline{\zeta''}) \ .
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\operatorname{pr}_F\left(\mathcal{R}^\vee(\zeta' \otimes \overline{\zeta'})\right) &= \mathcal{R}^\vee_{u\overline{u}}(\zeta' \otimes \overline{\zeta'}) \ , \quad \operatorname{pr}_F\left(\mathcal{R}^\vee(\zeta' \otimes \overline{\zeta''})\right) = \mathcal{R}^\vee_{u\overline{v}}(\zeta' \otimes \overline{\zeta''}) \ , \\
\operatorname{pr}_F\left(\mathcal{R}^\vee(\zeta'' \otimes \overline{\zeta'})\right) &= \mathcal{R}^\vee_{v\overline{u}}(\zeta'' \otimes \overline{\zeta'}) \ , \quad \operatorname{pr}_F\left(\mathcal{R}^\vee(\zeta'' \otimes \overline{\zeta''})\right) = \mathcal{R}^\vee_{v\overline{v}}(\zeta'' \otimes \overline{\zeta''}) \ .
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\operatorname{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta \otimes \overline{\zeta}) - \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{v\overline{v}}(\zeta' \otimes \overline{\zeta'}) =& \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{u\overline{u}}(\zeta' \otimes \overline{\zeta'}) + \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{u\overline{v}}(\zeta' \otimes \overline{\zeta''}) \\
&+ \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{v\overline{u}}(\zeta'' \otimes \overline{\zeta'}) + \operatorname{Tr}_{g,\eta} \mathcal{R}^\vee_{v\overline{v}}(\zeta'' \otimes \overline{\zeta''}) \\
=& \operatorname{Tr}_{g,\eta} \operatorname{pr}_F\left(\mathcal{R}^\vee(\zeta \otimes \overline{\zeta})\right)
\end{aligned}$$

and hence the lemma. $\square$

Let $g_F := \operatorname{pr}_F g$, and let $(g_F)^{\bar{j}j'}$'s for $1 \leq j, j' \leq m$ be the entries of the inverse of the matrix of g_F under the chosen coordinates. Denote by $(\cdot,\cdot)_{g,\eta,\chi}$ the pointwise inner product induced from $\left|\cdot\right|_{g,\eta,\chi}$. Write $\nabla^{(1,0)} = \nabla^{(1,0)}_u + \nabla^{(1,0)}_v$ as the splitting of $\nabla^{(1,0)}$ according to the decomposition (eq 2.4), and set $\nabla_u := \nabla^{(1,0)}_u$ and $\nabla_v := \nabla^{(1,0)}_v$ for convenience. The following integration by parts argument is put into a lemma for clarity.

Lemma 3.2.3. *For all* $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$, *one has*

$$\|\nabla_{\overline{v}}\zeta''\|^2_{K_c,\chi} = \|\nabla_v\zeta''\|^2_{K_c,\chi} - \int_{K_c} \left(\operatorname{Tr}_g \mathcal{R}_{v\overline{v}}\right) \left|\zeta''\right|^2_{g,\eta,\chi} \ .$$

Proof. Recall that $d\mu := \frac{\omega^{\wedge n}}{n!}$ is the volume element on K_c, while that on ∂K_c is given by $d\sigma := \left(\frac{(d\varphi)^\vee}{|d\varphi|_g} \lrcorner\, d\mu\right)\Big|_{\partial K_c}$. Einstein summation convention is applied in what follows. Fix a $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$. Let

$$Y := \left(\nabla_{\overline{v^j}}\zeta'', \zeta''\right)_{g,\eta,\chi} (g_F)^{\bar{j}j'} \frac{\partial}{\partial v^{j'}} \qquad \text{and} \qquad W := \left(\nabla_{v^{j'}}\zeta'', \zeta''\right)_{g,\eta,\chi} (g_F)^{\bar{j}j'} \frac{\partial}{\partial \overline{v^j}}$$

be two vector fields in $\mathscr{A}_{(0,1),(0,0)}(\overline{K}_c)$ and $\mathscr{A}_{(0,0),(0,1)}(\overline{K}_c)$ respectively. Then, using the fact that ∇ is compatible with $(\cdot,\cdot)_{g,\eta,\chi}$, it follows that

$$\begin{aligned}
\left|\nabla_{\overline{v}}\zeta''\right|^2_{g,\eta,\chi} d\mu =& \left(\partial_{v^{j'}}\left((g_F)^{\bar{j}j'}\nabla_{\overline{v^j}}\zeta'', \zeta''\right)_{g,\eta,\chi}\right) d\mu - \left((g_F)^{\bar{j}j'}\nabla_{v^{j'}}\nabla_{\overline{v^j}}\zeta'', \zeta''\right)_{g,\eta,\chi} d\mu \\
=& \ d\left(Y \lrcorner\, d\mu\right) - \left((g_F)^{\bar{j}j'}\nabla_{\overline{v^j}}\nabla_{v^{j'}}\zeta'', \zeta''\right)_{g,\eta,\chi} d\mu \\
& - \left((g_F)^{\bar{j}j'}\left[\nabla_{v^{j'}}, \nabla_{\overline{v^j}}\right]\zeta'', \zeta''\right)_{g,\eta,\chi} d\mu \\
=& \ d\left(Y \lrcorner\, d\mu\right) - \left(\partial_{\overline{v^j}}\left((g_F)^{\bar{j}j'}\nabla_{v^{j'}}\zeta'', \zeta''\right)_{g,\eta,\chi}\right) d\mu + \left|\nabla_v\zeta''\right|^2_{g,\eta,\chi} d\mu \\
& - \left(\operatorname{Tr}_g \mathcal{R}_{v\overline{v}}\right)\left|\zeta''\right|^2_{g,\eta,\chi} d\mu \\
=& \ d\left(Y \lrcorner\, d\mu\right) - d\left(W \lrcorner\, d\mu\right) + \left|\nabla_v\zeta''\right|^2_{g,\eta,\chi} d\mu - \left(\operatorname{Tr}_g \mathcal{R}_{v\overline{v}}\right)\left|\zeta''\right|^2_{g,\eta,\chi} d\mu \ .
\end{aligned}$$

Since $\partial K_c = \{\varphi = c\}$ and $(d\varphi)|_{\partial K_c} = 0$, it follows that for any vector field V such that $\left(V, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g = 0$, one has $(V \lrcorner\, d\mu)|_{\partial K_c} = 0$. The component of $Y - W$ in the direction of $\frac{(d\varphi)^\vee}{|d\varphi|_g}$ is $\left(Y - W, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g$. Therefore, by integrating over K_c and applying Stokes' theorem, it yields

$$\|\nabla_{\overline{v}}\zeta''\|^2_{K_c,\chi} = \int_{\partial K_c} \left(Y - W, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g d\sigma + \|\nabla_v \zeta''\|^2_{K_c,\chi} - \int_{K_c} (\operatorname{Tr}_g \mathcal{R}_{v\overline{v}})\, |\zeta''|^2_{g,\eta,\chi}\, d\mu\ .$$

But $(d\varphi)^\vee \in \mathscr{A}_{(1,0),(0,0)} \oplus \mathscr{A}_{(0,0),(1,0)}(X)$, so $\left(Y - W, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g = 0$ and hence the lemma. □

Combining the result above with (eq 3.9) yields

(eq 3.10)

$$\begin{aligned}\|S_q\zeta\|_3^2 + \left\|T^*_{q-1}\zeta\right\|_1^2 = {} & \operatorname{Bd}(\zeta,\zeta) + \left\|\overline{\partial}_{[u]}\zeta''\right\|_3^2 + \left\|\overline{\partial}_{[v]}\zeta'\right\|_3^2 \\ & + \|\nabla_{\overline{u}}\zeta'\|^2_{K_c,\chi} + \|\nabla_v\zeta''\|^2_{K_c,\chi} - \int_{K_c} (\operatorname{Tr}_g \mathcal{R}_{v\overline{v}})\, |\zeta''|^2_{g,\eta,\chi} \\ & + \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\mathcal{R}^\vee(\zeta \otimes \overline{\zeta})\right)\ .\end{aligned}$$

This formula is analogous to the usual ∇-Bochner–Kodaira formula (see [**Siu**, (2.2.1)]). However, it contains term involving $\nabla_{\overline{u}}$ and ∇_v but not ∇_u, and the boundary term is the same as the one in the $\overline{\nabla}$-Bochner–Kodaira formula.

Consider the boundary term $\operatorname{Bd}(\zeta,\zeta)$ in (eq 3.8). Since $\sqrt{-1}\partial\overline{\partial}\varphi$ is non-negative on ∂K_c, i.e. K_c is pseudoconvex, by choosing coordinates at any point in ∂K_c such that $\sqrt{-1}\partial\overline{\partial}\varphi$ and g are simultaneously diagonalized, one sees that $\operatorname{Bd}(\zeta,\zeta)$ is non-negative for all $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$. Noting that all other norm-square terms are also non-negative, one then obtains

$$\text{(eq 3.11)}\qquad \|S_q\zeta\|_3^2 + \left\|T^*_{q-1}\zeta\right\|_1^2 \geq \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\mathcal{R}^\vee(\zeta \otimes \overline{\zeta})\right)$$

and

$$\text{(eq 3.12)}\qquad \begin{aligned}\|S_q\zeta\|_3^2 + \left\|T^*_{q-1}\zeta\right\|_1^2 \geq {} & -\int_{K_c} (\operatorname{Tr}_g \mathcal{R}_{v\overline{v}})\, |\zeta''|^2_{g,\eta,\chi} \\ & + \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\mathcal{R}^\vee(\zeta \otimes \overline{\zeta})\right)\end{aligned}$$

for all $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\chi} T^*_{q-1}$. These are the *Bochner–Kodaira inequalities for T^*_{q-1} and S_q* which are used to obtain the required L^2 estimates.

3.3. Murakami's trick

From (eq 3.6) and $\log(\eta e^{-\chi}) = \log \eta_{\mathsf{t}} + \log \eta_{\mathsf{w}} - \chi$ (see §2.5 for notation), it follows that

$$\text{(eq 3.13)}\qquad \begin{aligned}\Theta &= \Theta_{\mathfrak{T}} + \Theta_{\mathfrak{W}} + \sqrt{-1}\partial\overline{\partial}\chi \\ &= \pi\sqrt{-1}\partial\overline{\partial}\mathcal{H} + 2\sqrt{-1}\partial\overline{\partial}\operatorname{Re}\hbar_\delta + \sqrt{-1}\partial\overline{\partial}\chi\end{aligned}$$

where $\Theta_{\mathfrak{T}}$ and $\Theta_{\mathfrak{W}}$ are respectively the tame and wild curvature forms of L defined in §2.5, and $\mathcal{H}$ is a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ associated to L. Therefore, by abusing

$\mathcal{H}$ to mean the associated hermitian form in $\mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1}(X)$, the curvature integral $\int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\mathcal{R}^\vee(\zeta \otimes \overline{\zeta})\right)$ in (eq 3.11) and (eq 3.12) can be split into the sum of

$$\begin{aligned}
\mathfrak{T}(\zeta,\zeta) &:= \pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\mathcal{H}^\vee(\zeta \otimes \overline{\zeta})\right) , \\
\mathfrak{W}(\zeta,\zeta) &:= \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\left(2\, \partial\overline{\partial} \operatorname{Re} \hbar_\delta\right)^\vee (\zeta \otimes \overline{\zeta})\right) , \\
\mathfrak{wt}(\zeta,\zeta) &:= \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\left(\partial\overline{\partial}\chi\right)^\vee (\zeta \otimes \overline{\zeta})\right)
\end{aligned} \tag{eq 3.14}$$

(recall that $\mathscr{A}^{1,1}$ is identified with $\mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1}$ via $dz^k \wedge \overline{dz^\ell} \mapsto dz^k \otimes \overline{dz^\ell}$ for all $1 \leq k, \ell \leq n$).

One of the essential ingredients for obtaining the required L^2 estimates for $q < s_F^-$ or $q > m - s_F^+$ is Murakami's trick used in [**Mur**]. The trick is applied to the part of the curvature integral $\mathfrak{T}(\zeta,\zeta)$ involving $\mathcal{H}_F := \mathcal{H}|_{F\times F}$. For any $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$ where $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L)$ and $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$, that part is given by

$$\begin{aligned}
\mathfrak{T}_F(\zeta,\zeta) &:= \pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\mathcal{H}_F^\vee(\zeta \otimes \overline{\zeta})\right) \\
&= \pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{H}_F^\vee(\zeta'' \otimes \overline{\zeta''}) = \mathfrak{T}_F(\zeta'',\zeta'') .
\end{aligned}$$

Definition 3.3.1. An $\mathcal{H}$-apt coordinate system is an apt coordinate system such that the matrix of $\mathcal{H}_F$ under such coordinate system is given by

$$H_F = D := \operatorname{diag}\big(\underbrace{1,\ldots,1}_{s_F^+}, \underbrace{-1,\ldots,-1}_{s_F^-}, \underbrace{0,\ldots,0}_{m-s_F^+-s_F^-}\big) . \tag{eq 3.15}$$

Under a chosen apt coordinate system, an $\mathcal{H}$-apt coordinate system can be obtained by a linear change of coordinates only in the variable v (which preserves the decomposition (eq 2.4)).

In what follows, write $\overline{dv^{J_q}} := \overline{dv^{j_1}} \wedge \cdots \wedge \overline{dv^{j_q}}$ for every q-multiindex $J_q = (j_1,\ldots,j_q)$. Moreover, let $\zeta''_{\overline{J_q}}$ be the component of $\zeta'' \in \mathscr{A}^{0,(0,q)}(L)$ corresponding to $\overline{dv^{J_q}}$, and $\zeta'_{\overline{i}\overline{J_{q-1}}}$ the component of $\zeta' \in \mathscr{A}^{0,(1,q-1)}(L)$ corresponding to $\overline{du^i} \wedge \overline{dv^{J_{q-1}}}$.

Lemma 3.3.2 (Murakami's trick for $q > m - s_F^+$). *For any $q > m - s_F^+$ and given any constant $M > 0$, one can choose the translational invariant hermitian metric g suitably such that $\mathfrak{T}_F(\zeta'',\zeta'') \geq \pi M \, \|\zeta''\|_2^2$ for every $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$.*

Proof. Fix an $\mathcal{H}$-apt coordinate system. Choose g such that it is diagonal in the chosen $\mathcal{H}$-apt coordinates and its matrix is given by

$$\operatorname{diag}\big(\underbrace{1,\ldots,1}_{n-m}, \underbrace{\frac{1}{g_+^F},\ldots,\frac{1}{g_+^F}}_{s_F^+}, \underbrace{\frac{1}{g_-^F},\ldots,\frac{1}{g_-^F}}_{s_F^-}, \underbrace{\frac{1}{g_0^F},\ldots,\frac{1}{g_0^F}}_{m-s_F^+-s_F^-}\big) , \tag{eq 3.16}$$

where g_+^F, g_-^F and g_0^F are positive numbers. Given $M > 0$, g_+^F, g_-^F and g_0^F are chosen as

$$g_+^F := s_F^- + M , \quad g_-^F := 1 \quad \text{and} \quad g_0^F := 1 .$$

Under the chosen $\mathcal{H}$-apt coordinates, since $\mathcal{H}_F$ and g are both diagonal, the monomial forms $\zeta''_{J_q} d\overline{v^{J_q}} \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$ with different multiindices J_q are orthogonal to one another with respect to $\mathfrak{T}_F$ and $\langle\cdot,\cdot\rangle_2$. Therefore, it suffices to show that

$$(*) \qquad \mathfrak{T}_F\left(\zeta''_{J_q} d\overline{v^{J_q}}, \zeta''_{J_q} d\overline{v^{J_q}}\right) \geq \pi M \left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|_2^2$$

for all monomial forms $\zeta''_{J_q} d\overline{v^{J_q}} \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$.

In fact, for each multiindex $J_q = (j_1, \dots, j_q)$, one has

$$\begin{aligned}\mathfrak{T}_F\left(\zeta''_{J_q} d\overline{v^{J_q}}, \zeta''_{J_q} d\overline{v^{J_q}}\right) &= \pi \int_{K_c} \left(\sum_{\nu=1}^{q} (g_F)^{\bar{j}_\nu j_\nu} (H_F)_{j_\nu \bar{j}_\nu}\right) \left|\zeta''_{J_q} d\overline{v^{J_q}}\right|^2_{g,\eta,\chi} \\ &= \pi \left(\sum_{\nu=1}^{q} (g_F)^{\bar{j}_\nu j_\nu} (H_F)_{j_\nu \bar{j}_\nu}\right) \left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|^2_2 ,\end{aligned}$$

where $(g_F)^{\bar{j}_\nu j_\nu}$'s are the diagonal components of $(g_F)^{-1} := (\operatorname{pr}_F g)^{-1}$, and $(H_F)_{j_\nu \bar{j}_\nu}$'s are the diagonal entries of H_F in (eq 3.15), which are either 1, -1 or 0. Define

$$\begin{aligned} R^+(J_q) &:= \#\left\{j_\nu \in J_q \ : \ 1 \leq j_\nu \leq s_F^+\right\} \\ R^-(J_q) &:= \#\left\{j_\nu \in J_q \ : \ s_F^+ + 1 \leq j_\nu \leq s_F^+ + s_F^-\right\} .\end{aligned}$$

Then, the sum in the parenthesis becomes

$$(\text{eq } 3.17) \qquad \begin{aligned}\sum_{\nu=1}^{q} (g_F)^{\bar{j}_\nu j_\nu} (H_F)_{j_\nu \bar{j}_\nu} &= g_+^F R^+(J_q) - g_-^F R^-(J_q) \\ &= (s_F^- + M) R^+(J_q) - R^-(J_q) .\end{aligned}$$

Since $q > m - s_F^+$, it follows that $R^+(J_q) \geq 1$ for any multiindex J_q. Note also that $R^-(J_q) \leq s_F^-$ for any J_q. Therefore, by the choice of g_+^F and g_-^F, one obtains $g_+^F R^+(J_q) - g_-^F R^-(J_q) \geq M$ and thus $(*)$ follows. This completes the proof. □

In order to apply Lemma 3.3.2 to (eq 3.11), note that for any $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$, one has

$$\|\zeta''\|_2^2 = \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \zeta'' \otimes \overline{\zeta''} .$$

Decompose $\mathcal{H} \in \mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1}(X)$ into $\mathcal{H}_E + \mathcal{H}_{u\bar{v}} + \mathcal{H}_{v\bar{u}} + \mathcal{H}_F$ according to the decomposition (eq 2.4) as is done to $\mathcal{R}$ (write $\mathcal{H}_E$ for $\mathcal{H}_{u\bar{u}}$ and $\mathcal{H}_F$ for $\mathcal{H}_{v\bar{v}}$ to respect previous notations). Now note that, for any $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$ such that $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L)$ and $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$, one has

$$\begin{aligned}\mathfrak{T}(\zeta,\zeta) = \pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \big(&\mathcal{H}_E^\vee(\zeta' \otimes \overline{\zeta'}) + \mathcal{H}_{u\bar{v}}^\vee(\zeta' \otimes \overline{\zeta''}) \\ &+ \mathcal{H}_{v\bar{u}}^\vee(\zeta'' \otimes \overline{\zeta'}) + \mathcal{H}_F^\vee(\zeta'' \otimes \overline{\zeta''})\big) .\end{aligned}$$

If $q > m - s_F^+$, Lemma 3.3.2 then implies that, given $M > 0$, g can be chosen such that

$$\begin{aligned}\mathfrak{T}(\zeta,\zeta) \geq \pi \int_{K_c} e^{-\chi} \big[&\operatorname{Tr}_{g,\eta}\left(\mathcal{H}_E^\vee(\zeta' \otimes \overline{\zeta'}) + \mathcal{H}_{u\bar{v}}^\vee(\zeta' \otimes \overline{\zeta''}) + \mathcal{H}_{v\bar{u}}^\vee(\zeta'' \otimes \overline{\zeta'})\right) \\ &+ M \operatorname{Tr}_{g,\eta} \zeta'' \otimes \overline{\zeta''}\big] .\end{aligned}$$

Define $\widetilde{\mathcal{H}}(M)$ to be an element in $\mathscr{A}^{0,0} \oplus (\mathscr{A}^{1,0} \otimes \mathscr{A}^{0,1})(X)$ such that

(eq 3.18)
$$\operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^{\vee}(\zeta\otimes\overline{\zeta})\right) = \mathcal{H}_E^{\vee}(\zeta'\otimes\overline{\zeta'}) + \mathcal{H}_{u\overline{v}}^{\vee}(\zeta'\otimes\overline{\zeta''}) + \mathcal{H}_{v\overline{u}}^{\vee}(\zeta''\otimes\overline{\zeta'}) + M\,\zeta''\otimes\overline{\zeta''}$$

for any $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$ (note that $\zeta' = 0$ when $q = 0$). Then one has

$$\mathfrak{T}(\zeta,\zeta) \geq \pi \int_{K_c} e^{-\chi}\operatorname{Tr}_{g,\eta}\operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^{\vee}(\zeta\otimes\overline{\zeta})\right)$$

when $q > m - s_F^+$. Therefore, the consequence of Lemma 3.3.2 applied to (eq 3.11) can be stated as follows.

Corollary 3.3.3. *Suppose $q > m - s_F^+$. Then, given any constant $M > 0$, the translational invariant hermitian metric g can be chosen suitably such that (eq 3.11) yields*

$$\|S_q\zeta\|_3^2 + \left\|T_{q-1}^*\zeta\right\|_1^2 \geq \pi\int_{K_c} e^{-\chi}\operatorname{Tr}_{g,\eta}\operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^{\vee}(\zeta\otimes\overline{\zeta})\right) + \mathfrak{W}(\zeta,\zeta) + \mathfrak{wt}(\zeta,\zeta)$$

for all $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom} T_{q-1}^*$.

Now consider the integral involving $\operatorname{Tr}_g \mathcal{R}_{v\overline{v}}$ in (eq 3.12). Note that

$$\operatorname{pr}_F\Theta = \pi\sqrt{-1}\partial_{[v]}\overline{\partial}_{[v]}\mathcal{H} + 2\sqrt{-1}\partial_{[v]}\overline{\partial}_{[v]}\left(\operatorname{Re}\hbar_\delta\right)\ .$$

Here no term involving χ appears since $\sqrt{-1}\partial_{[v]}\overline{\partial}_{[v]}\chi = 0$. Again, by abusing $\mathcal{H}_F$ to mean the associated hermitian form in $\mathscr{A}^{1,0}\otimes\mathscr{A}^{0,1}(X)$, the curvature integral $-\int_{K_c}(\operatorname{Tr}_g\mathcal{R}_{v\overline{v}})\,|\zeta''|^2_{g,\eta,\chi}$ in (eq 3.12) can be split into the sum of

(eq 3.19)
$$\mathfrak{T}'_F(\zeta'',\zeta'') := -\pi\int_{K_c}\left(\operatorname{Tr}_g\mathcal{H}_F\right)|\zeta''|^2_{g,\eta,\chi}\ ,$$
$$\mathfrak{W}'_F(\zeta'',\zeta'') := -\int_{K_c}\left(2\operatorname{Tr}_g\partial_{[v]}\overline{\partial}_{[v]}\operatorname{Re}\hbar_\delta\right)|\zeta''|^2_{g,\eta,\chi}\ .$$

Similar argument as in the proof of Lemma 3.3.2 yields

Lemma 3.3.4 (Murakami's trick for $q < s_F^-$). *Suppose that $q < s_F^-$. Then for any given constant $M > 0$, one can choose the translational invariant hermitian metric g suitably such that*

$$\mathfrak{T}'_F(\zeta'',\zeta'') + \mathfrak{T}_F(\zeta'',\zeta'') \geq \pi M\,\|\zeta''\|_2^2$$

for all $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$.

Proof. Fix an $\mathcal{H}$-apt coordinate system. Choose g as in (eq 3.16). Given $M > 0$, g_+^F, g_-^F and g_0^F are chosen as

$$g_+^F := 1\ ,\quad g_-^F := s_F^+ + M \quad\text{and}\quad g_0^F := 1\ .$$

Using the $\mathcal{H}$-apt coordinates, one sees that $\operatorname{Tr}_g\mathcal{H}_F = g_+^F s_F^+ - g_-^F s_F^-$ and therefore

$$\mathfrak{T}'_F(\zeta'',\zeta'') = \pi\left(g_-^F s_F^- - g_+^F s_F^+\right)\|\zeta''\|_2^2\ .$$

Again, since $\mathcal{H}_F$ and g are both diagonal under the chosen $\mathcal{H}$-apt coordinates, the monomial forms $\zeta''_{J_q}\overline{dv^{J_q}} \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$ with different multiindices J_q are orthogonal to one another with respect to $\mathfrak{T}_F$ and $\langle\cdot,\cdot\rangle_2$. Therefore, it suffices to show that

$$(**)\qquad \pi\left(g_-^F s_F^- - g_+^F s_F^+\right)\left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|_2^2 + \mathfrak{T}_F\left(\zeta''_{J_q} d\overline{v^{J_q}}, \zeta''_{J_q} d\overline{v^{J_q}}\right) \geq \pi M \left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|_2^2$$

for all monomial forms $\zeta''_{J_q} d\overline{v^{J_q}} \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$.

Taking into account (eq 3.17) and the expression of $\mathfrak{T}_F$ in the proof of Lemma 3.3.2, it follows that

$$\begin{aligned}
&\pi\left(g_-^F s_F^- - g_+^F s_F^+\right)\left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|_2^2 + \mathfrak{T}_F\left(\zeta''_{J_q} d\overline{v^{J_q}}, \zeta''_{J_q} d\overline{v^{J_q}}\right)\\
&= \pi\left(g_-^F\left(s_F^- - R^-(J_q)\right) - g_+^F\left(s_F^+ - R^+(J_q)\right)\right)\left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|_2^2\\
&= \pi\left(\left(s_F^+ + M\right)\left(s_F^- - R^-(J_q)\right) - \left(s_F^+ - R^+(J_q)\right)\right)\left\|\zeta''_{J_q} d\overline{v^{J_q}}\right\|_2^2 .
\end{aligned}$$

Since $q < s_F^-$, it follows that $s_F^- - R^-(J_q) \geq 1$ for any multiindex J_q. Note also that $s_F^+ - R^+(J_q) \leq s_F^+$ for any J_q. Therefore, by the choice of g_+^F and g_-^F, one obtains $g_-^F\left(s_F^- - R^-(J_q)\right) - g_+^F\left(s_F^+ - R^+(J_q)\right) \geq M$ and thus $(**)$ follows. This completes the proof. □

Considering the definition of $\widetilde{\mathcal{H}}(M)$ in (eq 3.18), Lemma 3.3.4 then implies that, if $q < s_F^-$, then, given $M > 0$, g can be chosen such that

$$\mathfrak{T}'_F(\zeta'', \zeta'') + \mathfrak{T}(\zeta, \zeta) \geq \pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^\vee (\zeta \otimes \overline{\zeta})\right)$$

for all $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$. Combining this with (eq 3.12) yields

Corollary 3.3.5. *Suppose $q < s_F^-$. Then, given any constant $M > 0$, the translational invariant hermitian metric g can be chosen suitably such that (eq 3.12) yields*

$$\begin{aligned}
\|S_q\zeta\|_3^2 + \left\|T_{q-1}^*\zeta\right\|_1^2 \geq \pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^\vee (\zeta \otimes \overline{\zeta})\right)&\\
+ \mathfrak{W}'_F(\zeta'', \zeta'') + \mathfrak{W}(\zeta, \zeta) + \mathfrak{wt}(\zeta, \zeta)&
\end{aligned}$$

for all $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom} T_{q-1}^*$, where $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$ and $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L) \cap \operatorname{Dom} \overline{\partial}^*_{[u]}$.

The remaining part of this section is devoted to getting a suitable estimate of the integral

$$\pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^\vee (\zeta \otimes \overline{\zeta})\right)$$

by varying $\mathcal{H}_E$ in $\widetilde{\mathcal{H}}(M)$ (see (eq 3.18)) according to Proposition 2.4.2.

Lemma 3.3.6. *Given a constant $M > 0$ and a fixed translational invariant hermitian metric g on X such that the decomposition (eq 2.4) is orthogonal, one can choose $\mathcal{H}_E$ sufficiently positive according to Proposition 2.4.2 such that*

$$\pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F\left(\left(\widetilde{\mathcal{H}}(M)\right)^\vee (\zeta \otimes \overline{\zeta})\right) \geq \tfrac{\pi}{4} M \|\zeta\|_2^2$$

for all $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$.

PROOF. For $q = 0$, it follows from (eq 3.18) that

$$\pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\left(\widetilde{\mathcal{H}}(M) \right)^{\vee} (\zeta \otimes \overline{\zeta}) \right) = \pi M \left\|\zeta\right\|_2^2 \geq \tfrac{\pi}{4} M \left\|\zeta\right\|_2^2 \ ,$$

so this case is done.

Assume $q \neq 0$. Since $\mathcal{H}_{u\overline{v}}^{\vee}$ is a bounded linear operator on $L^{0,(1,0)}_{2\,c,\chi} \otimes \overline{L^{0,(0,1)}_{2\,c,\chi}}$ (where $\overline{L^{0,(0,1)}_{2\,c,\chi}}$ here means the complex conjugate of $L^{0,(0,1)}_{2\,c,\chi}$), it follows that there is a bounded linear operator $\mathcal{N}\colon L^{0,(0,q)}_{2\,c,\chi} \to L^{0,(1,q-1)}_{2\,c,\chi}$ such that

$$\int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{H}_{u\overline{v}}^{\vee}(\zeta' \otimes \overline{\zeta''}) = \langle \zeta', \mathcal{N}\zeta'' \rangle_2$$

for all $\zeta' \in L^{0,(1,q-1)}_{2\,c,\chi}$ and $\zeta'' \in L^{0,(0,q)}_{2\,c,\chi}$. In fact, after a linear change of coordinates such that g becomes the Euclidean metric while keeping the decomposition (eq 2.4) orthogonal, one has

$$\operatorname{Tr}_{g,\eta} \mathcal{H}_{u\overline{v}}^{\vee}(\zeta' \otimes \overline{\zeta''}) = \eta \sideset{}{'}\sum_{J_{q-1}} \sum_{i=1}^{n-m} \sum_{j=1}^{m} \zeta'_{i\overline{J}_{q-1}} \overline{(\mathcal{H}_{v\overline{u}})_{j\overline{i}}\ \zeta''_{\overline{j}J_{q-1}}} \ ,$$

where $\sum'_{J_{q-1}}$ denotes summation over all ordered multiindices J_{q-1} such that $1 \leq j_1 < \cdots < j_{q-1} \leq m$, and $(\mathcal{H}_{v\overline{u}})_{j\overline{i}}$'s are the components of $\mathcal{H}_{v\overline{u}} = \overline{\mathcal{H}_{u\overline{v}}}$. Therefore, under such coordinates,

$$(\mathcal{N}\zeta'')_{i\overline{J}_{q-1}} = \sum_{j=1}^{m} (\mathcal{H}_{v\overline{u}})_{j\overline{i}}\ \zeta''_{\overline{j}J_{q-1}} \ .$$

Moreover,

$$\begin{aligned}
\left|\mathcal{N}\zeta''\right|^2_{g,\eta} &= \eta \sideset{}{'}\sum_{J_{q-1}} \sum_{i=1}^{n-m} \left| \sum_{j=1}^{m} (\mathcal{H}_{v\overline{u}})_{j\overline{i}}\ \zeta''_{\overline{j}J_{q-1}} \right|^2 \\
&\leq \eta \sideset{}{'}\sum_{J_{q-1}} \sum_{i=1}^{n-m} \left(\sum_{j=1}^{m} \left|(\mathcal{H}_{v\overline{u}})_{j\overline{i}}\right|^2 \right) \left(\sum_{j=1}^{m} \left|\zeta''_{\overline{j}J_{q-1}}\right|^2 \right) && \text{by Cauchy–Schwarz ineq.,} \\
&= \left|\mathcal{H}_{v\overline{u}}\right|^2_g \cdot q \left|\zeta''\right|^2_{g,\eta} = \left|\mathcal{H}_{u\overline{v}}\right|^2_g \cdot q \left|\zeta''\right|^2_{g,\eta} && \text{as } \mathcal{H}_{u\overline{v}} = \overline{\mathcal{H}_{v\overline{u}}} \ .
\end{aligned}$$

Since both $\mathcal{H}_{u\overline{v}}$ and g are translational invariant forms, $\left|\mathcal{H}_{u\overline{v}}\right|^2_g$ is a constant. Set $\nu := \sqrt{q}\left|\mathcal{H}_{u\overline{v}}\right|_g$. Then, one has

$$(*_\nu) \qquad \left\|\mathcal{N}\zeta''\right\|_2 \leq \nu \left\|\zeta''\right\|_2$$

for all $\zeta'' \in L^{0,(0,q)}_{2\,c,\chi}$. Note that ν depends only on q, $\mathcal{H}_{u\overline{v}}$ and g. It is independent of $\mathcal{H}_E$ in particular.

Since the decomposition (eq 2.4) is orthogonal with respect to g, g can be decomposed into $g_E + g_F$ such that g_E is a hermitian metric on $\mathbf{T}^{1,0}_u$ and g_F is that on $\mathbf{T}^{1,0}_v$. Choose a real number $\lambda > 0$ such that

$$(*_\lambda) \qquad \lambda \geq \max\left\{ \frac{M}{2}, \frac{2\nu^2}{M}, 4\nu \right\} \ .$$

Since ν is independent of $\mathcal{H}_E$, by varying the real part of the matrix of $\mathcal{H}_E$ under the chosen apt coordinates according to Proposition 2.4.2, $\mathcal{H}_E$ can be chosen such that

$$\mathcal{H}_E \geq \lambda g_E \ ,$$

and therefore,

$$\int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{H}_E^\vee(\zeta' \otimes \overline{\zeta'}) \geq \lambda \|\zeta'\|_2^2$$

for all $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L)$.

It follows from (eq 3.18) that, for any $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$,

$$\begin{aligned}
&\int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\widetilde{\mathcal{H}}(M)\right)^\vee (\zeta \otimes \overline{\zeta}) \\
&\geq \lambda \|\zeta'\|_2^2 + 2 \operatorname{Re} \langle \zeta', \mathcal{N}\zeta'' \rangle_2 + M \|\zeta''\|_2^2 \\
&= \lambda \left\| \zeta' + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 - \frac{1}{\lambda} \|\mathcal{N}\zeta''\|_2^2 + M \|\zeta''\|_2^2 && \text{by completing square ,} \\
&\geq \lambda \left\| \zeta' + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 - \frac{\nu^2}{\lambda} \|\zeta''\|_2^2 + M \|\zeta''\|_2^2 && \text{by } (*_\nu) \text{ ,} \\
&\geq \frac{M}{2} \left(\left\| \zeta' + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 + \|\zeta''\|_2^2 \right) && \text{by } (*_\lambda)\text{, thus } \frac{\nu^2}{\lambda} \leq \frac{M}{2} \text{ ,} \\
&= \frac{M}{2} \left\| \zeta + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 && \text{as } L^{2\,0,(1,q-1)}_{c,\chi} \perp L^{2\,0,(0,q)}_{c,\chi} \text{ .}
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
\left\| \zeta + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2 &\geq \|\zeta\|_2 - \frac{1}{\lambda} \|\mathcal{N}\zeta''\|_2 \\
&\geq \|\zeta\|_2 - \frac{\nu}{\lambda} \|\zeta''\|_2 && \text{by } (*_\nu) \text{ ,} \\
&\geq \left(1 - \frac{\nu}{\lambda}\right) \|\zeta\|_2 && \text{as } \|\zeta''\|_2 \leq \|\zeta\|_2 \text{ ,} \\
&\geq \frac{3}{4} \|\zeta\|_2 \geq 0 && \text{by } (*_\lambda) \text{ ,}
\end{aligned}$$

one has

$$\frac{M}{2} \left\| \zeta + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 \geq \frac{M}{2} \cdot \left(\frac{3}{4}\right)^2 \|\zeta\|_2^2 \geq \frac{M}{4} \|\zeta\|_2^2 \ .$$

This completes the proof. □

CHAPTER 4

The linearizable case

4.1. Proof of Theorem 1.1.1 for linearizable L

The proof of Theorem 1.1.1 for linearizable L is given here so that one can see clearly how the proof works without having to handle additional technicality required for the case of non-linearizable line bundles.

THEOREM 4.1.1. *Suppose L is linearizable and $q < s_F^-$ or $q > m - s_F^+$. Then, for any $\psi \in \mathscr{H}^{0,q}(X;L)$ such that $\overline{\partial}\psi = 0$, there exists $\xi \in \mathscr{H}^{0,q-1}(X;L)$ such that $\overline{\partial}\xi = \psi$ on X. (In case $q = 0 < s_F^-$, this means $\psi = 0$.) In other words, by virtue of Theorem 2.3.1, $H^q(X,L) = 0$ for any q in the given range.*

PROOF. Fix any $\psi \in \mathscr{H}^{0,q}(X;L) \cap \ker\overline{\partial}$.

An L^2-norm $\|\cdot\|_{X,\chi}$ is chosen as follows. Since L is linearizable, one can take $\hbar = 0$ (see §2.5 for the definition of $\hbar$). Then, choose $\delta = 0$ and thus $\hbar_\delta = \hbar - \delta = 0$. Choose the translational invariant hermitian metric g of the form as described in the proof of Lemma 3.3.2 for $q > m - s_F^+$ or Lemma 3.3.4 for $q < s_F^-$, with $M = 1$. For the hermitian form $\mathcal{H}$ associated to L, choose $\mathcal{H}_E := \mathcal{H}|_{E\times E}$ as described in the proof of Lemma 3.3.6. A hermitian metric η on L is then defined as in §2.5. Choose a convex increasing smooth function $\widetilde{\chi}$ (thus $\chi := \widetilde{\chi}\circ\varphi$ is plurisubharmonic, i.e. $\sqrt{-1}\partial\overline{\partial}\chi \geq 0$) such that $\|\psi\|_{X,\chi} < \infty$. An L^2-norm $\|\cdot\|_{X,\chi}$ is then fixed and $\psi \in L^{0,(0,q)}_{2\,\chi}(X;L)$.

Note that every $\zeta \in \mathscr{A}^{0,q}_{0\,<2>}(X;L)$ is contained in $\mathscr{A}^{0,q}_{0\,<2>}(K_c;L)$ for some sufficiently large but finite $c > 0$. Consequently, the conclusion of Corollary 3.3.3 when $q > m - s_F^+$ or Corollary 3.3.5 when $q < s_F^-$, as well as that of Lemma 3.3.6, holds for all $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{0\,<2>}(X;L)$, where $\zeta' \in \mathscr{A}^{0,(1,q-1)}_0(X;L)$ and $\zeta'' \in \mathscr{A}^{0,(0,q)}_0(X;L)$. Since $\hbar_\delta = 0$, $\mathfrak{W}(\zeta,\zeta)$ (see (eq 3.14)) and $\mathfrak{W}'_F(\zeta'',\zeta'')$ (see (eq 3.19)) both vanish for all $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{0\,<2>}(X;L)$.

Since χ is plurisubharmonic on X and $\overline{\partial}_{[v]}\chi = 0 = \partial_{[v]}\chi$, one can choose at every point $z \in X$ the coordinates such that both g and $\sqrt{-1}\partial_{[u]}\overline{\partial}_{[u]}\chi$ are simultaneously diagonalized while keeping the decomposition (eq 2.4) orthogonal, and see that

$$\operatorname{Tr}_{g,\eta}\operatorname{pr}_F\left(\left(\partial\overline{\partial}\chi\right)^\vee(\zeta\otimes\overline{\zeta})\right) = \operatorname{Tr}_{g,\eta}\left(\partial_{[u]}\overline{\partial}_{[u]}\chi\right)^\vee(\zeta'\otimes\overline{\zeta'}) \geq 0\ .$$

Therefore, $\mathfrak{wt}(\zeta,\zeta) \geq 0$ (see (eq 3.14)).

As a result, combining Lemma 3.3.6 as well as the above facts about $\mathfrak{W}$, $\mathfrak{W}'_F$ and $\mathfrak{wt}$ with Corollary 3.3.3 or Corollary 3.3.5, one obtains

$$\|S_q\zeta\|_3^2 + \left\|T_{q-1}^*\zeta\right\|_1^2 \geq \tfrac{\pi}{4}\|\zeta\|_2^2$$

for all $\zeta \in \mathscr{A}^{0,q}_{0\,<2>}(X;L)$. This is the required L^2 estimate. Proposition 3.1.5 and Remark 3.1.6 then assert that there exists $\xi \in \mathscr{H}^{0,q-1}(X;L)$ such that $\overline{\partial}\xi = \psi$ on X. □

CHAPTER 5

The non-linearizable case

For a non-linearizable line bundle L, the wild curvature terms $\mathfrak{W}$ (see (eq 3.14)) and $\mathfrak{W}'_F$ (see (eq 3.19)) are not identically zero. In order to get the estimates for these terms, Takayama's Weak $\partial\overline{\partial}$-Lemma (ref. [**Taka2**, Lemma 3.14]) is invoked. One is then forced to restrict attention to each of the K_c's and obtain the required L^2 estimates there. What then remains is to show that the existence of a solution of the $\overline{\partial}$-equation $\overline{\partial}\xi = \psi$ on every K_c implies the existence of a global solution. The argument for this latter part is essentially the same as the one in [**GR**, Ch. IV, §1, Thm. 7].

An apt coordinate system is fixed throughout this section.

5.1. Bounds on the wild curvature terms

Takayama proves in [**Taka2**] the following Weak $\partial\overline{\partial}$-Lemma.

WEAK $\partial\overline{\partial}$-LEMMA 5.1.1 (cf. [**Taka2**, Lemma 3.14]). *Let ω be a positive real $(1,1)$-form on X, and let θ be a smooth real 1-form on X such that $\theta = \overline{\beta} + \beta$ for some smooth $(0,1)$-form β, and $d\theta$ is of type $(1,1)$. Then for every positive number ε and every relatively compact open subset W of X, there exists a smooth function δ on X such that*

$$-\varepsilon\omega < d\theta - 2\sqrt{-1}\partial\overline{\partial}\operatorname{Re}\delta < \varepsilon\omega \quad \text{on } W\ .$$

Moreover, if $\beta \in \mathscr{H}^{0,1}(X)$, then δ can be chosen such that $\delta \in \mathscr{H}(X)$.

In the current situation, the role of β in Lemma 5.1.1 is taken by $\sqrt{-1}\ \overline{\partial}\hbar$ (therefore $d\theta = 2\sqrt{-1}\partial\overline{\partial}\operatorname{Re}\hbar$), and that of W by K_c.

REMARK 5.1.2. In Takayama's formulation, the assertion of the Weak $\partial\overline{\partial}$-Lemma is that there exists a smooth real valued function $f_{\varepsilon W} := 2(\operatorname{Im} f_0 + \operatorname{Im}\Psi_{M_0})$ on X such that $-\varepsilon\omega < d\theta - \sqrt{-1}\partial\overline{\partial}f_{\varepsilon W} < \varepsilon\omega$ on W, in which f_0 is a smooth function on X such that $\beta = \phi + \overline{\partial}f_0$ for some real analytic $(0,1)$-form ϕ in $\mathscr{H}^{0,1}(X)$, and Ψ_{M_0} is some real analytic function in $\mathscr{H}(X)$. Therefore, the smooth function δ here is given by $\delta := -\sqrt{-1}(f_0 + \Psi_{M_0})$ in Takayama's notation. If $\beta \in \mathscr{H}^{0,1}(X)$, then one has $f_0 \in \mathscr{H}(X)$ as $\overline{\partial}_{[u]}f_0 = 0$, so $\delta \in \mathscr{H}(X)$ also.

REMARK 5.1.3. As a side remark, following the construction of δ in [**Taka2**, Lemma 3.14], $\overline{\partial}\hbar_\delta = \overline{\partial}\hbar - \overline{\partial}\delta$ is real analytic on X, so $\hbar_\delta$ is real analytic on $\mathbb{C}^n$. It follows that the hermitian metric η on L is real analytic.

Suitable estimates for the wild curvature terms $\mathfrak{W}$ and $\mathfrak{W}'_F$ are obtained by choosing a proper $\delta \in \mathscr{H}(X)$ according to the Weak $\partial\overline{\partial}$-Lemma.

LEMMA 5.1.4. *Suppose a hermitian metric g on X and a choice of $\mathcal{H}_E$ are fixed. Then, on every K_c where $0 < c < \infty$, given any real number $\varepsilon_{\mathsf{w}} > 0$ and for any*

$q \geq 0$, *one can choose* $\delta_c \in \mathscr{H}(X)$ *which yields a hermitian metric* η_c *on* L *such that, for any given weight* χ,

$$|\mathfrak{W}(\zeta, \zeta)| \leq \varepsilon_{\mathsf{w}} q \left\|\zeta\right\|^2_{K_c,\eta_c,\chi} \tag{eq 5.1}$$

$$|\mathfrak{W}'_F(\zeta'', \zeta'')| \leq \varepsilon_{\mathsf{w}} m \left\|\zeta''\right\|^2_{K_c,\eta_c,\chi} \leq \varepsilon_{\mathsf{w}} m \left\|\zeta\right\|^2_{K_c,\eta_c,\chi} \tag{eq 5.2}$$

for all $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$ *where* $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L)$ *and* $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$.

Proof. First the estimate for $\mathfrak{W}$ is considered. Recall that ω is the $(1,1)$-form associated to g. The Weak $\partial\overline{\partial}$-Lemma asserts that, for any $\varepsilon_{\mathsf{w}} > 0$, there exists $\delta_c \in \mathscr{H}(X)$ such that

$$-2\varepsilon_{\mathsf{w}}\omega < 2\sqrt{-1}\partial\overline{\partial}\operatorname{Re}\hbar_{\delta_c} < 2\varepsilon_{\mathsf{w}}\omega \quad \text{on } K_c\ . \tag{eq 5.3}$$

Such δ_c yields a hermitian metric η_c on L given the fixed choice of $\mathcal{H}_E$. Then, it follows from (eq 3.14) that, for any weight χ,

$$-\varepsilon_{\mathsf{w}} \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta_c} \operatorname{pr}_F \left(g^\vee(\zeta \otimes \overline{\zeta})\right) \leq \mathfrak{W}(\zeta,\zeta) \leq \varepsilon_{\mathsf{w}} \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta_c} \operatorname{pr}_F \left(g^\vee(\zeta \otimes \overline{\zeta})\right)$$

for any $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$ (ε_{w} instead of $2\varepsilon_{\mathsf{w}}$ in the bounds because of the factor $\frac{1}{2}$ in $\omega = -\operatorname{Im} g = \frac{\sqrt{-1}}{2}\sum_{k,\ell} g_{k\overline{\ell}} dz^k \wedge d\overline{z^\ell}$). Note that

$$\int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta_c} \operatorname{pr}_F \left(g^\vee(\zeta \otimes \overline{\zeta})\right) = \left\|\zeta'\right\|^2_{K_c,\eta_c,\chi} + q\left\|\zeta''\right\|^2_{K_c,\eta_c,\chi} \leq q \left\|\zeta\right\|^2_{K_c,\eta_c,\chi}$$

when $q \geq 1$. When $q = 0$, the integral on the left hand side is zero, so the above inequality is still valid. As a result, one obtains

$$-\varepsilon_{\mathsf{w}} q \left\|\zeta\right\|^2_{K_c,\eta_c,\chi} \leq \mathfrak{W}(\zeta,\zeta) \leq \varepsilon_{\mathsf{w}} q \left\|\zeta\right\|^2_{K_c,\eta_c,\chi}$$

and hence (eq 5.1).

For the estimate for $\mathfrak{W}'_F$, note that (eq 5.3) implies

$$-2\varepsilon_{\mathsf{w}} \operatorname{pr}_F \omega < 2\sqrt{-1}\partial_{[v]}\overline{\partial}_{[v]} \operatorname{Re}\hbar_{\delta_c} < 2\varepsilon_{\mathsf{w}} \operatorname{pr}_F \omega \quad \text{on } K_c\ .$$

Then, one has $-\varepsilon_{\mathsf{w}} m < 2\operatorname{Tr}_g \partial_{[v]}\overline{\partial}_{[v]} \operatorname{Re}\hbar_{\delta_c} < \varepsilon_{\mathsf{w}} m$ with the same ε_{w} and δ_c as above. Therefore, it follows from (eq 3.19) that

$$-\varepsilon_{\mathsf{w}} m \left\|\zeta''\right\|^2_{K_c,\eta_c,\chi} \leq \mathfrak{W}'_F(\zeta'',\zeta'') \leq \varepsilon_{\mathsf{w}} m \left\|\zeta''\right\|^2_{K_c,\eta_c,\chi}$$

for any $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$, and hence (eq 5.2). □

5.2. Existence of weak solutions on K_c

With the bounds given in §5.1 for the wild curvature terms, it is easy to follow the proof of Theorem 4.1.1 and get the following

Proposition 5.2.1. *Suppose* L *is a holomorphic line bundle on* X *(which can possibly be non-linearizable), and suppose* $q < s_F^-$ *or* $q > m - s_F^+$*. Then, there exists a suitable hermitian metric* g *on* X *such that the following holds: for any* $0 < c < \infty$*, a hermitian metric* η_c *on* L *can be chosen such that, given any plurisubharmonic weight* χ*, the* L^2 *estimate*

$$\left\|S_q\zeta\right\|^2_{K_c,\eta_c,\chi} + \left\|T^*_{q-1}\zeta\right\|^2_{K_c,\eta_c,\chi} \geq \tfrac{\pi}{4}\left\|\zeta\right\|^2_{K_c,\eta_c,\chi}$$

for all $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\eta_c,\chi} T^*_{q-1}$ *is satisfied.*

Proof. Choose the translational invariant hermitian metric g as described in the proof of Lemma 3.3.2 for $q > m - s_F^+$ or Lemma 3.3.4 for $q < s_F^-$, with $M = 2$. For the hermitian form $\mathcal{H}$ associated to L, choose $\mathcal{H}_E$ as described in the proof of Lemma 3.3.6. These choices are independent of c.

Consider K_c for some fixed $0 < c < \infty$. Take any $\varepsilon_{\mathsf{w}} > 0$ such that

$$(*) \qquad \varepsilon_{\mathsf{w}}(q+m) \leq \frac{\pi}{4}$$

and choose $\delta_c \in \mathscr{H}(X)$ according to Lemma 5.1.4 such that, for any given weight χ, the inequalities (eq 5.1) and (eq 5.2) hold under the induced L^2-norm $\|\cdot\|_{K_c,\eta_c,\chi}$.

By the choices of the metrics, the conclusion of Corollary 3.3.3 when $q > m - s_F^+$ or Corollary 3.3.5 when $q < s_F^-$, as well as that of Lemma 3.3.6, holds for all $\zeta = \zeta' + \zeta'' \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c,\eta_c,\chi} T^*_{q-1}$, where $\zeta' \in \mathscr{A}^{0,(1,q-1)}(\overline{K}_c; L) \cap \operatorname{Dom}^{(1,q-1)}_{K_c,\eta_c,\chi} \overline{\partial}^*_{[u]}$ and $\zeta'' \in \mathscr{A}^{0,(0,q)}(\overline{K}_c; L)$.

Since χ is plurisubharmonic, $\mathfrak{wt}(\zeta,\zeta) \geq 0$ for all $\zeta \in \mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L)$ as in the proof of Theorem 4.1.1.

As a result, from Corollary 3.3.3 or 3.3.5 as well as Lemma 3.3.6, one obtains

$$\begin{aligned}
&\|S_q\zeta\|^2_{K_c,\eta_c,\chi} + \left\|T^*_{q-1}\zeta\right\|^2_{K_c,\eta_c,\chi} \\
&\geq \begin{cases} \frac{\pi}{2}\|\zeta\|^2_{K_c,\eta_c,\chi} + \mathfrak{W}(\zeta,\zeta) & \text{for } q > m - s_F^+ \\ \frac{\pi}{2}\|\zeta\|^2_{K_c,\eta_c,\chi} + \mathfrak{W}'_F(\zeta'',\zeta'') + \mathfrak{W}(\zeta,\zeta) & \text{for } q < s_F^- \end{cases} \\
&\geq \frac{\pi}{2}\|\zeta\|^2_{K_c,\eta_c,\chi} - \varepsilon_{\mathsf{w}}(m+q)\|\zeta\|_{K_c,\eta_c,\chi} \qquad \begin{array}{l}\text{by (eq 5.1) and (eq 5.2),}\\ \text{and } \varepsilon_{\mathsf{w}} q < \varepsilon_{\mathsf{w}}(m+q)\end{array} \\
&\geq \frac{\pi}{4}\|\zeta\|^2_{K_c,\eta_c,\chi} \qquad \text{by } (*)\,.
\end{aligned}$$

This gives the required L^2 estimate. □

Since, for any $\psi \in \mathscr{H}^{0,q}(X; L)$, one has $\psi|_{K_c} \in L_2^{0,(0,q)}(K_c; L)$ (unweighted) for any $0 < c < \infty$, it follows the following corollary of Propositions 3.1.5 and 5.2.1.

Corollary 5.2.2. *Consider the exhaustive sequence $\{K_\nu\}_{\nu\in\mathbb{N}_{>0}}$ of relatively compact open subsets of X. Suppose $q < s_F^-$ or $q > m - s_F^+$. Then one can choose a suitable hermitian metric g on X and a sequence of hermitian metrics $\{\eta_\nu\}_{\nu\in\mathbb{N}_{>0}}$ on L as in Proposition 5.2.1 such that, for any $\psi \in \mathscr{H}^{0,q}(X; L) \cap \ker\overline{\partial}$, there exists a sequence of solutions $\{\xi'_\nu\}_{\nu\in\mathbb{N}_{>0}}$ such that $\xi'_\nu \in L^{0,(0,q-1)}_{2\,\eta_\nu}(K_\nu; L)$ (unweighted) and $\overline{\partial}\xi'_\nu = \psi|_{K_\nu}$ in $L^{0,(0,q)}_{2\,\eta_\nu}(K_\nu; L)$.*

Remark 5.2.3. Since χ has to be smooth on a neighborhood of $\overline{K}_c$ (as required by [**Hör1**, Prop. 2.1.1] so that $\mathscr{A}^{0,q}_{<2>}(\overline{K}_c; L) \cap \operatorname{Dom} T^*_{q-1}$ is dense in $\operatorname{Dom} T^*_{q-1} \cap \operatorname{Dom} S_q$ under the suitable graph norm), if $\psi \in \mathscr{H}^{0,q}(K_c; L)$, there may not exist such χ such that $\|\psi\|_{K_c,\chi} < \infty$. To avoid technical difficulty, the author does not attempt to solve the $\overline{\partial}$-equation for any $\psi \in \mathscr{H}^{0,q}(K_c; L)$ such that $\overline{\partial}\psi = 0$ by means of L^2 estimates directly.

5.3. A Runge-type approximation

This section is devoted to proving a Runge-type approximation which is required to construct a global solution to the equation $\overline{\partial}\xi = \psi$ from the solutions on K_ν's given in Corollary 5.2.2.

In what follows, q is assumed to be $0 < q < s_F^-$ or $q > m - s_F^+$, and the hermitian metric g as well as the family of hermitian metrics $\{\eta_c\}_{c>0}$ as asserted by Proposition 5.2.1 is fixed. Then, according to the choices of the η_c's in the proof of Proposition 5.2.1, for any $c', c > 0$, one has

$$\eta_c = \eta_{c'} e^{2\operatorname{Re}(\delta_{c'} - \delta_c)} =: \eta_{c'} e^{\boldsymbol{\delta}_{c'c}} .$$

Note that $e^{\boldsymbol{\delta}_{c'c}} > 0$ on X. It is understood that the hermitian metric η_c on L is chosen when the L^2-norm on K_c is considered, so write $L^{0,(0,q)}_{2\ \eta_c,\chi}(K_c; L)$ as $L^{0,(0,q)}_{2\ \chi}(K_c; L)$, $\langle\cdot,\cdot\rangle_{K_c,\eta_c,\chi}$ as $\langle\cdot,\cdot\rangle_{K_c,\chi}$ and so on to simplify notation. When the weight χ is absent from the notation, e.g. $L^{0,(0,q)}_2(K_c; L)$ or $\langle\cdot,\cdot\rangle_{K_c}$, it is understood that the corresponding object is unweighted, i.e. $\chi = 0$.

For any finite $c' > c > 0$ and for any $\Psi \in L^{0,(0,q-1)}_2(K_c; L)$, if Ψ is extended by zero to a section in $L^{0,(0,q-1)}_2(K_{c'}; L)$, then it follows that

$$\langle\zeta, \Psi\rangle_{K_c} = \left\langle\zeta, \Psi e^{\boldsymbol{\delta}_{c'c}}\right\rangle_{K_{c'}} \tag{eq 5.4}$$

for any $\zeta \in L^{0,(0,q-1)}_2(K_{c'}; L)$.

Define $\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c}$ to be the image of $\ker_{K_{c'}} T_{q-1}$ under the restriction map $L^{0,(0,q-1)}_2(K_{c'}; L) \to L^{0,(0,q-1)}_2(K_c; L)$. Note that T_{q-1} commutes with the restriction map (as $c > 0$), so one has

$$\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c} \subset \ker_{K_c} T_{q-1} .$$

The following proof of the required Runge-type approximation is an analogue of the one for strongly pseudoconvex manifolds given in [**Hör3**, Lemma 4.3.1].

PROPOSITION 5.3.1. *Suppose $0 < q < s_F^-$ or $q > m - s_F^+$, and g and η_c's are chosen according to Proposition 5.2.1. Then, for any finite $c' > c > 0$, the closure of $\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c}$ in $L^{0,(0,q-1)}_2(K_c; L)$ is $\ker_{K_c} T_{q-1}$. In other words, $\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c}$ is dense in $\ker_{K_c} T_{q-1}$.*

PROOF. By virtue of the Hahn-Banach theorem, it suffices to show that for every $\Psi \in L^{0,(0,q-1)}_2(K_c; L)$, if the induced bounded linear functional

$$L^{0,(0,q-1)}_2(K_c; L) \ni \zeta \mapsto \langle\zeta, \Psi\rangle_{K_c}$$

vanishes on $\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c}$, then it also vanishes on $\ker_{K_c} T_{q-1}$. [1]

Suppose that $\Psi \in L^{0,(0,q-1)}_2(K_c; L)$ satisfies the above assumption. Extend Ψ by zero to $K_{c'}$ as a section in $L^{0,(0,q-1)}_2(K_{c'}; L)$. Now it suffices to show that there exists $\Xi \in L^{0,q}_{2\ <2>}(K_{c'}; L)$ such that $\Xi \equiv 0$ on $K_{c'} \setminus \overline{K}_c$ and

$$\left\langle\zeta, \Psi e^{\boldsymbol{\delta}_{c'c}}\right\rangle_{K_{c'}} = \langle T_{q-1}\zeta, \Xi\rangle_{K_{c'}} \tag{†}$$

for any $\zeta \in \operatorname{Dom}_{K_{c'}} T_{q-1}$, which then implies that

$$\langle\zeta, \Psi\rangle_{K_c} = \left\langle T_{q-1}\zeta, \Xi e^{-\boldsymbol{\delta}_{c'c}}\right\rangle_{K_c} \tag{‡}$$

for any $\zeta \in \operatorname{Dom}_{K_{c'}} T_{q-1}$ due to (eq 5.4). The equality (‡) holds true for $\zeta \in \mathscr{A}^{0,(0,q-1)}_0(K_{c'}; L)$ in particular, and $\mathscr{A}^{0,(0,q-1)}(\overline{K}_c; L)$ is dense in $\operatorname{Dom}_{K_c} T_{q-1}$ under the graph norm $\sqrt{\|\zeta\|^2_{K_c} + \|T_{q-1}\zeta\|^2_{K_c}}$ by [**Hör1**, Prop. 2.1.1], so (‡) also holds

[1] If there exists $\zeta \in \ker_{K_c} T_{q-1}$ which does not lie in the closure of $\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c}$ in $L^{0,(0,q-1)}_2(K_c; L)$, then the Hahn-Banach theorem asserts that there is a bounded linear functional Λ such that $\left(\ker_{K_{c'}} T_{q-1}\right)\big|_{K_c} \subset \ker\Lambda$ and $\Lambda\zeta = 1$.

true for $\zeta \in \mathrm{Dom}_{K_c} T_{q-1}$. It follows that

$$\langle \zeta, \Psi \rangle_{K_c} = \left\langle T_{q-1}\zeta, \Xi e^{-\boldsymbol{\delta}_{c'c}} \right\rangle_{K_c} = 0$$

for all $\zeta \in \ker_{K_c} T_{q-1} \subset \mathrm{Dom}_{K_c} T_{q-1}$ as required. It remains to show the existence of such Ξ.

Take a sequence of smooth convex increasing functions $\widetilde{\chi}_\nu \colon \mathbb{R} \to \mathbb{R}$ such that $\widetilde{\chi}_\nu(x) = 0$ for all $x \le c$, and $\widetilde{\chi}_\nu(x) \nearrow +\infty$ as $\nu \to \infty$ for every $x > c$. Note that $\widetilde{\chi}_\nu \ge 0$ for any $\nu \ge 0$ by such choice. Set $\chi_\nu := \widetilde{\chi}_\nu \circ \varphi$ as before. A sequence of weighted norms $\|\cdot\|_{c',\nu} := \|\cdot\|_{K_{c'},\chi_\nu}$ on $K_{c'}$ is then defined. Let the corresponding inner products, Hilbert spaces and Dom also be distinguished by using the subscripts c', ν, and the corresponding adjoint of T_{q-1} by $T^{*,\nu}_{q-1}$.

For any q in the given range, the L^2 estimate in Proposition 5.2.1 holds under each of the above weighted norms with T^*_{q-1} replaced by $T^{*,\nu}_{q-1}$. Since $\left\langle \zeta, \Psi e^{\boldsymbol{\delta}_{c'c}} e^{\chi_\nu} \right\rangle_{c',\nu} = \left\langle \zeta, \Psi e^{\boldsymbol{\delta}_{c'c}} \right\rangle_{K_{c'}}$ and the right hand side vanishes for all $\zeta \in \ker_{K_{c'}} T_{q-1} = \ker_{c',\nu} T_{q-1}$ by the assumption on Ψ, it follows that

$$\Psi e^{\boldsymbol{\delta}_{c'c}} e^{\chi_\nu} \in \left(\ker_{c',\nu} T_{q-1}\right)^\perp = \overline{\mathrm{im}_{c',\nu} T^{*,\nu}_{q-1}} \; .$$

Given the L^2 estimate, Theorem 3.1.1 (2) then asserts that there exists $\widetilde{\Xi}^\nu \in \mathrm{Dom}_{c',\nu} T^{*,\nu}_{q-1}$ such that $T^{*,\nu}_{q-1}\widetilde{\Xi}^\nu = \Psi e^{\boldsymbol{\delta}_{c'c}} e^{\chi_\nu}$. Therefore, one has

$$\begin{aligned} \left\langle \zeta, \Psi e^{\boldsymbol{\delta}_{c'c}} e^{\chi_\nu} \right\rangle_{c',\nu} &= \left\langle \zeta, T^{*,\nu}_{q-1}\widetilde{\Xi}^\nu \right\rangle_{c',\nu} \\ &= \left\langle T_{q-1}\zeta, \widetilde{\Xi}^\nu \right\rangle_{c',\nu} = \left\langle T_{q-1}\zeta, \widetilde{\Xi}^\nu e^{-\chi_\nu} \right\rangle_{K_{c'}} \end{aligned}$$

for all $\nu \in \mathbb{N}$ and for all $\zeta \in \mathrm{Dom}_{c',\nu} T_{q-1} = \mathrm{Dom}_{K_{c'}} T_{q-1}$. By defining $\Xi^\nu := \widetilde{\Xi}^\nu e^{-\chi_\nu}$, one obtains

$$(*) \qquad \left\langle \zeta, \Psi e^{\boldsymbol{\delta}_{c'c}} \right\rangle_{K_{c'}} = \left\langle T_{q-1}\zeta, \Xi^\nu \right\rangle_{K_{c'}} \; .$$

Moreover, notice that the constant in the L^2 estimate is independent of ν (which is chosen to be $\frac{\pi}{4}$ in Proposition 5.2.1). The estimate on the solution $\widetilde{\Xi}^\nu$ from Theorem 3.1.1 (2) then implies that

$$(**) \qquad \frac{\pi}{4} \int_{K_{c'}} |\Xi^\nu|^2_{g,\eta_{c'}} e^{\chi_\nu} \le \int_{K_{c'}} \left|\Psi e^{\boldsymbol{\delta}_{c'c}}\right|^2_{g,\eta_{c'}} e^{\chi_\nu} = \int_{K_c} |\Psi|^2_{g,\eta_c} e^{\boldsymbol{\delta}_{c'c}} e^{\chi_\nu} \; ,$$

where the last equality is due to the fact that Ψ vanishes on $K_{c'} \setminus \overline{K}_c$. Since $\widetilde{\chi}_\nu(\varphi)$ is independent of ν when $\varphi \le c$, the integral on the right hand side is independent of ν, so the left hand side is a bounded sequence in ν. This in turn implies that there exists a subsequence of $\{\Xi^\nu\}_{\nu\in\mathbb{N}}$ which converges to some $\Xi \in L^{0,q}_{2\,<2>}(K_{c'}; L)$ (unweighted) in the weak topology. From $(**)$, since $\widetilde{\chi}_\nu(\varphi) \nearrow +\infty$ for $\varphi > c$, it follows that $\Xi \equiv 0$ when $\varphi > c$, i.e. on $K_{c'} \setminus \overline{K}_c$. Moreover, from $(*)$ it follows that $(\dagger)$ holds for all $\zeta \in \mathrm{Dom}_{K_{c'}} T_{q-1}$. This is what is desired. $\square$

5.4. Proof of Theorem 1.1.1 for general L

First notice that, if $q = 0 < s_F^-$, then the L^2 estimate in Proposition 5.2.1 holds when the metrics are chosen suitably, and thus for any $\psi \in \mathscr{H}(X; L) \cap \ker \overline{\partial}$ one has

$$0 = \left\|\overline{\partial}\psi\right\|^2_{K_c} \ge \frac{\pi}{4} \left\|\psi\right\|^2_{K_c}$$

(note that $T^*_{-1}\zeta = 0$ for all $\zeta \in \mathscr{A}(\overline{K}_c; L)$). This means that $\psi|_{K_c} = 0$ for any $c > 0$, and thus $\psi = 0$ on X. Therefore, one has the following

THEOREM 5.4.1. *If $s_F^- > 0$, one has $H^0(X, L) = 0$.*

Assume $0 < q < s_F^-$ or $q > m - s_F^+$ in what follows. The metrics g and η_ν's from Corollary 5.2.2 are fixed for this section. Again, write $L^{0,(0,q)}_{2\,\eta_\nu,\chi}(K_\nu; L)$ as $L^{0,(0,q)}_{2\,\chi}(K_\nu; L)$ and so on, and notations like $L^{0,(0,q)}_2(K_c; L)$ or $\|\cdot\|_{K_c}$ are understood as unweighted objects, i.e. $\chi = 0$.

For every integer $\nu \geq 1$, as $\delta_{\nu+1} - \delta_\nu$ is smooth on X and $\overline{K}_{\nu+1}$ is compact, there exists a constant $M'_{\nu+1} \geq 1$ such that

$$\|\zeta\|_{K_\nu} \leq M'_{\nu+1} \|\zeta\|_{K_{\nu+1}} \tag{eq 5.5}$$

for all $\zeta \in L^{0,(0,q)}_2(K_{\nu+1}; L)$. Define also $M_1 := 1$ and $M_\nu := \prod_{k=2}^{\nu} M'_k$ for $\nu \geq 2$.

Proposition 5.3.1 is used to complete the proof of Theorem 1.1.1. The following argument is adopted from [**GR**, Ch. IV, §1, Thm. 7].

THEOREM 5.4.2. *Suppose $0 < q < s_F^-$ or $q > m - s_F^+$. Then one has $H^q(X, L) = 0$ for any q in the given range.*

PROOF. Given any $\psi \in \mathscr{H}^{0,q}(X; L) \cap \ker \overline{\partial}$, Corollary 5.2.2 provides a sequence of local solutions $\{\xi'_\nu\}_{\nu\geq 1}$ such that $\xi'_\nu \in L^{0,(0,q-1)}_2(K_\nu; L)$ and $\overline{\partial}\xi'_\nu = \psi|_{K_\nu}$ for all integers $\nu \geq 1$. First a sequence of local solutions $\{\xi_\nu\}_{\nu\geq 1}$ such that $\xi_\nu \in L^{0,(0,q-1)}_2(K_\nu; L)$, $\overline{\partial}\xi_\nu = \psi|_{K_\nu}$ and

$$\|\xi_{\nu+1} - \xi_\nu\|_{K_\nu} < \frac{1}{M_\nu 2^\nu} \tag{$*$}$$

for all $\nu \geq 1$ is defined inductively as follows. Set $\xi_1 := \xi'_1$. Suppose $\xi_1, \ldots, \xi_\nu$ are defined for some $\nu \geq 1$. Let $\gamma'_\nu := \xi'_{\nu+1}|_{K_\nu} - \xi_\nu$. Notice that $\gamma'_\nu \in \ker_{K_\nu} T_{q-1} \subset L^{0,(0,q-1)}_2(K_\nu; L)$. Proposition 5.3.1 then implies that there exists $\gamma_\nu \in \ker_{K_{\nu+1}} T_{q-1} \subset L^{0,(0,q-1)}_2(K_{\nu+1}; L)$ such that

$$\|\gamma'_\nu - \gamma_\nu\|_{K_\nu} < \frac{1}{M_\nu 2^\nu} \ .$$

Set $\xi_{\nu+1} := \xi'_{\nu+1} - \gamma_\nu$. Then one has $\overline{\partial}\xi_{\nu+1} = \overline{\partial}\xi'_{\nu+1} = \psi|_{K_{\nu+1}}$ and the inequality $(*)$ is satisfied. The required sequence $\{\xi_\nu\}_{\nu\geq 1}$ is therefore defined.

Notice that, for every $\nu \geq 1$, the sequence $\{\xi_\mu|_{K_\nu}\}_{\mu\geq\nu}$ converges in $L^{0,(0,q-1)}_2(K_\nu; L)$. Indeed, for any $\mu \geq \nu \geq 1$ and for any integer $k > 0$,

$$\begin{aligned}
\|\xi_{\mu+k} - \xi_\mu\|_{K_\nu} &\leq \sum_{r=0}^{k-1} \|\xi_{\mu+r+1} - \xi_{\mu+r}\|_{K_\nu} \\
&\leq \sum_{r=0}^{k-1} \frac{M_{\mu+r}}{M_\nu} \|\xi_{\mu+r+1} - \xi_{\mu+r}\|_{K_{\mu+r}} \quad \text{by (eq 5.5)} , \\
&\leq \frac{1}{M_\nu} \sum_{r=0}^{k-1} \frac{1}{2^{\mu+r}} \quad \text{by } (*) , \\
&\leq \frac{1}{M_\nu 2^{\mu-1}} ,
\end{aligned}$$

which tends to 0 as $\mu \to \infty$, so $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ is a Cauchy sequence in $L_2^{0,(0,q-1)}(K_\nu; L)$. Let $\xi^{(\nu)}$ be the limit of $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ in $L_2^{0,(0,q-1)}(K_\nu; L)$. Since $\overline{\partial}\xi_\mu|_{K_\nu} = \psi|_{K_\nu}$ for all $\mu \geq \nu$, and $\overline{\partial}$ is a closed operator, one has $\overline{\partial}\xi^{(\nu)} = \psi|_{K_\nu}$ for all $\nu \geq 1$. Now notice that restriction from $K_{\nu+1}$ to K_ν is continuous by (eq 5.5), so

$$\xi^{(\nu+1)}|_{K_\nu} - \xi^{(\nu)} = \lim_{\substack{\mu \geq \nu+1 \\ \mu \to \infty}} (\xi_\mu|_{K_\nu} - \xi_\mu|_{K_\nu}) = 0$$

in $L_2^{0,(0,q-1)}(K_\nu; L)$. On every K_ν, different choices of $\delta_\nu \in \mathscr{H}(X)$ yield equivalent norms. Therefore, by fixing one $\delta \in \mathscr{H}(X)$, one can consider $L_2^{0,q-1}(X; L; \mathrm{loc})$, the space of locally L^2 L-valued $(0, q-1)$-forms on X, and there exists $\xi' \in L_2^{0,q-1}(X; L; \mathrm{loc})$ such that

$$\begin{aligned} \xi'|_{K_\nu} &= \xi^{(\nu)} \quad \text{for all } \nu \geq 1\ , \text{ and} \\ \overline{\partial}\xi' &= \psi \qquad \text{in } L_2^{0,q-1}(X; L; \mathrm{loc})\ . \end{aligned}$$

Remark 3.1.6 then assures that there exists $\xi \in \mathscr{H}^{0,q-1}(X; L)$ such that $\overline{\partial}\xi = \psi$ on X.

Since $\psi \in \mathscr{H}^{0,q}(X; L) \cap \ker \overline{\partial}$ is arbitrary, this shows that $H^q(X, L) = 0$. This completes the proof. □

List of Symbols

$\mathscr{A}$ .... p. 5
$\mathscr{A}_{p,q}$ .... p. 16
$\mathscr{A}_{(p',p''),(q',q'')}$ .... p. 16
$\mathscr{A}_{(p',p''),0}$ .... p. 16
$\mathscr{A}^{(p',p''),(q',q'')}$ .... p. 9
$\mathscr{A}^{0,(q',q'')}$ .... p. 10
$\mathscr{A}_0^{0,(q',q'')}(K_c;L)$ .... p. 10
$\mathscr{A}_{<2>}^{0,q}(K_c;L)$ .... p. 12
$\mathscr{A}_{<3>}^{0,q+1}(K_c;L)$ .... p. 12
δ .... p. 8
$\partial_{\overline{k}}$ (resp. ∂_k) .... p. 16
$\partial_{\overline{v^j}}$ (resp. ∂_{v^j}) .... p. 16
$\overline{\partial}_{[u]}$ (resp. $\overline{\partial}_{[v]}$) .... p. 10
$\vartheta_{[u]}$ (resp. $\vartheta_{[v]}$) .... p. 10
$\overline{\partial}^*_{[u]}$ (resp. $\overline{\partial}^*_{[v]}$) .... p. 10
$\overline{dv^{J_q}}$ .... p. 21
η_{t} .... p. 7
η_{w} .... p. 8
η .... p. 9
$E \oplus F$ .... p. 4
F .... p. 2
f_γ .... p. 6
g .... p. 9
$\hbar$ .... p. 8
$\hbar_\delta$ .... p. 8
$\mathcal{H}$ .... p. 6
$\mathcal{H}_F$ .... p. 21
$\mathcal{H}_E + \mathcal{H}_{u\overline{v}} + \mathcal{H}_{v\overline{u}} + \mathcal{H}_F$ .... p. 22
$\mathscr{H}$ .... p. 5
$\mathscr{H}^{0,q}$ .... p. 5
$\mathscr{H}^{0,q}(U;V)$ .... p. 6
K .... p. 2
K_c .... p. 5
K_∞ (resp. K_0) .... p. 5
L .... p. 6
L_{t} (resp. L_{w}) .... p. 7
$L_{2\,c,\chi}^{0,q} := L_{2\,\chi}^{0,q}(K_c;L)$ .... p. 9
$L_{2\,c,\chi}^{0,(q',q'')} := L_{2\,\chi}^{0,(q',q'')}(K_c;L)$ .... p. 10
$L_{2\,c,\chi<2>}^{0,q}$ .... p. 12
$L_{2\,c,\chi<3>}^{0,q+1}$ .... p. 12
∇ .... p. 16
$\nabla_{\overline{k}}$ (resp. ∇_k) .... p. 16
$\nabla_{\overline{v^j}}$ (resp. ∇_{v^j}) .... p. 16
$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ .... p. 17
$\nabla^{(0,1)} = \nabla_{\overline{u}} + \nabla_{\overline{v}}$ .... p. 18
$\nabla^{(1,0)} = \nabla_u + \nabla_v$ .... p. 19
φ .... p. 5
pr_F .... p. 18
$\mathcal{R}$ .... p. 16
$\mathcal{R}_{u\overline{u}} + \mathcal{R}_{u\overline{v}} + \mathcal{R}_{v\overline{u}} + \mathcal{R}_{v\overline{v}}$ .... p. 17
s_F^+ (resp. s_F^-) .... p. 2
T^m .... p. 4
$\mathbf{T}_u^{*1,0}$ (resp. $\mathbf{T}_v^{*1,0}$) .... p. 5
$\mathfrak{T}(\zeta,\zeta)$, $\mathfrak{W}(\zeta,\zeta)$, $\mathfrak{wt}(\zeta,\zeta)$ .... p. 21
$\mathfrak{T}'_F(\zeta'',\zeta'')$, $\mathfrak{W}'_F(\zeta'',\zeta'')$ .... p. 23
$\mathfrak{T}_F(\zeta,\zeta)$ .... p. 21
Tr_g .... p. 17
T_{q-1} (resp. S_q) .... p. 12
Θ .... p. 16
u^i (resp. v^j) .... p. 4
$z = (u,v)$ .... p. 4
$\cdot^\vee$ .... p. 16
$|\cdot|_{g,\eta}$ .... p. 9
$\|\cdot\|_{K_c,\chi}$ (resp. $\langle\cdot,\cdot\rangle_{K_c,\chi}$) .... p. 9
$\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_3$ .... p. 12

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